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COMPARATIVE STUDY OF WAVELET METHODS FOR SOLVING BERNOULLI'S EQUATIONS

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Abstract

A comparative study of two numerical techniques is presented for solving nonlinear differential equations of the Bernoulli's type. Proposed techniques are based on the conversion of nonlinear differential equations into linear differential equations by substituting particular factor and utilization of Haar wavelet collocation method *(HWCM)* and Hermite wavelet collocation method *(HeWCM)* to these linear equations. Searching for numerical solutions of such equations has attracted a considerable amount of research work where computer symbolic systems facilitate the computational work.
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1 Introduction

As nonlinear differential equations have many applications in real life problems, several numerical methods have been developed like Adomian decomposition method *(ADM)*, Homotopy Perturbation method *(HPM)*, Homotopy Analysis method *(HAM)*, Laplace transform method, B-splines methods and finite difference methods *(FDM)*. All these methods have huge procedure for solving nonlinear differential equations. Wavelets are more powerful tools for solving differential as well as integral equations in comparison to pre-existing classical methods. Numerical solutions of inverse euler-Bernoulli problem with integral overdetermination and periodic boundary conditions have been presented in [1]. In [2], Chebyshev collocation method has been presented for solving Voltra-Fredholm integro-differential equations. New *ICI* self-cancellation technique has been investigated to mitigate the effect of *ICI* in *FFT-OFDM* and compared with *DCT* based *OFDM* system in terms of bit error rate (*BER*) and carrier to interference ratio (*CIR*) in [3]. Problem of determining the time-dependent leading coefficient to the time derivative of heat equation with nonlocal boundary and integral conditions has been discussed in [4]. In [5], a constructive approach has been developed for solving system of linear and nonlinear fractional differential equations with the help of modified differential transform method and Adomian polynomials. Haar wavelet is simplest and more reliable in comparison to other members of wavelet family. Haar wavelets are not applied directly for solving differential equations due to some shortcomings and these shortcomings were removed by regularizing the piecewise constant Haar functions with interpolation splines [8, 9] or by expanding the highest derivative appearing in the differential equation into the Haar series and other derivatives are obtained through integrations [10, 11]. The first possibility was discarded because by using this technique, it is difficult to find the solution easily and simplicity of Haar wavelets gets lost. Haar wavelet based numerical schemes have been discussed in [6, 14, 15, 18, 19, 20] for solving differential and integral equations. In [7, 12, 13, 16, 17, 21], numerical techniques based on Hermite wavelet and collocation points have been discussed for solving various variety of differential and integral equations.

The main objective of this research, is to compare the two wavelets based numerical techniques for solving differential equations of the Bernoulli type. For this purpose Haar and Hermite wavelets are utilized. The mathematical formulation of such differential equation is:

$$\frac{dy}{dx} + P(x)y = Q(x).F(y), \tag{1.1}$$

where $F(y)$ is nonlinear function in y. The initial conditions is $y(0) = a$, a is constant.

2 Haar wavelets and their operational matrices

Haar functions are an orthogonal family of switched rectangular waveforms where amplitudes can differ from one function to another. Haar wavelet is a sequence of rescaled square shaped functions which together forms a wavelet family or basis. The Haar wavelet function $h_i(x)$ is defined in the interval $[\alpha, \gamma]$ as

$$(2.1)\quad h_i(x) = \begin{cases} 1, & \alpha \le x < \beta, \\ -1, & \beta \le x < \gamma, \\ 0, & \text{elsewhere}, \end{cases}$$

where $\alpha = \frac{k}{m}$, $\beta = \frac{k+0.5}{m}$, $\gamma = \frac{k+1}{m}$, $m = 2^j$ and $j = 0, 1, 2, ..., J$. J denotes the level of resolution. The integer $k = 0, 1, 2, ..., m-1$ is the translation parameter. The index i is calculated as: $i = m + k + 1$. The minimal value of $i = 2$ and the maximal value of $i = 2^{j+1}$.
The collocation points are calculated as

$$(2.2)\quad x_l = \frac{l - 0.5}{2M}, \quad l = 1, 2, 3, ..., 2M.$$

The operational matrix P, which is $2M \times 2M$, is calculated as below

$$(2.3)\quad P_{1,i}(x) = \int_0^x h_i(x)dx,$$

and

$$(2.4)\quad P_{n+1,i}(x) = \int_0^x P_{n,i}(x)dx, \quad n = 1, 2, 3, ...$$

From (2.3), we obtain:

$$(2.5)\quad P_{1,i}(x) = \begin{cases} x - \alpha, & \alpha \le x < \beta, \\ \gamma - x, & \beta \le x < \gamma, \\ 0, & \text{elsewhere}, \end{cases}$$

3 Hermite wavelets and their operational matrices

Wavelets constitute a family of functions from dilation and translation of a single function known as mother wavelet. The continuous variation of dilation parameter α and translation parameter β, form a family of continuous wavelets as:

$$(3.1)\quad \psi_{\alpha,\beta}(x) = |\,\alpha\,|^{-\frac{1}{2}}\psi\Big(\frac{x-\beta}{\alpha}\Big), \quad \alpha, \beta \in R, \quad \alpha \neq 0,$$

if the dilation and translation parameters are restricted to discrete values by setting $\alpha = {\alpha_0}^{-k}, \beta = n\beta_0{\alpha_0}^{-k}$, $\alpha_0 > 1$, $\beta_0 > 0$, we obtain the following family of discrete wavelets:

$$(3.2)\quad \psi_{k,n}(x) = |\,\alpha\,|^{-\frac{1}{2}}\psi({\alpha_0}^k x - n\beta_0), \quad \alpha, \beta \in R, \quad \alpha \neq 0,$$

where $\psi_{k,n}$, form a wavelet basis for $L^2(R)$. For special case, if $\alpha_0 = 2$ and $\beta_0 = 1$, then $\psi_{k,n}(x)$ forms an orthonormal basis. Hermite wavelets are defined as:

$$(3.3)\quad \psi_{n,m}(x) = \begin{cases} \frac{2^{\frac{k+1}{2}}}{\sqrt{\pi}} H_m(2^k m - 2n + 1), & \frac{n-1}{2^{k-1}} \le x < \frac{n}{2^{k-1}}, \\ 0, & \text{Otherwise}, \end{cases}$$

where $m = 0, 1, 2, 3, ..., M - 1$ and $n = 1, 2, 3, ..., 2^{k-1}$ and k is assumed any positive integer. Also, H_m are Hermite polynomials of degree m with respect to weight function $W(x) = \sqrt{1 - x^2}$ on the real line R and satisfies the following recurrence relation

$$(3.4)\quad H_{m+2}(x) = 2xH_{m+1}(x) - 2(m + 1)H_m(x),$$

where $m = 0, 1, 2, ...,$ $H_0(x) = 1$ and $H_1(x) = 2x$.

3.1 Operational matrices of integration[21]

For $k = 1$ and $M = 6$, Assume the six basis functions on $[0, 1]$ as:

$$(3.5)\quad \begin{cases} \psi_{1,0}(x) = \frac{2}{\sqrt{\pi}}, \\ \psi_{1,1}(x) = \frac{2}{\sqrt{\pi}}(4x - 2), \\ \psi_{1,2}(x) = \frac{2}{\sqrt{\pi}}(16x^2 - 16x + 2), \\ \psi_{1,3}(x) = \frac{2}{\sqrt{\pi}}(64x^3 - 96x^2 + 36x - 2), \\ \psi_{1,4}(x) = \frac{2}{\sqrt{\pi}}(256x^4 - 512x^3 + 320x^2 - 64x + 2), \\ \psi_{1,5}(x) = \frac{2}{\sqrt{\pi}}(1024x^5 - 2560x^4 + 2240x^3 - 800x^2 + 100x - 2). \end{cases}$$

Let $\psi_6(x) = [\psi_{1,0}(x), \psi_{1,1}(x), \psi_{1,2}(x), \psi_{1,3}(x), \psi_{1,4}(x), \psi_{1,5}(x)]^T$. Integrating the above equations with respect to x, from 0 to x and after expressing in the matrix form, we obtain

$$(3.6)\quad \begin{cases} \int_0^x \psi_{1,0}(x)dx = \frac{2}{\sqrt{\pi}}x = \Big[\frac{1}{2}, \frac{1}{4}, 0, 0, 0, 0\Big]\psi_6(x), \\ \int_0^x \psi_{1,1}(x)dx = \frac{2}{\sqrt{\pi}}(2x^2 - 2x) = \Big[-\frac{1}{4}, 0, \frac{1}{8}, 0, 0, 0\Big]\psi_6(x), \\ \int_0^x \psi_{1,2}(x)dx = \frac{2}{\sqrt{\pi}}(\frac{16}{3}x^3 - 8x^2 + 2x) = \Big[-\frac{1}{3}, -\frac{1}{4}, 0, \frac{1}{12}, 0, 0\Big]\psi_6(x), \\ \int_0^x \psi_{1,3}(x)dx = \frac{2}{\sqrt{\pi}}(16x^4 - 32x^3 + 18x^2 - 2x) = \Big[\frac{1}{8}, 0, -\frac{1}{8}, 0, \frac{1}{16}, 0\Big]\psi_6(x), \\ \int_0^x \psi_{1,4}(x)dx = \frac{2}{\sqrt{\pi}}(\frac{256}{5}x^5 - \frac{512}{4}x^4 + \frac{320}{3}x^3 - 32x^2 + 2x) = \Big[-\frac{1}{15}, 0, 0, -\frac{1}{12}, 0, \frac{1}{20}\Big]\psi_6(x), \\ \int_0^x \psi_{1,5}(x)dx = \frac{2}{\sqrt{\pi}}(\frac{512}{3}x^6 - \frac{2560}{5}x^5 + \frac{1120}{2}x^4 - \frac{800}{3}x^3 + 50x^2 - 2x) = \Big[\frac{1}{24}, 0, 0, 0, -\frac{1}{16}, 0\Big]\psi_6(x). \end{cases}$$

Therefore,

$$(3.7)\quad \int_0^x \psi_6(x)dx = P_{6\times6}\psi_6(x) + \overline{\psi_6}(x),$$

where

$$(3.8)\quad P_{6\times6} = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & 0 & 0 & 0 & 0 \\ -\frac{1}{4} & 0 & \frac{1}{8} & 0 & 0 & 0 \\ -\frac{1}{3} & -\frac{1}{4} & 0 & \frac{1}{12} & 0 & 0 \\ \frac{1}{8} & 0 & -\frac{1}{8} & 0 & \frac{1}{16} & 0 \\ -\frac{1}{15} & 0 & 0 & -\frac{1}{12} & 0 & \frac{1}{20} \\ \frac{1}{24} & 0 & 0 & 0 & -\frac{1}{16} & 0 \end{pmatrix}$$

and

$$(3.9)\quad \bar{\psi_6}(x) = \Big(0, 0, 0, 0, 0, \frac{1}{24}\psi_{1,6}(x)\Big)^T.$$

Similarly integrating (3.7) with respect to x, from 0 to x, we obtain

$$(3.10)\quad \int_0^x \int_0^x \psi_6(x)dxdx = Q_{6\times6}\psi_6(x) + \bar{\bar{\psi_6}}(x),$$

where

$$(3.11)\quad Q_{6\times6} = \begin{pmatrix} \frac{3}{16} & \frac{1}{8} & \frac{1}{32} & 0 & 0 & 0 \\ -\frac{1}{6} & -\frac{3}{32} & 0 & \frac{1}{96} & 0 & 0 \\ -\frac{3}{32} & -\frac{1}{12} & -\frac{1}{24} & 0 & \frac{1}{192} & 0 \\ \frac{1}{10} & \frac{1}{16} & 0 & -\frac{1}{64} & 0 & \frac{1}{320} \\ -\frac{1}{24} & -\frac{1}{60} & \frac{1}{96} & 0 & -\frac{1}{120} & 0 \\ \frac{1}{42} & \frac{1}{96} & 0 & \frac{1}{192} & 0 & -\frac{1}{192} \end{pmatrix}$$

and

$$(3.12)\quad \bar{\bar{\psi_6}}(x) = \Big(0, 0, 0, 0, \frac{1}{480}\psi_{1,6}(x), \frac{1}{672}\psi_{1,7}(x)\Big)^T.$$

4 Function Approximation

4.1 Haar wavelet method

Consider any square integrable function $y(x)$ can be expanded in terms of infinite series of Haar basis functions as:

$$(4.1)\quad y(x) = \sum_{i=1}^{\infty} a_i h_i(x),$$

where a_i are constants of this infinite series, known as Haar wavelet coefficients. For numerical approximation the above infinite series is truncated upto finite terms as:

$$(4.2)\quad y(x) = \sum_{i=1}^{2M} a_i h_i(x) = A^T h(x),$$

where A and $h(x)$ are $2M \times 1$ matrices and are given by

$$(4.3)\quad A^T = [a_1, a_2, ..., a_{2M}],$$

and

$$(4.4)\quad h(x) = [h_1(x), h_2(x), ..., h_{2M}(x)]^T.$$

4.2 Hermite wavelet method

Consider any square integrable function $u(x)$ can be expanded in terms of infinite series of Hermite basis functions as:

$$(4.5)\quad u(x) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} C_{n,m}\psi_{n,m}(x),$$

where $C_{n,m}$ are constants of this infinite series, known as Hermite wavelet coefficients. For numerical approximation the above infinite series is truncated upto finite terms as:

$$(4.6)\quad u(x) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} C_{n,m}\psi_{n,m}(x) = C^T\Psi(x),$$

where C and Ψ are $2^{k-1}M \times 1$ matrices and are given by

$$(4.7)\quad C^T = [C_{1,0}, ..., C_{1,M-1}, ..., C_{2^{k-1},0}, ..., C_{2^{k-1},M-1}]$$

and

$$(4.8)\quad \Psi = [\psi_{1,0}, ..., \psi_{1,M-1}, ..., \psi_{2^{k-1},0}, ..., \psi_{2^{k-1},M-1}]^T.$$

5 Proposed methods for solving Bernoulli's equation

Consider the Bernoulli's equation

(5.1) $\frac{dy}{dx} + P(x)y = Q(x).F(y),$

where $F(y)$ is a nonlinear term. Above equation (5.1) is nonlinear differential equation. Divide (5.1) with function $F(y)$. The transformed equation is

(5.2) $F_1(y)\frac{dy}{dx} + P(x)F_2(y) = Q(x),$

where F_1, F_2 are functions of y. Putting $F_2(y) = z$ in (5.2), we get

(5.3) $\frac{dz}{dx} + P_1(x)z = Q_1(x),$

where P_1 and Q_1 are new functions of x. Equation (5.3) is linear differential equation with initial condition $z(0) = b$, where b is constant.

5.1 Haar wavelet collocation method

(*HWCM*) Consider the wavelet approximation

(5.4) $\frac{dz}{dx} = \sum_{i=1}^{2M} a_i h_i(x).$

Integrating (5.4) with respect to x, from 0 to x, we get

(5.5) $z(x) = z(0) + \sum_{i=1}^{2M} a_i P_{1,i}(x).$

Substituting (5.4) and (5.5) in (5.3), and applying initial conditions, we get

(5.6) $\sum_{i=1}^{2M} a_i\Big[h_i(x) + P_1(x)P_{1,i}(x)\Big] = Q_1(x) - b.P_1(x).$

From (5.6), we get Haar wavelet coefficient. The Haar wavelet solution $z(x)$ is obtained by substituting the values of wavelet coefficients into (5.5). The solution of (5.1) is obtained from the relation $y = {F_2}^{-1}(z)$.

5.2 Hermite wavelet collocation method

(HeWCM) Consider the wavelet approximation

(5.7) $\frac{dz}{dx} = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} C_{n,m}\psi_{n,m}(x) = C^T\Psi(x),$

Integrating (5.7) with respect to x, from 0 to x, we get

(5.8) $z(x) = z(0) + \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} C_{n,m}\int_0^x \psi_{n,m}(x)dx.$

Substituting (5.7) and (5.8) in (5.3), and applying initial conditions, we get

(5.9) $\sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} C_{n,m}\Big[\psi_{n,m}(x) + P_1(x)\int_0^x \psi_{n,m}(x)dx\Big] = Q_1(x) - b.P_1(x).$

From (5.9), we get Hermite wavelet coefficient. The Hermite wavelet solution $z(x)$ is obtained by substituting the values of wavelet coefficients into (5.8). The solution of (5.1) is obtained from the relation $y = {F_2}^{-1}(z)$.

6 Numerical Observations

We present here, numerical examples for solving some nonlinear differential equations, to illustrate the accuracy of the proposed method with the aid of two efficient techniques such as Haar wavelet method and Hermite wavelet method.

Example 6.1: Consider the nonlinear differential equation

(6.1) $\frac{dy}{dx} + x\sin 2y = x^3\cos^2 y,$

with initial condition $y(0) = 0$. The exact solution of the equation is

(6.2) $\tan y = \frac{1}{2}(x^2 - 1) + \frac{1}{2}e^{-x^2}.$

Table **6.1:** Numerical solutions of ***Example*** 6.1 for $2M = 8$.

x	Exact solution	Haar wavelet solution	Hermite wavelet solution
1/16	$3.8097e-006$	$1.5141e-005$	$4.5026e-006$
3/16	$3.0540e-004$	$4.3214e-004$	$3.0590e-004$
5/16	$2.3084e-003$	$2.6383e-003$	$2.3089e-003$
7/16	$8.6014e-003$	$9.1744e-003$	$8.6019e-003$
9/16	$2.2581e-002$	$2.3382e-002$	$2.2581e-002$
11/16	$4.7963e-002$	$4.8928e-002$	$4.7964e-002$
13/16	$8.8234e-002$	$8.9264e-002$	$8.8234e-002$
15/16	$1.4602e-001$	$1.4701e-001$	$1.4602e-001$

Table **6.2:** Comparison of absolute errors of ***Example*** 6.1 for $2M = 8$.

x	Absolute errors for Haar wavelet	Absolute errors for Hermite wavelet
1/16	$1.1331e-005$	$6.9287e-007$
3/16	$1.2674e-004$	$4.9578e-007$
5/16	$3.2985e-004$	$5.0176e-007$
7/16	$5.7292e-004$	$4.4160e-007$
9/16	$8.0075e-004$	$4.0076e-007$
11/16	$9.6413e-004$	$3.2832e-007$
13/16	$1.0300e-003$	$3.0043e-007$
15/16	$9.8735e-004$	$1.0734e-007$

***Table* 6.1** represents the comparison of numerical solutions obtained by Haar and Hermite wavelet methods with exact solution of ***Example* 6.1**. ***Table* 6.2** represents the comparison of absolute errors obtained by Haar wavelet method and Hermite wavelet method. ***Figure* 6.1** and ***Figure* 6.2** show the absolute errors of ***Example* 6.1** for $2M = 8$.

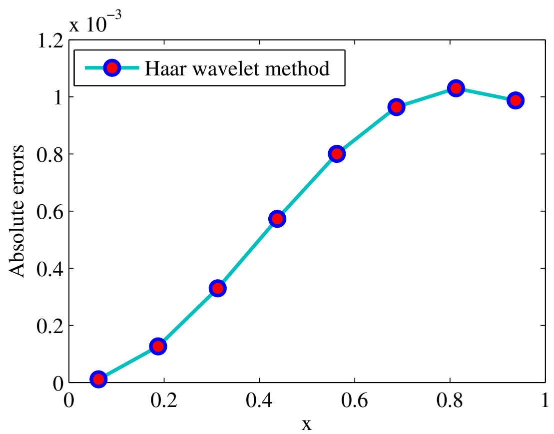

Figure **6.1:** Absolute errors of ***Example* 6.1** for $2M = 8$.

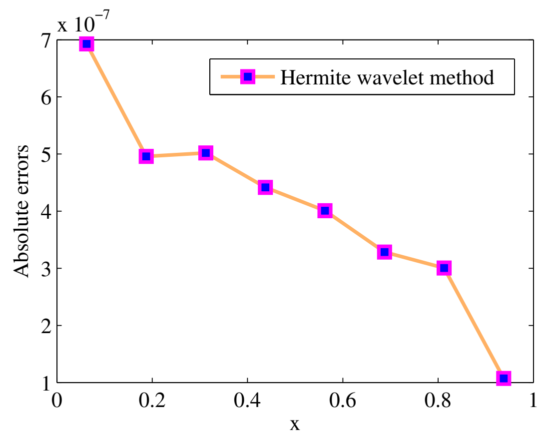

Figure **6.2:** Absolute errors of ***Example* 6.1** for $2M = 8$.

***Example* 6.2**: Consider the nonlinear differential equation

$$(6.3)\quad e^y\left(\frac{dy}{dx} + 1\right) = e^x,$$

with initial condition $y(0) = 0$. The exact solution of the equation is

$$(6.4)\quad e^y = \frac{1}{2}(e^x + e^{-x}).$$

Table **6.3:** Numerical solutions of ***Example* 6.2** for $2M = 8$.

x	Exact solution	Haar wavelet solution	Hermite wavelet solution
1/16	$1.9519e-003$	$3.7866e-003$	$1.9519e-003$
3/16	$1.7476e-002$	$1.9089e-002$	$1.7476e-002$
5/16	$4.8053e-002$	$4.9471e-002$	$4.8053e-002$
7/16	$9.2797e-002$	$9.4047e-002$	$9.2797e-002$
9/16	$1.5050e-001$	$1.5161e-001$	$1.5050e-001$
11/16	$2.1977e-001$	$2.2076e-001$	$2.1977e-001$
13/16	$2.9910e-001$	$3.0000e-001$	$2.9910e-001$
15/16	$3.8703e-001$	$3.8786e-001$	$3.8703e-001$

Table 6.4: Comparison of absolute errors of ***Example*** **6.2** for $2M = 8$.

x	Absolute errors for Haar wavelet	Absolute errors for Hermite wavelet
1/16	$1.8348e-003$	$1.1023e-010$
3/16	$1.6130e-003$	$6.6898e-011$
5/16	$1.4174e-003$	$6.3810e-011$
7/16	$1.2496e-003$	$5.1481e-011$
9/16	$1.1095e-003$	$4.5878e-011$
11/16	$9.9544e-004$	$3.6280e-011$
13/16	$9.0478e-004$	$3.7286e-011$
15/16	$8.3425e-004$	$5.4857e-012$

Table **6.3** represents the comparison of numerical solutions obtained by Haar and Hermite wavelet methods with exact solution of ***Example*** **6.2**. ***Table*** **6.4** represents the comparison of absolute errors obtained by Haar wavelet method and Hermite wavelet method. ***Figure*** **6.3** and ***Figure*** **6.4** show the absolute errors of ***Example*** **6.2** for $2M = 8$.

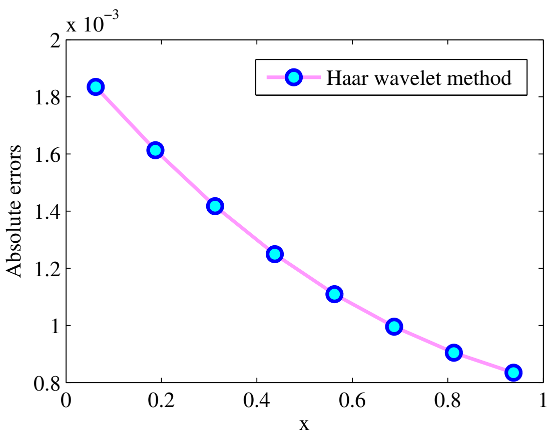

Figure **6.3**: Absolute errors of ***Example*** 6.2 for $2M = 8$.

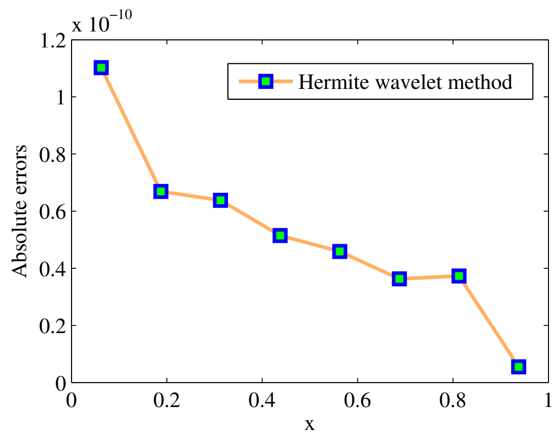

Figure **6.4**: Absolute errors of ***Example*** 6.2 for $2M = 8$.

Example **6.3**: Consider the nonlinear differential equation

$$(6.5)\quad xy(1 + xy^2)\frac{dy}{dx} = 1,$$

with initial condition $x(0) = 1$. The exact solution of the equation is

$$(6.6)\quad \frac{1}{x} = (2 - y^2) - e^{-\frac{y^2}{2}}.$$

Table 6.5: Numerical solutions of ***Example*** **6.3** for $2M = 8$.

y	Exact solution	Haar wavelet solution	Hermite wavelet solution
1/16	1.0020	1.0039	1.0020
3/16	1.0181	1.0201	1.0181
5/16	1.0526	1.0549	1.0526
7/16	1.1113	1.1139	1.1113
9/16	1.2049	1.2081	1.2049
11/16	1.3553	1.3594	1.3553
13/16	1.6104	1.6162	1.6104
15/16	2.0977	2.1075	2.0977

Table **6.6**: Comparison of absolute errors of ***Example*** **6.3** for $2M = 8$.

y	Absolute errors for Haar wavelet	Absolute errors for Hermite wavelet
1/16	$1.9627e-003$	$7.1279e-008$
3/16	$2.0568e-003$	$5.3094e-008$
5/16	$2.2577e-003$	$5.9452e-008$
7/16	$2.5976e-003$	$6.1041e-008$
9/16	$3.1475e-003$	$6.9398e-008$
11/16	$4.0705e-003$	$7.7959e-008$
13/16	$5.7923e-003$	$1.1056e-007$
15/16	$9.7275e-003$	$8.6224e-008$

Table **6.5** represents the comparison of numerical solutions obtained by Haar and Hermite wavelet methods with exact solution of ***Example*** **6.3**. ***Table*** **6.6** represents the comparison of absolute errors obtained by Haar wavelet method and Hermite wavelet method. ***Figure*** **6.5** and ***Figure*** **6.6** show the absolute errors of ***Example*** **6.3** for $2M = 8$.

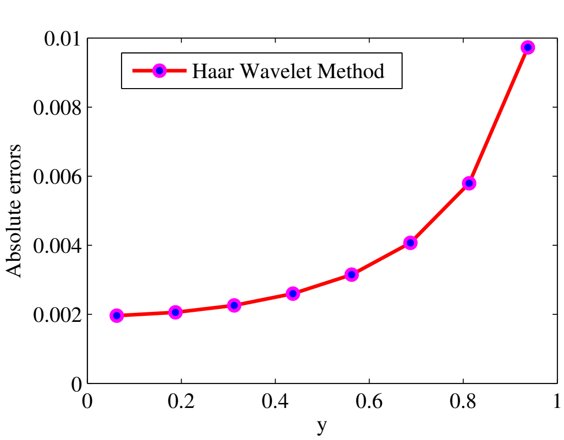

Figure **6.5**: Absolute errors of ***Example*** **6.3** for $2M = 8$.

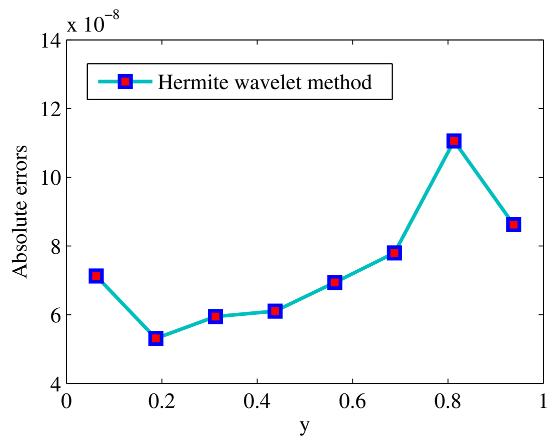

Figure **6.6**: Absolute errors of ***Example*** **6.3** for $2M = 8$.

7 Conclusion

From above discussion, it is concluded that the Hermite wavelet based collocation is much better in comparison to Haar wavelet based collocation method for solving nonlinear differential equations of the Bernoulli's type. For getting the necessary accuracy the number of collocation points may be increased.

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