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*(Dedicated to Honor Dr. R. C. Singh Chandel on His $75^{th}$ Birth Anniversary Celebrations)*

# AN UNRELIABLE BATCH ARRIVAL *G*-QUEUE WITH WORKING VACATION, VACATION INTERRUPTION AND MULTI-OPTIONAL SERVICES

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**Abstract**

This paper predicts the performance of an unreliable $M^X/G/1$ $G$-queue with delayed repair. The negative customers stop functioning of the server and force the server to undergo repair. The failed server takes a random amount of time called **delay time** before going for repair. It is assumed that the positive customers arrive in batches and negative customers arrive singly, according to Poisson process. The server provides first phase of regular service to all arriving customers whereas it provides $l$ types of optional services to only those who demand the same. The working vacation period of the server starts either if queue becomes empty or repairing of the server finishes. Numerical experiments are provided to show the effects of various critical system parameters on performance measures.

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## 1 Introduction

This work is motivated by modeling general service non-Markovian queueing systems having two types of customers, ***positive and negative***, depending upon their nature. The positive customers are usual customers which enter the queue for receiving service if it not immediate, otherwise depart from the system after getting service, while negative customers are those who remove the positive customer in service and force the server for immediate repair. There are many real life congestion situations in which we found these type of customers. For example, in distributed computer systems or databases, there are some commands which delete some transaction because of locking of data or because of inconsistency. In the paper by Gelenbe [3], the author provides analogy of queueing networks with neural networks wherein each queue represents a neuron and customers represent excitation (positive) or inhibition (negative) signals. This type of queueing system which involves negative customers is termed as **"*G*-queues"**. Applications of these types of queues can be found in computer networks, data communication systems, distributed systems, manufacturing systems, neural networks and many more. A survey on $G$-queues has been presented by Do [2]. Some useful performance characteristics of an unreliable $M/G/1$ retrial $G$-queue under priority scheme have been obtained by Wu and Lian [8]. Further Peng et al. [5] examined the performance characteristics of an unreliable $M/G/1$ retrial G-queue with preemptive resume priority and collisions under delayed repairs. Recently, Kirupa and Chandrika [4] throw light on the performance prediction of $M^X/G/1$ $G$-queue with heterogeneous service, setup time and reserved time.

The concept of working vacation was introduced by Servi and Finn [7], according to which server may provide service during vacation with comparatively slower rate rather than completely stops service during vacation as found in the past literature. Working vacation queues with negative arrivals find wide applicability in the working of computer networks, web servers, file transfer systems and email servers. Do et al. [1] studied an $M/M/1$ retrial $G$-queue with working vacation. Further, using $SVT$ (Supplementary Variable Technique), Rajadurai et al. [6] examined the performance of an $M/G/1$ retrial queue with balking under working vacation policy. Zhang and Liu [9] investigated the behavior of an $M/G/1$ queue with negative arrivals and server breakdowns, working vacation and vacation interruption, using both $SVT$ and matrix analytic method.

Due to scarcity of work done on batch arrival $G$-queues with working vacation and vacation interruption, we put forward our effort to analysze it. The rest of the work done is as follows. Section 2 describes the model by stating requisite assumptions and notations. The queue size distribution has been explored in ***Section* 3** via probability generating function method. Various useful performance measures of our model are explored in ***Section* 4** using queue size distribution obtained in ***Section* 3**. In ***Section* 5**, we show that the stochastic decomposition results holds good for the developed model. Some special cases have been deduced in ***Section* 6**. The cost function is constructed in ***Section* 7**. A numerical example and the effect of some sensitive parameters on various performance measures are explored in ***Section* 8**. Finally concluding remarks and future scope have been outlined in ***Section* 9**.

## 2 System Description

Consider a single server batch arrival $G$-queue with unreliable server and delayed repair. The server renders first phase of essential service to all the arriving customers whereas it renders any type of $l$ optional services to only those who demand for the same. The server may undergo breakdown due to the presence of a negative arrival. To formulate the model, we make following assumptions as given below:

The arrival stream is composed of two types of customers' positive customers and negative customers. The positive customers arrive in groups or batches with rate $p^+$. The batch size is an arbitrary distributed independent random variable denoted by $\gamma$ with probability distribution $Pr\{\gamma = k\} = \epsilon_k,\ k \geq 0,\ pgf$ (probability generating function) $\gamma(z) = \sum_{k=1}^{\infty} \epsilon_k z^k$ and first and second finite moments $\epsilon_1$ and $\epsilon_2$. The arrival of a negative customer with rate $p^-$ not only removes the positive customer in service but also leads the server to undergo repair. The failed server waits for the repair to start; this waiting time of the server is called as ***delay time*** which follows a general distribution with distribution function $D_i(x)$, *LST* (Laplace Steiljes Transform) $D_i^*(s)$ and $r^{th}$ factorial moments $\omega_i^{(r)}(r \geq 1)$, where $0 \leq i \leq l$. The repair time of the server is assumed to be arbitrarily distributed random variable with distribution function $\Im_i(x)$, *LST* (Laplace Steiljes Transform) $\Im_i^*(s)$ and $r^{th}$ factorial moments $\omega_i^{(r)}(r \geq 1)$, where $0 \leq i \leq l$. The negative customer affects the server if and only if it is busy. The server renders first phase of essential service denoted by *FES* to all positive customers whereas only few of them will receive second phase of optional service *(SOS)* based upon their demand. After finishing *FES* by the customer, he may demand for any of $l$ different kinds of optional services with probability $\zeta_i(1 \leq i \leq l)$, otherwise he leaves the service area with complementary probability $\bar{\zeta}_i(= 1 - \zeta_i), 1 \leq i \leq l$. The service discipline is *FCFS* (first come first served). The service times during *FES* and *SOS* are *i.i.d.* (independent and identically distributed) random variables having probability distribution $E_0(x)$ and $E_i(x)$; *LST* $E_0{}^*(x)$ and $E_i{}^*(x)$ and $r^{th}$ factorial moments $\zeta_0^{(r)}$ and $\zeta_i^{(r)}(r \geq 1)(0 \leq i \leq l)$, respectively.

During regular busy period, the server serves the positive customers with rate $\mu_B$. After each service completion or repair completion, the server enters into a working vacation period of random length *'W'* which follows an exponential distribution with parameter $\nu$. When the server finds that the queue is non-empty, it immediately changes the service rate from $\mu_\nu$ to $\mu_B$ and initiates a regular busy period no matter whether or not the vacation has ended. The vacation time is assumed to be *i.i.d* random variable with probability distribution $H(x)$; *LST* $H^*(x)$ and $r^{th}$ factorial moments $\vartheta^{(r)}$ with $r \geq 1$. The service discipline during both regular busy period and working vacation period is *FCFS* (first come first served). We assume that the hazard rates for service time, delay time and repair time are $\Psi_i(x),\ \eta_i(x)$ and $\phi_i(x)(0 \leq i \leq l)$, respectively whereas that of working vacation time is $\gamma(x)$. So we can define service time, delay time, repair time and working vacation time distribution by

$$E_i(x) = 1 - \exp\left\{-\int_0^x \varphi_i(t)dt\right\},\ 0 \leq i \leq l;\ D_i(x) = 1 - \exp\left\{-\int_0^x \eta_i(t)dt\right\};\ 0 \leq i \leq l;$$

$$\Im_i(x) = 1 - \exp\left\{-\int_0^x \phi_i(t)dt\right\};\ 0 \leq i \leq l;$$

$$H(x) = 1 - \exp\left\{-\int_0^x \gamma(t)dt\right\}.$$

The *LST* for service time, delay time, repair time and working vacation time distribution are defined as

$$E_i^*(s) = \int_0^\infty \exp\{-sx\}dE_i(x),\ D_i^*(x) = \int_0^\infty \exp\{-sx\}dD_i(x),\ \Im_i^*(s) = \int_0^\infty \exp\{-sx\}d\Im_i(x),$$

$$H_\nu^*(s) = \int_0^\infty \exp\{-sx\}dH_\nu(x);\ 0 \leq i \leq l.$$

Also, where, '$A$' denotes random variable for either service time or repair time or vacation time distribution.

## 3 Queue Size Distribution

At time $t \geq 0$, we define the system state by forming a Markov process $\lambda(t) = \{\xi(t),\ \pi(t),\ \kappa(t)\}$, , where

$$\pi(t) = \begin{cases} 0, \text{ if the server is in working vacation period at time } t \\ 1, \text{ if the server is busy with } FES \text{ at time } t \\ 2, \text{ if the server is busy with } SOS \text{ at time } t \\ 3, \text{ if the server is waiting for repair at time } t \\ 4, \text{ if the server is under repair at time } t. \end{cases}$$

Here $\xi(t)$ represents the number of customers in the queue. When $\pi(t) = 0$ and $\xi(t) \geq 0,\ \kappa(t)$ denotes the elapsed working vacation time of the server. If $\pi(t) = 1, 2$ and $\xi(t) \geq 0,\ \kappa(t)$ denotes the elapsed service time either during *FES* or during *SOS* at time $t$. If $\pi(t) = 3$ and $\xi(t) \geq 0,\ \kappa(t)$ denotes the elapsed delay time at time $t$. If $\pi(t) = 4$ and $\xi(t) \geq 0,\ \kappa(t)$ denotes the elapsed repair time at time $t$. In order to construct an embedded Markov chain of our developed model, we assume that $\xi(t_n^+)$ be the number of customers in the queue, respectively just after the time $t_n$. Then the sequence of random variables $\{M_n, n \in N\}$ forms an embedded Markov chain of our developed model.

***Lemma* 3.1** *The embedded Markov chain $\{M_n, n \in N\}$is ergodic iff $\rho < 1$, which is the necessary and sufficient condition for the stability of the system, where $\rho$ is given by*

(3.1) $\rho = p^+\varepsilon_1[\nu_0 + E_0^*(p^-)\sum_{i=1}^l \varsigma_i\nu_i + p^-\{\nu_0\varpi_0^{(1)} + E_0^*(p^-)\sum_{i=1}^l \varsigma_i\nu_i\varpi_i^{(1)} + \nu_0\omega_0^{(1)} + E_0^*(p^-)\sum_{i=1}^l \varsigma_i\nu_i\omega_i^{(1)}\}]$.

***Proof.*** To prove the sufficient condition one can use Foster's criterion. According to this, an irreducible and aperiodic Markov chain $M_n$with state space $\varphi$is ergodic if there exists a non-negative function $f(j),\ j \in \varphi$ called test function, and $\delta > 0$ such that the mean drift $N_j = E[f(M_{n+1}) - f(M_n)/M_n = j]$ is finite for all $j \in \varphi$ and $M_j \leq -\delta$ for all $j \in \varphi$ except perhaps a finite number. For this model, it is easy to show that $M_n$is irreducible and aperiodic. Further one can prove the necessary condition by proceeding in the same way as done by Zhang and Liu ([9], p. 262-263, ***Theorem* 3.1**).

Under the established stability condition, we define some limiting probabilities for the Markov process $\{M(t), t \geq 0\}$ as given below:

$H_n = \underset{t\to\infty}{Lim} Pr\{\Gamma(t) = 0, M(t) = n\}, n \geq 0,$

$E_{0,n}(x) = \underset{t\to\infty}{Lim} Pr\{\Gamma(t) = 1, M(t) = n, x \leq \Phi(t) \leq x + dx\}, n \geq 0,$

$E_{i,n}(x) = \underset{t\to\infty}{Lim} Pr\{\Gamma(t) = 2, M(t) = n, x \leq \Phi(t) \leq x + dx\}, n \geq 0, 1 \leq i \leq l,$

$D_{0,n}(x) = \underset{t\to\infty}{Lim} Pr\{\Gamma(t) = 3, M(t) = n, x \leq \Phi(t) \leq x + dx\}, n \geq 0,$

$D_{i,n}(x) = \underset{t\to\infty}{Lim} Pr\{\Gamma(t) = 3, M(t) = n, x \leq \Phi(t) \leq x + dx\}, n \geq 0, 1 \leq i \leq l,$

$\Im_{0,n}(x) = \underset{t\to\infty}{Lim} Pr\{\Gamma(t) = 4, M(t) = n, x \leq \Phi(t) \leq x + dx\}, n \geq 0,$

$\Im_{i,n}(x) = \underset{t\to\infty}{Lim} Pr\{\Gamma(t) = 4, M(t) = n, x \leq \Phi(t) \leq x + dx\}, n \geq 0, 1 \leq i \leq l.$

State Governing equations

(3.2) $\frac{d}{dx}H_n(x) = -\{p^+ + \theta + \varphi_V(x)\}H_n(x) + p^+(1 - \delta_{0,j})\sum_{k=1}^j \varepsilon_k H_{n-k}(x); n \geq 1,$

(3.3) $\frac{d}{dx}E_{0,n}(x) = -\{p + \varphi_0(x)\}E_{0,n}(x) + p^+(1 - \delta_{1,n})\sum_{k=1}^n \varepsilon_k E_{0,n-k}(x); n \geq 0,$

(3.4) $\frac{d}{dx}E_{i,n}(x) = -\{p + \varphi_i(x)\}E_{i,n}(x) + p^+(1 - \delta_{1,n})\sum_{k=1}^n \varepsilon_k E_{i,n-k}(x); n \geq 0;\ 1 \leq i \leq l,$

(3.5) $\frac{d}{dx}D_{0,n}(x) = -\{p^+ + \eta_0(x)\}D_{0,n}(x) + p^+(1 - \delta_{1,n})\sum_{k=1}^n \varepsilon_k D_{0,n-k}(x);\ n \geq 0,$

(3.6) $\frac{d}{dx}D_{i,n}(x) = -\{p^+ + \eta_i(x)\}D_{i,n}(x) + p^+(1 - \delta_{1,n})\sum_{k=1}^n \varepsilon_k D_{i,n-k}(x);\ n \geq 0,\ 1 \leq i \leq l,$

(3.7) $\frac{d}{dx}\Im_{0,n}(x) = -\{p^+ + \phi_0(x)\}\Im_{0,n}(x) + p^+(1 - \delta_{1,n})\sum_{k=1}^n \varepsilon_k \Im_{0,n-k}(x);\ n \geq 0,$

(3.8) $\frac{d}{dx}\Im_{i,n}(x) = -\{p^+ + \phi_i(x)\}\Im_{i,n}(x) + p^+(1 - \delta_{1,n})\sum_{k=1}^n \varepsilon_k \Im_{i,n-k}(x); n \geq 0;\ 1 \leq i \leq l.$

Boundary Conditions

(3.9) $p^+\varepsilon_1 H_0 = \int_0^\infty H_1(x)\varphi_V(x)dx + \varsigma_0\int_0^\infty E_{0,1}(x)\varphi_0(x)dx + \sum_{i=1}^l \bar{\varsigma}_i\int_0^\infty E_{i,n+1}(x)\varphi_i(x)dx +$
$\int_0^\infty \Im_{0,0}(x)\phi_0(x)dx + \sum_{i=1}^l \int_0^\infty \Im_{i,0}(x)\phi_i(x)dx,$

(3.10) $H_1(0) = p^+\varepsilon_1 H_0;\ H_n(0) = 0;\ n \geq 2,$

(3.11) $E_{0,n}(0) = \varsigma_0\int_0^\infty E_{0,n+1}(x)\varphi_0(x)dx + \sum_{i=1}^l \bar{\varsigma}_i\int_0^\infty E_{i,n+1}(x)\varphi_i(x)dx + \int_0^\infty H_{n+1}(x)\varphi_V(x)dx +$
$\theta\int_0^\infty H_n(x)dx + \int_0^\infty \Im_{0,n}(x)\phi_0(x)dx + \sum_{i=1}^l \int_0^\infty \Im_{i,n}(x)\phi_i(x)dx,$

(3.12) $E_{i,n}(0) = \varsigma_i\int_0^\infty E_{0,n}(x)\varphi_0(x)dx; 1 \leq i \leq l, j \geq 0, n \geq 1,$

(3.13) $D_{0,n}(0) = p^-\int_0^\infty E_{0,n+1}(x)dx;\ n] \geq 0,$

(3.14) $D_{i,n}(0) = p^-\int_0^\infty E_{i,n+1}(x)dx;\ 1 \leq i \leq l;\ n \geq 0,$

(3.15) $\Im_{0,n}(0) = \int_0^\infty D_{0,n}(x)\eta_0(x)dx;\ n \geq 0,$

(3.16) $\Im_{i,n}(0) = \int_0^\infty \Im_{i,n}(x)\eta_i(x)dx;\ n \geq 0,$

The normalizing condition of the system is given by

(3.17) $H_0 + \sum_{n=1}^\infty \int_0^\infty H_n(x)dx + \sum_{i=0}^l \sum_{n=1}^\infty \left[\int_0^\infty E_{i,n}(x)dx + \int_0^\infty D_{i,n}(x)dx + \int_0^\infty \Re_{i,n}(x)dx\right] = 1.$

The probability generating functions of sequences of the above probabilities are given by

$A(z) = H_V^*(p^+(1-\gamma(z)) + \theta); D(z) = \tilde{H}_V(p^+(1-\gamma(z)) + \theta)$

$E_i(z) = E_i^*(p - p^+\gamma(z)); V_i(z) = \tilde{E}_i(p - p^+\gamma(z)); F_i(z) = D_i^*(p^+(1-\gamma(z))); 0 \leq i \leq l.$

For solution purpose, we define following generating functions for different states of the server:

$H(x,z) = \sum_{n=1}^\infty H_n(x)z^n, E_i(x,z) = \sum_{n=0}^\infty E_{i,n}(x)z^n; 0 \leq i \leq l, D_i(x,z) = \sum_{n=0}^\infty D_{i,n}(x)z^n; 0 \leq i \leq l,$

$\Im_i(x,z) = \sum_{n=0}^\infty \Im_{i,n}(x)z^n; 0 \leq i \leq l.$

Using probability generating function method, we multiply each of the equations (3.2)-(3.8) by $z^n$ and then summing over all values of $n$, we get

(3.18) $\dfrac{\partial}{\partial x}H(x,z) = -\{p^+(1-\gamma(z)) + \theta + \varphi_V(x)\}H(x,z),$

(3.19) $\dfrac{\partial}{\partial x}E_i(x,z) = -\{p - p^+(1-\gamma(z)) + \varphi_i(x)\}E_i(x,z);\ 0 \leq i \leq l,$

(3.20) $\dfrac{\partial}{\partial x}D_i(x,z) = -\{p^+(1-\gamma(z)) + \eta_i(x)\}D_i(x,z);\ 0 \leq i \leq l,$

(3.21) $\dfrac{\partial}{\partial x}\Im_i(x,z) = -\{p^+(1-\gamma(z)) + \phi_i(x)\}\Im_i(x,z);\ 0 \leq i \leq l.$

On solving equations (3.17)-(3.20), we obtain

(3.22) $H(x,z) = H(0,z) \times \exp[-\{p^+(1-\gamma(z)) + \theta\}x]\overline{H}_V(x),$

(3.23) $E_i(x,z) = E_i(0,z) \times \exp[-\{p - p^+\gamma(z)\}x]\overline{E}_i(x);\ 0 \leq i \leq l,$

(3.24) $D_i(x,z) = D_i(0,z) \times \exp[-\{p^+(1-\gamma(z))\}x]\bar{D}_i(x);\ 0 \leq i \leq l,$

(3.25) $\Im_i(x,z) = \Im_i(0,z) \times \exp[-\{p^+(1-\gamma(z))\}x]\Im_i(x);\ 0 \leq i \leq l.$

Again multiplying each of the equations (3.8)-(3.15) by $z^n$ and then summing over all values of $n$, we get

(3.26) $H(0,z) = p^+\varepsilon_1 H_0 z,$

(3.27) $E_0(0,z) = \dfrac{\varsigma_0 E_0(z)E_0(0,z)}{z} + \sum_{i=1}^l \frac{E_i(z)E_i(0,z)}{z} + \frac{A(z)H(0,z)}{z} + \theta D(z)H(0,z) + E_0(z)\Im_0(0,z) - (\varsigma_0 \int_0^\infty E_{0,1}(x)\varphi_0(x)dx$
$+ \sum_{i=1}^l \bar{\varsigma}_i \int_0^\infty E_{i,1}(x)\varphi_i(x)dx + \int_0^\infty H_1(x)\varphi_V(x)dx) - (\int_0^\infty H_n(x)\varphi_V(x)dx + \int_0^\infty \Im_{0,0}(x)\phi_0(x)dx + \sum_{i=1}^l \int_0^\infty \Im_{i,0}(x)\phi_i(x)dx),$

(3.28) $E_i(0,z) = \varsigma_i E_0(0,z)E_i(z);\ 1 \leq i \leq l,$

(3.29) $D_i(0,z) = \dfrac{p^- V_i(z)E_i(0,z)}{z};\ 0 \leq i \leq l,$

(3.30) $\Im_i(0,z) = D_i(0,z)F_i(z);\ 0 \leq i \leq l.$

Now, substituting equations (3.25) and (3.27)-(3.29) in equation (3.26) and solving, we get

(3.31) $E_0(0,z) = \dfrac{p^+\varepsilon_1 z H_0[\theta(z-1) + p^+\{\gamma(z)-1\}]D(z)}{[z - \{\varsigma_0 + \sum_{i=1}^l \varsigma_i E_i(z)\}E_0(z) - p^-\{E_0(z)V_0(z)F_0(z) + E_0(z)\sum_{i=1}^l \varsigma_i E_i(z)V_i(z)F_i(z)\}]}.$

Using equation (3.30) equation (3.27) yields

(3.32) $E_i(0,z) = \dfrac{p^+\varepsilon_1\varsigma_i z H_0[\theta(z-1) + p^+\{\gamma(z)-1\}]D(z)E_0(z)}{[z - \{\varsigma_0 + \sum_{i=1}^l \varsigma_i E_i(z)\}E_0(z) - p^-\{E_0(z)V_0(z)F_0(z) + E_0(z)\sum_{i=1}^l \varsigma_i E_i(z)V_i(z)F_i(z)\}]} 1 \leq i \leq l.$

Applying (3.30)-(3.31) in equation (3.28), we get two equations given below for $i = 0$ and $1 \leq i \leq l$, respectively

(3.33) $D_0(0,z) = \dfrac{p^+p^-\varepsilon_1 H_0[\theta(z-1) + p^+\{\gamma(z)-1\}]D(z)V_0(z)}{[z - \{\varsigma_0 + \sum_{i=1}^l \varsigma_i E_i(z)\}E_0(z) - p^-\{E_0(z)V_0(z)F_0(z) + E_0(z)\sum_{i=1}^l \varsigma_i E_i(z)V_i(z)F_i(z)\}]},$

(3.34) $D_i(0,z) = \dfrac{p^+p^-\varepsilon_1\varsigma_i H_0[\theta(z-1) + p^+\{\gamma(z)-1\}]D(z)E_0(z)V_i(z)}{[z - \{\varsigma_0 + \sum_{i=1}^l \varsigma_i E_i(z)\}E_0(z) - p^-\{E_0(z)V_0(z)F_0(z) + E_0(z)\sum_{i=1}^l \varsigma_i E_i(z)V_i(z)F_i(z)\}]};\ 1 \leq i \leq l.$

Using (3.32)-(3.33) in equation (3.29), we get two equations given below for $i = 0$ and $1 \leq i \leq l$, respectively

(3.35) $\Im_0(0,z) = \dfrac{p^+p^-\varepsilon_1 H_0[\theta(z-1) + p^+\{\gamma(z)-1\}]D(z)V_0(z)F_0(z)}{[z - \{\varsigma_0 + \sum_{i=1}^l \varsigma_i E_i(z)\}E_0(z) - p^-\{E_0(z)V_0(z)F_0(z) + E_0(z)\sum_{i=1}^l \varsigma_i E_i(z)V_i(z)F_i(z)\}]},$

(3.36) $\Im_i(0,z) = \dfrac{p^+p^-\varepsilon_1\varsigma_i H_0[\theta(z-1) + p^+\{\gamma(z)-1\}]D(z)E_0(z)V_i(z)F_i(z)}{[z - \{\varsigma_0 + \sum_{i=1}^l \varsigma_i E_i(z)\}E_0(z) - p^-\{E_0(z)V_0(z)F_0(z) + E_0(z)\sum_{i=1}^l \varsigma_i E_i(z)V_i(z)F_i(z)\}]};\ 1 \leq i \leq l.$

***Lemma* 3.2** *The probability generating functions of the stationary joint distribution of the system size and server state of the Markov process under the established stability condition are given by*

$$H(x,z) = p^{+}\varepsilon_1 z H_0 \exp[-\{p^{+}(1-\gamma(z))+\theta\}x]\overline{H}_V(x),$$

$$E_0(x,z) = \frac{p^{+}\varepsilon_1 z H_0[\theta(z-1)+p^{+}\{\gamma(z)-1\}]D(z)}{[z-\{\varsigma_0+\sum_{i=1}^{l}\varsigma_i E_i(z)\}E_0(z)-p^{-}\{E_0(z)V_0(z)F_0(z)+E_0(z)\sum_{i=1}^{l}\varsigma_i E_i(z)V_i(z)F_i(z)\}]}\exp[-\{p-p^{+}\gamma(z)\}x]\overline{E}_0(x),$$

$$E_i(x,z) = \frac{p^{+}\varepsilon_1 \varsigma_i z H_0[\theta(z-1)+p^{+}\{\gamma(z)-1\}]D(z)E_0(z)}{[z-\{\varsigma_0+\sum_{i=1}^{l}\varsigma_i E_i(z)\}E_0(z)-p^{-}\{E_0(z)V_0(z)F_0(z)+E_0(z)\sum_{i=1}^{l}\varsigma_i E_i(z)V_i(z)F_i(z)\}]}\exp[-\{p-p^{+}\gamma(z)\}x]\overline{E}_i(x);\ 1\le i\le l,$$

$$D_0(x,z) = \frac{p^{+}p^{-}\varepsilon_1 H_0[\theta(z-1)+p^{+}\{\gamma(z)-1\}]D(z)V_0(z)}{[z-\{\varsigma_0+\sum_{i=1}^{l}\varsigma_i E_i(z)\}E_0(z)-p^{-}\{E_0(z)V_0(z)F_0(z)+E_0(z)\sum_{i=1}^{l}\varsigma_i E_i(z)V_i(z)F_i(z)\}]}\exp[-\{p^{+}(1-\gamma(z))\}x]\bar{D}_0(x),$$

$$D_i(x,z) = \frac{p^{+}p^{-}\varepsilon_1 \varsigma_i H_0[\theta(z-1)+p^{+}\{\gamma(z)-1\}]D(z)E_0(z)V_i(z)}{[z-\{\varsigma_0+\sum_{i=1}^{l}\varsigma_i E_i(z)\}E_0(z)-p^{-}\{E_0(z)V_0(z)F_0(z)+E_0(z)\sum_{i=1}^{l}\varsigma_i E_i(z)V_i(z)F_i(z)\}]}\exp[-\{p^{+}(1-\gamma(z))\}x]\bar{D}_i(x);\ 1\le i\le l,$$

$$\Im_0(x,z) = \frac{p^{+}p^{-}\varepsilon_1 H_0[\theta(z-1)+p^{+}\{\gamma(z)-1\}]D(z)V_0(z)F_0(z)}{[z-\{\varsigma_0+\sum_{i=1}^{l}\varsigma_i E_i(z)\}E_0(z)-p^{-}\{E_0(z)V_0(z)F_0(z)+E_0(z)\sum_{i=1}^{l}\varsigma_i E_i(z)V_i(z)F_i(z)\}]}\exp[-\{p^{+}(1-\gamma(z))\}x]\Im_0(x),$$

$$\Im_i(x,z) = \frac{p^{+}p^{-}\varepsilon_1 \varsigma_i H_0[\theta(z-1)+p^{+}\{\gamma(z)-1\}]D(z)E_0(z)V_i(z)F_i(z)}{[z-\{\varsigma_0+\sum_{i=1}^{l}\varsigma_i E_i(z)\}E_0(z)-p^{-}\{E_0(z)V_0(z)F_0(z)+E_0(z)\sum_{i=1}^{l}\varsigma_i E_i(z)V_i(z)F_i(z)\}]}\exp[-\{p^{+}(1-\gamma(z))\}x]\Im_i(x);\ 1\le i\le l.$$

***Lemma* 3.3** *The marginal probability generating functions of the system size when the server is on working vacation, busy with FES, busy with SOS, waiting for repair due to failure in FES, waiting for repair due to failure in SOS, under repair due to failure in FES and under repair due to failure in SOS, respectively are given by*

$$H(z) = (1+p^{+}\varepsilon_1 z)D(z)H_0,$$

$$E_0(z) = \frac{p^{+}\varepsilon_1 z H_0[\theta(z-1)+p^{+}\{\gamma(z)-1\}]D(z)V_0(z)}{[z-\{\varsigma_0+\sum_{i=1}^{l}\varsigma_i E_i(z)\}E_0(z)-p^{-}\{E_0(z)V_0(z)F_0(z)+E_0(z)\sum_{i=1}^{l}\varsigma_i E_i(z)V_i(z)F_i(z)\}]},$$

$$E_i(z) = \frac{p^{+}\varepsilon_1 \varsigma_i z H_0[\theta(z-1)+p^{+}\{\gamma(z)-1\}]D(z)E_0(z)V_i(z)}{[z-\{\varsigma_0+\sum_{i=1}^{l}\varsigma_i E_i(z)\}E_0(z)-p^{-}\{E_0(z)V_0(z)F_0(z)+E_0(z)\sum_{i=1}^{l}\varsigma_i E_i(z)V_i(z)F_i(z)\}]},\ 1\le i\le l,$$

$$D_0(z) = \frac{p^{-}\varepsilon_1 H_0[\theta(z-1)+p^{+}\{\gamma(z)-1\}]D(z)V_0(z)}{[z-\{\varsigma_0+\sum_{i=1}^{l}\varsigma_i E_i(z)\}E_0(z)-p^{-}\{E_0(z)V_0(z)F_0(z)+E_0(z)\sum_{i=1}^{l}\varsigma_i E_i(z)V_i(z)F_i(z)\}]}\frac{[1-F_0(z)]}{[1-C(z)]},$$

$$D_i(z) = \frac{p^{-}\varepsilon_1 \varsigma_i H_0[\theta(z-1)+p^{+}\{\gamma(z)-1\}]D(z)E_0(z)V_i(z)}{[z-\{\varsigma_0+\sum_{i=1}^{l}\varsigma_i E_i(z)\}E_0(z)-p^{-}\{E_0(z)V_0(z)F_0(z)+E_0(z)\sum_{i=1}^{l}\varsigma_i E_i(z)V_i(z)F_i(z)\}]}\frac{[1-F_i(z)]}{[1-C(z)]};\ 1\le i\le l,$$

$$\Im_0(z) = \frac{p^{-}\varepsilon_1 H_0[\theta(z-1)+p^{+}\{\gamma(z)-1\}]D(z)V_0(z)F_0(z)}{[z-\{\varsigma_0+\sum_{i=1}^{l}\varsigma_i E_i(z)\}E_0(z)-p^{-}\{E_0(z)V_0(z)F_0(z)+E_0(z)\sum_{i=1}^{l}\varsigma_i E_i(z)V_i(z)F_i(z)\}]}\frac{[1-E_0(z)]}{[1-C(z)]},$$

$$\Im_i(z) = \frac{p^{-}\varepsilon_1 \varsigma_i H_0[\theta(z-1)+p^{+}\{\gamma(z)-1\}]D(z)E_0(z)V_i(z)F_i(z)}{[z-\{\varsigma_0+\sum_{i=1}^{l}\varsigma_i E_i(z)\}E_0(z)-p^{-}\{E_0(z)V_0(z)F_0(z)+E_0(z)\sum_{i=1}^{l}\varsigma_i E_i(z)V_i(z)F_i(z)\}]}\frac{[1-E_i(z)]}{[1-C(z)]};\ 1\le i\le l.$$

*where,* $H_0 = \dfrac{(1-\rho)}{(1-\rho)+(p^{+}\varepsilon_1+\theta\rho)\tilde{H}(\theta)}$ *which is obtained by using normalizing condition* (3.16).

***Theorem* 3.1** *The probability generating function of system size denoted by* $G_S(Z)$*, is given by*

$$G_S(z) = \frac{\theta}{p^{+}}H_0 + H(z) + \sum_{i=0}^{k} E_i(z) + \sum_{i=0}^{k} D_i(z) + \sum_{i=0}^{k} \Im_i(z).$$

# 4 Performance Indices

The performance prediction of a queueing system can be done by analyzing the system. For this, first we find the steady state probabilities for different states of the server which we use to obtain various useful performance measures. Various useful queueing and reliability indices of an unreliable $M^X/G/1/G$-queue with delayed repair where server follows working vacation policy, are given by

(a) Steady state probabilities

1. Probability that the server is on working vacation, denoted by $P[H]$ and is given by

$$P[H] = H(1) = \left[1 + \frac{p^{+}\varepsilon_1\{1-H_V^{*}(\theta)\}}{\theta}\right]H_0.$$

2. Probability that the server is busy in providing *FES* to the customers denoted by $P[E_0]$ and is obtained as $P[E_0] = E_0(1) = \dfrac{p^+\varepsilon_1[\theta + p^+\varepsilon_1][1 - E_0^*(p^-)][1 - H_V^*(\theta)]H_0}{\theta p^-(1-\rho)}$.
3. Probability that the server is busy in providing *SOS* to the customers denoted by $P[E_i]$ for $1 \le i \le l$ and is found to be $P[E_i] = E_i(1) = \dfrac{p^+\varepsilon_1\varsigma_i[\theta + p^+\varepsilon_1]E_0^*(p^-)[1 - E_i^*(p^-)][1 - H_V^*(\theta)]H_0}{\theta p^-(1-\rho)}; 1 \le i \le l.$
4. Probability that the server is waiting for repair due to failure in *FES* of the customers denoted by $P[D_0]$ and is given by $P[D_0] = D_0(1) = \dfrac{p^+\varepsilon_1\omega_0^{(1)}[\theta + p^+\varepsilon_1][1 - E_0^*(p^-)][1 - H_V^*(\theta)]H_0}{\theta p^-(1-\rho)}$.
5. Probability that the server is waiting for repair due to failure in *SOS* of the customers denoted by $P[D_i]$ and is given by $P[D_i] = D_i(1) = \dfrac{p^+\varepsilon_1\varsigma_i\omega_i^{(1)}[\theta + p^+\varepsilon_1]E_0^*(p^-)[1 - E_0^*(p^-)][1 - H_V^*(\theta)]H_0}{\theta p^-(1-\rho)};\ 1 \le i \le l.$
6. Probability that the server is under repair due to failure in *FES* of the customers denoted by $P[\Im_0]$ and is given by $P[\Im_0] = \Im_0(1) = \dfrac{p^+p^-\varepsilon_1\varpi_0^{(1)}[\theta + p^+\varepsilon_1][1 - E_0^*(p^-)][1 - H_V^*(\theta)]H_0}{\theta p^-(1-\rho)}$.
7. Probability that the server is under repair due to failure in *SOS* of the customers denoted by $P[\Im_i]$ and is given by $P[\Im_i] = \Im_i(1) = \dfrac{p^+p^-\varepsilon_1\varsigma_i\varpi_i^{(1)}[\theta + p^+\varepsilon_1]E_0^*(p^-)[1 - E_0^*(p^-)][1 - H_V^*(\theta)]H_0}{\theta p^-(1-\rho)}; 1 \le i \le l.$

(b) Average number and mean waiting time of customers in the system

The average number and mean waiting time of customers in the system denoted by $L_S$ and $W_S$, respectively are given by

$L_S = G_S'(1) = H'(1) + E_0{}'(1) + \sum_{i=1}^{l} E_i{}'(1) + D_0{}'(1) + \sum_{i=1}^{l} D_i{}'(1) + \Im_0{}'(1) + \sum_{i=1}^{l} \Im_i{}'(1)$ and $W_S = \frac{L_S}{p^+\varepsilon_1}$.

For, $0 \le i \le l$ we have

$$b_i = E_i'^*(p^-), \nu_i = \frac{[1 - E_i^*(p^-)]}{p^-}, e_i = f_i = 1, e_i' = p^+\varepsilon_1\varpi_i^{(1)}, f_i' = p^+\varepsilon_1\omega_i^{(1)},$$

$$b_i' = -p^+\varepsilon_1 E_i^{*\prime}(p^-), \nu_i' = \frac{p^+\varepsilon_1[p^-E_i^{*\prime}(p^-)1 - E_i^*(p^-)]}{(p^-)^2},$$

$$e_i'' = p^+\varepsilon_2\varpi_i^{(1)} + (p^+\varepsilon_1)^2\varpi_i^{(2)}, f_i'' = p^+\varepsilon_2\omega_i^{(1)} + (p^+\varepsilon_1)^2\omega_i^{(2)}, b_i'' = -p^+\varepsilon_2 E_i^{*\prime}(p^-) + (p^+\varepsilon_1)^2 E_i^{*\prime\prime}(p^-),$$

$$\nu_i'' = \frac{[p^+p^-\{p^-\varepsilon_2 + 2p^+\varepsilon_1^2\}\{p^-E_i^{*\prime}(p^-)1 - E_i^*(p^-)\} - (p^-)^3(p^+\varepsilon_1)^2E_i^{*\prime\prime}(p^-)]}{(p^-)^4},$$

$$d_1 = \frac{1 - H_V^*(\theta)}{\theta}; d_1' = \frac{p^+\varepsilon_1[d_1 + H_V^{*\prime}(\theta)]}{\theta}; T' = (\theta + p^+\varepsilon_1); T'' = p^+\varepsilon_2,$$

$$\Psi_0' = b_0\sum_{i=1}^{l} b_i' + b_0'(\varsigma_0 + \sum_{i=1}^{l}\varsigma_i b_i), \Psi_0'' = b_0\sum_{i=1}^{l}\varsigma_i b_i'' + b_0''(\varsigma_0 + \sum_{i=1}^{l}\varsigma_i b_i) + 2b_0'\sum_{i=1}^{l}\varsigma_i b_i',$$

$$\Psi_1' = e_0'\nu_0 f_0 + e_0\nu_0' f_0 + e_0\nu_0 f_0', \Lambda'' = -2\varepsilon_1\Omega'; \Lambda''' = -3(\varepsilon_1\Omega'' + \varepsilon_2\Omega')$$

$$\Psi_2' = b_0\sum_{i=1}^{l}\varsigma_i(e_i'\nu_i f_i + e_i\nu_i' f_i + e_i\nu_i f_i')\Psi_1'' = 2(e_0'\nu_0' f_0 + e_0'\nu_0 f_0' + e_0\nu_0' f_0') + e_0''\nu_0 f_0 + e_0\nu_0'' f_0 + e_0\nu_0 f_0'',$$

$$\Psi_2'' = 2b_0'\sum_{i=1}^{l}\varsigma_i(e_0'\nu_0' f_0 + e_0'\nu_0 f_0' + e_0\nu_0' f_0') + \sum_{i=1}^{l}[2(e_i'\nu_i' f_i + e_i'\nu_i f_i' + e_i\nu_i' f_i') + e_i''\nu_i f_i + e_i\nu_i'' f_i + e_i\nu_i f_i''],$$

$$\Omega' = 1 - \psi_0' - p^-(\psi_1' + \psi_2'); \Omega'' = -\psi_0'' - p^-(\psi_1'' + \psi_2''),$$

$$H'(1) = \frac{p^+\varepsilon_1[(p^+\varepsilon_1 + \theta)d_1 + p^+\varepsilon_1 H_V^*(\theta)]H_0}{\theta}; E_0'(1) = \frac{p^+\varepsilon_1 H_0[Num''\Omega' - Num'\Omega'']}{2(\Omega')^2},$$

$$E_i'(1) = \frac{p^+\varepsilon_1\varsigma_i H_0[Num_1''\Omega' - Num_1'\Omega'']}{2(\Omega')^2},$$

$$D_0'(1) = \frac{p^-\varepsilon_1 H_0[Num_2'''\Lambda'' - Num_2''\Lambda''']}{3(\Lambda'')^2}; D_i'(1) = \frac{p^-\varepsilon_1\varsigma_i H_0[Num_3'''\Lambda'' - Num_3''\Lambda''']}{3(\Lambda'')^2},$$

$$\mathfrak{I}_0'(1) = \frac{p^-\varepsilon_1 H_0[Num_4'''\Lambda'' - Num_4''\Lambda''']}{3(\Lambda'')^2}; \mathfrak{I}_i'(1) = \frac{p^-\varepsilon_1\varsigma_i H_0[Num_5'''\Lambda'' - Num_5''\Lambda''']}{3(\Lambda'')^2},$$

$$Num' = d_1\nu_0 T'; Num'' = 2(d_1\nu_0 T' + d_1'\nu_0 T' + d_1\nu_0' T') + d_1\nu_0 T'',$$

$$Num_1' = d_1\nu_i b_0 T'; Num_1'' = 2(d_1\nu_i b_0 T' + d_1'\nu_i b_0 T' + d_1\nu_i' b_0 T' + d_1\nu_i b_0' T') + d_1\nu_i b_0 T'',$$

$$Num_2'' = -2d_1\nu_0 f_0' T'; Num_2''' = -6(d_1\nu_0' + d_1'\nu_0 + d_1\nu_0')f_0' T' - 3d_1\nu_0(T' f_0'' + T'' f_0'),$$

$$Num_3'' = -2d_1\nu_0 f_i' T'; Num_3''' = -6(d_1\nu_i' b_0 + d_1'\nu_i b_0 + d_1\nu_i b_0')f_i' T' - 3d_1\nu_i b_0(T' f_i'' + T'' f_i'),$$

$$Num_4'' = -2d_1\nu_0 e_0' f_0 T'; Num_4''' = -6(d_1\nu_0' f_0 + d_1'\nu_0 f_0 + d_1\nu_0 f_0')e_0' T' - 3d_1\nu_0 f_0(T' e_0'' + T'' e_0'),$$

$$Num_5'' = -2d_1\nu_i e_0' b_0 T'; Num_5''' = -6[d_1 b_0(\nu_i f_i' + f_i\nu_i') + \nu_i(d_1' b_0 + d_1 b_0')]e_i' T' - 3d_1\nu_i b_0(T' e_i'' + T'' e_i').$$

(c) Reliability Indices

Let $A(t)$ represents the point wise availability of the server at time '$t$' i.e. probability that the server is either serving a customer in *FES* or in *SOS* or the server is free. Let $\tilde{A} = \underset{t\to\infty}{Lim} A(t)$ denotes the steady state availability of the server, then we have

$$\tilde{A} = \underset{t\to\infty}{Lim} A(t) = 1 - \mathfrak{I}_0(1) - \sum_{i=1}^{l} \mathfrak{I}_i(1) = 1 - P[\mathfrak{I}_0] - \sum_{i=1}^{l} P[\mathfrak{I}_i].$$

Let $\tilde{F}$ denotes the steady state failure frequency of the server and it is given by

$$\tilde{F} = p^-[E_0(1,1) + \sum_{i=1}^{l} E_i(1,1)] = p^-[P[E_0] + \sum_{i=1}^{l} P[E_i]].$$

(d) Busy Period

***Theorem* 4.1** *Under the steady state conditions, let the expected length of busy period and busy cycle be denoted by $E[B]$and $E[C]$, respectively then we have*

$$E[C] = \frac{1}{p^+\varepsilon_1 H_0} \text{ and } E[B] = \frac{(1-H_0)}{p^+\varepsilon_1 H_0}.$$

***Proof.*** By applying the arguments of an alternating renewal process, we can use the following results directly as

$$H_0 = \frac{E[I]}{E[I]+E[B]}, E[B] = \frac{(1-H_0)}{p^+\varepsilon_1 H_0} \text{ and } E[C] = \frac{1}{p^+\varepsilon_1 H_0} = E[I] + E[B],$$

where, $E[I]$is the time length that the system is in empty state. Since the inter arrival time between two customers follows exponential distribution with parameter $p^+$, we have $E[I] = \frac{1}{p^+\varepsilon_1}$.

## 5 Special Cases

In this section, we deduce some special cases to validate our results with that of the developed model from the literature:

***Case* I:** $M/G/1$ *G*-queue with unreliable server, working vacations and vacation interruption (i.e. No batch arrival, No two phase service and No delayed repair)

Setting $\varepsilon_1 = 1,\ \varepsilon_n = 0\ \forall n \geq 1,\ C(z) = z,\ \varsigma_i = 0 (1 \leq i \leq l), \varpi_i^{(1)} = 1,\ \varpi_i^{(r)} = 0 (1 \leq i \leq l), (r \geq 2)$, we have

$$H(1) = [1 + \frac{p^+\{1 - H_V^*(\theta)\}}{\theta}]H_0,$$

$$E_0(1) = \frac{p^+[\theta + p^+][1 - E_0^*(p^-)][1 - H_V^*(\theta)]H_0}{\theta p^-(1-\rho)},$$

$$\mathfrak{I}_0(1) = \frac{p^+ p^-[\theta + p^+][1 - E_0^*(p^-)][1 - H_V^*(\theta)]H_0}{\theta p^-(1-\rho)}.$$

The above results coincide with that of Zhang and Liu [5].

***Case* II:** $M/G/1$ queue with unreliable server, working vacations and vacation interruption (i.e. No negative arrival, No batch arrival, No two phase service and No delayed repair)

Setting $p \to 0,\ \varepsilon_1 = 1,\ \varepsilon_n = 0\ \forall n \geq 1,\ C(z) = z,\ \varsigma_i = 0 (1 \leq i \leq l),\ \varpi_i^{(1)} = 1,\ \varpi_i^{(r)} = 0 (1 \leq i \leq l)(r \geq 2)$, our results coincides with that of Zhang and Hou [10].

## 6 Cost Function

In this section, we have constructed a cost function to make the system cost effective. The various cost elements associated with different activities are given as follows:

$C_1$ : Holding cost per unit time of each customer present in the system,

$C_2$ : Cost per unit time when the server is on and in operation,

$C_3$ : Setup cost per busy cycle,

$C_4$ : Startup cost per unit time for the preliminary work done by the server before initiating the service.

We construct the function for the expected total cost per unit time as follows:

$$E[TC] = C_1 L_S + C_2[1 - H_0] + C_3 p^+ \varepsilon_1 + C_4 H_0.$$

## 7 Numerical Illustration

This section explores the queueing congestion situation of a call/contact center of a certain mobile company. For illustration purpose, we assume that all distribution functions used in this paper are exponential i.e. $H(x)$, $E_i(x)$, $D_i(x)$, $\Im_i(x)(0 \leq i \leq l)$ are all exponential with rates $q$ (working vacation completion rate), $\mu_B$ (service rate during *FES*), $\mu_{B_1}, \mu_{B_2}, \mu_{B_3}$ (service rate during *SPS* with $l = 3$), $g_1$ (delayed repair rate during *FES*), $g_{21}, g_{22}, g_{23}$ (delayed repair rate during *SPS* with $l = 3$), $b_1$ (repair rate during *FES*), $b_{21}$, $b_{22}$, $b_{23}$ (repair rate during *SPS* with $l = 3$, respectively. There are company service representatives *(CSR's)* whose primary job is to attend and respond to customer's calls. When *CSR* is free, he/she may perform some secondary task during which he/she can make phone calls to promote company's service and products, but if a fresh call arrives, the *CSR* resumes its primary job and assist the call. This situation is called as *working vacation and vacation interruption* in queueing terminology. The calls arrive to the call/contact center in batches in Poisson fashion with rate $p^+ = 4$ messages/min.. The batch size is assumed to be geometrically distributed with $\varepsilon_1 = 0.67$ and $\varepsilon_2 = 0.89$. The calls are served by the *CSR* with rate $\mu_B = 5$ calls/min.. If the customer's problem is not solved by that *CSR*, he/she passes the request to another *CSR's* with rates $\mu_{B_1} = \mu_{B_2} = \mu_{B_3} = 4.5$ calls /min.. The *CSR* can perform secondary task during working vacation with rate $h = 0.2$ calls/min. and if there is some primary call, it interrupts its vacation and returns to the system with rate $q = 0.4$ calls/min. There are various harmful activities such as stealing hard disk space or *CPU* time, retrieving secret information, distorting data, etc. (called negative customers in queueing terminology) which arrives with rate $p- = 0.5$ messages/min. and forces the server to leave the system for repair. Before repair, the repairman took some time for preliminary settings; the *CSR* is said to be under delayed repair state. The delayed repair (repair) rates are taken as $g_1 = 8, g_{21} = 6 = g_{22} = g_{23}$ $(b_1 = 12, b_{21} = 8 = b_{22} = b_{23})$. For cost effective model, we consider cost sets as $C_1 = \$5; C_2 = \$20; C_3 = \$100; C_4 = \$20$. By coding a program in MATLAB software, we obtain following performance measures:$P[H] = 0.1483$; $P[E_0] = 0.3109$; $P[E_S] = \sum_{i=1}^{3} P[E_i] = 0.0821$; $P[D_0] = 0.1360$; $P[D_S] = \sum_{i=1}^{3} P[D_i] = 0.1541$; $P[\Im_0] = 0.0907$; $P[\Im_S] = \sum_{i=1}^{3} P[\Im_i] = 0.1156$; $L_S = 13.33$; $WS = 4.99$ min.; $\tilde{A} = 0.7937$; $\tilde{F} = 1.3756$ and $E[TC] = \$353.31$.

## 8 Sensitivity Analysis

In this section, we explore the effects of some critical system parameters on various queueing and reliability indices by coding a program in MATLAB software. We assume that batch size is geometric distributed whereas service time, delayed repair time, repair time and vacation time are assumed to be exponential distributed. The results are summarized in ***Figures* 8.1-8.2** and in ***Tables* 8.1-8.5**. We have made the system cost effective by providing trends in expected system cost by varying some critical system parameters and cost elements. The default parameters for ***Table* 8.1** are taken as $k = 3, p^- = 3.5, p^+ = 4, \mu_B = 5, \mu_{B_1} = \mu_{B_2} = \mu_{B_3} = 4.5, \theta = 0.2, \mu_v = 0.2, \gamma_1 = 8, \gamma_{21} = \gamma_{22} = \gamma_{23} = 6, \beta_1 = 12, \beta_{21} = 8, \beta_{22} = 8, \beta_{23} = 8$; for ***Tables* 8.2-8.5** are taken as $k = 3, p^- = 3.5, p^+ = 4, \mu_B = 5, \mu_{B_1} = \mu_{B_2} = \mu_{B_3} = 4.5, \theta = 0.2, \mu_v = 0.2, \gamma_1 = 8, \gamma_{21} = \gamma_{22} = \gamma_{23} = 6, \beta_1 = 12, \beta_{21} = 8, \beta_{22} = 8, \beta_{23} = 8$ and for ***Figs* 8.1-8.2** are taken as $k = 3, p_- = 3, p^+ = 3, \mu_B = 4.5, \mu_{B_1} = \mu_{B_2} = \mu_{B_3} = 4, \theta = 0.2, \mu_v = 0.2, \gamma_1 = 8, \gamma_{21} = \gamma_{22} = \gamma_{23} = 6, \beta_1 = 12, \beta_{21} = \beta_{22} = \beta_{23} = 8$ and for ***Figs* 8.1** are taken as $k = 3, p^- = 3, p^+ = 3, \mu_B = 4.5, \mu_{B_1} = \mu_{B_2} = \mu_{B_3} = 4.5, \theta = 0.2, \mu_v = 0.2, \gamma_1 = 8, \gamma_{21} = \gamma_{22} = \gamma_{23} = 6, \beta_1 = 12, \beta_{21} = 8, \beta_{22} = 8, \beta_{23} = 8$ and for ***Figs* 8.2** are taken as $k = 3, p^- = 3, p^+ = 3, \mu_B = 4.5, \mu_{B_1} = \mu_{B_2} = \mu_{B_3} = 4, \theta = 0.2, \mu_v = 0.2, \gamma_1 = 8, \gamma_2 1 = \gamma_{22} = \gamma_{23} = 6, \beta_1 = 12, \beta_{21} = \beta_{22} = \beta_{23} = 8$.

***Tables* 8.1-8.2** show the effect of some parameters such as $p^-, \mu_B, \gamma_1$ and $\beta_1$ on various performance measures. ***Table* 8.1** depicts that the negative arrival rate $p^-$ has significant effect on the server when the server is either on working vacation or under delayed repair state or under repair state; we see an increasing trend in the long run probabilities $P[H]$, $P[D_0]$, $P[D_S]$, $P[\Im_0]$ and $P[\Im_S]$ for increasing values of negative arrival rate $p^-$. On the other hand $P[E_0]$ and $P[E_S]$ decrease with the increase in the values of $p^-$. This is due to the fact that the presence of negative customer leads to server failure which forces the server to leave the system for repair. Moreover we observe from ***Table* 8.1** that the server is more prone to failure and is less available in the system when the negative customer enters the system with higher rate. From ***Table* 8.1**, we also notice that the long run probabilities $P[E_0]$, $P[E_S]$, $P[D_0]$, $P[D_S]$, $P[\Im_0]$ and $P[\Im_S]$ decrease while that of $P[H]$ increase with the increase in service rate $\mu_B$. Also $\tilde{F}$ decreases but $\tilde{A}$ increases on increasing the values of $\mu_B$.

***Table* 8.1:** Effect of (a) $p^-$ and (b) $\mu_B$ on various performance measures

| $p^-$ | $P[H]$ | $P[E_0]$ | $P[E_S]$ | $P[D_0]$ | $P[D_S]$ | $P[\Im_0]$ | $P[\Im_S]$ | $\tilde{A}$ | $\tilde{F}$ |
|---|---|---|---|---|---|---|---|---|---|
| 3.2 | 0.0345 | 0.3244 | 0.1121 | 0.1297 | 0.1842 | 0.1038 | 0.1382 | 0.7579 | 1.3970 |
| 3.4 | 0.0421 | 0.3165 | 0.1065 | 0.1345 | 0.1907 | 0.1076 | 0.1430 | 0.7492 | 1.4387 |
| 3.6 | 0.0494 | 0.3090 | 0.1014 | 0.1390 | 0.1968 | 0.11128 | 0.1476 | 0.7410 | 1.4778 |
| 3.8 | 0.0564 | 0.3019 | 0.0966 | 0.1434 | 0.2027 | 0.1147 | 0.1520 | 0.7332 | 1.5145 |
| $\mu_B$ | $P[H]$ | $P[E_0]$ | $P[E_S]$ | $P[D_0]$ | $P[D_S]$ | $P[\Im_0]$ | $P[\Im_S]$ | $\tilde{A}$ | $\tilde{F}$ |
| 5.0 | 0.0458 | 0.3127 | 0.1039 | 0.1368 | 0.1938 | 0.1094 | 0.1453 | 0.7451 | 1.4585 |
| 5.2 | 0.0527 | 0.3054 | 0.1014 | 0.1336 | 0.1937 | 0.1069 | 0.1453 | 0.7477 | 1.4242 |
| 5.4 | 0.0593 | 0.2984 | 0.0991 | 0.1305 | 0.1936 | 0.1044 | 0.1452 | 0.7502 | 1.3914 |
| 5.6 | 0.0656 | 0.2917 | 0.0968 | 0.1276 | 0.1936 | 0.1021 | 0.1452 | 0.7526 | 1.3602 |

As clear from ***Table* 8.2**, the average system size, average waiting time and the expected system cost decreases with the increase in the values of either $\gamma_1$ or $\beta_1$. This is because on increasing either $\gamma_1$ or $\beta_1$, customers spent less time under delayed repair or repair state, which in turn decreases each of $LS, WS$ and $E[TC]$. In ***Tables* 8.3-8.5**, we explore the effect of various cost elements viz. $C_1, C_2, C_3$ and $C_4$ on $E[TC]$.

***Figures* 8.1-8.2** display the trends in average system size and expected system cost by varying some parameters such as $p^+$, *mB, mV* and $\mu_B 1$. From ***Figs* 8.1-8.2**, we observe that both $L_S$ and *E[TC]* increases linearly with the increase in values of arrival rate of positive customers whereas these decrease first slowly then sharply for increasing the values of either of service rates *mB, mV* and $\mu_B 1$. This feature matches with many real life congestion situations wherein if the server provides service during working vacation with higher rate, customers tend to accumulate more in the system which increases both average system size and expected system cost. Moreover the average system size decreases on increasing the values of vacation rate $q$. Also, ***Fig.* 8.2** depict that the negative arrival rate tends to increase the total system cost.

***Table* 8.2:** Effect of (a) $\gamma_i$ and (b) $\beta_1$ on $L_S$, $W_S$ and $E[TC]$ for cost sets $C_1$ = \$5, $C_2$ = \$20, $C_3$ = \$100, $C_4$ = \$20.

| | $q = 0.4$ | | | $q = 0.5$ | | |
|---|---|---|---|---|---|---|
| $\gamma_1$ | $L_S$ | $W_S$ | $E[TC]$ | $L_S$ | $W_S$ | $E[TC]$ |
| 6 | 13.5522 | 5.0835 | \$354.44 | 12.8803 | 4.8301 | \$351.06 |
| 7 | 13.4869 | 5.0576 | \$354.10 | 12.7379 | 4.7767 | \$350.35 |
| 8 | 13.3281 | 4.9980 | \$353.30 | 12.5360 | 4.7010 | \$349.34 |
| 9 | 13.1784 | 4.9419 | \$352.55 | 12.3580 | 4.6342 | \$348.45 |
| $\beta_1$ | $L_S$ | $W_S$ | $E[TC]$ | $L_S$ | $W_S$ | $E[TC]$ |
| 9 | 13.5225 | 5.0709 | \$354.27 | 12.8014 | 4.8005 | \$350.67 |
| 10 | 13.4657 | 5.0496 | \$353.99 | 12.7136 | 4.7676 | \$350.23 |
| 11 | 13.3963 | 5.0236 | \$353.64 | 12.6215 | 4.7330 | \$349.77 |
| 12 | 13.3281 | 4.9980 | \$353.30 | 12.5360 | 4.7010 | \$349.34 |

***Table* 8.3:** Effects of cost elements $(C_1,\ C_2)$ on $E[TC]$ with fixed $(C_3,\ C_4)$ = (\$100, \$20).

| $(C_1, C_2)$ | (\$5, \$20) | (\$5,\$ 40) | (\$5, \$60) | (\$10, \$40) | (\$15, \$40) |
|---|---|---|---|---|---|
| $E[TC]$ | \$353.307276 | \$372.762397 | \$392.217519 | \$439.403006 | \$506.043615 |

*Table* **8.4:** Effect of cost elements $(C_2, C_4)$ on $E[TC]$ with fixed $(C_1, C_3)$ = (\$5, \$100).

| $(C_2, C_4)$ | (\$40, \$10) | (\$40,\$20) | (\$40, \$30) | (\$50, \$20) | (\$60, \$20) |
|---|---|---|---|---|---|
| $E[TC]$ | \$372.489958 | \$372.762397 | \$373.034836 | \$382.489958 | \$392.217519 |

*Table* **8.5:** Effects of cost elements $(C_3, C_4)$ on $E[TC]$ with fixed $(C_1, C_2)$ = (\$5, \$40).

| $(C_2, C_4)$ | (\$110, \$10) | (\$110,\$20) | 399.701503 | (\$120, \$20) | (\$130, \$20) |
|---|---|---|---|---|---|
| $E[TC]$ | \$399.156625 | \$399.429064 | 399.701503 | \$426.095731 | \$452.762397 |

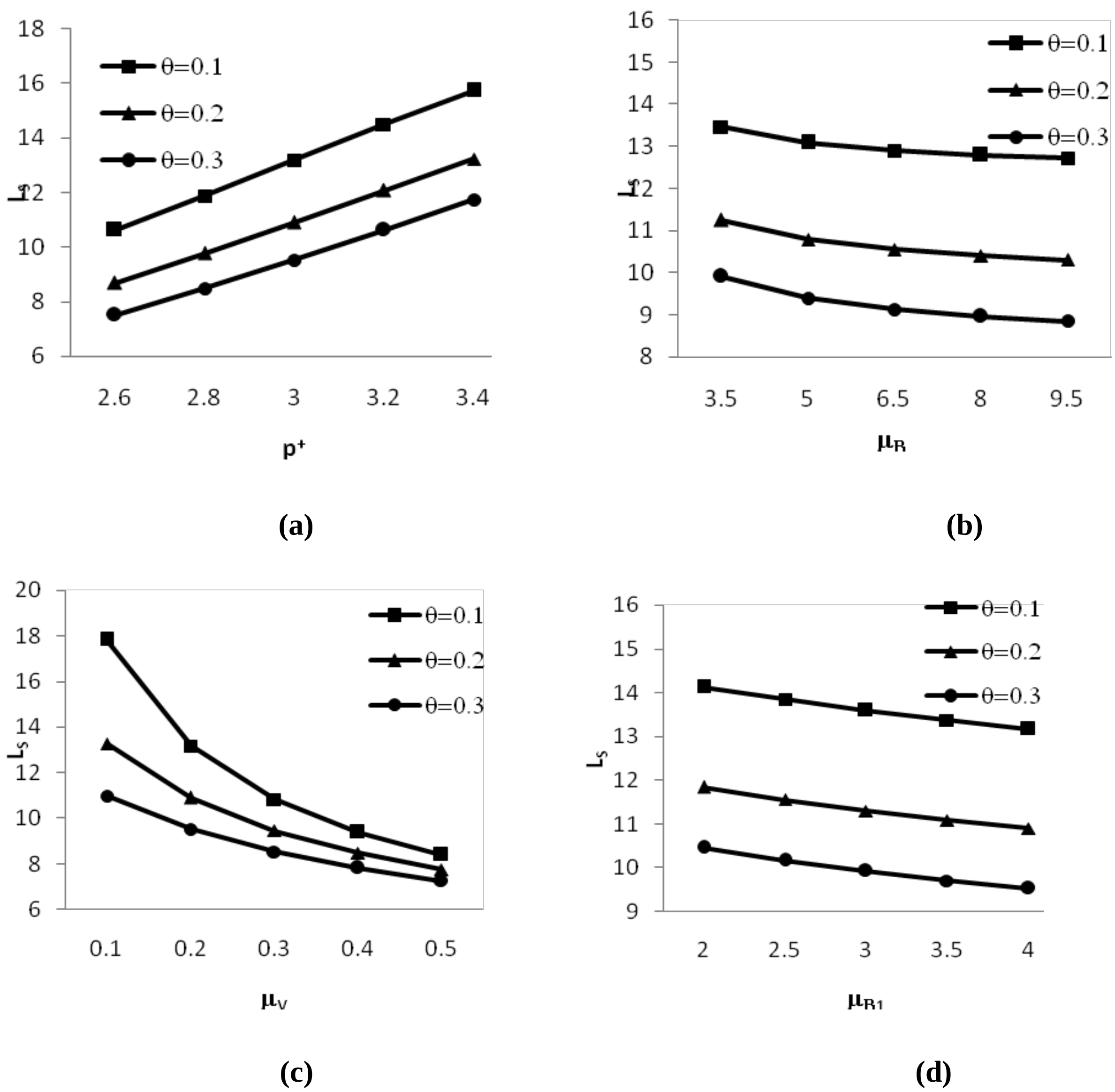
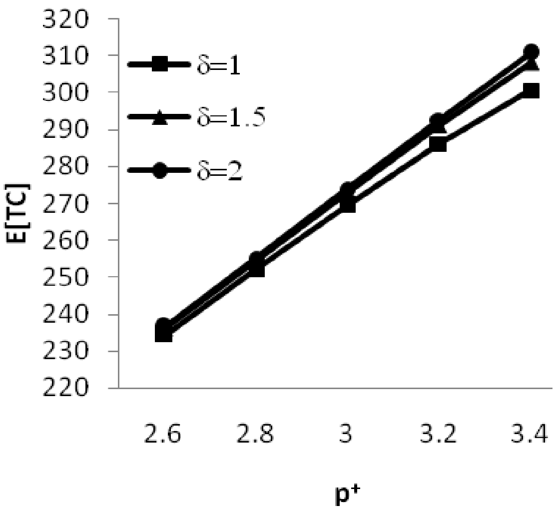
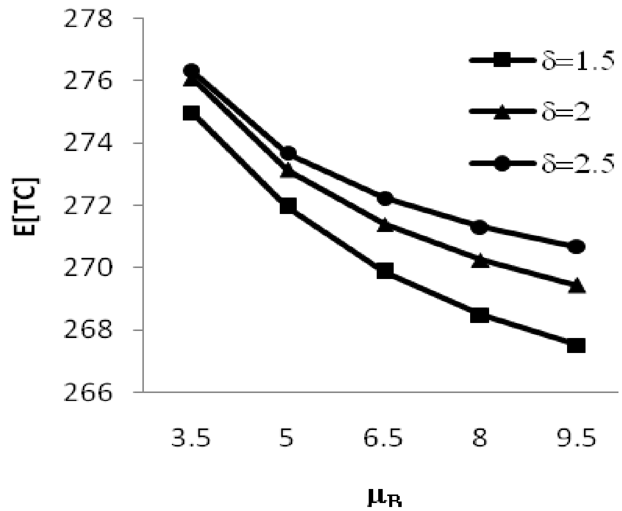


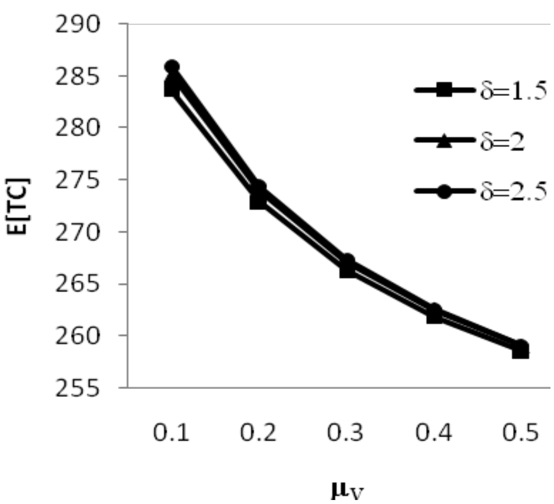

*Figure* **8.1:** $L_S$ versus (a) $p^+$ (b) $\mu_B$ (c) $\mu_v$ (d) $\mu_{B_1}$



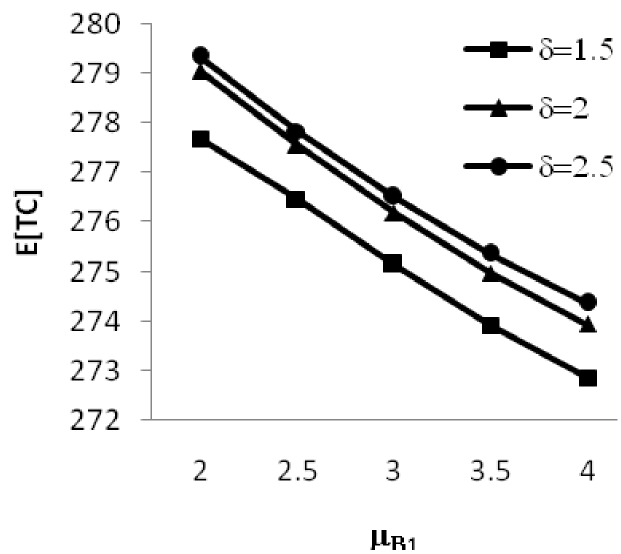

**(a)**




**(b)**




**(c)**




**(d)**

***Figure* 8.2:** $E[TC]$ versus (a) $p^+$ (b) $\mu_B$ (c) $\mu_v$ (d) $\mu_{B_1}$ for cost sets $C_1 = 5$, $C_2 = 20$, $C_3 = 100$, $C_4 = 20$.

Finally we conclude that

- The system designers and decision makers can build a cost effective system by controlling some critical system parameters such as delayed repair rate, repair rate, service rate and various cost elements as our results show a significant effect of these parameters on expected system cost.
- As observed from these results, the negative arrival rate has a significant impact on average system size, average waiting time and expected system cost; this parameter must be controlled in an effective manner.
- The average queue length, average waiting time and expected system cost can be reduced to some extent with the provision of working vacation as we have noticed the decreasing trends in these indices by increasing $\theta$.

## 9 Conclusion

In this paper, we have investigated an $M^X/G/1$ $G$-queue with working vacation and vacation interruption under delayed repair. Such queueing systems can be used to model many real life congestion situations wherein servers are not always available for rendering service; during the idle time, the servers may leave for more economical type of vacation called working vacation. On the other hand our model can be used to examine the effect of negative customers (virus, unwanted programs, distorted data, etc.) in *CCN's*, packet switching networks, telecommunication networks and many more wherein the presence of some unwanted arrivals forces the server to fail and therefore the server requires immediate repair. A numerical illustration for the proposed model has been provided and sensitivity analysis is carried out to observe the influence of some critical system parameters on various performance indices. Moreover, stochastic decomposition results have been derived for the proposed model.

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