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on His 75th Birth Anniversary Celebrations

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Foreword

It gives me immense pleasure to write the Foreword to this Special Dedication Issue of *Jñānābha* in honour of Dr. R. C. Singh Chandel, who is now a retired member of the teaching faculty in the Department of Mathematics of Dayanand Vedic Post-Graduate College at Orai in the Province of Uttar Pradesh in India, on the occasion of his seventy-fifth birthday.

My close professional association with Dr. Chandel dates back to the year 1971 when he almost single-handedly launched and founded this 50- year-old journal. My personal as well professional friendship and editorial collaboration with Dr. Chandel has developed remarkably ever since the establishment of *Jñānābha*.

Finally, on my own behalf as well as on behalf of the members of the *Jñānābha* fraternity, I am exceedingly delighted to thank all of the authors for their invaluable and active participation toward the notable success of this Special Dedication Issue.

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This Special Volume of
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is Being Dedicated to Honor
DR. R. C. SINGH CHANDEL
on His 75th Birth Anniversary Celebrations

DR. RAM CHARAN SINGH CHANDEL
(Born : July 07, 1945)

Jñānābha, **Vol. 50(2) (2020), 1-8**
(Dedicated to Honor Dr. R. C. Singh Chandel on His 75^{th} Birth Anniversary Celebrations)

DR. R. C. SINGH CHANDEL (RAM CHARAN SINGH CHANDEL) :
A DOWN-TO-EARTH PERSON WITH DELIGENCE, DEDICATION AND MODESTY

By

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I know Dr. R. C. Singh Chandel since July 1964 when both of us were first year students in Master's Program in Mathematics at Government Science College, Gwalior, a premier institute of learning of Madhya Pradesh those days. When the university center of learning were not established he was a shy young lad and innocent looks as with few words as he came from a rural background of the Nagla Fauzi a Village of bordering Uttar Pradesh State, India, where he born on 7th July, 1945. He did his schooling at Sirsaganj, Jasrana and graduation form Narain College, Shikohabad (Agra University, Agra) of Uttar Pradesh.

During Masters' program he was a silent and deligdent student unlike two other top position competitors students, myself and one Gopal Das Lakhani, who is in USA at present. We two were together again when joined Ph.D. program in 1966 jointly at SATI, Vidisha registered at Vikram University, Ujjain, MP, India under the same supervisor Professor P. M. Gupta but on different topics. We shared the same apartment in Vidisha and were couleague round the clock. He got married during this period with Ms. Madhavi Chandel, a silent well educated and dedicated house wife who stood solid behind Dr. Chandel througout her life till she expired about 9 year back. Still RCS Continued his hard work for the VPI and Journal *Jñānābha* even having responsibility of house hold.

After completing Ph.D. he joined as Lecturer at D. V. Postgraduate College, Orai (Bundelkhand University, Jhansi) Uttar Pradesh located at Jhansi-Kanpur highway, while I parted his company to join as postdoctoral fellow at SATI first and then at M.A.C.T. (MANIT) Bhopal. I was little bit surprised by his decision to join a semi Government institution, that too at a town not known as an educational destination. But I proved wrong as he became asset to the institution by developing it as center of Mathematical Education and Research. He dedicated his entire teaching carrier to D. V. Postgraduate College, Orai, working at successive higher positions till he retired as Associate Professor of Mathematics in 2008. His unofficial affiliation to the college still continues due to establishment of a Research society called 'Vijñāna Parishad of India' (VPI) and launching simultaneously its research journal named as' '*Jñānābha*' pronounced as ज्ञानाभ in Hindi. I was keenly and anxiously supporting him and contributed a landmark research articles in it.

Vijñāna Parishad of India has become synonymous with Dr. R. C. Singh Chandel and D. V. Postgraduate College, Orai, which is now completing fifty years of glorious journey with support from Professor H. M. Srivastava, University of Victoria, B. C., Canada as Foreign Secretary and Chief Editor, *Jñānābha*. Eminent mathematicians are on the Editorial Board from all over the world while Dr. Chandel is its Founder Executive Editor. This journal is recognized internationally and reviewed regularly among others by Zentralblatt für Mathematik Germany and Mathematical Reviews, USA, This journal is already recognized by the University Grant Commission of India. With Dr. Chandel's almost single handed efforts *Jñānābha* has achieved new highlights and circulated throughout the country and abroad, VPI saw a new dimension in terms of National and International conferences hosted at leading Institutions of North India including MANIT, Bhopal, Netaji Subhas Institute Delhi and BITS, Pilani with participants from different parts of India and abroad. It grew fast and collaborated with other leading societies like Gwalior Academy of Mathematical Sciences and Society for Special Functions and Applications of which Dr. Chandel is an integral part. VPI is also a member of prestigious TMC (The Mathematics Consortium).

At the same time R. C. S. Chandel created his own pitch in the field of Mathematics by publishing large number of research papers and books (List enclosed) and delivering Lectures all India and abroad. He supervised about twenty Ph.D. scholars and visited following Institutions : Memorial University of Newfoundland, St. John's, Canada, University of Victoria, Victoria, B. C., Canada. He was also offered as Associate Professor in University of Minnesota, Duluth, USA but he did not join due to some personal circumstances.

He participated, gave invited talks and chaired sessions in International Conference at University of Newfoundland St. John's, Canada, Summer Meeting of American Mathematical Society (Chicago), Conference of American Math.

Society and Mathematics Association of America at Wyoming and delivered lecture at Fourth ISAAC Congress at York University, Toronto, Canada.

He has several prestigious awards and honours to his credit including life membership of several academic societies. Some of them are given below:

Awards and Honors:

1. *Life-Long Achievements Award* 2019 : Vijñāna Parishad of India for Life-Long Outstanding Contribution to his subject and Life-Time Whole Hearted Distinguished and Dedicated Services to Vijñāna Parishad of India and its Journal *Jñānābha*.
2. *Bharat Ratn Mother Teresa Gold Medal Award*, 2019 : Global Economic Progress Research Association, Tamil Nadu, India.
3. *Senior Rotary Distinguished Service Award*, 2019 :Rotary club of Orai (RI District 3110).
4. *Biography Pubished by : The Marquis Who's Who in the World*- 20th Edition, 2003.
5. *Best Teacher Award and Gold Medal*, 2001 : Bundelkhand University, Jhansi, Uttar Pradesh, India for outstanding contribution in Teaching, Research and National Development.
6. *The Twentieth Century Award for Achievement*, 2000 : International Biographical Institute, Cambridge, England.
7. *Outstanding Man of the* 20^{th} *Century*, 1999: American Biographical Institute, USA.
8. *Distinguished Service Award* 1996 : Vijñāna Parishad of India for the outstanding contribution to Mathematics and Distinguished Services rendered to Vijñāna Parishad of India.
9. *Outstanding Youth Person of District Jalaun*, Uttar Pradesh, India, 1977: JCI, Orai.

Fellowship Award:

1. *Paul Harris Fellow Award (PHF)*, 2018 : Rotary International .
2. *Honorary Elected Fellow (FVPI),* 2008 : Vijñāna Parishad of India.

Life Member of various National or International Societies:

1. National Academy of Sciences, India.
2. Vijñāna Parishad of India.
3. Indian Math, Society.
4. BHU Math, Society.
5. SSFA (Society for Special Functions and Applications, India).
6. GAMS (Gwalior Academy of Mathematical Sciences).
7. CONSORTIUM.
8. Annual Member of American Math, Society for last 20 years.

Social Services: Rotary International since last 38 years

1. Chairman Literacy Volunteer Management : RI District 3110 (2020-21).
2. Rotary District Secretary (Youth Promotion) : RI District 3110 (2019- 20).
3. President : Rotary Club of Orai, RI District 3110 [1996-97, 2016-17, 2017-18].
4. Rotary District Management Committee, District Health and Family Welfare Authority Jalaun, Uttar Pradesh, India, (2000).
5. Rotary International, District 3110, Pulse Polio Coordinator (1997-98).

Books Authored (See Appendix)

Single Authored : 4 Books for Graduate classes.
Co-authored : 1 Book for B.E. Classes.
8 Books for Graduate classes.

List of Research Publications (See Appendix)

We his colleagues and friends feel proud at the achievements of an un parallel personality of Dr. R. C. Singh Chandel. I am sure his parents Late Shri Bridawan Singh and Late Smt. Champa Devi will shower heavenly blessings as well as his worthsons Parmatma, Onkar Singh (Washington, USA) and Ram Chandel (California, USA) will be bestowing their heart felt admiration on the completion of seventy five gracious years of age of Dr. Ram Charan Singh Chandel.

APPENDIX

List of Publications

1. Generalized Laguerre polynomials and the polynomials related to them, *Indian J. Math.,* **11** (1969). 57-66.
2. A short note on generalized Laguerre polynomials and the polynomials related to them, *Indian J. Math.,* **13** (1971), 25-27.
3. Generalized Laguerre polynomials and the polynomials related to them II, *Indian J. Math.,* **14** (1972), 149-155.
4. Generalized Laguerre polynomials and the polynomials related to them III. *Jñānābha Sect. A,* **2** (1972), 49-58.
5. Operational representations and hypergeometric functions of three variables, *Proc. Nat. Acad. Sci., India,* **39** (A) (1969), 217-222.
6. The products of certain classical polynomials and the generalized Laplacian operator, *Ganita,* **20** (1969), 79-87. Corrigendum *Ganita,* **23** (1972), 90.
7. Fractional integration and integral representations of certain generalized hypergeometric functions of several variables, *Jñānābha*, *Sect. A.,* **1** (1971), 45-56.
8. On the *G*-function of two variables, *Jñānābha*, *Sect. A.,* **1** (1971), 84-91 [With R. D. Agrawal].
9. Generalized Hermite polynomials, *Jñānābha, Sect. A.* **2** (1972), 19-27.
10. On some multiple hypergeometric functions related to Lauricella functions, *Jñānābha Sect. A.* **3** (1973), 119-136.
11. A new class of polynomials, *Indian J. Math.,* **15** (1973), 41-49.
12. A further note on the class of polynomials $T_n^{(\alpha,k)}(x, r, p)$, *Indian J. Math.,* **16** (1974), 39-48.
13. A further generalization of the class of polynomials $T_n^{(\alpha,k)}(x, r, p)$, *Kyunpook Math. J.,* **14** (1974), 45-54.
14. Operational representations of certain generalized hypergeometric functions in several variables, *Ranchi Univ. Math. J.,* **7** (1976), 56-60.
15. On some generalized Jacobi polynomials, *Ranchi Univ. Math. J.,* **6** (1975), 54-61 [with H. C. Agrawal].
16. Generalized Rice polynomials, *Jour. Maulana Azad College Tech.* **8** (1975), 67-71 [with R. S. Pal].
17. A note on Stirling numbers and polynomials, *Jour. Maulana Azad College Tech.* **9** (1976), 143-146 [H. C. Yadava].
18. On some operational relationships, *Indian J. Math.* **19** (1977), 173-179 [with H. C. Agrawal].
19. Generalized Stirling numbers and polynomials, *Pub. del. Institute Mathematique, tome* **22** (36) (1977), 145-149.
20. Some polynomials of R. Panda and the polynomials related to them, *Bul. Inst. Math. Acad. Sinica.* **7** (1979), 145-149. [with S. K. Bhargava]
21. Some generating functions for certain polynomials systems in several variables, *Proc. Nat. Acad. Sci. India,* **51** (1981), 133-138 [with H. C. Yadava]
22. A note on binomial and exponential identities, *Ranchi Univ. Math. J.,* **10** (1979), 33-38 [With B. N. Dwivedi]
23. Corrigendum to "On a new class of polynomials and the polynomials related to them" by Sunil Kumar Sinha. *Indian J. Math.,* **19** (1977), 141-148; ibid **21** (1979), 207-208.
24. A note on generating functions for certain polynomial systems *Ranchi Univ. Math. J.,* **10** (1979), 62-66 [with H. C. Yadava]
25. Generalized Whittaker transforms of hypergeometric functions of several variables, *Bul. Inst Math. Acad. Sinica, China,* **8** No. 4 (1980), 595-601 [with B. N. Dwivedi].
26. Srivastava and Daoust functions of several variables, *Pure Appl. Math. Sci.,* **14** No. 1-2 Sept (1981), 53-59 [with B. N. Dwivedi]
27. Operational representations of hypergeometric functions of four variables, *Pure Appl. Math. Sci.,* **16** (1982), 43-52 [with B. N. Dwivedi].
28. A generalization of certain classes of polynomials. *Indian J. Pure Appl. Math.,* **12** (1981), 103-110 [with S. K. Bhargava]
29. A further note on the polynomials of R. Panda and the polynomials related to them, *Ranchi Univ. Math. J.,* **10** (1979), 74-80 [with S. K. Bhargava]
30. Multidimensional Whittaker transforms, *Indian J. Math.,* **24** (1982), 49-53 [with B. N. Dwivedi]
31. A class of polynomials and the polynomials related to them, *Indian J. Math.,* **24** (1982), 41-48. [with S. K. Bhargava]
32. A note on some generating functions, *Indian J. Math.,* **25** (1983), 185-188.
33. A problem on heat conduction, *The Math. Student,* **46** (1978), 240-247.
34. On some associated polynomials, *Ranchi Univ. Math. J.,* **11** (1980), 13-19 [with B. N. Dwivedi]
35. A note on some generating functions for a certain class of polynomials, *Vijñāna Parishad Anusandhan Patrika,* **25** (1982), 25-30 [with B. N. Dwivedi]
36. A problem on the cooling of a heated cylinder, *Jour. MACT,* **15** (1982), 99-103] (with S. K. Bhargava)

37. A new class of polynomials and the polynomials related to them, *Rev. Tec. Ing, Univ. Zulia,* 7 No. 1 (1984), 63-67 [with R. S. Chandel]
38. Two transformation formulas for the generalized multiple hypergeometric function of Srivastava and Daoust, *Indian J. Pure Appl. Math.,* **15** (6) (1984), 633-640 [with Anil Kumar Gupta]
39. A binomial analogue of Srivastava's Theorem, *Indian J. Pure Appl. Math.* **15** (4) (1984), 383-386 [with H. C. Yadava]
40. Heat conduction and multiple hypergeometric function of Srivastava and Daoust, *Indian J. Pure Appl. Math.,* **15**(4) (1984), 371-376 [with H. C. Yadava]
41. Applications of Srivastava Theorem, *Indian J. Pure Appl. Math.,* **15** (1984), 1315-1318 [with H. C. Yadava]
42. Additional applications of binomial analogue of Srivastava's Theorem, *Indian J. Pure Appl. Math.,* **27** (1985), 137-141 [with H. C. Yadava]
43. Heat conduction and generalized Kamp de Friet function of Srivastava and Daoust, *Ranchi Univ. Math. J.,* **14** (1983), 1-10 [with S. K. Bhargava]
44. Recurrence relations of multiple hypergeometric functions of several variables, *Pure Appl. Math. Sci.,* **21** (1985), 65-70 [with Anil Kumar Gupta]
45. Heat conduction and H-function of several variables *Jour. MACT,* **12** (1984), 85-92 [with Anil Kumar Gupta]
46. Applications of Srivastava's hypergeometric function of three variables, *Jñānābha*, **15**, (1985), 65-69 [with B. N. Dwivedi]
47. Multiple hypergeometric functions related to Lauricella functions, *Jñānābha*, **16** (1986), 195-209 [with Anil Kumar Gupta]
48. Use of multivariable H-function of Srivastava and Panda in cooling of a heated cylinder, *Pure Appl. Math. Sci.,* **25** (1987), 43-48 [with Anil Kumar Gupta]
49. Recurrence relations of multiple hypergeometric function of Srivastava and Daoust and the multivariable H-function of Srivastava and Panda, *Indian J. Pure Appl. Math.,* **18** (1987), 347-359. [with Anil Kumar Gupta].
50. A problem on heat conduction in a finite bar, *Jour. MACT,* **19** (1986), 91-95 [with Anil Kumar Gupta]
51. Further applications and extensions of the addition theorems of Srivastava, Lavoie and Tremblay, *Indian J. Pure Appl. Math.,* **18** (1987), 830-834 [with S. Sahgal].
52. Karlsson's multiple hypergeometric function and its confluent forms, *Jñānābha*,**19** (1989), 173-185 [with P. K. Vishwakarma]
53. Fractional integration and integral representations of Karlsson's multiple hypergeometric function and it's confluent forms, *Jñānābha*,**20** (1990), 101-110 [with P. K. Vishwakarma].
54. A multivariable H-function of Srivastava and Panda and it's applications in a problem on electrostatic potential in spherical regions, *Jour. MACT.* **23** (1990), 39-46 [with R. D. Agarwal and H. Kumar]
55. A multivariable analogue of Panda's polynomials, *Indian J. Pure Appl. Math.,* **21** (12) (1990), 1101-1106 [with S. Sahgal]
56. A class of polynomials in several variables, *Ganita Sandesh,* **4** (1990), 27-32 [with R. D. Agarwal and H. Kumar].
57. A multivariable analogue of Gould and Gould-Hooper's polynomials, *Indian J. Pure Appl. Math.,* **22** (3) (1991), 225-229 [with S. Sahgal].
58. A binomial analogue of the class of addition theorems of Srivastava, Lavoie and Tremblay and its applications, *Proc. VPI,* **1** (1989), 145-148.
59. A remark on " Hypergeometric functions of four variables I" by Chhaya Sharma and C. L. Parihar, {*Indian Acad. Math.,* **11** (2) (1989), 121-133}, *Proc. VPI,* **2** (1990), 113-115 [with H. Kumar]
60. Another multivariable analogue of Gould and Hopper's polynomials, *Pure Math. Manuscript,* **9** (1990-1991), 125-135 [with Abha Tiwari].
61. Unified study of two general classes of functions, *Pure Math. Manuscript,* **9** (1990-1991), 111-123 [with Shashi Agrawal]
62. A generalization of a class of polynomials, *Jñānābha*, **21** (1991), 19-25 [with Shashi Agrawal]
63. Generating relations involving hypergeometric functions of four variables, *Pure Appl. Math. Sci.,* **34** (1991), 15-25 [with Abha Tiwari]
64. Multivariable analogue of Gould and Hooper's polynomials defined by Rodrigues' formula, *Indian J. Pure Appl. Math.,* **22** (1991), 757-761 [with Abha Tiwari]
65. A multivariable analogue of Hermite polynomials, *Ganita Sandesh,* **5** (1991), 92-95 [with Abha Tiwari]
66. Fractional derivatives of confluent hypergeometric forms of Karlsson's multiple hypergeometric function, *Pure Appl. Math. Sci.,* **35** (1992), 31-39 [with P. K. Vishwakarma]

67. An integral involving sign functions, exponential functions, the Kamp de Friet function and the multivariable H-function of Srivastava and Panda, and its applications in potential problem on a circular disk, *Pure Appl. Math. Sci.,* **35** (1992), 59-69 [with R. D. Agrawal and H. Kumar].
68. Hypergeometric functions of four variables and their integral representations, *Math. Education,* **26** (1992), 76-94 [with R. D. Agrawal and H. Kumar].
69. Two variables analogue of Gould and Hopper's polynomials, *Jour. MACT,* **25** (1992), 63-69 [with R. D. Agrawal and H. Kumar].
70. Unified presentation of two general sequences of functions, *Jñānābha*, **22** (1992), 13-22 [with Shashi Agrawal]
71. Binomial analogues of the class of addition theorems of Srivastva, Lovoie and Tremblay, *Jñānābha*, **22** (1992), 23-29 [with Shashi Agrawal]
72. Fourier series involving the multivariable H- function of Srivastava and Panda, *Indian J. Pure Appl. Math.,* **23** (1992), 343-357 [with R. D. Agrawal and H. Kumar]
73. Velocity coefficient of chemical reaction and Lauricella's multiple hypergeometric function, *Math. Student,* **63** (1993), 1-4 [with R. D. Agrawal and H. Kumar]
74. Multiple hypergeometric function of Srivastava and Daoust and it's applications in two boundary value problems, *Jñānābha* **23** (1993), 97-103 [with Abha Tiwari].
75. Multivariable analogues of a class of polynomials, *Jñānābha* **23** (1993), 105-113 [with Shashi Agrawal]
76. Multidimensional fractional derivatives of multiple hypergeometric functions of several variables, *Jñānābha*,**24** (1994), 19-27 [with P. K. Vishwakarma]
77. Fractional derivatives of certain generalized hypergeometric functions of several variables, *Jour. Math. Anal. Appl.,* **184** (1994), 560-572 [with H. M. Srivastava and P. K. Vishwakarma].
78. A multilinear generating function, *Math Ed. (Siwan),* **28** (1994), 32-37 [with Abha Tiwari]
79. On some relations between hypergeometric functions of three and four variables, *Jñānābha*,**26** (1996), 72-82 [with P. K. Vishwakarma]
80. Fractional derivatives of the multiple hypergeometric functions of four variables, *Jñānābha*, **26** (1996), 83-87 [with P. K. Vishwakarma].
81. Some probability distributions and expectations associated with multivariate beta and gamma distributions involving multiple hypergeometric functions of Srivastava and Daoust, *Jñānābha*,**27** (1997), 131-137 [with P. K. Vishwakarma].
82. Some more inequalities involving Fox's H-function, *Jñānābha*,**28** (1998), 133-140 [with H. Kumar].
83. Determination of phase shift difference for binomial potential function, *Jñānābha*,**28** (1998). 141-146 [with H. Kumar].
84. Phase shifts involving multiple hypergeometric functions of Srivastava and Daoust, *Jñānābha*, **29** (1999), 117-122 [with H. Kumar and R. D. Agarwal]
85. Some expectations associated with Multivariate Gamma and Beta Distributions involving the multiple hypergeometric function of Srivastava and Daoust, *Jñānābha*, **30** (2000), 9-16 [with P. K. Vishwakarma].
86. On some multidimensional integral transforms of Srivastava and Panda's H-function of several complex variables, *Jñānābha*, **30** (2000), 125-130 [With Kamlendra Kumar]
87. Integrals involving multiple hypergeometric functions of several variables through difference operator approach, *Jñānābha***31/32** (2002) 151-157. (with S. S. Chauhan)
88. Remarks on " Certain integrals involving hypergeometric functions of three and four variables" by Sunil Joshi and S. S. Bhati (*Jñānābha*, 27 (1997), 93-98), *Jñānābha*, **31/32** (2002) 167 (with S. S. Chauhan)
89. Multidimensional Laguerre transforms, *Jour. Pure Math.,* **20** (2003) 59-72 (with S. S. Chauhan)
90. On two boundary value problems, *Jñānābha*, **31/32** (2002), 89-104. (with S. Sengar)
91. A problem on heat conduction in a rod under the Robin condition, *Jñānābha*, **33** (2003), 131-138. (with S. Sengar)
92. Hypergeometric functions of four variables, *Pure Appl. Math. Sci.,* **58** (2003), 7-18, (with S. Sharma)
93. Temperature distribution due to population growth of interacting multispecies in the limited environment, *Mathematics and Information Theory*: *Recent topics and applications*, (Editor V. K. Kanpur), Anannya Publishers, New Delhi, India, 2004 [with Hemant Kumar]
94. Fractional derivatives of our hypergeometric functions of four variables, *Jñānābha*,**34** (2004), 113-132. (with S. Sharma).
95. Two boundary value problems, *Indian J. Theoretical Physics,* **53** (4), (2005), 339-350. (with Yogesh Kumar)
96. Generalized multidimensional Laguerre transforms, *Jnānābha*, **35** (2005), 17-27 (with Kamalendra Kumar)
97. On a general class of generating functions and its applications, *Jñānābha*, **35** (2005), 67-72 (with H. Kumar and S. Sengar)
98. On some generalized results of fractional derivatives, *Jñānābha*, **36** (2006), 105-112 (with Yogesh Kumar)

99. Applications of the generalized polynomials of several variables of Srivastava and Multivariable H-function of Srivsatava-Panda in boundary value problems, *Jñānābha*, **36** (2006), 125-134. (with Swapna Dwivedi)
100. A new multidimensional integral transform, *Jour. Pure. Math.,* **23,** (2006), 101-119. (with S. S. Chauhan)
101. Transformations of some multiple hypergeometric functions of several variables, *Jour. Pure Math.,* **24** (2007), 49-58. [Also presented in ISAAC congress, 2003 held at York Univ. Toronto, Canada [August 11-16, 2003] in special session on Non-Linear Analysis]
102. Applications of multiple hypergeometric function of Srivastava and Daoust and generalized polynomials of Srivastava in two boundary value problems, *Indian J. Nat. Acad. Math.* **21** (2007), 19-38 [with S. S. Chauhan]
103. On some new multiple hypergeometric functions related to Lauricella's functions, *Jñānābha*, **37** (2007), 107-122. [with Vandana Gupta]
104. A generalization of multivariable polynomials, *Jñānābha*, **38** (2008), 153-160. (with K. P. Tiwari)
105. Laplace integral representations and recurrence relations of multiple hypergeometric functions related to Laurecella's functions, *Jñānābha*, **39** (2009), 121-154 (with Vandana Gupta)
106. On some multivariable generalized truesdell polynomials, *Jour. Pure Math.,* **26** (2009), 9-13. [with S. S. Chauhan]
107. Multivariable generalized polynomials defined through their generating function, *Jñānābha*, **40** (2010) 105-112. [with Shailja Sengar]
108. On some expectations associated with probability density functions of various multivariable distributions, *Jñānābha*, **40** (2010), 129-146. [with K. P. Tiwari]
109. Fourier series involving generalized Srivastava polynomials of several variables and the maltivariable H-function of Srivastava-Panda, *Jñānābha*, **40** (2010), 147-156. [With Yogesh Kumar]
110. On the distributions of the densities involving non-zero Zeros of Bessel and Legendre functions and their infinite series, *Proyeciones Journal of Mathematics,* **29** (2010), 165-182. [with H. Kumar and M. A. Pathan]
111. Applications of Bessel function, multivariable generalized Srivastava polynomials and multivariable H-function of Srivastava-Panda in a problem on cooling of a heated cylinder, *Indian Journal of Theoretical Physics,* **59(2)** (2011), 127-140. [with Vandana Gupta]
112. A remark on "Fractional Calculus operator involving the product of hypergeometric functions of several variables" by V. B. L. Chaurasia and Hari Singh Parihar, *Jñānābha*, **35** (2005), 175-187, *Jñānābha*, **41** (2011) 40. [with S. S. Chauhan]
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AN UNRELIABLE BATCH ARRIVAL *G*-QUEUE WITH WORKING VACATION, VACATION INTERRUPTION AND MULTI-OPTIONAL SERVICES

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Abstract

This paper predicts the performance of an unreliable $M^X/G/1$ G-queue with delayed repair. The negative customers stop functioning of the server and force the server to undergo repair. The failed server takes a random amount of time called **delay time** before going for repair. It is assumed that the positive customers arrive in batches and negative customers arrive singly, according to Poisson process. The server provides first phase of regular service to all arriving customers whereas it provides l types of optional services to only those who demand the same. The working vacation period of the server starts either if queue becomes empty or repairing of the server finishes. Numerical experiments are provided to show the effects of various critical system parameters on performance measures.

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1 Introduction

This work is motivated by modeling general service non-Markovian queueing systems having two types of customers, ***positive and negative***, depending upon their nature. The positive customers are usual customers which enter the queue for receiving service if it not immediate, otherwise depart from the system after getting service, while negative customers are those who remove the positive customer in service and force the server for immediate repair. There are many real life congestion situations in which we found these type of customers. For example, in distributed computer systems or databases, there are some commands which delete some transaction because of locking of data or because of inconsistency. In the paper by Gelenbe [3], the author provides analogy of queueing networks with neural networks wherein each queue represents a neuron and customers represent excitation (positive) or inhibition (negative) signals. This type of queueing system which involves negative customers is termed as **"*G*-queues"**. Applications of these types of queues can be found in computer networks, data communication systems, distributed systems, manufacturing systems, neural networks and many more. A survey on G-queues has been presented by Do [2]. Some useful performance characteristics of an unreliable $M/G/1$ retrial G-queue under priority scheme have been obtained by Wu and Lian [8]. Further Peng et al. [5] examined the performance characteristics of an unreliable $M/G/1$ retrial G-queue with preemptive resume priority and collisions under delayed repairs. Recently, Kirupa and Chandrika [4] throw light on the performance prediction of $M^X/G/1$ G-queue with heterogeneous service, setup time and reserved time.

The concept of working vacation was introduced by Servi and Finn [7], according to which server may provide service during vacation with comparatively slower rate rather than completely stops service during vacation as found in the past literature. Working vacation queues with negative arrivals find wide applicability in the working of computer networks, web servers, file transfer systems and email servers. Do et al. [1] studied an $M/M/1$ retrial G-queue with working vacation. Further, using SVT (Supplementary Variable Technique), Rajadurai et al. [6] examined the performance of an $M/G/1$ retrial queue with balking under working vacation policy. Zhang and Liu [9] investigated the behavior of an $M/G/1$ queue with negative arrivals and server breakdowns, working vacation and vacation interruption, using both SVT and matrix analytic method.

Due to scarcity of work done on batch arrival G-queues with working vacation and vacation interruption, we put forward our effort to analysze it. The rest of the work done is as follows. Section 2 describes the model by stating requisite assumptions and notations. The queue size distribution has been explored in ***Section* 3** via probability generating function method. Various useful performance measures of our model are explored in ***Section* 4** using queue size distribution obtained in ***Section* 3**. In ***Section* 5**, we show that the stochastic decomposition results holds good for the developed model. Some special cases have been deduced in ***Section* 6**. The cost function is constructed in ***Section* 7**. A numerical example and the effect of some sensitive parameters on various performance measures are explored in ***Section* 8**. Finally concluding remarks and future scope have been outlined in ***Section* 9**.

2 System Description

Consider a single server batch arrival G-queue with unreliable server and delayed repair. The server renders first phase of essential service to all the arriving customers whereas it renders any type of l optional services to only those who demand for the same. The server may undergo breakdown due to the presence of a negative arrival. To formulate the model, we make following assumptions as given below:

The arrival stream is composed of two types of customers' positive customers and negative customers. The positive customers arrive in groups or batches with rate p^+. The batch size is an arbitrary distributed independent random variable denoted by γ with probability distribution $Pr\{\gamma = k\} = \epsilon_k,\ k \geq 0,\ pgf$ (probability generating function) $\gamma(z) = \sum_{k=1}^{\infty} \epsilon_k z^k$ and first and second finite moments ϵ_1 and ϵ_2. The arrival of a negative customer with rate p^- not only removes the positive customer in service but also leads the server to undergo repair. The failed server waits for the repair to start; this waiting time of the server is called as ***delay time*** which follows a general distribution with distribution function $D_i(x)$, *LST* (Laplace Steiljes Transform) $D_i^*(s)$ and r^{th} factorial moments $\omega_i^{(r)}(r \geq 1)$, where $0 \leq i \leq l$. The repair time of the server is assumed to be arbitrarily distributed random variable with distribution function $\Im_i(x)$, *LST* (Laplace Steiljes Transform) $\Im_i^*(s)$ and r^{th} factorial moments $\omega_i^{(r)}(r \geq 1)$, where $0 \leq i \leq l$. The negative customer affects the server if and only if it is busy. The server renders first phase of essential service denoted by *FES* to all positive customers whereas only few of them will receive second phase of optional service *(SOS)* based upon their demand. After finishing *FES* by the customer, he may demand for any of l different kinds of optional services with probability $\zeta_i(1 \leq i \leq l)$, otherwise he leaves the service area with complementary probability $\bar{\zeta}_i(= 1 - \zeta_i), 1 \leq i \leq l$. The service discipline is *FCFS* (first come first served). The service times during *FES* and *SOS* are *i.i.d.* (independent and identically distributed) random variables having probability distribution $E_0(x)$ and $E_i(x)$; *LST* $E_0{}^*(x)$ and $E_i{}^*(x)$ and r^{th} factorial moments $\zeta_0^{(r)}$ and $\zeta_i^{(r)}(r \geq 1)(0 \leq i \leq l)$, respectively.

During regular busy period, the server serves the positive customers with rate μ_B. After each service completion or repair completion, the server enters into a working vacation period of random length *'W'* which follows an exponential distribution with parameter ν. When the server finds that the queue is non-empty, it immediately changes the service rate from μ_ν to μ_B and initiates a regular busy period no matter whether or not the vacation has ended. The vacation time is assumed to be *i.i.d* random variable with probability distribution *H(x)*; *LST* $H^*(x)$ and r^{th} factorial moments $\vartheta^{(r)}$ with $r \geq 1$. The service discipline during both regular busy period and working vacation period is *FCFS* (first come first served). We assume that the hazard rates for service time, delay time and repair time are $\Psi_i(x)$, $\eta_i(x)$ and $\phi_i(x)(0 \leq i \leq l)$, respectively whereas that of working vacation time is $\gamma(x)$. So we can define service time, delay time, repair time and working vacation time distribution by

$$E_i(x) = 1 - \exp\left\{-\int_0^x \varphi_i(t)dt\right\},\ 0 \leq i \leq l;\ D_i(x) = 1 - \exp\left\{-\int_0^x \eta_i(t)dt\right\};\ 0 \leq i \leq l;$$

$$\Im_i(x) = 1 - \exp\left\{-\int_0^x \phi_i(t)dt\right\};\ 0 \leq i \leq l;$$

$$H(x) = 1 - \exp\left\{-\int_0^x \gamma(t)dt\right\}.$$

The *LST* for service time, delay time, repair time and working vacation time distribution are defined as

$$E_i^*(s) = \int_0^\infty \exp\{-sx\}dE_i(x),\ D_i^*(x) = \int_0^\infty \exp\{-sx\}dD_i(x),\ \Im_i^*(s) = \int_0^\infty \exp\{-sx\}d\Im_i(x),$$

$$H_\nu^*(s) = \int_0^\infty \exp\{-sx\}dH_\nu(x);\ 0 \leq i \leq l.$$

Also, where, 'A' denotes random variable for either service time or repair time or vacation time distribution.

3 Queue Size Distribution

At time $t \geq 0$, we define the system state by forming a Markov process $\lambda(t) = \{\xi(t),\ \pi(t),\ \kappa(t)\}$, , where

$$\pi(t) = \begin{cases} 0, \text{ if the server is in working vacation period at time } t \\ 1, \text{ if the server is busy with } FES \text{ at time } t \\ 2, \text{ if the server is busy with } SOS \text{ at time } t \\ 3, \text{ if the server is waiting for repair at time } t \\ 4, \text{ if the server is under repair at time } t. \end{cases}$$

Here $\xi(t)$ represents the number of customers in the queue. When $\pi(t) = 0$ and $\xi(t) \geq 0,\ \kappa(t)$ denotes the elapsed working vacation time of the server. If $\pi(t) = 1, 2$ and $\xi(t) \geq 0,\ \kappa(t)$ denotes the elapsed service time either during *FES* or during *SOS* at time t. If $\pi(t) = 3$ and $\xi(t) \geq 0,\ \kappa(t)$ denotes the elapsed delay time at time t. If $\pi(t) = 4$ and $\xi(t) \geq 0,\ \kappa(t)$ denotes the elapsed repair time at time t. In order to construct an embedded Markov chain of our developed model, we assume that $\xi(t_n^+)$ be the number of customers in the queue, respectively just after the time t_n. Then the sequence of random variables $\{M_n, n \in N\}$ forms an embedded Markov chain of our developed model.

***Lemma* 3.1** *The embedded Markov chain $\{M_n, n \in N\}$is ergodic iff $\rho < 1$, which is the necessary and sufficient condition for the stability of the system, where ρ is given by*

(3.1) $\rho = p^+\varepsilon_1[\nu_0 + E_0^*(p^-)\sum_{i=1}^{l}\varsigma_i\nu_i + p^-\{\nu_0\varpi_0^{(1)} + E_0^*(p^-)\sum_{i=1}^{l}\varsigma_i\nu_i\varpi_i^{(1)} + \nu_0\omega_0^{(1)} + E_0^*(p^-)\sum_{i=1}^{l}\varsigma_i\nu_i\omega_i^{(1)}\}]$.

Proof. To prove the sufficient condition one can use Foster's criterion. According to this, an irreducible and aperiodic Markov chain $M_n$with state space $\varphi$is ergodic if there exists a non-negative function $f(j),\ j \in \varphi$ called test function, and $\delta > 0$ such that the mean drift $N_j = E[f(M_{n+1}) - f(M_n)/M_n = j]$ is finite for all $j \in \varphi$ and $M_j \leq -\delta$ for all $j \in \varphi$ except perhaps a finite number. For this model, it is easy to show that $M_n$is irreducible and aperiodic. Further one can prove the necessary condition by proceeding in the same way as done by Zhang and Liu ([9], p. 262-263, ***Theorem* 3.1**).

Under the established stability condition, we define some limiting probabilities for the Markov process $\{M(t), t \geq 0\}$ as given below:

$H_n = \underset{t\to\infty}{Lim} Pr\{\Gamma(t) = 0, M(t) = n\}, n \geq 0,$

$E_{0,n}(x) = \underset{t\to\infty}{Lim} Pr\{\Gamma(t) = 1, M(t) = n, x \leq \Phi(t) \leq x + dx\}, n \geq 0,$

$E_{i,n}(x) = \underset{t\to\infty}{Lim} Pr\{\Gamma(t) = 2, M(t) = n, x \leq \Phi(t) \leq x + dx\}, n \geq 0, 1 \leq i \leq l,$

$D_{0,n}(x) = \underset{t\to\infty}{Lim} Pr\{\Gamma(t) = 3, M(t) = n, x \leq \Phi(t) \leq x + dx\}, n \geq 0,$

$D_{i,n}(x) = \underset{t\to\infty}{Lim} Pr\{\Gamma(t) = 3, M(t) = n, x \leq \Phi(t) \leq x + dx\}, n \geq 0, 1 \leq i \leq l,$

$\Im_{0,n}(x) = \underset{t\to\infty}{Lim} Pr\{\Gamma(t) = 4, M(t) = n, x \leq \Phi(t) \leq x + dx\}, n \geq 0,$

$\Im_{i,n}(x) = \underset{t\to\infty}{Lim} Pr\{\Gamma(t) = 4, M(t) = n, x \leq \Phi(t) \leq x + dx\}, n \geq 0, 1 \leq i \leq l.$

State Governing equations

(3.2) $\frac{d}{dx}H_n(x) = -\{p^+ + \theta + \varphi_V(x)\}H_n(x) + p^+(1 - \delta_{0,j})\sum_{k=1}^{j}\varepsilon_k H_{n-k}(x); n \geq 1,$

(3.3) $\frac{d}{dx}E_{0,n}(x) = -\{p + \varphi_0(x)\}E_{0,n}(x) + p^+(1 - \delta_{1,n})\sum_{k=1}^{n}\varepsilon_k E_{0,n-k}(x); n \geq 0,$

(3.4) $\frac{d}{dx}E_{i,n}(x) = -\{p + \varphi_i(x)\}E_{i,n}(x) + p^+(1 - \delta_{1,n})\sum_{k=1}^{n}\varepsilon_k E_{i,n-k}(x); n \geq 0;\ 1 \leq i \leq l,$

(3.5) $\frac{d}{dx}D_{0,n}(x) = -\{p^+ + \eta_0(x)\}D_{0,n}(x) + p^+(1 - \delta_{1,n})\sum_{k=1}^{n}\varepsilon_k D_{0,n-k}(x);\ n \geq 0,$

(3.6) $\frac{d}{dx}D_{i,n}(x) = -\{p^+ + \eta_i(x)\}D_{i,n}(x) + p^+(1 - \delta_{1,n})\sum_{k=1}^{n}\varepsilon_k D_{i,n-k}(x);\ n \geq 0,\ 1 \leq i \leq l,$

(3.7) $\frac{d}{dx}\Im_{0,n}(x) = -\{p^+ + \phi_0(x)\}\Im_{0,n}(x) + p^+(1 - \delta_{1,n})\sum_{k=1}^{n}\varepsilon_k\Im_{0,n-k}(x);\ n \geq 0,$

(3.8) $\frac{d}{dx}\Im_{i,n}(x) = -\{p^+ + \phi_i(x)\}\Im_{i,n}(x) + p^+(1 - \delta_{1,n})\sum_{k=1}^{n}\varepsilon_k\Im_{i,n-k}(x); n \geq 0;\ 1 \leq i \leq l.$

Boundary Conditions

(3.9) $p^+\varepsilon_1 H_0 = \int_0^\infty H_1(x)\varphi_V(x)dx + \varsigma_0\int_0^\infty E_{0,1}(x)\varphi_0(x)dx + \sum_{i=1}^{l}\bar{\varsigma}_i\int_0^\infty E_{i,n+1}(x)\varphi_i(x)dx +$
$\int_0^\infty \Im_{0,0}(x)\phi_0(x)dx + \sum_{i=1}^{l}\int_0^\infty \Im_{i,0}(x)\phi_i(x)dx,$

(3.10) $H_1(0) = p^+\varepsilon_1 H_0;\ H_n(0) = 0;\ n \geq 2,$

(3.11) $E_{0,n}(0) = \varsigma_0\int_0^\infty E_{0,n+1}(x)\varphi_0(x)dx + \sum_{i=1}^{l}\bar{\varsigma}_i\int_0^\infty E_{i,n+1}(x)\varphi_i(x)dx + \int_0^\infty H_{n+1}(x)\varphi_V(x)dx +$
$\theta\int_0^\infty H_n(x)dx + \int_0^\infty \Im_{0,n}(x)\phi_0(x)dx + \sum_{i=1}^{l}\int_0^\infty \Im_{i,n}(x)\phi_i(x)dx,$

(3.12) $E_{i,n}(0) = \varsigma_i\int_0^\infty E_{0,n}(x)\varphi_0(x)dx; 1 \leq i \leq l, j \geq 0, n \geq 1,$

(3.13) $D_{0,n}(0) = p^-\int_0^\infty E_{0,n+1}(x)dx;\ n] \geq 0,$

(3.14) $D_{i,n}(0) = p^-\int_0^\infty E_{i,n+1}(x)dx;\ 1 \leq i \leq l;\ n \geq 0,$

(3.15) $\Im_{0,n}(0) = \int_0^\infty D_{0,n}(x)\eta_0(x)dx;\ n \geq 0,$

(3.16) $\Im_{i,n}(0) = \int_0^\infty \Im_{i,n}(x)\eta_i(x)dx;\ n \geq 0,$

The normalizing condition of the system is given by

(3.17) $H_0 + \sum_{n=1}^\infty \int_0^\infty H_n(x)dx + \sum_{i=0}^l \sum_{n=1}^\infty \left[\int_0^\infty E_{i,n}(x)dx + \int_0^\infty D_{i,n}(x)dx + \int_0^\infty \Re_{i,n}(x)dx\right] = 1.$

The probability generating functions of sequences of the above probabilities are given by

$A(z) = H_V^*(p^+(1-\gamma(z)) + \theta); D(z) = \tilde{H}_V(p^+(1-\gamma(z)) + \theta)$

$E_i(z) = E_i^*(p - p^+\gamma(z)); V_i(z) = \tilde{E}_i(p - p^+\gamma(z)); F_i(z) = D_i^*(p^+(1-\gamma(z))); 0 \leq i \leq l.$

For solution purpose, we define following generating functions for different states of the server:

$H(x,z) = \sum_{n=1}^\infty H_n(x)z^n, E_i(x,z) = \sum_{n=0}^\infty E_{i,n}(x)z^n; 0 \leq i \leq l, D_i(x,z) = \sum_{n=0}^\infty D_{i,n}(x)z^n; 0 \leq i \leq l,$

$\Im_i(x,z) = \sum_{n=0}^\infty \Im_{i,n}(x)z^n; 0 \leq i \leq l.$

Using probability generating function method, we multiply each of the equations (3.2)-(3.8) by z^n and then summing over all values of n, we get

(3.18) $\dfrac{\partial}{\partial x}H(x,z) = -\{p^+(1-\gamma(z)) + \theta + \varphi_V(x)\}H(x,z),$

(3.19) $\dfrac{\partial}{\partial x}E_i(x,z) = -\{p - p^+(1-\gamma(z)) + \varphi_i(x)\}E_i(x,z);\ 0 \leq i \leq l,$

(3.20) $\dfrac{\partial}{\partial x}D_i(x,z) = -\{p^+(1-\gamma(z)) + \eta_i(x)\}D_i(x,z);\ 0 \leq i \leq l,$

(3.21) $\dfrac{\partial}{\partial x}\Im_i(x,z) = -\{p^+(1-\gamma(z)) + \phi_i(x)\}\Im_i(x,z);\ 0 \leq i \leq l.$

On solving equations (3.17)-(3.20), we obtain

(3.22) $H(x,z) = H(0,z) \times \exp[-\{p^+(1-\gamma(z)) + \theta\}x]\overline{H}_V(x),$

(3.23) $E_i(x,z) = E_i(0,z) \times \exp[-\{p - p^+\gamma(z)\}x]\overline{E}_i(x);\ 0 \leq i \leq l,$

(3.24) $D_i(x,z) = D_i(0,z) \times \exp[-\{p^+(1-\gamma(z))\}x]\bar{D}_i(x);\ 0 \leq i \leq l,$

(3.25) $\Im_i(x,z) = \Im_i(0,z) \times \exp[-\{p^+(1-\gamma(z))\}x]\Im_i(x);\ 0 \leq i \leq l.$

Again multiplying each of the equations (3.8)-(3.15) by z^n and then summing over all values of n, we get

(3.26) $H(0,z) = p^+\varepsilon_1 H_0 z,$

(3.27) $E_0(0,z) = \dfrac{\varsigma_0 E_0(z)E_0(0,z)}{z} + \sum_{i=1}^l \frac{E_i(z)E_i(0,z)}{z} + \frac{A(z)H(0,z)}{z} + \theta D(z)H(0,z) + E_0(z)\Im_0(0,z) - (\varsigma_0 \int_0^\infty E_{0,1}(x)\varphi_0(x)dx$
$+ \sum_{i=1}^l \bar{\varsigma}_i \int_0^\infty E_{i,1}(x)\varphi_i(x)dx + \int_0^\infty H_1(x)\varphi_V(x)dx) - (\int_0^\infty H_n(x)\varphi_V(x)dx + \int_0^\infty \Im_{0,0}(x)\phi_0(x)dx + \sum_{i=1}^l \int_0^\infty \Im_{i,0}(x)\phi_i(x)dx),$

(3.28) $E_i(0,z) = \varsigma_i E_0(0,z)E_i(z);\ 1 \leq i \leq l,$

(3.29) $D_i(0,z) = \dfrac{p^- V_i(z)E_i(0,z)}{z};\ 0 \leq i \leq l,$

(3.30) $\Im_i(0,z) = D_i(0,z)F_i(z);\ 0 \leq i \leq l.$

Now, substituting equations (3.25) and (3.27)-(3.29) in equation (3.26) and solving, we get

(3.31) $E_0(0,z) = \dfrac{p^+\varepsilon_1 z H_0[\theta(z-1) + p^+\{\gamma(z)-1\}]D(z)}{[z - \{\varsigma_0 + \sum_{i=1}^l \varsigma_i E_i(z)\}E_0(z) - p^-\{E_0(z)V_0(z)F_0(z) + E_0(z)\sum_{i=1}^l \varsigma_i E_i(z)V_i(z)F_i(z)\}]}.$

Using equation (3.30) equation (3.27) yields

(3.32) $E_i(0,z) = \dfrac{p^+\varepsilon_1 \varsigma_i z H_0[\theta(z-1) + p^+\{\gamma(z)-1\}]D(z)E_0(z)}{[z - \{\varsigma_0 + \sum_{i=1}^l \varsigma_i E_i(z)\}E_0(z) - p^-\{E_0(z)V_0(z)F_0(z) + E_0(z)\sum_{i=1}^l \varsigma_i E_i(z)V_i(z)F_i(z)\}]} 1 \leq i \leq l.$

Applying (3.30)-(3.31) in equation (3.28), we get two equations given below for $i = 0$ and $1 \leq i \leq l$, respectively

(3.33) $D_0(0,z) = \dfrac{p^+p^-\varepsilon_1 H_0[\theta(z-1) + p^+\{\gamma(z)-1\}]D(z)V_0(z)}{[z - \{\varsigma_0 + \sum_{i=1}^l \varsigma_i E_i(z)\}E_0(z) - p^-\{E_0(z)V_0(z)F_0(z) + E_0(z)\sum_{i=1}^l \varsigma_i E_i(z)V_i(z)F_i(z)\}]},$

(3.34) $D_i(0,z) = \dfrac{p^+p^-\varepsilon_1 \varsigma_i H_0[\theta(z-1) + p^+\{\gamma(z)-1\}]D(z)E_0(z)V_i(z)}{[z - \{\varsigma_0 + \sum_{i=1}^l \varsigma_i E_i(z)\}E_0(z) - p^-\{E_0(z)V_0(z)F_0(z) + E_0(z)\sum_{i=1}^l \varsigma_i E_i(z)V_i(z)F_i(z)\}]};\ 1 \leq i \leq l.$

Using (3.32)-(3.33) in equation (3.29), we get two equations given below for $i = 0$ and $1 \leq i \leq l$, respectively

(3.35) $\Im_0(0,z) = \dfrac{p^+p^-\varepsilon_1 H_0[\theta(z-1) + p^+\{\gamma(z)-1\}]D(z)V_0(z)F_0(z)}{[z - \{\varsigma_0 + \sum_{i=1}^l \varsigma_i E_i(z)\}E_0(z) - p^-\{E_0(z)V_0(z)F_0(z) + E_0(z)\sum_{i=1}^l \varsigma_i E_i(z)V_i(z)F_i(z)\}]},$

(3.36) $\Im_i(0,z) = \dfrac{p^+p^-\varepsilon_1 \varsigma_i H_0[\theta(z-1) + p^+\{\gamma(z)-1\}]D(z)E_0(z)V_i(z)F_i(z)}{[z - \{\varsigma_0 + \sum_{i=1}^l \varsigma_i E_i(z)\}E_0(z) - p^-\{E_0(z)V_0(z)F_0(z) + E_0(z)\sum_{i=1}^l \varsigma_i E_i(z)V_i(z)F_i(z)\}]};\ 1 \leq i \leq l.$

***Lemma* 3.2** *The probability generating functions of the stationary joint distribution of the system size and server state of the Markov process under the established stability condition are given by*

$$H(x,z) = p^{+}\varepsilon_1 z H_0 \exp[-\{p^{+}(1-\gamma(z))+\theta\}x]\overline{H}_V(x),$$

$$E_0(x,z) = \frac{p^{+}\varepsilon_1 z H_0[\theta(z-1)+p^{+}\{\gamma(z)-1\}]D(z)}{[z-\{\varsigma_0+\sum_{i=1}^{l}\varsigma_i E_i(z)\}E_0(z)-p^{-}\{E_0(z)V_0(z)F_0(z)+E_0(z)\sum_{i=1}^{l}\varsigma_i E_i(z)V_i(z)F_i(z)\}]}\exp[-\{p-p^{+}\gamma(z)\}x]\overline{E}_0(x),$$

$$E_i(x,z) = \frac{p^{+}\varepsilon_1 \varsigma_i z H_0[\theta(z-1)+p^{+}\{\gamma(z)-1\}]D(z)E_0(z)}{[z-\{\varsigma_0+\sum_{i=1}^{l}\varsigma_i E_i(z)\}E_0(z)-p^{-}\{E_0(z)V_0(z)F_0(z)+E_0(z)\sum_{i=1}^{l}\varsigma_i E_i(z)V_i(z)F_i(z)\}]}\exp[-\{p-p^{+}\gamma(z)\}x]\overline{E}_i(x);\ 1\le i\le l,$$

$$D_0(x,z) = \frac{p^{+}p^{-}\varepsilon_1 H_0[\theta(z-1)+p^{+}\{\gamma(z)-1\}]D(z)V_0(z)}{[z-\{\varsigma_0+\sum_{i=1}^{l}\varsigma_i E_i(z)\}E_0(z)-p^{-}\{E_0(z)V_0(z)F_0(z)+E_0(z)\sum_{i=1}^{l}\varsigma_i E_i(z)V_i(z)F_i(z)\}]}\exp[-\{p^{+}(1-\gamma(z))\}x]\bar{D}_0(x),$$

$$D_i(x,z) = \frac{p^{+}p^{-}\varepsilon_1 \varsigma_i H_0[\theta(z-1)+p^{+}\{\gamma(z)-1\}]D(z)E_0(z)V_i(z)}{[z-\{\varsigma_0+\sum_{i=1}^{l}\varsigma_i E_i(z)\}E_0(z)-p^{-}\{E_0(z)V_0(z)F_0(z)+E_0(z)\sum_{i=1}^{l}\varsigma_i E_i(z)V_i(z)F_i(z)\}]}\exp[-\{p^{+}(1-\gamma(z))\}x]\bar{D}_i(x);\ 1\le i\le l,$$

$$\Im_0(x,z) = \frac{p^{+}p^{-}\varepsilon_1 H_0[\theta(z-1)+p^{+}\{\gamma(z)-1\}]D(z)V_0(z)F_0(z)}{[z-\{\varsigma_0+\sum_{i=1}^{l}\varsigma_i E_i(z)\}E_0(z)-p^{-}\{E_0(z)V_0(z)F_0(z)+E_0(z)\sum_{i=1}^{l}\varsigma_i E_i(z)V_i(z)F_i(z)\}]}\exp[-\{p^{+}(1-\gamma(z))\}x]\Im_0(x),$$

$$\Im_i(x,z) = \frac{p^{+}p^{-}\varepsilon_1 \varsigma_i H_0[\theta(z-1)+p^{+}\{\gamma(z)-1\}]D(z)E_0(z)V_i(z)F_i(z)}{[z-\{\varsigma_0+\sum_{i=1}^{l}\varsigma_i E_i(z)\}E_0(z)-p^{-}\{E_0(z)V_0(z)F_0(z)+E_0(z)\sum_{i=1}^{l}\varsigma_i E_i(z)V_i(z)F_i(z)\}]}\exp[-\{p^{+}(1-\gamma(z))\}x]\Im_i(x);\ 1\le i\le l.$$

***Lemma* 3.3** *The marginal probability generating functions of the system size when the server is on working vacation, busy with FES, busy with SOS, waiting for repair due to failure in FES, waiting for repair due to failure in SOS, under repair due to failure in FES and under repair due to failure in SOS, respectively are given by*

$$H(z) = (1+p^{+}\varepsilon_1 z)D(z)H_0,$$

$$E_0(z) = \frac{p^{+}\varepsilon_1 z H_0[\theta(z-1)+p^{+}\{\gamma(z)-1\}]D(z)V_0(z)}{[z-\{\varsigma_0+\sum_{i=1}^{l}\varsigma_i E_i(z)\}E_0(z)-p^{-}\{E_0(z)V_0(z)F_0(z)+E_0(z)\sum_{i=1}^{l}\varsigma_i E_i(z)V_i(z)F_i(z)\}]},$$

$$E_i(z) = \frac{p^{+}\varepsilon_1 \varsigma_i z H_0[\theta(z-1)+p^{+}\{\gamma(z)-1\}]D(z)E_0(z)V_i(z)}{[z-\{\varsigma_0+\sum_{i=1}^{l}\varsigma_i E_i(z)\}E_0(z)-p^{-}\{E_0(z)V_0(z)F_0(z)+E_0(z)\sum_{i=1}^{l}\varsigma_i E_i(z)V_i(z)F_i(z)\}]},\ 1\le i\le l,$$

$$D_0(z) = \frac{p^{-}\varepsilon_1 H_0[\theta(z-1)+p^{+}\{\gamma(z)-1\}]D(z)V_0(z)}{[z-\{\varsigma_0+\sum_{i=1}^{l}\varsigma_i E_i(z)\}E_0(z)-p^{-}\{E_0(z)V_0(z)F_0(z)+E_0(z)\sum_{i=1}^{l}\varsigma_i E_i(z)V_i(z)F_i(z)\}]}\frac{[1-F_0(z)]}{[1-C(z)]},$$

$$D_i(z) = \frac{p^{-}\varepsilon_1 \varsigma_i H_0[\theta(z-1)+p^{+}\{\gamma(z)-1\}]D(z)E_0(z)V_i(z)}{[z-\{\varsigma_0+\sum_{i=1}^{l}\varsigma_i E_i(z)\}E_0(z)-p^{-}\{E_0(z)V_0(z)F_0(z)+E_0(z)\sum_{i=1}^{l}\varsigma_i E_i(z)V_i(z)F_i(z)\}]}\frac{[1-F_i(z)]}{[1-C(z)]};\ 1\le i\le l,$$

$$\Im_0(z) = \frac{p^{-}\varepsilon_1 H_0[\theta(z-1)+p^{+}\{\gamma(z)-1\}]D(z)V_0(z)F_0(z)}{[z-\{\varsigma_0+\sum_{i=1}^{l}\varsigma_i E_i(z)\}E_0(z)-p^{-}\{E_0(z)V_0(z)F_0(z)+E_0(z)\sum_{i=1}^{l}\varsigma_i E_i(z)V_i(z)F_i(z)\}]}\frac{[1-E_0(z)]}{[1-C(z)]},$$

$$\Im_i(z) = \frac{p^{-}\varepsilon_1 \varsigma_i H_0[\theta(z-1)+p^{+}\{\gamma(z)-1\}]D(z)E_0(z)V_i(z)F_i(z)}{[z-\{\varsigma_0+\sum_{i=1}^{l}\varsigma_i E_i(z)\}E_0(z)-p^{-}\{E_0(z)V_0(z)F_0(z)+E_0(z)\sum_{i=1}^{l}\varsigma_i E_i(z)V_i(z)F_i(z)\}]}\frac{[1-E_i(z)]}{[1-C(z)]};\ 1\le i\le l.$$

where, $H_0 = \dfrac{(1-\rho)}{(1-\rho)+(p^{+}\varepsilon_1+\theta\rho)\tilde{H}(\theta)}$ *which is obtained by using normalizing condition* (3.16).

***Theorem* 3.1** *The probability generating function of system size denoted by* $G_S(Z)$*, is given by*

$$G_S(z) = \frac{\theta}{p^{+}}H_0 + H(z) + \textstyle\sum_{i=0}^{k} E_i(z) + \sum_{i=0}^{k} D_i(z) + \sum_{i=0}^{k} \Im_i(z).$$

4 Performance Indices

The performance prediction of a queueing system can be done by analyzing the system. For this, first we find the steady state probabilities for different states of the server which we use to obtain various useful performance measures. Various useful queueing and reliability indices of an unreliable $M^X/G/1/G$-queue with delayed repair where server follows working vacation policy, are given by

(a) Steady state probabilities

1. Probability that the server is on working vacation, denoted by $P[H]$ and is given by

$$P[H] = H(1) = \left[1+\frac{p^{+}\varepsilon_1\{1-H_V^{*}(\theta)\}}{\theta}\right]H_0.$$

2. Probability that the server is busy in providing *FES* to the customers denoted by $P[E_0]$ and is obtained as $P[E_0] = E_0(1) = \dfrac{p^+\varepsilon_1[\theta + p^+\varepsilon_1][1 - E_0^*(p^-)][1 - H_V^*(\theta)]H_0}{\theta p^-(1-\rho)}$.
3. Probability that the server is busy in providing *SOS* to the customers denoted by $P[E_i]$ for $1 \le i \le l$ and is found to be $P[E_i] = E_i(1) = \dfrac{p^+\varepsilon_1\varsigma_i[\theta + p^+\varepsilon_1]E_0^*(p^-)[1 - E_i^*(p^-)][1 - H_V^*(\theta)]H_0}{\theta p^-(1-\rho)}; 1 \le i \le l.$
4. Probability that the server is waiting for repair due to failure in *FES* of the customers denoted by $P[D_0]$ and is given by $P[D_0] = D_0(1) = \dfrac{p^+\varepsilon_1\omega_0^{(1)}[\theta + p^+\varepsilon_1][1 - E_0^*(p^-)][1 - H_V^*(\theta)]H_0}{\theta p^-(1-\rho)}$.
5. Probability that the server is waiting for repair due to failure in *SOS* of the customers denoted by $P[D_i]$ and is given by $P[D_i] = D_i(1) = \dfrac{p^+\varepsilon_1\varsigma_i\omega_i^{(1)}[\theta + p^+\varepsilon_1]E_0^*(p^-)[1 - E_0^*(p^-)][1 - H_V^*(\theta)]H_0}{\theta p^-(1-\rho)};\ 1 \le i \le l.$
6. Probability that the server is under repair due to failure in *FES* of the customers denoted by $P[\Im_0]$ and is given by $P[\Im_0] = \Im_0(1) = \dfrac{p^+p^-\varepsilon_1\varpi_0^{(1)}[\theta + p^+\varepsilon_1][1 - E_0^*(p^-)][1 - H_V^*(\theta)]H_0}{\theta p^-(1-\rho)}$.
7. Probability that the server is under repair due to failure in *SOS* of the customers denoted by $P[\Im_i]$ and is given by $P[\Im_i] = \Im_i(1) = \dfrac{p^+p^-\varepsilon_1\varsigma_i\varpi_i^{(1)}[\theta + p^+\varepsilon_1]E_0^*(p^-)[1 - E_0^*(p^-)][1 - H_V^*(\theta)]H_0}{\theta p^-(1-\rho)}; 1 \le i \le l.$

(b) Average number and mean waiting time of customers in the system

The average number and mean waiting time of customers in the system denoted by L_S and W_S, respectively are given by

$L_S = G_S'(1) = H'(1) + E_0{}'(1) + \sum_{i=1}^{l} E_i{}'(1) + D_0{}'(1) + \sum_{i=1}^{l} D_i{}'(1) + \Im_0{}'(1) + \sum_{i=1}^{l} \Im_i{}'(1)$ and $W_S = \frac{L_S}{p^+\varepsilon_1}$.

For, $0 \le i \le l$ we have

$$b_i = E_i'^*(p^-), \nu_i = \frac{[1 - E_i^*(p^-)]}{p^-}, e_i = f_i = 1, e_i' = p^+\varepsilon_1\varpi_i^{(1)}, f_i' = p^+\varepsilon_1\omega_i^{(1)},$$

$$b_i' = -p^+\varepsilon_1 E_i^{*\prime}(p^-), \nu_i' = \frac{p^+\varepsilon_1[p^-E_i^{*\prime}(p^-)1 - E_i^*(p^-)]}{(p^-)^2},$$

$$e_i'' = p^+\varepsilon_2\varpi_i^{(1)} + (p^+\varepsilon_1)^2\varpi_i^{(2)}, f_i'' = p^+\varepsilon_2\omega_i^{(1)} + (p^+\varepsilon_1)^2\omega_i^{(2)}, b_i'' = -p^+\varepsilon_2 E_i^{*\prime}(p^-) + (p^+\varepsilon_1)^2 E_i^{*\prime\prime}(p^-),$$

$$\nu_i'' = \frac{[p^+p^-\{p^-\varepsilon_2 + 2p^+\varepsilon_1^2\}\{p^-E_i^{*\prime}(p^-)1 - E_i^*(p^-)\} - (p^-)^3(p^+\varepsilon_1)^2E_i^{*\prime\prime}(p^-)]}{(p^-)^4},$$

$$d_1 = \frac{1 - H_V^*(\theta)}{\theta}; d_1' = \frac{p^+\varepsilon_1[d_1 + H_V^{*\prime}(\theta)]}{\theta}; T' = (\theta + p^+\varepsilon_1); T'' = p^+\varepsilon_2,$$

$$\Psi_0' = b_0\sum_{i=1}^{l} b_i' + b_0'(\varsigma_0 + \sum_{i=1}^{l}\varsigma_i b_i), \Psi_0'' = b_0\sum_{i=1}^{l}\varsigma_i b_i'' + b_0''(\varsigma_0 + \sum_{i=1}^{l}\varsigma_i b_i) + 2b_0'\sum_{i=1}^{l}\varsigma_i b_i',$$

$$\Psi_1' = e_0'\nu_0 f_0 + e_0\nu_0' f_0 + e_0\nu_0 f_0', \Lambda'' = -2\varepsilon_1\Omega'; \Lambda''' = -3(\varepsilon_1\Omega'' + \varepsilon_2\Omega')$$

$$\Psi_2' = b_0\sum_{i=1}^{l}\varsigma_i(e_i'\nu_i f_i + e_i\nu_i' f_i + e_i\nu_i f_i')\Psi_1'' = 2(e_0'\nu_0' f_0 + e_0'\nu_0 f_0' + e_0\nu_0' f_0') + e_0''\nu_0 f_0 + e_0\nu_0'' f_0 + e_0\nu_0 f_0'',$$

$$\Psi_2'' = 2b_0'\sum_{i=1}^{l}\varsigma_i(e_0'\nu_0' f_0 + e_0'\nu_0 f_0' + e_0\nu_0' f_0') + \sum_{i=1}^{l}[2(e_i'\nu_i' f_i + e_i'\nu_i f_i' + e_i\nu_i' f_i') + e_i''\nu_i f_i + e_i\nu_i'' f_i + e_i\nu_i f_i''],$$

$$\Omega' = 1 - \psi_0' - p^-(\psi_1' + \psi_2'); \Omega'' = -\psi_0'' - p^-(\psi_1'' + \psi_2''),$$

$$H'(1) = \frac{p^+\varepsilon_1[(p^+\varepsilon_1 + \theta)d_1 + p^+\varepsilon_1 H_V^*(\theta)]H_0}{\theta}; E_0'(1) = \frac{p^+\varepsilon_1 H_0[Num''\Omega' - Num'\Omega'']}{2(\Omega')^2},$$

$$E_i'(1) = \frac{p^+\varepsilon_1\varsigma_i H_0[Num_1''\Omega' - Num_1'\Omega'']}{2(\Omega')^2},$$

$$D_0'(1) = \frac{p^-\varepsilon_1 H_0[Num_2'''\Lambda'' - Num_2''\Lambda''']}{3(\Lambda'')^2}; D_i'(1) = \frac{p^-\varepsilon_1\varsigma_i H_0[Num_3'''\Lambda'' - Num_3''\Lambda''']}{3(\Lambda'')^2},$$

$$\mathfrak{I}_0'(1) = \frac{p^-\varepsilon_1 H_0[Num_4'''\Lambda'' - Num_4''\Lambda''']}{3(\Lambda'')^2}; \mathfrak{I}_i'(1) = \frac{p^-\varepsilon_1\varsigma_i H_0[Num_5'''\Lambda'' - Num_5''\Lambda''']}{3(\Lambda'')^2},$$

$$Num' = d_1\nu_0 T'; Num'' = 2(d_1\nu_0 T' + d_1'\nu_0 T' + d_1\nu_0' T') + d_1\nu_0 T'',$$

$$Num_1' = d_1\nu_i b_0 T'; Num_1'' = 2(d_1\nu_i b_0 T' + d_1'\nu_i b_0 T' + d_1\nu_i' b_0 T' + d_1\nu_i b_0' T') + d_1\nu_i b_0 T'',$$

$$Num_2'' = -2d_1\nu_0 f_0' T'; Num_2''' = -6(d_1\nu_0' + d_1'\nu_0 + d_1\nu_0')f_0' T' - 3d_1\nu_0(T' f_0'' + T'' f_0'),$$

$$Num_3'' = -2d_1\nu_0 f_i' T'; Num_3''' = -6(d_1\nu_i' b_0 + d_1'\nu_i b_0 + d_1\nu_i b_0')f_i' T' - 3d_1\nu_i b_0(T' f_i'' + T'' f_i'),$$

$$Num_4'' = -2d_1\nu_0 e_0' f_0 T'; Num_4''' = -6(d_1\nu_0' f_0 + d_1'\nu_0 f_0 + d_1\nu_0 f_0')e_0' T' - 3d_1\nu_0 f_0(T' e_0'' + T'' e_0'),$$

$$Num_5'' = -2d_1\nu_i e_0' b_0 T'; Num_5''' = -6[d_1 b_0(\nu_i f_i' + f_i\nu_i') + \nu_i(d_1' b_0 + d_1 b_0')]e_i' T' - 3d_1\nu_i b_0(T' e_i'' + T'' e_i').$$

(c) Reliability Indices

Let $A(t)$ represents the point wise availability of the server at time 't' i.e. probability that the server is either serving a customer in *FES* or in *SOS* or the server is free. Let $\tilde{A} = \underset{t\to\infty}{Lim} A(t)$ denotes the steady state availability of the server, then we have

$$\tilde{A} = \underset{t\to\infty}{Lim} A(t) = 1 - \mathfrak{I}_0(1) - \textstyle\sum_{i=1}^{l} \mathfrak{I}_i(1) = 1 - P[\mathfrak{I}_0] - \sum_{i=1}^{l} P[\mathfrak{I}_i].$$

Let $\tilde{F}$ denotes the steady state failure frequency of the server and it is given by

$$\tilde{F} = p^-[E_0(1,1) + \textstyle\sum_{i=1}^{l} E_i(1,1)] = p^-[P[E_0] + \sum_{i=1}^{l} P[E_i]].$$

(d) Busy Period

***Theorem* 4.1** *Under the steady state conditions, let the expected length of busy period and busy cycle be denoted by $E[B]$and $E[C]$, respectively then we have*

$$E[C] = \frac{1}{p^+\varepsilon_1 H_0} \textit{ and } E[B] = \frac{(1 - H_0)}{p^+\varepsilon_1 H_0}.$$

Proof. By applying the arguments of an alternating renewal process, we can use the following results directly as

$$H_0 = \frac{E[I]}{E[I] + E[B]}, E[B] = \frac{(1 - H_0)}{p^+\varepsilon_1 H_0} \text{ and } E[C] = \frac{1}{p^+\varepsilon_1 H_0} = E[I] + E[B],$$

where, $E[I]$is the time length that the system is in empty state. Since the inter arrival time between two customers follows exponential distribution with parameter p^+, we have $E[I] = \frac{1}{p^+\varepsilon_1}$.

5 Special Cases

In this section, we deduce some special cases to validate our results with that of the developed model from the literature:

***Case* I:** $M/G/1$ G-queue with unreliable server, working vacations and vacation interruption (i.e. No batch arrival, No two phase service and No delayed repair)

Setting $\varepsilon_1 = 1,\ \varepsilon_n = 0\ \forall n \geq 1,\ C(z) = z,\ \varsigma_i = 0(1 \leq i \leq l), \varpi_i^{(1)} = 1,\ \varpi_i^{(r)} = 0(1 \leq i \leq l), (r \geq 2)$, we have

$$H(1) = [1 + \frac{p^+\{1 - H_V^*(\theta)\}}{\theta}]H_0,$$

$$E_0(1) = \frac{p^+[\theta + p^+][1 - E_0^*(p^-)][1 - H_V^*(\theta)]H_0}{\theta p^-(1-\rho)},$$

$$\mathfrak{I}_0(1) = \frac{p^+ p^-[\theta + p^+][1 - E_0^*(p^-)][1 - H_V^*(\theta)]H_0}{\theta p^-(1-\rho)}.$$

The above results coincide with that of Zhang and Liu [5].

***Case* II:** $M/G/1$ queue with unreliable server, working vacations and vacation interruption (i.e. No negative arrival, No batch arrival, No two phase service and No delayed repair)

Setting $p \to 0,\ \varepsilon_1 = 1,\ \varepsilon_n = 0\ \forall n \geq 1,\ C(z) = z,\ \varsigma_i = 0(1 \leq i \leq l),\ \varpi_i^{(1)} = 1,\ \varpi_i^{(r)} = 0(1 \leq i \leq l)(r \geq 2)$, our results coincides with that of Zhang and Hou [10].

6 Cost Function

In this section, we have constructed a cost function to make the system cost effective. The various cost elements associated with different activities are given as follows:

C_1 : Holding cost per unit time of each customer present in the system,

C_2 : Cost per unit time when the server is on and in operation,

C_3 : Setup cost per busy cycle,

C_4 : Startup cost per unit time for the preliminary work done by the server before initiating the service.

We construct the function for the expected total cost per unit time as follows:

$$E[TC] = C_1 L_S + C_2[1 - H_0] + C_3 p^+ \varepsilon_1 + C_4 H_0.$$

7 Numerical Illustration

This section explores the queueing congestion situation of a call/contact center of a certain mobile company. For illustration purpose, we assume that all distribution functions used in this paper are exponential i.e. $H(x)$, $E_i(x)$, $D_i(x)$, $\Im_i(x)(0 \leq i \leq l)$ are all exponential with rates q (working vacation completion rate), μ_B (service rate during *FES*), $\mu_{B_1}, \mu_{B_2}, \mu_{B_3}$ (service rate during *SPS* with $l = 3$), g_1 (delayed repair rate during *FES*), g_{21}, g_{22}, g_{23} (delayed repair rate during *SPS* with $l = 3$), b_1 (repair rate during *FES*), b_{21}, b_{22}, b_{23} (repair rate during *SPS* with $l = 3$, respectively. There are company service representatives *(CSR's)* whose primary job is to attend and respond to customer's calls. When *CSR* is free, he/she may perform some secondary task during which he/she can make phone calls to promote company's service and products, but if a fresh call arrives, the *CSR* resumes its primary job and assist the call. This situation is called as *working vacation and vacation interruption* in queueing terminology. The calls arrive to the call/contact center in batches in Poisson fashion with rate $p^+ = 4$ messages/min.. The batch size is assumed to be geometrically distributed with $\varepsilon_1 = 0.67$ and $\varepsilon_2 = 0.89$. The calls are served by the *CSR* with rate $\mu_B = 5$ calls/min.. If the customer's problem is not solved by that *CSR*, he/she passes the request to another *CSR's* with rates $\mu_{B_1} = \mu_{B_2} = \mu_{B_3} = 4.5$ calls /min.. The *CSR* can perform secondary task during working vacation with rate $h = 0.2$ calls/min. and if there is some primary call, it interrupts its vacation and returns to the system with rate $q = 0.4$ calls/min. There are various harmful activities such as stealing hard disk space or *CPU* time, retrieving secret information, distorting data, etc. (called negative customers in queueing terminology) which arrives with rate $p- = 0.5$ messages/min. and forces the server to leave the system for repair. Before repair, the repairman took some time for preliminary settings; the *CSR* is said to be under delayed repair state. The delayed repair (repair) rates are taken as $g_1 = 8, g_{21} = 6 = g_{22} = g_{23}$ ($b_1 = 12, b_{21} = 8 = b_{22} = b_{23}$). For cost effective model, we consider cost sets as $C_1 = \$5; C_2 = \$20; C_3 = \$100; C_4 = \$20$. By coding a program in MATLAB software, we obtain following performance measures:$P[H] = 0.1483$; $P[E_0] = 0.3109$; $P[E_S] = \sum_{i=1}^{3} P[E_i] = 0.0821$; $P[D_0] = 0.1360$; $P[D_S] = \sum_{i=1}^{3} P[D_i] = 0.1541$; $P[\Im_0] = 0.0907$; $P[\Im_S] = \sum_{i=1}^{3} P[\Im_i] = 0.1156$; $L_S = 13.33$; $WS = 4.99$ min.; $\tilde{A} = 0.7937$; $\tilde{F} = 1.3756$ and $E[TC] = \$353.31$.

8 Sensitivity Analysis

In this section, we explore the effects of some critical system parameters on various queueing and reliability indices by coding a program in MATLAB software. We assume that batch size is geometric distributed whereas service time, delayed repair time, repair time and vacation time are assumed to be exponential distributed. The results are summarized in ***Figures* 8.1-8.2** and in ***Tables* 8.1-8.5**. We have made the system cost effective by providing trends in expected system cost by varying some critical system parameters and cost elements. The default parameters for ***Table* 8.1** are taken as $k = 3, p^- = 3.5, p^+ = 4, \mu_B = 5, \mu_{B_1} = \mu_{B_2} = \mu_{B_3} = 4.5, \theta = 0.2, \mu_v = 0.2, \gamma_1 = 8, \gamma_{21} = \gamma_{22} = \gamma_{23} = 6, \beta_1 = 12, \beta_{21} = 8, \beta_{22} = 8, \beta_{23} = 8$; for ***Tables* 8.2-8.5** are taken as $k = 3, p^- = 3.5, p^+ = 4, \mu_B = 5, \mu_{B_1} = \mu_{B_2} = \mu_{B_3} = 4.5, \theta = 0.2, \mu_v = 0.2, \gamma_1 = 8, \gamma_{21} = \gamma_{22} = \gamma_{23} = 6, \beta_1 = 12, \beta_{21} = 8, \beta_{22} = 8, \beta_{23} = 8$ and for ***Figs* 8.1-8.2** are taken as $k = 3, p_- = 3, p^+ = 3, \mu_B = 4.5, \mu_{B_1} = \mu_{B_2} = \mu_{B_3} = 4, \theta = 0.2, \mu_v = 0.2, \gamma_1 = 8, \gamma_{21} = \gamma_{22} = \gamma_{23} = 6, \beta_1 = 12, \beta_{21} = \beta_{22} = \beta_{23} = 8$ and for ***Figs* 8.1** are taken as $k = 3, p^- = 3, p^+ = 3, \mu_B = 4.5, \mu_{B_1} = \mu_{B_2} = \mu_{B_3} = 4.5, \theta = 0.2, \mu_v = 0.2, \gamma_1 = 8, \gamma_{21} = \gamma_{22} = \gamma_{23} = 6, \beta_1 = 12, \beta_{21} = 8, \beta_{22} = 8, \beta_{23} = 8$ and for ***Figs* 8.2** are taken as $k = 3, p^- = 3, p^+ = 3, \mu_B = 4.5, \mu_{B_1} = \mu_{B_2} = \mu_{B_3} = 4, \theta = 0.2, \mu_v = 0.2, \gamma_1 = 8, \gamma_2 1 = \gamma_{22} = \gamma_{23} = 6, \beta_1 = 12, \beta_{21} = \beta_{22} = \beta_{23} = 8$.

***Tables* 8.1-8.2** show the effect of some parameters such as p^-, μ_B, γ_1 and β_1 on various performance measures. ***Table* 8.1** depicts that the negative arrival rate p^- has significant effect on the server when the server is either on working vacation or under delayed repair state or under repair state; we see an increasing trend in the long run probabilities $P[H]$, $P[D_0]$, $P[D_S]$, $P[\Im_0]$ and $P[\Im_S]$ for increasing values of negative arrival rate p^-. On the other hand $P[E_0]$ and $P[E_S]$ decrease with the increase in the values of p^-. This is due to the fact that the presence of negative customer leads to server failure which forces the server to leave the system for repair. Moreover we observe from ***Table* 8.1** that the server is more prone to failure and is less available in the system when the negative customer enters the system with higher rate. From ***Table* 8.1**, we also notice that the long run probabilities $P[E_0]$, $P[E_S]$, $P[D_0]$, $P[D_S]$, $P[\Im_0]$ and $P[\Im_S]$ decrease while that of $P[H]$ increase with the increase in service rate μ_B. Also $\tilde{F}$ decreases but $\tilde{A}$ increases on increasing the values of μ_B.

***Table* 8.1:** Effect of (a) p^- and (b) μ_B on various performance measures

p^-	$P[H]$	$P[E_0]$	$P[E_S]$	$P[D_0]$	$P[D_S]$	$P[\Im_0]$	$P[\Im_S]$	$\tilde{A}$	$\tilde{F}$
3.2	0.0345	0.3244	0.1121	0.1297	0.1842	0.1038	0.1382	0.7579	1.3970
3.4	0.0421	0.3165	0.1065	0.1345	0.1907	0.1076	0.1430	0.7492	1.4387
3.6	0.0494	0.3090	0.1014	0.1390	0.1968	0.11128	0.1476	0.7410	1.4778
3.8	0.0564	0.3019	0.0966	0.1434	0.2027	0.1147	0.1520	0.7332	1.5145
μ_B	$P[H]$	$P[E_0]$	$P[E_S]$	$P[D_0]$	$P[D_S]$	$P[\Im_0]$	$P[\Im_S]$	$\tilde{A}$	$\tilde{F}$
5.0	0.0458	0.3127	0.1039	0.1368	0.1938	0.1094	0.1453	0.7451	1.4585
5.2	0.0527	0.3054	0.1014	0.1336	0.1937	0.1069	0.1453	0.7477	1.4242
5.4	0.0593	0.2984	0.0991	0.1305	0.1936	0.1044	0.1452	0.7502	1.3914
5.6	0.0656	0.2917	0.0968	0.1276	0.1936	0.1021	0.1452	0.7526	1.3602

As clear from ***Table* 8.2**, the average system size, average waiting time and the expected system cost decreases with the increase in the values of either γ_1 or β_1. This is because on increasing either γ_1 or β_1, customers spent less time under delayed repair or repair state, which in turn decreases each of LS, WS and $E[TC]$. In ***Tables* 8.3-8.5**, we explore the effect of various cost elements viz. C_1, C_2, C_3 and C_4 on $E[TC]$.

***Figures* 8.1-8.2** display the trends in average system size and expected system cost by varying some parameters such as p^+, *mB, mV* and $\mu_B 1$. From ***Figs* 8.1-8.2**, we observe that both L_S and *E[TC]* increases linearly with the increase in values of arrival rate of positive customers whereas these decrease first slowly then sharply for increasing the values of either of service rates *mB, mV* and $\mu_B 1$. This feature matches with many real life congestion situations wherein if the server provides service during working vacation with higher rate, customers tend to accumulate more in the system which increases both average system size and expected system cost. Moreover the average system size decreases on increasing the values of vacation rate q. Also, ***Fig.* 8.2** depict that the negative arrival rate tends to increase the total system cost.

***Table* 8.2:** Effect of (a) γ_i and (b) β_1 on L_S, W_S and $E[TC]$ for cost sets $C_1 = \$5$, $C_2 = \$20$, $C_3 = \$100$, $C_4 = \$20$.

	$q = 0.4$			$q = 0.5$		
γ_1	L_S	W_S	$E[TC]$	L_S	W_S	$E[TC]$
6	13.5522	5.0835	$354.44	12.8803	4.8301	$351.06
7	13.4869	5.0576	$354.10	12.7379	4.7767	$350.35
8	13.3281	4.9980	$353.30	12.5360	4.7010	$349.34
9	13.1784	4.9419	$352.55	12.3580	4.6342	$348.45
β_1	L_S	W_S	$E[TC]$	L_S	W_S	$E[TC]$
9	13.5225	5.0709	$354.27	12.8014	4.8005	$350.67
10	13.4657	5.0496	$353.99	12.7136	4.7676	$350.23
11	13.3963	5.0236	$353.64	12.6215	4.7330	$349.77
12	13.3281	4.9980	$353.30	12.5360	4.7010	$349.34

***Table* 8.3:** Effects of cost elements $(C_1,\ C_2)$ on $E[TC]$ with fixed $(C_3,\ C_4) = (\$100,\ \$20)$.

(C_1, C_2)	($5, $20)	($5,$ 40)	($5, $60)	($10, $40)	($15, $40)
$E[TC]$	$353.307276	$372.762397	$392.217519	$439.403006	$506.043615

***Table* 8.4:** Effect of cost elements (C_2, C_4) on $E[TC]$ with fixed (C_1, C_3) = (\$5, \$100).

(C_2, C_4)	(\$40, \$10)	(\$40,\$20)	(\$40, \$30)	(\$50, \$20)	(\$60, \$20)
$E[TC]$	\$372.489958	\$372.762397	\$373.034836	\$382.489958	\$392.217519

***Table* 8.5:** Effects of cost elements (C_3, C_4) on $E[TC]$ with fixed (C_1, C_2) = (\$5, \$40).

(C_2, C_4)	(\$110, \$10)	(\$110,\$20)	399.701503	(\$120, \$20)	(\$130, \$20)
$E[TC]$	\$399.156625	\$399.429064	399.701503	\$426.095731	\$452.762397

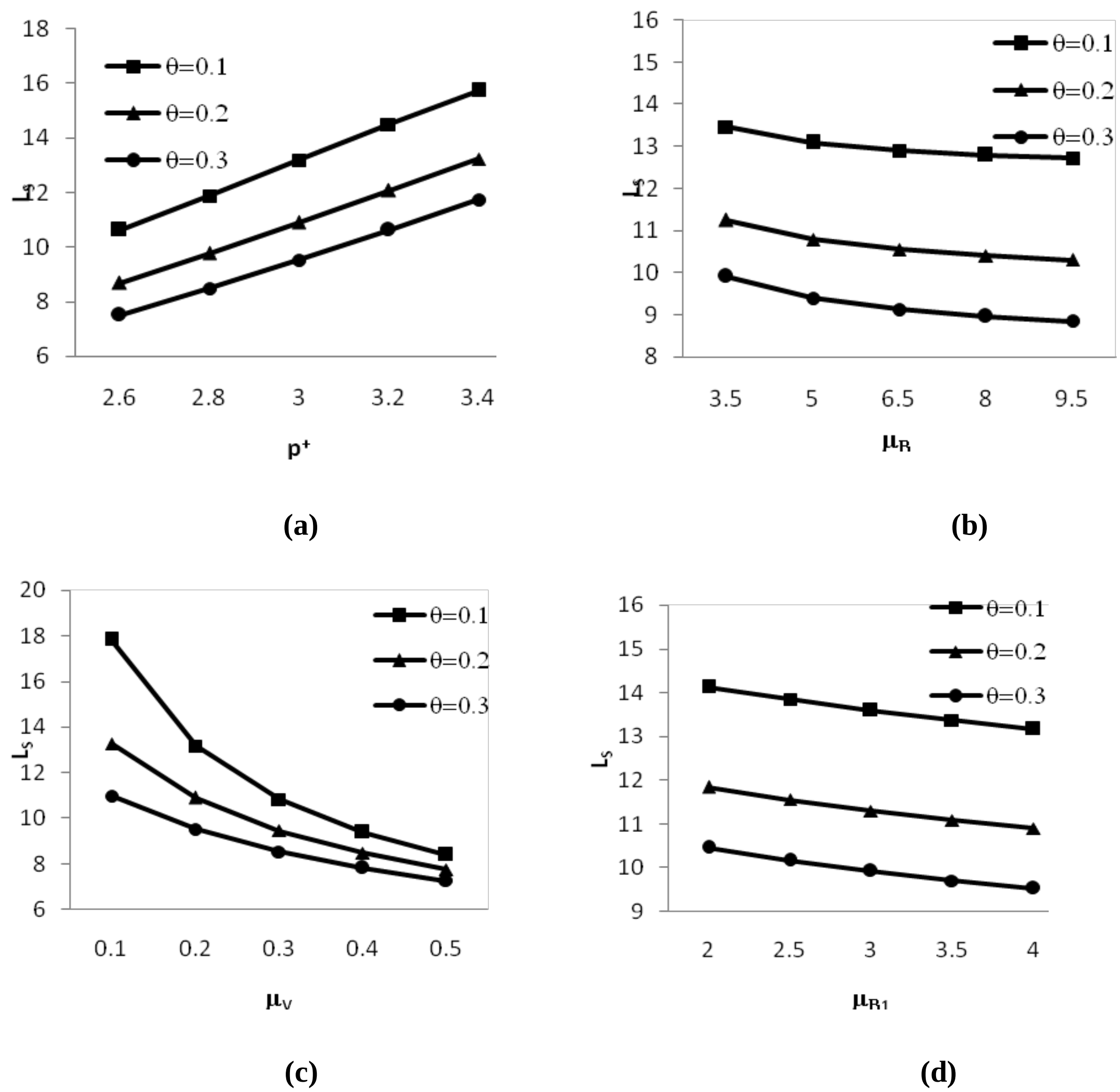

***Figure* 8.1:** L_S versus (a) p^+ (b) μ_B (c) μ_v (d) μ_{B_1}

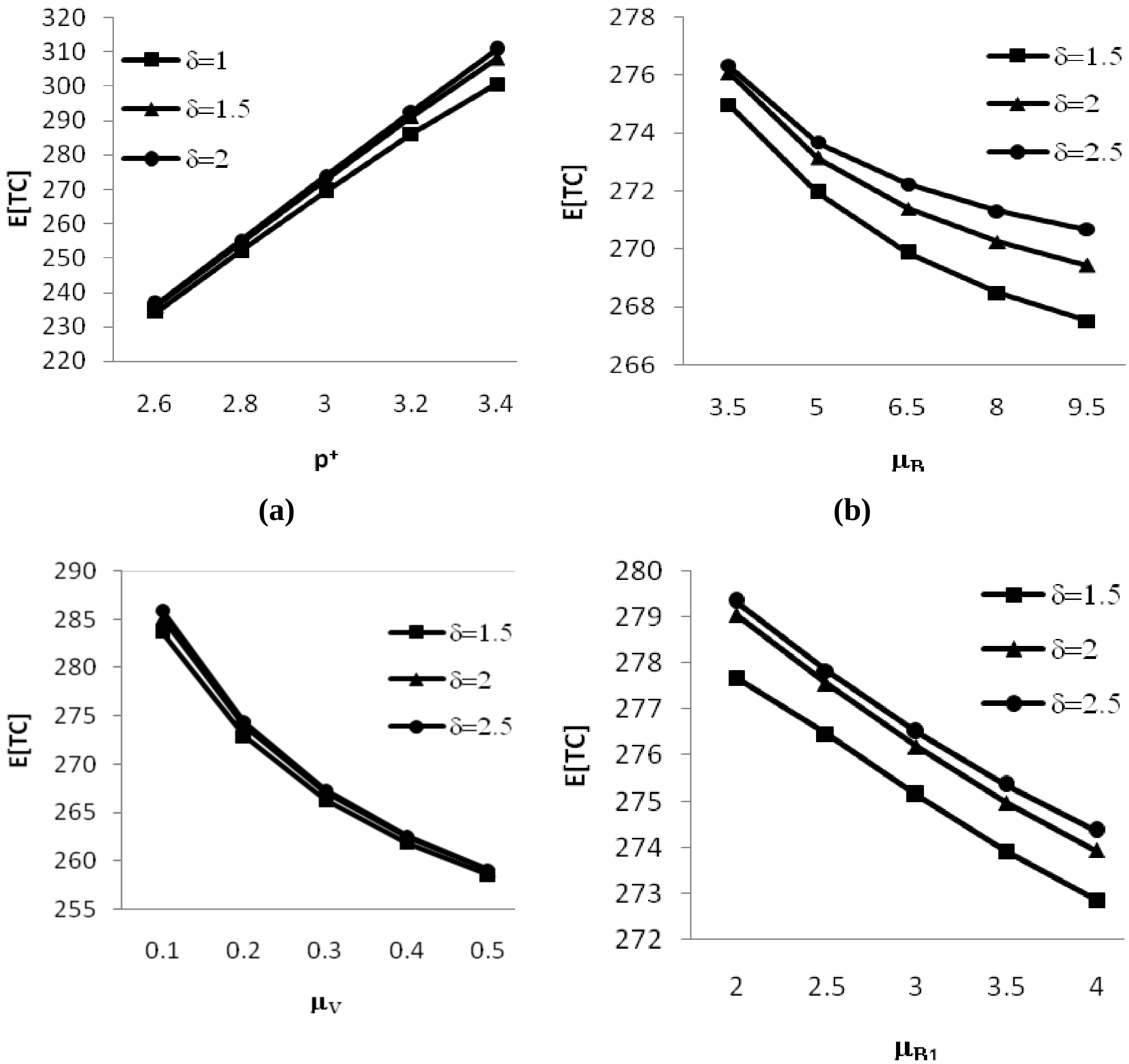

***Figure* 8.2:** $E[TC]$ versus (a) p^+ (b) μ_B (c) μ_v (d) μ_{B_1} for cost sets $C_1 = 5$, $C_2 = 20$, $C_3 = 100$, $C_4 = 20$.

Finally we conclude that

- The system designers and decision makers can build a cost effective system by controlling some critical system parameters such as delayed repair rate, repair rate, service rate and various cost elements as our results show a significant effect of these parameters on expected system cost.
- As observed from these results, the negative arrival rate has a significant impact on average system size, average waiting time and expected system cost; this parameter must be controlled in an effective manner.
- The average queue length, average waiting time and expected system cost can be reduced to some extent with the provision of working vacation as we have noticed the decreasing trends in these indices by increasing θ.

9 Conclusion

In this paper, we have investigated an $M^X/G/1$ G-queue with working vacation and vacation interruption under delayed repair. Such queueing systems can be used to model many real life congestion situations wherein servers are not always available for rendering service; during the idle time, the servers may leave for more economical type of vacation called working vacation. On the other hand our model can be used to examine the effect of negative customers (virus, unwanted programs, distorted data, etc.) in *CCN's*, packet switching networks, telecommunication networks and many more wherein the presence of some unwanted arrivals forces the server to fail and therefore the server requires immediate repair. A numerical illustration for the proposed model has been provided and sensitivity analysis is carried out to observe the influence of some critical system parameters on various performance indices. Moreover, stochastic decomposition results have been derived for the proposed model.

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ANALYSIS OF A FINITE MARKOVIAN QUEUE WITH SERVER BREAKDOWN, SETUP TIME AND STATE DEPENDENT RATE UNDER *N*-POLICY STRATEGY

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Abstract

This investigation analyzes a breakdown service with setup time under *N*-policy. Only in working condition, the server may fail and dispatch immediately for the repair job which can be performed according to exponential distribution by a repairman. The server goes on vacation if the system does not have job to perform, i.e. the system is empty and later on switches on when the system accumulates *N* jobs. The arrivals at service station of the customers follow Poisson fashion with rate dependent on the server's status which may be idle, busy or brokendown state. By using generating function method, we derive the queue size distribution. The expressions for various performance characteristics including average queue length, probabilities of long-term fraction of time for which the server is idle, busy, broken down and in repair state. The optimal value of threshold parameter *N* which minimizes the total average cost is determined analytically. The parameters' sensitivity on different performance measures is examined by facilitating the numerical results.

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1 Introduction

Queueing problems with server breakdown have found increasing attention of research workers to portray realistic congestion scenarios of routine as well as commercial/industrial systems. Queueing systems subject to server breakdown have many applications in practical problems which ranges from computer networks to high speed distributed networks, call centers to cloud computing centers, Hospitals to malls, mobile computing to Internet of Things (IoT). Several authors have contributed towards server vacation model with threshold-based service policies. In *N*-policy queueing systems, the server starts the service as soon as the customers' count reached to a threshold level (say *N*). In recent past, Gaver [2], Grey et al. [3], Hersh and Brosh [4] and many more authors analyzed *N*- policy for a variant of queueing problems. Mitrans and Wright [10], Jain [5], Wang [12], Wang and Hung [13] analyzed a queueing model for *N*-parallel servers with server breakdowns and repairs.

Jain et al. [6] suggested (*N, F*) policy for the control of service and arrivals in case of an unreliable server queue in which repair is performed in multi-optional phase along with start-up. Kumar and Jain [8] proposed threshold '*N*' based policy for the performance prediction of (*M, m*) degraded machine repair problem with spare provisioning by including the concepts of the multi-heterogeneous servers, standby switching failure and multiple vacations. In recent years, Jiang et al. [6], Lan and Tang [9], Bu et al. [1] and Yen et al. [14] developed some queueing models for congestion problems under *N*-policy strategy. Sethi et al.. [11] developed Markovian model for unreliable single server queue by considering the impatience behavior of the customers and system operating under *N*-policy. They have also proposed the *ANFIS* soft computing approach and cost optimization.

The period for which the server is engaged in service is known as busy period. Due to unreliable server, there may be breakdown period during which the server cannot provide service to the customers. The time taken by the repairman before starting the repairing of the breakdown server is called setup time. A busy cycle is called the amount of idle time, busy period, breakdown period and restored period. In this paper, we investigate a queueing model subject to random breakdowns and repairs by incorporating the state dependent rate and setup time. The method of generating function is used to for establishing the probabilities for various states. The key performance measures are derived by using probabilities generating function (*PGF*). The optimum value of threshold parameter '*N'* is obtained to minimize the average total cost. The rest of the paper is arranged as follows. ***Section* 2** provides the model description and formulation of Chapman-Kolmogorov equations for steady state using appropriate transition rates. ***Section* 3** provides results for *PGF*. In Section 4 we calculate the value of $P_0(0)$ by using the normalizing conditions. In ***Section* 5,** we derive various performance measures in order to establish optimal N-policy. In ***Section* 6**, cost analysis is suggested so as to evaluate optimal value of threshold parameter based on different parameter values. ***Section* 7** discusses special cases while ***Section* 8** discusses sensitivity analysis. Finally, ***Section* 9** concludes the work done by highlighting the novel features.

2 Model Description

To formulate Markov's finite queueing model with server breakdown, setup time under N-policy and state dependent rates, we denote the server's status any time epoch 't' by

$$J(t) = \begin{cases} 0, & \text{server is idle} \\ 1, & \text{server is turned on and operating} \\ 2, & \text{server is turned on and breakdown} \\ 3, & \text{server is turned on and under repair} \end{cases}$$

The customers arrive independent to each other at the service stations in Poisson fashion. The server performs service in the first come first out *(FIFO)* order. The input rate of the customers is assumed to be state-dependent based on the status of the server. When in working condition, the server is subject to random failure and is returned to its previous state after repair. The repairman needs some time called set up time before beginning repair of the failed server. The repairman's setup process starts as soon as the server fails. The customers' service time as well as server's life, maintenance time and installation time are governed by exponential distributions.

The notations used for mathematical formulation of the model are described as below:

$\lambda, \lambda_1, \lambda_2, \lambda_3$	Customers' arrival rate when $J(t) = 0, 1, 2, 3$, respectively
$\alpha(\beta)$	Failure (repair) rate of the server
$\mu_0(\mu_1)$	Service rate of the server for $1 \le n \le N(N+1 \le n \le K)$
$\nu_0(\nu_1)$	Setup rate of repairmen for $1 \le n \le N(N+1 \le n \le K)$
$E[I](E[B])$	Expected idle (busy) time length.
$E[D](E[R])$	Expected breakdown (repair) time length.
$E[C]$	Expected cyclic time length.
$P_I(P_B)$	The long-term portion of the time for which the server is idle (busy)
P_D (P_R)	The long-term portion of the time for which server is in breakdown (under repair) state.
$P_{i,n}$	The steady state probability that the service will have n customers when the server is in state $J(t) = i, (i = 0, 1, 2, 3.)$

To formulate the mathematical model, the governing difference equations are constructed using appropriate transition rates. Chapman-Kolmogorov equations for formulating the mathematical model of concerned system for steady state are framed as follows:

(2.1) $0 = -\lambda p_{0,0} + \mu_0 p_{1,1},$

(2.2) $0 = -\lambda p_{0,n} + \lambda p_{0,n-1}, \quad 1 \le n \le N-1,$

(2.3) $0 = -(\lambda_1 + \alpha + \mu_0)P_{1,1} + \mu_0 P_{1,2} + \beta P_{3,1},$

(2.4) $0 = -(\lambda_1 + \alpha + \mu_0)P_{1,n} + \lambda_1 P_{1,n-1} + \mu_0 P_{1,n+1} + \beta P_{3,n}, \quad 2 \le n \le N-1,$

(2.5) $0 = -(\lambda_1 + \alpha + \mu_0)P_{1,N} + \lambda_1 P_{1,N-1} + \mu_1 P_{1,N+1} + \beta P_{3,N} + \lambda P_{0,N-1},$

(2.6) $0 = -(\lambda_1 + \alpha + \mu_1)P_{1,n} + \lambda_1 P_{1,n-1} + \mu_1 P_{1,n+1} + \beta P_{3,n}, \quad N+1 \le n \le K-1,$

(2.7) $0 = -(\mu_1 + \alpha)P_{1,K} + \lambda_1 P_{1,K-1} + \beta P_{3,K},$

(2.8) $0 = -(\lambda_2 + \nu_0)P_{2,1} + \alpha p_{1,1},$

(2.9) $0 = -(\lambda_2 + \nu_0)P_{2,n} + \lambda_2 P_{2,n-1} + \alpha p_{1,n}, \quad 2 \le n \le N,$

(2.10) $0 = -(\lambda_2 + \nu_1)P_{2,n} + \lambda_2 P_{2,n-1} + \alpha p_{1,n}, \quad N+1 \le n \le K-1,$

(2.11) $0 = -\nu_1 P_{2,K} + \alpha P_{1,K} + \lambda_2 P_{2,K-1},$

(2.12) $0 = -(\lambda_3 + \beta)P_{3,1} + \nu_0 p_{2,1},$

(2.13) $0 = -(\lambda_3 + \beta)P_{3,n} + \nu_0 p_{2,n} + \lambda_3 P_{3,n-1}, \quad 2 \le n \le N,$

(2.14) $0 = -(\lambda_3 + \beta)P_{3,n} + \nu_1 p_{2,n} + \lambda_3 P_{3,n-1}, \quad N+1 \le n \le K-1,$

(2.15) $0 = -\beta P_{3,K} + \nu_1 p_{2,K} + \lambda_3 P_{3,K-1}.$

3 Queue Size Distribution

The generating functions are defined as:

(3.1) $G_0(z) = \sum_{n=0}^{N-1} P_{0,n} z^n$,

(3.2) $G_i(z) = \sum_{n=1}^{K} P_{i,n} z^n, \qquad i = 1,\ 2,\ 3..$

Using (2.1)-(2.15) and (3.1)-(3.2), we get

(3.3) $G_0(z) = \dfrac{1-z^N}{1-z} P_{0,0}$,

(3.4) $$G_1(z) = \frac{\begin{bmatrix} [\lambda z(1-z^N)P_{0,0} + (z-1)a(z)]L_2(z)L_3(z) + \lambda_2\beta z(z-1)b(z) - \lambda_1(1-z) \\ z^{K+1}L_2(z)L_3(z)P_{1,K} - \beta z(1-z)\lambda_2\nu_1 z^{K+1}P_{2,K} + \lambda_3\beta z z^{K+1}(1-z)L_2(z)P_{3,K} \end{bmatrix}}{[\alpha\beta\nu_1 z + L_1(z)L_2(z)L_3(z)]},$$

(3.5) $$G_2(z) = \frac{\begin{bmatrix} [\alpha\lambda z(z^N-1)P_{0,0} + \alpha(1-z)a(z)]L_3(z) - [\alpha\beta z - L_1(z)L_3(z)]b(z) + \alpha\lambda_1(1-z) \\ z^{K+1}L_3(z)P_{1,K} - \lambda_2 L_1(z)L_3(z)z^{K+1}(1-z)P_{2,K} - \alpha\beta z\lambda_3 z^{K+1}(1-z)P_{3,K} \end{bmatrix}}{[\alpha\beta\nu_1 z + L_1(z)L_2(z)L_3(z)]},$$

(3.6) $$G_3(z) = \frac{\begin{bmatrix} \alpha\nu_1[\lambda z(1-z^N)P_{0,0} + (z-1)a(z)] - \lambda_2(z-1)L_1(z)b(z) - \alpha\nu_1\lambda_1(1-z)z^{K+1}P_{1,K} \\ +\lambda_2\nu_1(1-z)L_1(z)z^{K+1}P_{2,K} - \lambda_3 L_1(z)L_2(z)(1-z)z^{K+1}P_{3,K} \end{bmatrix}}{[\alpha\beta\nu_1 z + L_1(z)L_2(z)L_3(z)]},$$

where,

$L_1(z) = \left[\lambda_1 z^2 - (\lambda_1 + \alpha + \mu_1)z + \mu_1\right]$,

$L_2(z) = (\lambda_2 z - \lambda_2 - \nu_1)$,

$L_3(z) = (\lambda_3 z - \lambda_3 - \beta)$,

$a(z) = (\mu_0 - \mu_1)\sum_{n=1}^{N} P_{1,n} z^n$,

$b(z) = (\nu_0 - \nu_1)\sum_{n=1}^{N} P_{2,n} z^n$.

4 Computation of $P_{0,\,0}$

Using normalizing condition stated as

$G(1) = \sum_{i=1}^{3} G_i(1) = 1$,

we get,

(4.1) $$P_{0,0} = \frac{\begin{bmatrix} \gamma + [\alpha\beta + \nu_1(\alpha+\beta)]a(1) + [\alpha(\lambda_2 - \lambda_3) + \beta(\lambda_2 - \lambda_1 + \mu_1)]b(1) \\ +[\alpha\beta + \nu_1(\alpha+\beta)]\sum_{i=1}^{3}\lambda_i P_{i,K} \end{bmatrix}}{N[\gamma + \lambda\{\alpha\beta + \nu_1(\alpha+\beta)\}]}.$$

5 Performance Characteristics

Various performance characteristics are obtained explicitly as follows:
The long-term portion of the time for which the server is idle, busy, in breakdown and repair states respectively are

(5.1) $P_I = G_0(1) = NP_{0,0}$.

(5.2) $$P_B = G_1(1) = \frac{\beta\lambda\nu_1 - \beta\nu_1 a(1) - \beta(\lambda - \lambda_2)b(1) - \beta\nu_1\sum_{i=1}^{3}\lambda_i P_{i,K}}{[\gamma + \lambda\{\alpha\beta + \nu_1(\alpha+\beta)\}]}.$$

(5.3) $$P_D = G_2(1) = \frac{\alpha\lambda\beta - \alpha\beta a(1) - [\alpha(\lambda_3 - \lambda) + \beta(\lambda_1 - \lambda - \mu_1)]b(1) - \alpha\beta\sum_{i=1}^{3}\lambda_i P_{i,K}}{[\gamma + \lambda\{\alpha\beta + \nu_1(\alpha+\beta)\}]}.$$

(5.4) $$P_R = G_3(1) = \frac{\alpha\lambda\nu_1 - \alpha\nu_1 a(1) - \alpha(\lambda - \lambda_2)b(1) - \alpha\nu_1\sum_{i=1}^{3}\lambda_i P_{i,K}}{[\gamma + \lambda\{\alpha\beta + \nu_1(\alpha+\beta)\}]}.$$

We derive the formulae for the average number of customers in various states namely when idle [$E(N_0)$] busy [$E(N_1)$], broken down [$E(N_2)$], and under repair [$E(N_3)$] as follows:

(5.5) $E(N_0) = \dfrac{N(N-1)}{2} \mathrm{P}_{0,0}$, (5.6) $E(N_i) \ = \ \lim_{z\to 1} G_i'(z) = \lim_{z\to 1} \dfrac{N_i''(1)D'(1) - D''(1)N_i'(1)}{2(D'(1))^2}$, $\quad i = 1,\ 2,\ 3$

where

$$D'(1) = (\mu_1 - \lambda_1)\beta\nu_1 - \alpha\beta\lambda_2 - \alpha\nu_1\lambda_3 = \gamma,$$
$$D''(1) = 2[\lambda_1\beta\nu_1 - (\lambda_1 - \alpha - \mu_1)(\nu_1\lambda_3 + \beta\lambda_2) - \alpha\lambda_2\lambda_3] = 2\delta,$$
$$N_1'(1) = [-\beta\lambda\nu_1 NP_{0,0} + \beta\nu_1 a(1) + \beta\lambda_2 b(1) + \beta\nu_1 \textstyle\sum_{i=1}^{3} \lambda_i P_{i,K}],$$
$$\begin{aligned} N_1''(1) = {} & [-\beta\lambda\nu_1 N(N+1)P_{0,0} + 2\beta\nu_1 a'(1) + 2(\beta\lambda_2 + \lambda_3\nu_1)\{\lambda NP_{0,0} - a(1)\} \\ & +2\beta\lambda_2\{b(1) + b'(1)\} + 2\lambda_1\{(K+1)\beta\nu_1 - \nu_1\lambda_3 - \beta\lambda_2\}P_{1,K} \\ & +2\lambda_2\beta\nu_1(K+2)P_{2,K} + 2\lambda_3\beta\{(K+2)\nu_1 - \lambda_2\}P_{3,K}], \end{aligned}$$
$$\begin{aligned} N_2''(1) = {} & [\alpha\lambda NP_{0,0}\{2\lambda_3 - \beta(N+1)\} - 2\alpha\lambda_3 a(1) + 2\alpha\beta a'(1) - 2(\lambda_1\beta - \lambda_1\lambda_3 \\ & +\alpha\lambda_3 + \mu_1\lambda_3)b(1) + 2(\mu_1\beta - \lambda_1\beta - \alpha\lambda_3)b'(1) + 2\alpha\lambda_1\{(K+1)\beta - \lambda_3\}P_{1,K} \\ & +\lambda_2\{2(K+1)\alpha\beta - 2\alpha\lambda_3 - (\lambda_1 - \alpha - \mu_1)\}P_{2,K} + 2\alpha\beta\lambda_3(K+2)P_{3,K}], \end{aligned}$$
$$N_3'(1) = [-\alpha\lambda\nu_1 NP_{0,0} + \alpha\nu_1 a(1) + \alpha\lambda_2 b(1) + \alpha\nu_1 \textstyle\sum_{i=1}^{3} \lambda_i P_{i,K}],$$
$$\begin{aligned} N_3''(1) = {} & [-\alpha\lambda\nu_1 N(N+1)P_{0,0} + 2\alpha\nu_1 a'(1) - 2\lambda_2(\lambda_1 - \alpha - \mu_1)b(1) \\ & +2\alpha\lambda_2 b'(1) + 2\alpha\lambda_1\nu_1(K+1)P_{1,K} + \lambda_2\nu_1\{2(K+1)\alpha - 2(\lambda_1 - \alpha - \mu_1)\} \\ & P_{2,K} - 2\lambda_3\{-\alpha\nu_1(K+1) + \alpha\lambda_2 + \nu_1(\lambda_1 - \alpha - \mu_1)\}P_{3,K}]. \end{aligned}$$

Using (5.5)-(5.6) the average number of customers in the system obtained

(5.7) $E(N_S) = E(N_0) + \sum_{i=1}^{3} E(N_i)$.

Some More important Results are obtained as follows:

(5.8) $E[I] = \dfrac{N}{\lambda}$.

(5.9) $E[C] = E[I] + E[D] + E[B] + E[R]$,

(5.10) $P_I = \dfrac{E[I]}{E[C]}, P_D = \dfrac{E[D]}{E[C]}, P_B = \dfrac{E[B]}{E[C]}, P_R = \dfrac{E[R]}{E[C]}$.

Expected cyclic time length is now obtained as

(5.11) $$E[C] = \frac{E[I]}{P_I} = \frac{N[\gamma + \lambda\{\alpha\beta + \nu_1(\alpha+\beta)\}]}{\lambda\begin{bmatrix} \gamma + [\alpha\beta + \nu_1(\alpha+\beta)]a(1) + [\alpha(\lambda_2 - \lambda_3) + \beta(\lambda_2 - \lambda_1 + \mu_1)]b(1) \\ +[\alpha\beta + \nu_1(\alpha+\beta)]\sum_{i=1}^{3}\lambda_i P_{i,K} \end{bmatrix}}.$$

Expected busy time length is

(5.12) $$E[B] = E[C]P_B = \frac{N\left[\beta\lambda\nu_1 - \beta\nu_1 a(1) - \beta(\lambda - \lambda_2)b(1) - \beta\nu_1\sum_{i=1}^{3}\lambda_i P_{i,K}\right]}{\lambda\begin{bmatrix} \gamma + [\alpha\beta + \nu_1(\alpha+\beta)]a(1) + [\alpha(\lambda_2 - \lambda_3) + \beta(\lambda_2 - \lambda_1 + \mu_1)]b(1) \\ +[\alpha\beta + \nu_1(\alpha+\beta)]\sum_{i=1}^{3}\lambda_i P_{i,K} \end{bmatrix}}.$$

Also, expected bkeakdown time length is given by

(5.13) $$E[D] = E[C]P_D = \frac{N\left[\alpha\lambda\beta - \alpha\beta a(1) - [\alpha(\lambda_3 - \lambda) + \beta(\lambda_1 - \lambda - \mu_1)]b(1) - \alpha\beta\sum_{i=1}^{3}\lambda_i P_{i,K}\right]}{\lambda\begin{bmatrix} \gamma + [\alpha\beta + \nu_1(\alpha+\beta)]a(1) + [\alpha(\lambda_2 - \lambda_3) + \beta(\lambda_2 - \lambda_1 + \mu_1)]b(1) \\ +[\alpha\beta + \nu_1(\alpha+\beta)]\sum_{i=1}^{3}\lambda_i P_{i,K} \end{bmatrix}}.$$

Expected repair time length is obtained as

(5.14) $$E[R] = E[C]P_R = \frac{N\left[\alpha\lambda\nu_1 - \alpha\nu_1 a(1) - \alpha(\lambda - \lambda_2)b(1) - \alpha\nu_1\sum_{i=1}^{3}\lambda_i P_{i,K}\right]}{\lambda\begin{bmatrix} \gamma + [\alpha\beta + \nu_1(\alpha+\beta)]a(1) + [\alpha(\lambda_2 - \lambda_3) + \beta(\lambda_2 - \lambda_1 + \mu_1)]b(1) \\ +[\alpha\beta + \nu_1(\alpha+\beta)]\sum_{i=1}^{3}\lambda_i P_{i,K} \end{bmatrix}}.$$

6 Cost Analysis

In order to determine the optimal value of control parameter N, total expected cost per unit time is evaluated using different cost components.

The total cost per unit time is estimated

$$E\{TC(N)\} = (C_1 + C_2)\frac{1}{E[C]} + C_3P_B + C_4P_I + C_5P_D + C_6P_R + C_7E(N_s)$$

where

C_1 Start-up costs when the system is up and running,
C_2 Shutdown costs when the server is turned off,
C_3 Cost per unit time for server servicing,
C_4 Cost per unit time to shut down the server,
C_5 Price for a broken down server per unit time,
C_6 Price per system failure server repair time,
C_7 Holding costs per unit time per system customer.

We observe that P_B, P_I, P_D, P_R are not the function of decision variable N, hence the effective expected total cost per unit time is given by

$$E\{C(N)\} = (C_1 + C_2)\frac{1}{E[C]} + C_7E(N_s). \tag{6.1}$$

The aim is to minimize $E\{C(N)\}$ to determine the optimal value (say $N*$) of the N factor for decision. A heuristic approach based on a discrete distribution can be used to measure $N*$.

7 Special Cases

Now we deduce results for some special cases by setting appropriate parameter as follows:

I. When $\mu_0 = \mu_1, v_0 = v_1,\ \lambda = \lambda_1 = \lambda_2 = \lambda_3$ and $K \to \infty$ then our results matches with Wang [12].
II. If $\lambda = \lambda_1 = \lambda_2 = \lambda_3$ and service station is perfect i.e. $\beta = \alpha = 0$, then the results coincide with Wang and Huang [13].
III. If we take service rate constant, then the model is without setup time. It is noticed that $G_3(z) \to G_2(z), K \to \infty$ and $\lambda_3 \to \lambda_2$ so that our model matches with Jain [5].

8 Sensitivity Analysis

We conduct statistical experiments using MATLAB to show the effect of different parameters on the average queue size. ***Figures*** **8.1-8.6** depict the comparison of total queue size for homogenous arrival rate ($\lambda = \lambda_1 = \lambda_2 = \lambda_3 = 0.5$) with heterogeneous arrival rates ($\lambda_1 = 0.9\lambda, \lambda_2 = 0.5\lambda, \lambda_3 = 0.3\lambda$) and ($\lambda_1 = 0.7\lambda, \lambda_2 = 0.4\lambda, \lambda_3 = 0.3\lambda$) by varying other parameters. From ***Table*** **8.1** which displayed the effects of parameters $\lambda, (\alpha, \beta)$ and N on cost function, we note that the cost function decreases first and then increases by increasing the value of N. The minimum costs corresponding to different values of $\lambda, (\alpha, \beta)$ are shown by bold letters. ***Table*** **8.2** exhibits the effect of parameters $(\lambda, \mu, \alpha, \beta, v)$ on queue length in the idle state, busy state, breakdown state, under repair state and in the cycle for homogenous and heterogeneous cases.

***Table* 8.1:** The threeshold value of *N* and corresponding expected cost by varying different parameters

λ	*N*	*E{C(N)}*			
		α=.1	α=.1	α=.5	α=.5
		β=2	β=4	β=2	β=4
0.5	5	14.36	14.44	13.13	13.52
	6	13.74	13.81	12.71	13.04
	7	13.44	13.49	**12.55**	12.84
	8	**13.33**	**13.39**	12.56	**12.81**
	9	13.37	13.41	12.68	12.90
0.6	5	15.44	15.54	13.85	14.33
	6	14.78	14.87	13.45	13.85
	7	14.46	14.53	**13.30**	13.64
	8	**14.34**	**14.40**	13.31	**13.61**
	9	14.35	14.41	13.43	13.70
0.7	5	16.25	16.38	14.38	14.91
	6	15.61	15.71	14.04	14.48
	7	15.29	15.38	**13.94**	14.30
	8	**15.18**	**15.25**	13.98	**14.30**
	9	15.20	15.27	14.12	14.40
0.8	5	16.85	16.99	14.87	15.37
	6	16.27	16.38	14.64	15.03
	7	15.99	16.09	**14.61**	**14.92**
	8	**15.91**	**15.99**	14.71	14.96
	9	15.96	16.03	14.89	15.10

***Table* 8.2:** The expected number of customers for homogenous and heterogeneous arrival rates

	Homogenous arrival rate					Heterogeneous arrival rate				
λ	*E[I]*	*E[B]*	*E[D]*	*E[R]*	*E[C]*	*E[I]*	*E[B]*	*E[D]*	*E[R]*	*E[C]*
0.1	20.51	2.44	0.81	0.61	24.37	20.62	2.30	0.77	0.57	24.25
0.2	13.07	3.83	1.28	0.96	19.13	13.38	3.36	1.12	0.84	18.69
0.3	9.35	5.35	1.78	1.34	17.82	9.95	4.29	1.43	1.07	16.75
0.4	6.77	7.39	2.46	1.85	18.46	7.78	5.24	1.75	1.31	16.08
0.5	4.56	10.95	3.65	2.74	21.91	6.19	6.29	2.10	1.57	16.16
μ	*E[I]*	*E[B]*	*E[D]*	*E[R]*	*E[C]*	*E[I]*	*E[B]*	*E[D]*	*E[R]*	*E[C]*
1.0	4.56	10.95	3.65	2.74	21.91	6.19	6.29	2.10	1.57	16.16
1.5	6.87	4.85	1.62	1.21	14.55	7.46	3.76	1.25	0.94	13.41
2.0	7.77	3.22	1.07	0.80	12.87	8.08	2.71	0.90	0.68	12.37
2.5	8.27	2.42	0.81	0.60	12.10	8.46	2.12	0.71	0.53	11.82
3.0	8.58	1.94	0.65	0.49	11.66	8.71	1.75	0.58	0.44	11.48
α	*E[I]*	*E[B]*	*E[D]*	*E[R]*	*E[C]*	*E[I]*	*E[B]*	*E[D]*	*E[R]*	*E[C]*
0.4	5.16	9.68	2.58	1.94	19.36	6.42	6.25	1.67	1.25	15.58
0.5	4.56	10.95	3.65	2.74	21.91	6.19	6.29	2.10	1.57	16.16
0.6	3.87	12.91	5.16	3.87	25.82	5.97	6.35	2.54	1.90	16.76
0.7	3.03	16.51	7.71	5.78	33.03	5.75	6.41	2.99	2.24	17.40
0.8	1.83	27.39	14.61	10.95	54.77	5.53	6.48	3.46	2.59	18.07
β	*E[I]*	*E[B]*	*E[D]*	*E[R]*	*E[C]*	*E[I]*	*E[B]*	*E[D]*	*E[R]*	*E[C]*
2.0	4.56	10.95	3.65	2.74	21.91	6.19	6.29	2.10	1.57	16.16
3.0	5.00	10.00	3.33	1.67	20.00	6.32	6.32	2.11	1.05	15.81
4.0	5.20	9.61	3.20	1.20	19.22	6.39	6.34	2.11	0.79	15.64
5.0	5.32	9.39	3.13	0.94	18.79	6.44	6.35	2.12	0.64	15.54
6.0	5.40	9.26	3.09	0.77	18.52	6.46	6.36	2.12	0.53	15.47
ν	*E[I]*	*E[B]*	*E[D]*	*E[R]*	*E[C]*	*E[I]*	*E[B]*	*E[D]*	*E[R]*	*E[C]*
1.5	4.56	10.95	3.65	2.74	21.91	6.19	6.29	2.10	1.57	16.16
2.0	5.00	10.00	2.50	2.50	20.00	6.37	6.22	1.55	1.55	15.70
2.5	5.24	9.53	1.91	2.38	19.07	6.48	6.17	1.23	1.54	15.43
3.0	5.40	9.26	1.54	2.31	18.52	6.55	6.14	1.02	1.54	15.26
3.5	5.51	9.07	1.30	2.27	18.15	6.61	6.12	0.87	1.53	15.14
4.0	5.59	8.94	1.12	2.24	17.89	6.65	6.11	0.76	1.53	15.05

In ***Fig.*** **8.1** we display the effect of arrival rate λ on the average queue length $E(N_S)$ for fixed values $N = 5$, $\alpha = .1$, $\beta = 5$, $\nu = 2.5$, $\mu = 1$. It is observed that when λ increases, the average queue length increases, the effect is more prevalent for higher λ value. In ***Fig.*** **8.2**, we fix $N = 5$, $\lambda = .5$, $\nu = 2.5$, $\beta = 2$, $\alpha = .1$ and plot the graph for average queue length by varying μ from 0.7 to 2.8. From the graph we examine that $E(N_S)$ decreases with the increase in value of μ. As we expect, $E(N_S)$ decreases sharply for lower value of μ.

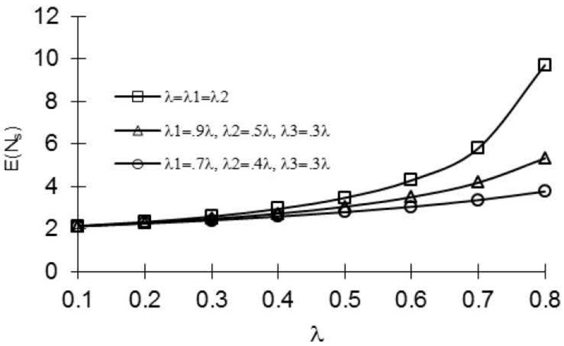

Figure **8.1:** Average Queue Length vs. λ
$(N = 5, \mu = 1, \nu = 2.5, \beta = 2, \alpha = 1)$

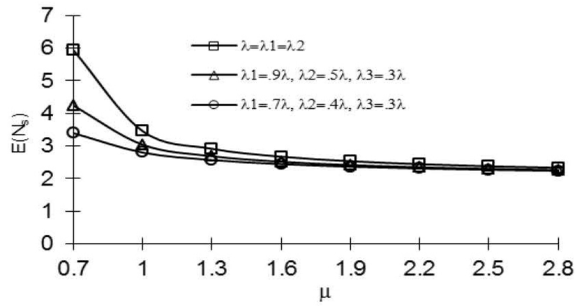

Figure **8.2:** Average Queue Length vs. μ
$(N = 5, \lambda = .5, \nu = 2.5, \beta = 2, \alpha = 1)$

In ***Fig.*** **8.3** and **8.4**, we vary $\alpha(\beta)$ for fix values $N = 5$, $\lambda = .5$, $\nu = 2.5$, $\mu = .1$ and find that the average queue length increases (decreases) as the value of $\alpha(\beta)$ increases. By fixing $\lambda = 0.5$, $\alpha = 0.1$, $\beta = 5$, $\mu = 1$, we plot graphs for the average queue length $E(N_S)$ vs. ν and N in ***Figures*** **8.5** and **8.6**, respectively. It is seen that $E(N_S)$ decreases asymptotically with the increase in ν. Also, $E(N_S)$ increases linearly with N.

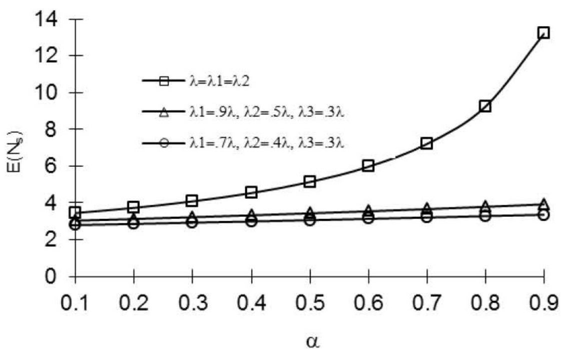

Figure **8.3:** Average Queue Length vs. α
$(N = 5, \lambda = .5, \nu = 2.5, \beta = 2, \mu = 1)$

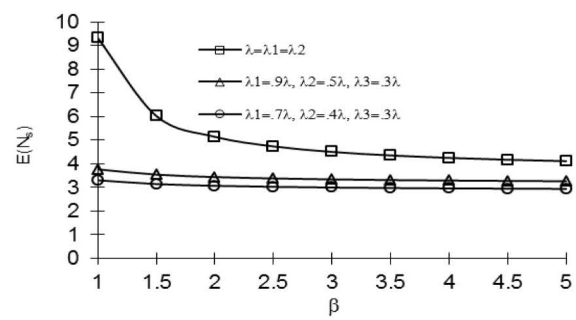

Figure **8.4:** Average Queue Length vs. β
$(N = 5, \lambda = .5, \nu = 2.5, \alpha = .5, \mu = 1)$

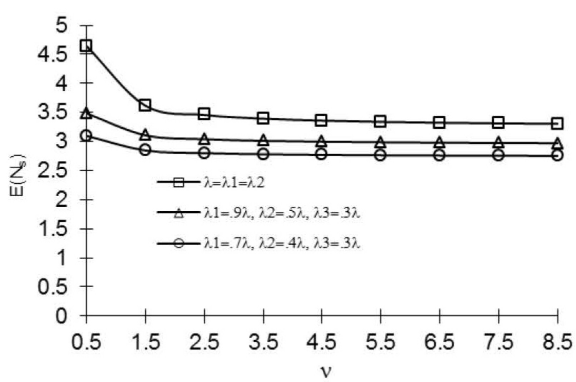

Figure **8.5:** Average Queue Length vs. ν
$(N = 5, \lambda = .5, \beta = 2, \alpha = .1, \mu = 1)$

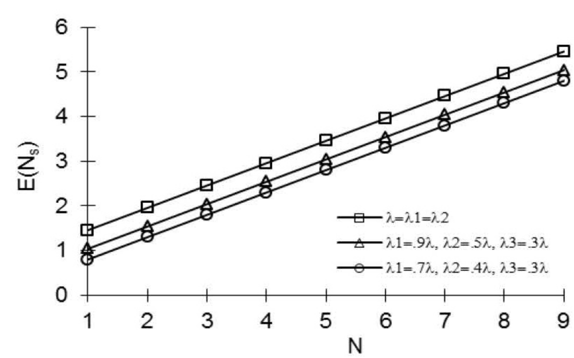

Figure **8.6:** Average Queue Length vs. N
$(\lambda = 5, \beta = 2, \alpha = .1, \mu = 1, \nu = 2.5)$

We conclude numerical results based on the sensitivity analysis carried as follows:

- that minimum expected length increases for both homogenous and heterogeneous arrival rate.
- As we expect, average queue lengths, indicates increasing (decreasing) trend with (λ, μ) and (α, β), but for threshold parameter N, $E(N_S)$ linearly increase.
- By improving the rate of service station to some specific threshold level, the average queue length can be reduced to some extend only.
- For constant input rate system, all results seem to be better in terms of reduced average queue length.

9 Concluding Remarks

In this investigation, we have analyzed a finite queue operating under N-policy by including some realistic features namely setup time, server breakdown and state dependent rates. The incorporation of setup time, which can be realized in many real time systems makes our model more versatile than previous existing models in the literature. The explicit expressions provided for various staging measures in term of steady state probabilities can be easily computed as illustrated by taking numerical examples. The sensitivity analysis facilitated may be helpful in exploring the effect of different parameters on key indices describing the system dynamic.

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(Dedicated to Honor Dr. R. C. Singh Chandel on His 75th Birth Anniversary Celebrations)

STATISTICAL AND TIME SERIES ANALYSIS OF BLACK CARBON IN THE MAJOR COAL MINES OF INDIA

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Abstract

Time series analysis has been widely used by the researchers in the field of mathematical forecasting; it has been mainly used to obtain the forecast of time series dealing with pollutants, groundwater level, and stock exchange so as to study their future behavior of such time series. The present research work deals with the black carbon concentrations in three major coal mines of India namely, Bokaro, Jharia and Raniganj. In this study, a time series data last 38 years (from 1980 to 2018) obtained from a reliable source (*NASA*) have been considered by statistical analysis tools like mean, median, mode, standard deviation, skewness, kurtosis, coefficient of variation and time series (*ARIMA* (Autoregressive Integrated Moving Average)) model at 95% confidence limits have been applied.The validation of the model is tested using *R*-square, stationary *R*-square, root mean square error *(RMSE)*, normalized Bayesian information criterion *(BIC)*. It is observed that the model fitted very well, based on these past observations, *ARIMA* model is applied to obtain the prediction of the amount of black carbon emission for next 7 years 5 months (from Jun 2018 to Oct 2025). These results will help to develop new policies and preventive measures in future by the government agencies, NGOs in these areas and take a note of the seriousness and impact of such huge concentration of black carbon emission in these areas.
2010 Mathematics Subject Classifications: 93A30, 97M10.
Keywords and phrases: *ARIMA*, Black Carbon, *RMSE*, Mathematical Modeling.

1 Introduction

Black carbon (*BC*) has emerged as an alarming area of interest among the researchers, in recent times due to its share in global warming and severe health impacts. Black carbon is black sooty material produced as substantial particle of the carbonaceous aerosol released due incomplete combustion of biofuels, fossil fuels and biomass in coal-fired power plants, steel plants, petroleum industries and oil refineries. In indoor conditions it mainly released due to cooking and burning of fuels like wood, coal, animal manure, residues of crops [3, 38]. In Asia, the contribution of open biomass burning from fossil fuels is nearly 40%, and that from burning of biofuels is 20% in the overall *BC* emission [29]. It is a global problem as it has negative impact on human health such as Inhalation of *BC* leads to problems related to respiratory such as chronic bronchitis and asthma, lung disease, damage to eye sights, cardio vascular disease, cancer and even leads to birth defects. It gets mixed with air, water and soil thus entering the food chain and enters the human body. Carbonaceous aerosols have received a great attention of the researchers recently due to its severe impact on human health [14, 17, 21, 26, 37], agriculture [7] and the quality of air [10, 13, 39].

Black carbon is the major absorber of solar heat radiation in the atmosphere, *BC* leads to the heating of the Earths atmosphere as it results into the reduction in incoming short-wave solar radiation at the Earths surface [8, 11, 12], and thus leading to the change in the temperature of the troposphere, which affects the microphysical properties of the clouds and thus affecting the rainfall mechanisms [20]. *BC* aerosol affects the rainfall pattern by influencing the cloud formation and precipitation process [28]. India is the worlds second largest producer of coal in the world and various mining activities performed in the coal mining regions are leading to the spontaneous emission of black carbon along with various other harmful gases in these regions [35].

Time series modeling has been largely used [9, 24, 30] to study the fluctuations and making good modeling forecasts, it is very beneficial in decision-making of climatic conditions and estimation of future data. The Auto regressive integrated moving average (*ARIMA*) models have been used in various studies of time series modeling of air pollution [1, 2, 4, 6, 7, 16, 27] and water pollution [22, 24, 25, 36, 40].Time series modeling methods have also been used to study the emission of black carbon [1, 5, 15, 31, 32, 33, 34].

Thus due to the above discussed severe impact of black carbon on human life and environment both nationally and globally, the future study of black carbon is very important for framing national and international policies for prediction of the level of black carbon emission in the future. Coal mines region being one of the major sources for *BC* emission, the aim of this study is to calculate the amount of *BC* mass concentrations in the major coal fields of India viz. coal field area of Raniganj, Jharia and Bokaro and making future forecast for these regions using statistical and time series analysis.

2 Research methodology

2.1 Statistical analysis

Statistical analysis consists of mean, median, mode, standard deviation, kurtosis, skewness and coefficient of variation, the spreadness or variability of the data in the sample is explained by standard deviation, to determine the nature of the distribution curve it is classified as platykurtic, mesokurtic and leptokurtic which depends on the peakedness or flatness of the curve we use kurtosis, skewness refers to the symmetry of the sample, the relative measure of the series is termed as coefficient of variation (CV) [22, 23] and is defined as:

(2.1) $CV\% = \frac{\sigma}{\mu} \times 100\%,$

where, σ is the standard deviation and μ is the mean of the series. It is used to find the total variation in the *BC* concentration.

2.2 Time series

A time series is a sequential set of data points measured over successive time intervals arranged in a proper chronological order. It is one of the most widely used mathematical technique developed by researchers in the field of mathematical modeling for studying fluctuations, extracting meaningful statistics and making good forecasts of the time series. It is very beneficial in decision-making of climatic conditions and estimation of future values [18, 19].

2.2.1 Autoregressive Moving Average *ARMA* (p, q) Model

The Autoregressive (*AR*) and the Moving average (*MA*) are effectively combined together to form the Autoregressive Moving average (*ARMA*) model. Mathematically it is represented as,

(2.2) $y_t = c + \epsilon_t + \sum_{i=1}^{p} \phi_i y_{t-i} + \sum_{j=1}^{q} \theta_j \epsilon_{t-j}.$

2.2.2 *ARIMA* Model

The most widely used time series model is the Box Jenkins based *ARIMA* (Autoregressive Integrated Moving) model. In recent years the *ARIMA* model has been widely used in the fields of medicine, engineering, stock markets, weather forecasting, economics, business, finance etc. In *ARIMA* model a non-stationary time series can be converted to stationary by using the finite differencing technique. Mathematically the *ARIMA* (p, d, q) model is expressed,

(2.3) $\phi(L) = (1 - L)^d y_t = \theta(L)\epsilon_t,$

i.e.,

(2.4) $\left(1 - \sum_{i=1}^{p} \phi_i L^i\right)(1 - L)^d y_t = \left(1 + \sum_{j=1}^{q} \theta_j L^j\right)\epsilon_t.$

Here p, d and q are the order of the autoregressive, integrated and moving average parts and these are non-negative integers greater than or equal to zero. If any of these values become zero than it becomes the basic *AR*, *MA* or the *ARMA* model of the time series.

The level of differencing is defined by the parameter d and it keeps a check on the level of differencing. The value of $d = 1$ in most of the cases and if $d = 0$ then the model gets reduced to the *ARMA* (p, q) model.

If $d = q = 0$, then $ARIMA(p, 0, 0)$ reduces to the $AR(p)$ model and if $p = d = 0$, then $ARIMA(0, 0, q)$ reduces to the $MA(q)$ model.

If $p = q = 0$ and $d = 1$, then *ARIMA* $(0, 1, 0)$ becomes $y_t = y_{t-1} + \epsilon_t$ which is known as the Random walk model.

2.3 Root Mean Square error *(RMSE)*

It is the coefficient of error representing the standard deviation of the difference of actual values of the data from the values predicted by the time series model also termed as the residual values, it is used to determine amount of spreadness of the values from the line of best fit for a model and to determine the accuracy of the forecasted values. Mathematically it is given by,

(2.5) $RMSE = \sqrt{\frac{1}{n}\sum_{i=1}^{n} e_i^2}$,

where n denotes the time period and e_i denotes the error of forecasting.

2.4 *R*-squared and stationary *R*-squared values

These values are used as a measures for goodness of fit for a time series model; they are used as a coefficient of determination of a model. The value of R square ranges from 0 to 1 while that of stationary R-squared ranges from $-\infty$ to 1, higher values indicate that the model considered is better than the baseline model.

3 Results and discussion

3.1 Sample sites

Raniganj ($23^o\,40'\,N\,87^o\,05'\,E$) in West Bengal, Jharia ($23^o\,50'\,N 86^o\,33'\,E$) and Bokaro ($23^o\,46'\,N\,85^o\,55'\,E$) in Jharkhand as shown in ***Fig.* 3.1** are main focused sites for our current study, these are among the major coal mines of India.The data of *BC* is obtained by *NASA* and processing the data is done via Giovanni website (http://nasa.gov/). Using statistical and time series analysis the concentration of black carbon at these three sites have been discussed. The results are based on long term trend analysis of the concentration of black carbon expressed in volume(magnitude) as $e^{-11}\,kgm^{-3}$ units over the past 38 year, 05 months data from Jan 1980 to May 2018. *IBMSPSS* Statistics software has been used for testing and training the data for choosing an appropriate time series model. Further the statistical and time series results have been obtained using the same.

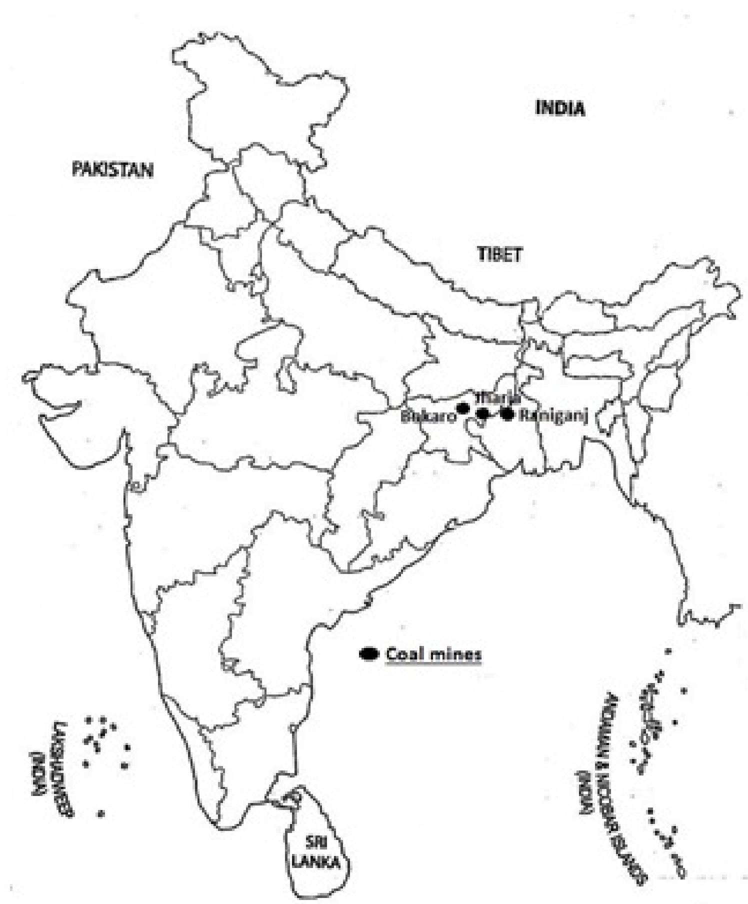

***Figure* 3.1:** Map of India showing coal mines of Raniganj, Jharia and Bokaro.

3.2 Statistical Analysis of Black carbon

Statistical parameters such as mean, mode, median, standard deviation, variance, skewness, kurtosis, range, coefficient of variation have been used to study the behavior of our parameter i.e. *BC* concentration. The results of statistical analysis have been shown in ***Table* 3.1** and bar chart depiction of the observed results has been shown in ***Fig.* 3.2**.

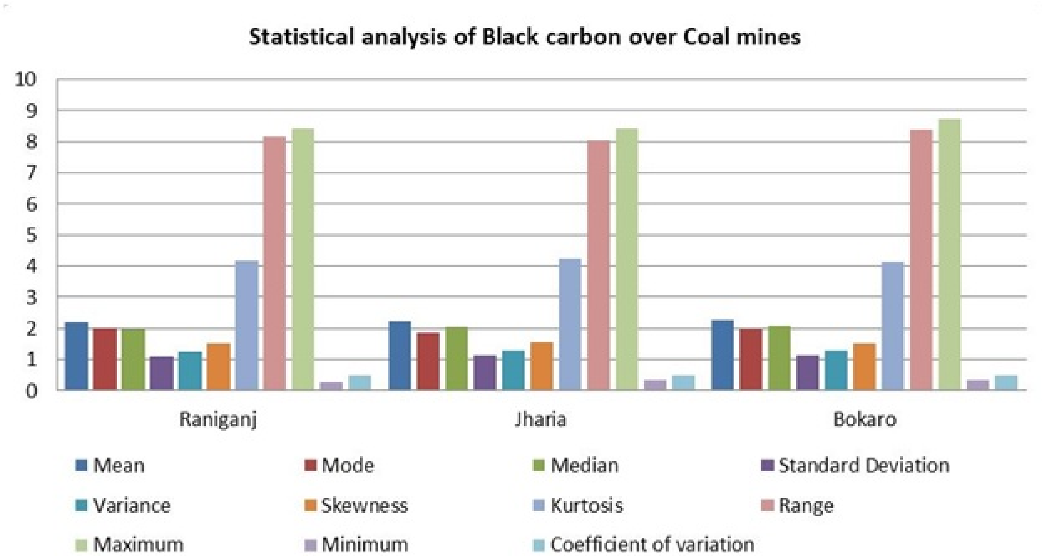

***Figure* 3.2:** Statistical analysis of black carbon at Raniganj, Jharia and Bokaro.

3.2.1 Raniganj ($23^o\ 40'\ N\ 87^o\ 05'\ E$)

The mean, mode and median value of *BC* concentration are at 2.192875807, 2.002202643 and 1.983193277, these values are close to 2 depicting that the data distribution curve is symmetrical and follows a normal distribution. The value of standard deviation and skewness are at 1.113654555 and 1.507609253 indicating that the data points are distributed close to each other along the mean and the distribution curve is moderately skewed to the right. The curve is leptokurtic as indicated by the value 4.192888602 in the ***Table* 3.1**.

***Table* 3.1:** Time series and *ARIMA* forecast of Jharia

	Raniganj	Jharia	Bokaro
Mean	2.192875807	2.232444485	2.269679113
Mode	2.002202643	1.846846847	1.962719298
Median	1.983193277	2.059907834	2.078581871
Standard Deviation	1.113654555	1.130771534	1.142111571
Variance	1.240226469	1.278644263	1.30441884
Skewness	1.507609253	1.560077228	1.529151854
Kurtosis	4.192888602	4.245785802	4.152665796
Range	8.145594714	8.065878378	8.391812865
Maximum	8.423127753	8.414977477	8.742690058
Minimum	0.27753304	0.349099099	0.350877193
Coeff. of variation	0.507851175	0.506517202	0.503203983

3.2.2 Jharia ($23^o\,50'\,N 86^o\,33'\,E$)

As shown in ***Table* 3.1** the mean, mode and median value of *BC* concentration stand at 2.232444485, 1.846846847 and 2.059907834 all near to 2 as shown in the ***Table* 3.1** indicating that the data exhibit normal distribution and the distribution curve is symmetrical. Standard deviation is 1.130771534 and skewness as 1.560077228 indicating moderately positive skewness of the data with data value near to each other. The curve is leptokurtic as the value of kurtosis is 4.245785802.

3.2.3 Bokaro ($23^o\,46'\,N\,85^o\,55'\,E$)

The value of mean, mode and median value of *BC* concentration are 2.269679113, 1.962719298 and 2.078581871 all near to 2 as shown in ***Table* 3.1** indicating that the data curve exhibit normal distribution and is symmetrical. Small value of standard deviation along with skewness at 1.142111571 and 1.529151854 respectively indicate that the data values are closely distributed with the mean and the data is positively skewed moderately towards the right. The curve is leptokurtic as the value of kurtosis is greater than 3.

3.3 Time series prediction of Black carbon

For all the sample sites, it is observed that time series *ARIMA* (1,0,1) (0,1,1) model fitted very well to the data at 95% confidence limits with 460 degree of freedom. As shown in ***Table* 3.2**, the values of stationary R^2 and *R*-squared which are the measures for goodness of fit, are both close to 1 depicting that the applied model fitted very well to the data.

***Table* 3.2:** Time series and *ARIMA* forecast of Bokaro

Fit Statistic	Raniganj	Jharia	Bokaro
Stationary R^2	0.703	0.678	0.663
R-squared	0.63	0.624	0.619
RMSE	0.64	0.662	0.677
Normalized BIC	-0.851	-0.783	-0.739
DF	460	460	460
ARIMA Model	(1,0,1)(0,1,1)	(1,0,1)(0,1,1)	(1,0,1)(0,1,1)
Predicted value	2.33769715	2.369926804	2.426624731
LCL	1.247798074	1.271560365	1.272485047
UCL	3.761368167	3.82001922	3.945160031
Residual	0.004329502	0.004394215	0.004352878

Small value of RMSE show that the actual time series is very near to the model predicted, it can also be seen from the ***Fig.* 3.3** , ***Fig.* 3.4** and ***Fig.* 3.5** that the actual time series of the data and the predicted time series obtained using *ARIMA* (1,0,1) (0,1,1) model are nearly coinciding with each other, along with the *LCL* (lower confidence limit) and *UCL* (upper confidence limit) values presented in the figures. The figures also represent the forecasting for next 7 years and 5 months starting from Jun 2018 to Oct 2025 obtained using this model.The numerical values of normalized *BIC*, mean predicted, lower confidence limit (*LCL*), upper confidence limit (*UCL*) and residual values for each of the sample sites is discussed below.

3.3.1 Raniganj ($23^o\,40'\,N\,87^0\,05'\,E$)

The value of normalized *BIC* is -0.851. The mean predicted, lower confidence limit (*LCL*), upper confidence limit (*UCL*) and residual values are observed to be 2.33769715, 1.247798074, 3.761368167 and 0.004329502.

3.3.2 Jharia ($23^o\,50'\,N\,86^o\,33'\,E$)

Normalized *BIC* value is at -0.783. The mean predicted, lower confidence limit (*LCL*), upper confidence limit (*UCL*) and residual values are observed to be as 2.369926804, 1.271560365, 3.82001922 and 0.004394215.

3.3.3 Bokaro ($23^o\,46'\,N\,85^o\,55'\,E$)

For normalized BIC the value is -0.739. The mean predicted, lower confidence limit (LCL), upper confidence limit (UCL) and residual values are observed to be 2.426624731, 1.272485047, 3.945160031and 0.004352878.

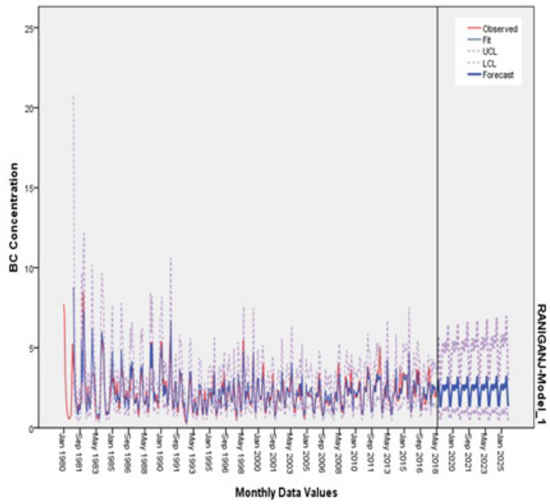

***Figure* 3.3:** Statistical analysis of black carbon

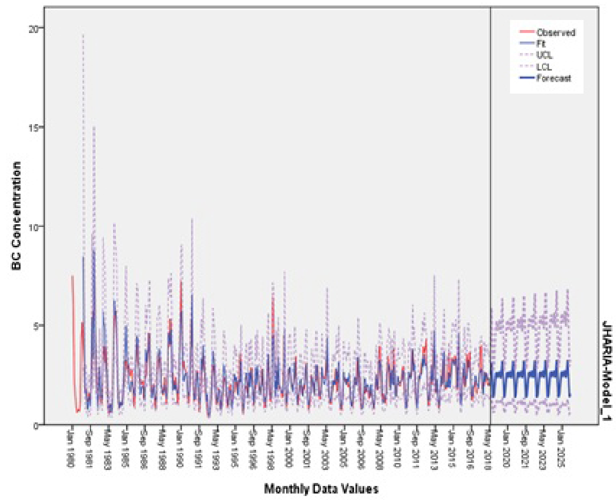

***Figure* 3.4:** Time series analysis of black carbon

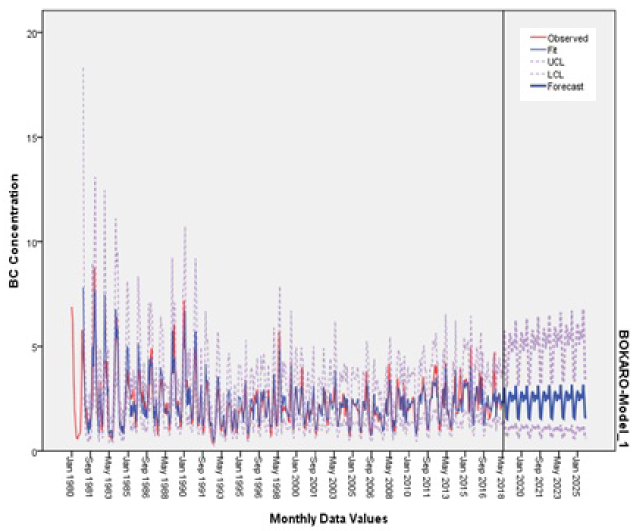

***Figure* 3.5:** Time series and *ARIMA* forecast of Raniganj

4 Conclusion

Considering a long term data for black carbon concentration of 38 years, 05 months from Jan 1980 to May 2018for coal mine regions of Raniganj, Jharia and Bokaro,time series *ARIMA*(1,0,1) (0,1,1) model is used to obtain a prediction of next 7 years and 5 months starting from Jun 2018 to Oct 2025. The shape of the distribution curve is leptokurtic at all the three sites. Small value of $RMSE$ in all the three cases indicates that the values of the original time series and the predicted model are very close to each other. It can been seen from ***Fig.* 3.3** , ***Fig.* 3.4** and ***Fig.* 3.5** that *ARIMA*(1,0,1) (0,1,1) model fitted quite well to the data over the three sample sites, the curves representing the observed and predicted values of black carbon concentration coincide with each other depicting that the difference between the values is very small. The figure also represents the future prediction made from Jun 2018 to Oct. 2025. Thus the model applied gave quite reliable results and it can be used as a future forecasting tool over the coal mines to measure the BC concentration over these regions and help in framing policies necessary for controlling air pollution and its adverse effect due to black carbon in the coal mine regions of India.

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HYERS - ULAM STABILITY OF FIRST AND SECOND ORDER PARTIAL DIFFERENTIAL EQUATIONS

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Abstract

In this paper, we prove the Hyers-Ulam (HU) stability of the first and second order partial differential equations: $u_x(x,t) + K(x,u(x,t)) = 0$ and $u_{xx}(x,t) + F(x,u)u_x(x,t) + H(x,u) = 0$ respectively.
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1 Introduction

Hyers Ulam *(HU)* stability of differential equation has drawn much attention since Ulam's [16] presentation of the problem on stability of group homomorphism in 1940 and Hyers' [5] partial solution to it in 1941. For ordinary differential equations one can refer [3, 15, 6, 7] and [8, 9] for partial differential equations. Its various extensions have been named with additional word. One such extension is Hyers Ulam Rassias *(HUR)* stability. *HUR* stability for linear differential operators of n^{th} order with non-constant coefficients is studied in [10] and [11]. *HUR* stability for special types of non-linear equations has been studied in [1, 2, 12, 13, 14]. In 2011, Gordji et al. [4], proved the *HUR* stability of non-linear partial differential equations by using Banach's Contraction Principle. In this paper, we prove the *HU* stability of first and second order partial differential equations:

(1.1) $u_x(x,t) + K(x,u(x,t)) = 0,$

where $K : J \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $u(x,t) \in C^1(J \times J)$, $J = [a,b]$ be a closed interval and

(1.2) $u_{xx}(x,t) + F(x,u)u_x(x,t) + H(x,u) = 0,$

where $F, H : J \times \mathbb{R} \to \mathbb{R}$ are continuous functions. Here $u(x,t) \in C^2(J \times J)$.

First we define HU stability.

***Definition* 1.1** *The equation* (1.1) *is said to be HU stable if the following holds:*
Let $\epsilon \geq 0$. *Assume that , for any function* $u(x,t) \in C^1$ *satisfying the differential inequality*

(1.3) $|u_x(x,t) + K(x,u(x,t))| \leq \epsilon, \ \forall x, t \in J,$

there exists a solution $u_0(x,t) \in C^1$ *of equation* (1.1) *and* $M(\epsilon) > 0$ *such that*

$$u(x,t) - u_0(x,t)| \leq M(\epsilon), \ \forall (x,t) \in J \times J.$$

Similarly we can define HU stability for equation (1.2).

We need the following result.

***Theorem* 1.1** *(Banach Contraction Principle)* [4] *: Let* (X,d) *be a complete metric space and* $T : X \to X$ *be a contraction , that is, there exists* $\alpha \in (0,1)$ *such that* $d(Tx,Ty) \leq \alpha d(x,y), \forall x, y \in X$. *Then* $\exists$ *a unique* $a \in X$ *such that* $Ta = a$. *Moreover,* $a = \lim_{n\to\infty} T^n x$ *and* $d(a,x) \leq \frac{1}{(1-\alpha)} d(x,Tx), \forall x \in X$.

2 Main Results

In this section we prove the *HU* stability of first and second order partial differential equations (1.1) and (1.2) respectively.

Theorem 2.1 *Let $x_0 \in J$ and $K : J \times R \to R$ be a continuous function such that*

(2.1) $|K(x, v(x,t)) - K(x, w(x,t))| \leq \lambda |v(x,t) - w(x,t)|, \forall x, t \in J,$

where $\lambda > 0, \lambda \in R$ and $v(x,t), w(x,t) \in C^1$.

Let

(2.2) $M_1 = \sup_{x \in J} \left| \int_{x_0}^{x} ds \right|,$

with $0 < \lambda M_1 < 1$. Let $u(x,t) \in C^1$ satisfy

(2.3) $|u_x(x,t) + K(x, u(x,t))| \leq \epsilon, \forall x, t \in J,$

then there exists a unique function $u_0(x,t) \in C^1$, such that

$$\left| \frac{\partial u_0}{\partial x}(x,t) + K(x, u_0(x,t)) \right| < \epsilon \quad \text{and } |u(x,t) - u_0(x,t)| \leq \frac{M_1}{1 - \lambda M_1} \epsilon.$$

Proof. Consider the differential equation

(2.4) $u_x(x,t) + K(x, u(x,t)) = 0, \ \forall x, t \in J.$

We define a metric d and an operator P on C^1, respectively by

(2.5) $d(\zeta, \eta) = \sup_{x,t \in J} |\zeta(x,t) - \eta(x,t)|$

and

$$(P\zeta)(x,t) = u(x_0, t) - \int_{x_0}^{x} K(s, \zeta(s,t)) ds, \forall \zeta \in C^1.$$

Consider,

$$\begin{aligned} d(P\zeta, P\eta) = & \sup_{x,t \in J} |(P\zeta)(x,t) - (P\eta)(x,t)| \\ = & \sup_{x,t \in J} \left| - \int_{x_0}^{x} K(s, \zeta(s,t)) ds + \int_{x_0}^{x} K(s, \eta(s,t)) ds \right| \\ = & \sup_{x,t \in J} \left| \int_{x_0}^{x} K(s, \zeta(s,t)) ds - \int_{x_0}^{x} K(s, \eta(s,t)) ds \right| \\ \leq & \sup_{x,t \in J} \left| \int_{x_0}^{x} |K(s, \zeta(s,t)) - K(s, \eta(s,t))| ds \right| \\ \leq & \sup_{x,t \in J} \left| \int_{x_0}^{x} \lambda |\zeta(s,t)) - \eta(s,t)| ds \right| \quad \text{(by equation (2.1))} \\ \leq & \lambda \sup_{x,t \in J} \left| \int_{x_0}^{x} \sup_{s,t \in J} |\zeta(s,t)) - \eta(s,t)| ds \right| \\ \leq & \lambda d(\zeta, \eta) \times \sup_{x \in J} \left| \int_{x_0}^{x} ds \right| \\ \leq & \lambda d(\zeta, \eta) \times M_1 \quad \text{(by equation (2.2)).} \end{aligned}$$

Then by using Banach Contraction Principle, there exists a unique $u_0(x,t) \in C^1$ such that $Pu_0(x,t) = u_0(x,t)$. Thus $u_0(x,t)$ satisfy $u(x_0,t) - \int_{x_0}^{x} K(s, u_0(s,t)) ds = u_0(x,t)$ and

(2.6) $d(u_0, u) \leq \dfrac{1}{1 - \lambda M_1} d(u, Pu).$

Now by inequality (2.3) we get,

$$-\epsilon \leq \frac{\partial u}{\partial x}(x,t) + K(x, u(x,t)) \leq \epsilon, \forall x, t \in J.$$

Integrating from x_0 to x we get,

$$-\epsilon \int_{x_0}^{x} ds \leq \int_{x_0}^{x} \left\{ \tfrac{\partial u}{\partial s}(s,t) + K(s, u(s,t)) \right\} ds \leq \epsilon \int_{x_0}^{x} ds,$$

$$\Rightarrow -\epsilon \int_{x_0}^{x} ds \leq \{u(x,t) - u(x_0,t) + \int_{x_0}^{x} K(s, u(s,t))\} ds \leq \epsilon \int_{x_0}^{x} ds.$$

$$\Rightarrow -\epsilon \sup_{x\in J} \int_{x_0}^{x} ds \le -\epsilon \int_{x_0}^{x} ds \le \{u(x,t) - u(x_0,t) + \int_{x_0}^{x} K(s,u(s,t))\}ds \le \epsilon \int_{x_0}^{x} ds \le \sup_{x\in J} \int_{x_0}^{x} ds.$$

$$\Rightarrow -\epsilon \sup_{x\in J}\left| \int_{x_0}^{x} ds\right| \le -\epsilon \sup_{x\in J} \int_{x_0}^{x} ds \le -\epsilon \int_{x_0}^{x} ds \le \{u(x,t) - u(x_0,t) + \int_{x_0}^{x} K(s,u(s,t))\}ds$$

$$\le \epsilon \int_{x_0}^{x} ds \le \sup_{x\in J} \int_{x_0}^{x} ds \le \epsilon \sup_{x\in J}\left| \int_{x_0}^{x} ds\right|.$$

$$\Rightarrow -\epsilon M_1 \le \{u(x,t) - u(x_0,t) + \int_{x}^{x_0} K(s,u(s,t))\}ds \le \epsilon M_1.$$

$$\Rightarrow |u(x,t) - u(x_0,t) + \int_{x}^{x_0} K(s,u(s,t))ds| \le \epsilon M_1.$$

$$\Rightarrow |u(x,t) - (Pu)(x,t)| \le \epsilon M_1.$$

$$\Rightarrow \sup_{x,t\in J}|u(x,t) - (Pu)(x,t)| \le \epsilon M_1.$$

$$\Rightarrow \quad d(u,Pu) \le \epsilon M_1.$$

Using this inequality and equation (2.6) we get,

$$\begin{aligned}
|u(x,t) - u_0(x,t)| &= |u_0(x,t) - u(x,t)| \\
&\le \sup_{x,t\in J}|u_0(x,t) - u(x,t)| \\
&= d(u_0(x,t), u(x,t)) \\
&\le \frac{1}{1-\lambda M_1} d(u,Pu) \\
&\le \frac{M_1}{1-\lambda M_1}\epsilon.
\end{aligned}$$

We now prove the *HU* stability of equation (1.2).

Theorem 2.2 *Let $x_0 \in J$ and $F, H : J \times \mathbb{R} \to \mathbb{R}$ be a continuous functions such that*

(2.7) $|F(x,v(x,t))v_x(x,t) - F(x,w(x,t))w_x(x,t)| \le \lambda_1|v(x,t) - w(x,t)|,$

and

(2.8) $|H(x,v(x,t)) - H(x,w(x,t))| \le \lambda_2|v(x,t) - w(x,t)|, \ \forall x,t \in J,$

where $\lambda_1, \lambda_2 > 0, \lambda_1, \lambda_2 \in \mathbb{R}$ and $v(x,t), w(x,t) \in C^2(J\times J)$.

Let

(2.9) $M_2 = \sup_{x,y\in J}\left| \int_{x_0}^{x}\int_{x_0}^{y} dsdy\right|,$

with $0 < \{\lambda_1+\lambda_2\}M_2 < 1$. If $u(x,t) \in C^2(J\times J)$ satisfy

(2.10) $|u_{xx}(x,t) + F(x,u)u_x(x,t) + H(x,u)| \le \epsilon, \ \forall x,t \in J,$

then there exists, a unique function, $u_0(x,t) \in C^2(J\times J)$,
such that $\frac{\partial^2 u_0}{\partial x^2}(x,t) + F(x,u_0(x,t))\frac{\partial u_0}{\partial x}(x,t) + H(x,u_0(x,t)) = 0$ and $|u(x,t) - u_0(x,t)| \le \frac{M_2}{1-\{\lambda_1+\lambda_2\}M_2}\epsilon$.

Proof. Consider the differential equation

(2.11) $$\frac{\partial^2 u}{\partial x^2}(x,t) + F(x,u(x,t))\frac{\partial u}{\partial x}(x,t) + H(x,u(x,t)) = 0, \forall x,t \in J.$$

We define a metric d and an operator P on $C^2(J\times J)$, respectively by

(2.12) $d(\zeta,\eta) = \sup_{x,t\in J}|\zeta(x,t) - \eta(x,t)|,$

and

$$(P\zeta)(x,t) = u(x_0,t) - \int_{x_0}^{x}\int_{x_0}^{y} F(s,\zeta(s,t))\zeta_s(s,t)dsdy - \int_{x_0}^{x}\int_{x_0}^{y} H(s,\zeta(s,t))dsdy$$

$\forall \ \zeta \in C^2(J\times J).$

Consider

$$d(P\zeta, P\eta) = \sup_{x,t\in J}|(P\zeta)(x,t) - (P\eta)(x,t)|$$

$$
\begin{aligned}
&= \sup_{x,t\in J} \Big| -\int_{x_0}^{x}\int_{x_0}^{y} F(s,\zeta(s,t))\zeta_s(s,t)dsdy - \int_{x_0}^{x}\int_{x_0}^{y} H(s,\zeta(s,t))dsdy \\
&+ \int_{x_0}^{x}\int_{x_0}^{y} F(s,\eta(s,t))\eta_s(s,t)dsdy + \int_{x_0}^{x}\int_{x_0}^{y} H(s,\eta(s,t))dsdy\Big| \\
&= \sup_{x,t\in J} \Big| \int_{x_0}^{x}\int_{x_0}^{y} F(s,\zeta(s,t))\zeta_s(s,t)dsdy + \int_{x_0}^{x}\int_{x_0}^{y} H(s,\zeta(s,t))dsdy \\
&- \int_{x_0}^{x}\int_{x_0}^{y} F(s,\eta(s,t))\eta_s(s,t)dsdy - \int_{x_0}^{x}\int_{x_0}^{y} H(s,\eta(s,t))dsdy\Big| \\
&= \sup_{x,t\in J} \Big| \int_{x_0}^{x}\int_{x_0}^{y} F(s,\zeta(s,t))\zeta_s(s,t)dsdy - \int_{x_0}^{x}\int_{x_0}^{y} F(s,\eta(s,t))\eta_s(s,t)dsdy \\
&+ \int_{x_0}^{x}\int_{x_0}^{y} H(s,\zeta(s,t))dsdy - \int_{x_0}^{x}\int_{x_0}^{y} H(s,\eta(s,t))dsdy\Big| \\
&\le \sup_{x,t\in J} \Big| \int_{x_0}^{x}\int_{x_0}^{y} F(s,\zeta(s,t))\zeta_s(s,t)dsdy - \int_{x_0}^{x}\int_{x_0}^{y} F(s,\eta(s,t))\eta_s(s,t)dsdy\Big| \\
&+ \sup_{x,t\in J} \Big| \int_{x_0}^{x}\int_{x_0}^{y} H(s,\zeta(s,t))dsdy - \int_{x_0}^{x}\int_{x_0}^{y} H(s,\eta(s,t))dsdy\Big| \\
&\le \sup_{x,t\in J} \Big| \int_{x_0}^{x}\int_{x_0}^{y} |F(s,\zeta(s,t))\zeta_s(s,t) - F(s,\eta(s,t))\eta_s(s,t)|dsdy\Big| \\
&+ \sup_{x,t\in J} \Big| \int_{x_0}^{x}\int_{x_0}^{y} |H(s,\zeta(s,t)) - H(s,\eta(s,t))|dsdy\Big| \\
&\le \sup_{x,t\in J} \Big| \int_{x_0}^{x}\int_{x_0}^{y} \lambda_1|\zeta(s,t)) - \eta(s,t)|dsdy\Big| + \sup_{x,t\in J} \Big| \int_{x_0}^{x}\int_{x_0}^{y} \lambda_2|\zeta(s,t)) - \eta(s,t)|dsdy\Big|
\end{aligned}
$$

(by equation (2.7) and (2.8))

$$
\begin{aligned}
&\le \lambda_1 \sup_{x,t\in J} \Big| \int_{x_0}^{x}\int_{x_0}^{y} \sup_{s,t\in J}|\zeta(s,t)) - \eta(s,t)|dsdy\Big| + \\
&\lambda_2 \sup_{x,t\in J} \Big| \int_{x_0}^{x}\int_{x_0}^{y} \sup_{s,t\in J}|\zeta(s,t)) - \eta(s,t)|dsdy\Big| \\
&\le \lambda_1 \sup_{x,t\in J} \Big| \int_{x_0}^{x}\int_{x_0}^{y} d(\zeta,\eta)dsdy\Big| + \lambda_2 \sup_{x,t\in J} \Big| \int_{x_0}^{x}\int_{x_0}^{y} d(\zeta,\eta)dsdy\Big| \\
&\le \{\lambda_1+\lambda_2\}d(\zeta,\eta) \sup_{x,t\in J} \Big| \int_{x_0}^{x}\int_{x_0}^{y} dsdy\Big| \\
&\le \{\lambda_1+\lambda_2\}d(\zeta,\eta)M_2 \qquad \text{(by equation (2.9))} \\
&\le \{\lambda_1+\lambda_2\}M_2 \times d(\zeta,\eta).
\end{aligned}
$$

Therefore by using ***Theorem* 1.1** , there exists, a unique, $u_0(x,t) \in C^2(J\times J)$ such that $Pu_0(x,t) = u_0(x,t)$. Thus $u_0(x,t)$ satisfies

(2.13) $u(x_0,t) - \int_{x_0}^{x}\int_{x_0}^{y} F(s,u_0(s,t))u_s(s,t))dsdy - \int_{x_0}^{x}\int_{x_0}^{y} H(s,u_0(s,t))dsdy = u_0(x,t)$

and

$$d(u_0,u) \le \frac{1}{1-(\lambda_1+\lambda_2)M_2} d(u,Pu).$$

Now by inequality (2.10) we get,

$$-\epsilon \le \frac{\partial^2 u}{\partial x^2}(x,t) + F(x,u)\frac{\partial u}{\partial x}(x,t) + H(x,u) \le \epsilon, \quad \forall x,t\in J.$$

Integrating from x_0 to x we derive,

$$-\epsilon\int_{x_0}^{x} ds \le \tfrac{\partial u}{\partial x}(x,t) - \tfrac{\partial u}{\partial x}(x_0,t) + \int_{x_0}^{x} F(s,u(s,t))u_s(s,t)ds + \int_{x_0}^{x} H(s,u(s,t))ds \le \epsilon\int_{x_0}^{x} ds.$$

Again integrating from x_0 to x we obtain,

$$
\begin{aligned}
&-\epsilon\int_{x_0}^{x}\int_{x_0}^{y} dsdy \le u(x,t) - u(x_0,t) - [u(x_0,t) - u(x_0,t)] + \int_{x_0}^{x}\int_{x_0}^{y} F(s,u(s,t))u_s(s,t)dsdy \\
&+ \int_{x_0}^{x}\int_{x_0}^{y} H(s,u(s,t))dsdy \le \epsilon\int_{x_0}^{x}\int_{x_0}^{y} dsdy.
\end{aligned}
$$

By using the equation (2.12) we get,

$$
\begin{aligned}
&-\epsilon\int_{x_0}^{x}\int_{x_0}^{y} dsdy \le u(x,t) - (Pu)(x,t) \le \epsilon\int_{x_0}^{x}\int_{x_0}^{y} dsdy. \\
&\Rightarrow -\epsilon \sup_{x,y\in J}\int_{x_0}^{x}\int_{x_0}^{y} dsdy \le -\epsilon\int_{x_0}^{x}\int_{x_0}^{y} dsdy \le u(x,t) - (Pu)(x,t) \le \epsilon\int_{x_0}^{x}\int_{x_0}^{y} dsdy \le \epsilon \sup_{x,y\in J}\int_{x_0}^{x}\int_{x_0}^{y} dsdy. \\
&\Rightarrow -\epsilon \sup_{x,y\in J}\Big|\int_{x_0}^{x}\int_{x_0}^{y} dsdy\Big| \le -\epsilon \sup_{x,y\in J}\int_{x_0}^{x}\int_{x_0}^{y} dsdy \le -\epsilon\int_{x_0}^{x}\int_{x_0}^{y} dsdy \le u(x,t) - (Pu)(x,t)
\end{aligned}
$$

$$\leq \epsilon \int_{x_0}^{x}\int_{x_0}^{y} dsdy \leq \epsilon \sup_{x,y\in J}\int_{x_0}^{x}\int_{x_0}^{y} dsdy \leq \epsilon \sup_{x,y\in J}\left|\int_{x_0}^{x}\int_{x_0}^{y} dsdy\right|.$$

$$\Rightarrow -\epsilon \sup_{x,y\in J}\left|\int_{x_0}^{x}\int_{x_0}^{y} dsdy\right| \leq u(x,t)-(Pu)(x,t) \leq \epsilon \sup_{x,y\in J}\left|\int_{x_0}^{x}\int_{x_0}^{y} dsdy\right|.$$

$$\Rightarrow -\epsilon M_2 \leq u(x,t)-(Pu)(x,t) \leq \epsilon M_2.$$

$$\Rightarrow |u(x,t)-(Pu)(x,t)| \leq \epsilon M_2.$$

$$\Rightarrow \sup_{x,t\in J}|u(x,t)-(Pu)(x,t)| \leq \epsilon M_2.$$

$$\Rightarrow d(u,Pu) \leq \epsilon M_2,$$

which with equation (2.13) yields

$$\begin{aligned}|u(x,t)-u_0(x,t)| &= |u_0(x,t)-u(x,t)|\\ &\leq \sup_{x,t\in J}|u_0(x,t)-u(x,t)|\\ &= d(u_0(x,t),u(x,t)),\\ &\leq \frac{1}{1-\{\lambda_1+\lambda_2\}M_2}d(u,Pu),\\ &\leq \frac{M_2}{1-\{\lambda_1+\lambda_2\}M_2}\epsilon.\end{aligned}$$

Hence the result.

3 Conclusion

In this paper, we have proved the Hyers - Ulam stability of first and second order partial differential equations (1.1) and (1.2) respectively by employing Banach's Contraction Principle.

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FUZZY PREOPEN SETS AND FUZZY PRE-CONTINUITY

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Abstract

In the present paper, we introduce fuzzy preopen (closed) sets and fuzzy pre-continuity in Sostak fuzzy topological space. Also we investigate their significant characteristic properties.
2010 Mathematics Subject Classifications: 54A40.
Keywords and phrases: Fuzzy sets, fuzzy topological space, gradation of openness.

1 Introduction

The concept of fuzzy sets was introduced by Zadeh [9] and later Chang [2] defined fuzzy topological spaces. Sostak [8] introduced a new fuzzy topological space exploiting the idea of partial openness of fuzzy sets. This generalized fuzzy topological space was later rephrased by Chattopadhyay et.al. [3]. Several mathematicians have worked on this space (see [4], [5]).

The concepts of fuzzy preopen sets, fuzzy strong preopen sets and strong pre continuity (see [6], [7]) have been introduced in case of classical fuzzy topological spaces introduced by Chang [2]. In the present paper, we introduce fuzzy preopen (closed) sets and fuzzy pre continuity in the Sostak fuzzy topological space redefined by Chattopdhyay [3]. Further we establish their significant properties.

2 Preliminaries

Let X be a non-empty set and $I \equiv [0, 1]$ be the unit closed interval of real line. Let I^X denote the set of all fuzzy sets on X. A fuzzy set A on X is a mapping $A : X \to I$, where for any $x \in X$, $A(x)$ denotes the degree of membership of element x in fuzzy set A. The null fuzzy set 0 and whole fuzzy set 1 are the constant mappings from X to $\{0\}$ and $\{1\}$ respectively.

A family τ of fuzzy sets on X is called a fuzzy topology (see [2]) on X if (i) 0 and 1 belong to τ, (ii) Any union of members of τ is in τ, (iii) a finite intersection of members of τ is in τ. The system consisting of X equipped with fuzzy topology τ defined on it, is called a fuzzy topological space and is denoted as (X, τ). Now we define the So-fuzzy topological space (see [3], [8]).

A So-fuzzy topology on a non-empty set X is a family τ of fuzzy sets on X satisfying the following axioms with respect to a mapping $\tau : I^X \to I$ such that

1. $\tau(0) = \tau(1) = 1$;
2. $\tau(A \cap B) \geq \tau(A) \wedge \tau(B)$; for any $A, B \in I^X$;
3. $\tau(\cup_{i\in J} A_i) \geq \wedge_{i\in J} \tau(A_i)$, for any arbitrary family $\{A_i : i \in J\} \subseteq I^X$.

The system (X, τ) is called So-fuzzy topological space and the real number $\tau(A)$ is called the degree (or grade) of openness of fuzzy set A. We note that

***Proposition* 2.1** *Let X be a non-empty set. Then the map* $\tau : I^X \to I$ *given by* $\tau(0) = 1$ *and* $\tau(A) = \inf\{A(x) : x \in suppA\}$, *if* $A \neq 0$, *satisfies the axioms of gradation of openness.*

If (X, τ) *is a So-fuzzy topological space, then we observe that (see [2]) for any* $\rho \in [0, 1]$, *the family* $\tau_\rho \equiv \{ A \in I^X : \tau(A) \geq \rho\}$ *is actually a fuzzy topology in sense of Chang [2] and it is called* ρ*-level fuzzy topology on X with respect to the gradation of openness* τ. *All fuzzy sets belonging to* τ_ρ *are called fuzzy-*ρ*-open sets and their complements are called fuzzy-*ρ*-closed sets.*

For any fuzzy set A, the interior and closure of A with respect to τ_ρ *are defined as follows:*

$$Int_\rho(A) = \cup\{G \in I^X : G \subseteq A \text{ and } G \in \tau_\rho\}$$

$$Cl_\rho(A) = \cap\{K \in I^X : A \subseteq K \text{ and } K^c \in \tau_\rho\}$$

3 Fuzzy-ρ-Pre Open (Closed) Sets

In this section, we define fuzzy-ρ-pre open sets and fuzzy-ρ-pre closed sets in So-fuzzy topological space and investigate their properties.

***Definition* 3.1** *Let $(X,\ \tau)$ be a So-fuzzy topological space and $A \in I^X$ be a fuzzy set. Then for any $\rho \in I$, a fuzzy set A is said to be a*

(i) *Fuzzy-ρ-pre open set in X iff $A \subseteq Int_\rho(Cl_\rho(A))$,*

(ii) *Fuzzy-ρ-pre closed set in X iff $A \supseteq Cl_\rho(Int_\rho(A))$.*

Clearly fuzzy sets 0 *and* 1 *are both trivially fuzzy ρ-pre open as well as fuzzy ρ-pre closed sets in X.*

***Remark* 3.1** *It is clear that every fuzzy-ρ-open (closed) set is a fuzzy-ρ-pre open (closed) set, but converse of these may not be true in general.*

***Example* 3.1** *Let $X = \{a,\ b\}$ and $A,\ B,\ C \in I^X$ be fuzzy sets defined as follows:*

$A = \{(a,\ 0.6),\ (b,\ 0.3)\}$; $B = \{(a,\ 0.4),\ (b,\ 0.2)\}$; $C = \{(a,\ 0.8),\ (b,\ 0.5)\}$.

Define a map $\tau : I^X \to I$ as follows:

$$\tau(F) = \{ \begin{array}{ll} 1, & if\ F = 0,\ 1 \\ 0.3, & if\ F = A \\ 0.2, & if\ F = B \\ 0, & otherwise. \end{array} .$$

Suppose $\rho = 0.1$. *We see that fuzzy set C is a fuzzy-ρ-pre open set because $Int_\rho(Cl_\rho(C)) = 1 \supseteq C$. But it is not a fuzzy-$\rho$-open set (because $\tau(C) = 0 \not\geq 0.1$).*

***Theorem* 3.1** *Let (X, τ) be a So-fuzzy topological space. Then for any $\rho \in I$,*

(a) Any union of fuzzy-ρ-pre open sets is a fuzzy-ρ-pre open set;

(b) Any intersection of fuzzy-ρ-pre closed sets is a fuzzy-ρ-pre closed set.

Proof. **(a)** Let $\{A_i\ : i \in J\}$ be an arbitrary collection of fuzzy-ρ-pre open sets in So-fuzzy topological space $(X,\ \tau)$. Then for each $i \in J$, we have $A_i \subseteq Int_\rho(Cl_\rho(A_i))$. Hence

$$\cup_{i\in J} A_i \subseteq\ \cup_{i\in J}\ Int_\rho(Cl_\rho(A_i)) \subseteq Int_\rho(\cup_{i\in J}\ Cl_\rho(A_i)) \subseteq Int_\rho(Cl_\rho(\cup_{i\in J}\ A_i)).$$

Thus $\cup_{i\in J}A_i$ is a fuzzy-ρ-pre open set. We can prove (b) similarly.

***Definition* 3.2** *Let $(X,\ \tau)$ be a So-fuzzy topological space and $A \in I^X$ be a fuzzy set. Then for each $\rho \in I$, fuzzy-ρ-pre interior and fuzzy-ρ-pre closure of fuzzy set A denoted as P-$int_\rho(A)$ and P-$cl_\rho(A)$ are defined as follows:*

$$P - int_\rho(A) = \cup\{G \in I^X : G \subseteq A \text{ and } G \text{ is a } fuzzy - \rho - pre\ open\ set\ in\ X\},$$

$$P - cl_\rho(A) \ \ = \cap\{K \in I^X : K \supseteq A \text{ and } K \text{ is a } fuzzy - \rho - pre\ closed\ set\ in\ X\}.$$

***Theorem* 3.2** *Let $(X,\ \tau)$ be a So-fuzzy topological space and $A \in I^X$ be a fuzzy set. Then for any $\rho \in I$,*

(i) P-$cl_\rho(1 - A) = 1 - P$-$int_\rho(A)$,

(ii) P-$int_\rho(1 - A) = 1 - P$-$cl_\rho(A)$.

Proof. **(i)** Suppose $\{G_i\}_{i\in J}$ is the family of all fuzzy-ρ-preopen sets in X contained in A. Then

$$P - int_\rho(A) = \ \cup_{i\in J}G_i = 1 - \cap_{i\in J}\ G_i^c.$$

Since $G_i \subseteq A$, we have $G_i^c \supseteq A^c,\ \forall\ i \in J$. Thus $\{G_i^c\}_{i\in J}$ is the collection of all fuzzy-ρ-preclosed sets containing A^c. Hence $\cap_{i\in J}G_i^c = P$-$cl_\rho(A^c) = P$-$cl_\rho(1 - A)$. Thus P-$int_\rho(A) = 1 - P$-$cl_\rho(1 - A)$. Hence P-$cl_\rho(1 - A) = 1 - P$-$int_\rho(A)$.

Proof. **(ii)** It can be proved in a similar manner.

***Theorem* 3.3** *Let $(X,\ \tau)$ be a So-fuzzy topological space. Then for any $\rho \in I$, a fuzzy set $A \in I^X$ is a*

(a) Fuzzy-ρ-pre open set iff P-$int_\rho(A) = A$;

(b) Fuzzy-ρ-pre closed set iff P-$cl_\rho(A) = A$.

Proof. **(a)** Let A be fuzzy-ρ-pre open set in X. Let $\{G_i\}_{i\in J}$ be the family of all fuzzy ρ-pre open sets contained in A. Since each $G_i \subseteq A$, $i \in J$, we have $\cup_{i\in J} G_i \subseteq A$. Therefore

(3.1) $P - int_\rho = \cup_{i\in J}\{G_i \in I^X : G_i \subseteq A$ and G_i is a fuzzy $- \rho -$ *preopen set*$\} \subseteq A$.

Since $A \subseteq A$ and A is a fuzzy-ρ-preopen set in X, hence $A \in \{G_i\}_{i\in J}$. Therefore

(3.2) $A \subseteq \cup_{i\in J} G_i \equiv P - int_\rho(A)$.

From equations (3.3.1) and (3.3.2), $A = P\text{-}int_\rho(A)$.

Conversely; suppose A is a fuzzy set in So-fuzzy topological space (X, τ) such that $A = P\text{-}int_\rho(A)$. Then

(3.3) $A = P - int_\rho(A) = \cup\{G_i \in I^X : G_i \subseteq A$ and $G_i$is a fuzzy $- \rho -$ pre open set$\}$.

Since any union of fuzzy-ρ-preopen sets is a fuzzy-ρ-preopen set, in view of (3.3.3), set A is a fuzzy-ρ-pre open set in X.

Proof. **(b)** This can be proved in a similar manner.

***Theorem* 3.4** *Let (X, τ) be a So-fuzzy topological space. Then for any $\rho \in I$, the following properties hold for fuzzy-ρ-pre closure:*

(i) $P\text{-}cl_\rho(0) = 0$;
(ii) $P\text{-}cl_\rho(A)$ is a fuzzy-ρ-pre closed set in X;
(iii) $P\text{-}cl_\rho(A) \subseteq P\text{-}cl_\rho(B)$, if $A \subseteq B$;
(iv) $P\text{-}cl_\rho(P\text{-}cl_\rho(A)) = P\text{-}cl_\rho(A)$;
(v) $P\text{-}cl_\rho(A \cup B) \supseteq P\text{-}cl_\rho(A) \cup P\text{-}cl_\rho(B)$;
(vi) $P\text{-}cl_\rho(A \cap B) \subseteq P\text{-}cl_\rho(A) \cap P\text{-}cl_\rho(B)$.

Proof. It is easy to prove.

***Theorem* 3.5** *Let (X, τ) be a So-fuzzy topological space and $A, B \in I^X$ be fuzzy sets. Then for any $\rho \in I$,*

(i) $P\text{-}int_\rho(1) = 1$;
(ii) $P\text{-}int_\rho(A)$ is a fuzzy-ρ-pre open set in X;
(iii) $P\text{-}int_\rho(A) \subseteq P\text{-}int_\rho(B)$, if $A \subseteq B$;
(iv) $P\text{-}int_\rho(P\text{-}int_\rho(A)) = P\text{-}int_\rho(A)$;
(v) $P\text{-}int_\rho(A \cup B) \supseteq P\text{-}int_\rho(A) \cup P\text{-}int_\rho(B)$;
(vi) $P\text{-}int_\rho(A \cap B) \subseteq P\text{-}int_\rho(A) \cap P\text{-}int_\rho(B)$.

4 Fuzzy-ρ-Pre Continuous Map

In this section, we define a fuzzy-ρ-pre continuous map from one So-fuzzy topological space to another and investigate its characteristic properties. We know fuzzy-ρ-continuous map is defined (see [3]) as follows:

***Definition* 4.1** *Let (X, τ) and (Y, σ) be two So-fuzzy topological spaces. A map $f : X \to Y$ is said to be a fuzzy-ρ-continuous map if $\tau(f^{-1}(B)) \geq \sigma(B)$, for each fuzzy set $B \in I^Y$ such that $\sigma(B) \geq \rho$.*

Now we define fuzzy-ρ-pre continuous map as follow:

***Definition* 4.2** *Let (X, τ) and (Y, σ) be two So-fuzzy topological spaces. A map f from X to Y is called a fuzzy-ρ-pre continuous map iff $f^{-1}(B)$ is a fuzzy-ρ-pre open set for any fuzzy set $B \in I^Y$ such that $\sigma(B) \geq \rho$.*

***Remark* 4.1** *It is obvious that every fuzzy-ρ-continuous map is a fuzzy-ρ-pre continuous map, but converse may not be true.*

***Example* 4.1** *Let $X = \{a, b\}$, $Y = \{u, v\}$ and $A, B \in I^X$, $C \in I^Y$ be fuzzy sets defined as follows:*

$A = \{(a, 0.7), (b, 0.2)\}$; $B = \{(a, 0.5), (b, 0.6)\}$; $C = \{(a, 0.7), (b, 0.6)\}$;
$D = \{(a, 0.5), (b, 0.2)\}$; $E = \{(u, 0.8), (v, 0.7)\}$.

We define fuzzy topologies $\tau : I^X \to I$ and $\sigma : I^Y \to I$ as follows:

$$\tau(F) = \begin{cases} 1, & \text{if } F = 0,\ 1 \\ 0.2, & \text{if } F = A,\ D \\ 0.5, & \text{if } F = B \\ 0.6 & \text{if } F = C \\ 0, & \text{otherwise,} \end{cases}$$

$$\sigma(F) = \begin{cases} 1, & \text{if } F = 0,\ 1 \\ 0.7, & \text{if } F = E \\ 0, & \text{otherwise.} \end{cases}$$

Consider a map $f : (X, \tau) \to (Y, \sigma)$ defined as $f(A) = u$, $f(b) = v$. Suppose $\rho = 0.1$. We see that $f^{-1}(E) \subseteq Int_\rho(Cl_\rho(f^{-1}(E)))$. Hence $f^{-1}(E)$ is a fuzzy-ρ-pre open set. Similarly $f^{-1}(0) \equiv 0$ and $f^{-1}(1) \equiv 1$ are also fuzzy-ρ-pre open sets. Thus f is a fuzzy-ρ-pre continuous map. But f is not a fuzzy-ρ-continuous map because $f^{-1}(E)$ is not a fuzzy-ρ-open set.

***Theorem* 4.1** *Let $f : (X. \tau) \to (Y, \sigma)$ be a map from one So-fuzzy topological space to another such that $\tau^*(f^{-1}(B)) \geq \rho$, for each $B \in I^Y$ with $\sigma^*(B) \geq \rho$, then f is a fuzzy-ρ-pre continuous map.*

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ be a map such that $\tau^*(f^{-1}(B)) \geq \rho$, for each $B \in I^Y$ for which $\sigma^*(B) \geq \rho$. Since $f^{-1}(B) \in I^X$ and $\tau^*(f^{-1}(B)) = \tau((f^{-1}(B))^c) = \tau(f^{-1}(B^c)) \geq \rho$, we conclude that $f^{-1}(B^c)$ is a fuzzy-ρ-open set in X. Since every fuzzy ρ-open set is a fuzzy ρ-pre open set, $f^{-1}(B^c)$ is a fuzzy-ρ-pre open set in X. Further $\sigma(B^c) = \sigma^*(B) \geq \rho$. Thus $f^{-1}(B^c)$ is a fuzzy-ρ-pre open set in X for each $B^c \in I^Y$ such that $\sigma(B^c) \geq \rho$. Therefore f is a fuzzy-ρ-pre continuous map.

***Theorem* 4.2** *Let $f : (X. \tau) \to (Y, \sigma)$ be a map from one So-fuzzy topological space to another. Then for any $\rho \in I$, following statements are equivalent:*

(a) f is a fuzzy-ρ-pre continuous map;

(b) $f^{-1}(B)$ is a fuzzy-ρ-pre closed set for each fuzzy-ρ-closed set B in Y;

(c) $Cl_\rho(Int_\rho(f^{-1}(B))) \subseteq f^{-1}(Cl_\rho(B))$, for each fuzzy set B in Y;

(d) $f(Cl_\rho(Int_\rho(A))) \subseteq Cl_\rho(f(A))$, for each fuzzy set A in X.

Proof. Let (X, τ) and (Y, σ) be two So-fuzzy topological spaces. We will prove this theorem in following steps:

(i) $(a)arrow(b)$: Let $f : X \to Y$ be a fuzzy-ρ-pre continuous map for any $\rho \in I$. Let B be a fuzzy-ρ-closed set in Y. Then B^c is a fuzzy-ρ-open set in Y so that $\sigma(B^c) \geq \rho$. Since f is a fuzzy ρ-continuous map, we find that $f^{-1}(B^c)$ is a fuzzy-ρ-pre open set in X. Therefore $(f^{-1}(B^c))^c = f^{-1}(B)$ is a fuzzy-ρ-pre closed set in X. Similarly we can prove (b)$arrow$(a).

(ii) $(b)arrow(c)$: Let B be a fuzzy set in Y, then $Cl_\rho(B)$ is a fuzzy-ρ-closed set in Y and hence by (b), $f^{-1}(Cl_\rho(B))$ is a fuzzy-ρ-pre closed set in X. Therefore by definition, $f^{-1}(Cl_\rho(B)) \supseteq Cl_\rho(Int_\rho(f^{-1}(Cl_\rho(B)))) \supseteq Cl_\rho(Int_\rho(f^{-1}(B)))$. Thus $Cl_\rho(Int_\rho(f^{-1}(B))) \subseteq f^{-1}(Cl_\rho(B))$.

(iii) $(c)arrow(d)$: Let $A \in I^X$ be any fuzzy set, then $f(A) \in I^Y$. Now by (c), $Cl_\rho(Int_\rho(f^{-1}(f(A)))) \subseteq f^{-1}(Cl_\rho(f(A)))$. It implies that $Cl_\rho(Int_\rho(A)) \subseteq f^{-1}(Cl_\rho(f(A)))$. Hence $f(Cl_\rho(Int_\rho(A))) \subseteq f(f^{-1}(Cl_\rho(f(A)))) \subseteq Cl_\rho(f(A))$.

(iv) $(d)arrow(b)$: can be proved easily.

***Theorem* 4.3** *Let (X, τ), (Y, σ) and (Z, δ) be three So-fuzzy topological spaces and let $\rho \in I$ be any real number. If $f : X \to Y$ is a fuzzy-ρ-pre continuous map and $g : Y \to Z$ is a fuzzy-ρ-continuous map, then $g \circ f : X \to Z$ is a fuzzy-ρ-pre continuous map.*

Proof. Let C be a fuzzy-ρ-open set in Z so that $\delta(C) \geq \rho$, then $\sigma(g^{-1}(C)) \geq \delta(C) \geq \rho$. Thus by hypothesis $g^{-1}(C)$ is a fuzzy-ρ-open set in Y. Since f is a fuzzy-ρ-pre continuous map, we get that $f^{-1}(g^{-1}(C))$ is a fuzzy-ρ-pre open set in X. Now $f^{-1}(g^{-1}(C)) = (g \circ f)^{-1}(C)$. Hence $(g \circ f)^{-1}(C)$ is a fuzzy-ρ-pre open set in X. Now $g \circ f : (X, \tau) \to (Z, \delta)$ is a map and we have derived that for any fuzzy-ρ-open set C in Z, fuzzy set $(g \circ f)^{-1}(C)$ is a fuzzy-ρ-pre open set in X. Hence $(g \circ f)$ is a fuzzy-ρ-pre continuous map.

5 Conclusion

In the present paper, we have defined fuzzy pre open (closed) sets and fuzzy pre-continuity in Sostak fuzzy topological space. The concept is introduced as an extension of concepts of fuzzy preopen sets introduced in [6]. Several significant results have been obtained.

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(Dedicated to Honor Dr. R. C. Singh Chandel on His 75^{th} Birth Anniversary Celebrations)

A COMPUTATIONAL METHOD FOR SOLVING STOCHASTIC INTEGRAL EQUATIONS USING HAAR WAVELETS

By

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Abstract

In this article, we have developed a new technique for solving stochastic integral equations. A new Haar wavelets stochastic operational matrix of integration (*HWSOMI*) is developed in order to obtain efficient and accurate solution for stochastic integral equations. In the beginning we study the properties of stochastic integrals and Haar wavelets. Convergence and error analysis of Haar wavelet method is presented. Accuracy of the method investigated is justified through some examples.

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Keywords and phrases: Stochastic operational matrix of integration, Haar wavelets, stochastic integral equations.

1 Introduction

Wavelet is a newly emerging area of mathematics. Wavelets have a number of applications in signal processing [11]. Integral equations are the most important tools describing knowledge models. Since many a times, the exact solution of integral equations does not exist, the numerical approximation of these equations become necessary. Different methods are used for approximating these equations and different basis functions are used.

Modeling various phenomena in science, engineering and physics requires stochastic integrals [4]. Numerical computations of stochastic integral equations have been studied by various authors. Some of which are Claeden and Platen [6], Oksendal [8], Maleknejad et al. [7], Cortes et al. [3], Douglas et al. [4], and Zhang [12].

Due to the large number of applications of Haar wavelets in solving differential, integral and integro differential equations, many authors have studied the computational methods for the solution of these equations using Haar wavelets. Some of which are found in [9], [10], [1] and [2]. In the present investigation with the help of Haar wavelets we are developing a novel stochastic operational matrix of Haar wavelets through which we can obtain an accurate solution for the stochastic integral equations. Here, we consider the following stochastic integral equation,

(1.1) $U(t) = g(t) + \int_0^t k_1(s,t)\, U(s) ds + \int_0^t k_2(s,t)\, U(s) dB(s), \quad t \in [0,T),$

where $U(t)$, $g(t)$, $k_1(s,t)$ and $k_2(s,t)$ for $s, t \in [0,T)$ are the stochastic processes on the same probability space (Ω, F, P) and $U(t)$ is unknown. Also $B(t)$ is a Brownian motion process and $\int_0^t k_2(s,t)\ U(s)\, dB(s)$ is the Itô integral [8].

The article is organized in the following way. Some definitions of stochastic calculus, properties of Haar wavelets and operational matrix of integration of Haar wavelets are studied. Also, *HWSOMI* is derived in ***Section*** **2**. Method of solution is given in ***Section*** **3**. In ***Section*** **4**, convergence and error analysis of the proposed method is studied. ***Section*** **5** presents some examples which shows the efficiency of the presented method. Lastly, ***Section*** **6** gives the conclusion.

2 Stochastic Calculus and Wavelets

Here we examine some definitions existing in stochastic calculus. And we study the properties of Haar wavelets and operational matrix of integration of Haar wavelets (*HWOMI*). Stochastic operational matrix of integration of Haar wavelet is derived. Lastly, some results which will are used in further sections are mentioned.

2.1 Stochastic calculus

Definition **2.1** *A standard Brownian motion defined on the interval* $[0,\ T)$ *is a random variable* $B(t)$ *which depends on* $t \in [0,\ T)$ *and satisfies the following conditions:*

1. $B(0) = 0$ *with probability* 1.
2. *For* $0 \le s < t \le T$*, the random variable given by increment* $B(t) - B(s)$ *is distributed normally with mean zero and variance* $t - s$*, equivalently,* $B(t) - B(0) \sim \sqrt{t-s}N(0,1)$*, where* $N(0,1)$ *is a random variable distributed normally with mean zero and variance* 1.

3. *The increments $B(t) - B(s)$ and $B(v) - B(u)$ are independent for $0 \leq s < t < u < v \leq T$.*

Definition 2.2 [5] *The sequence U_n converge to U in L^2 if for each n, $E\left(|U_n|^2\right) < \infty$. Let us assume that $0 \leq s \leq T$, let $v = v(s,T)$ be the class of functions that $g(t,w) : [0,\infty] \times \Omega \to R^n$, satisfy,*

1. *the function $(t,w) \to g(t,w)$ is $\beta \times G$ measurable, where β is Borel algebra.*
2. *g is adapted to G_t.*
3. *$E\left[\int_s^T g(t,w)^2\, dt\right] < \infty$.*

Definition 2.3 *(The Itô-integral [8]) Let $g \in v(s,T)$, then the Itô-integral of g is defined by*

$$\int_s^T g(t,w)dB(t)(w) = \lim_{n\to\infty} \int_s^T \varphi_{t,w} dB(t)(w),$$

where, $\{\varphi\}$ is the sequence of elementary functions such that

$$E\left[\int_s^T (g - \varphi_n)^2\, dt\right] \to 0 \quad \text{a.s,} \quad n \to \infty.$$

2.2 Haar Wavelets

Haar wavelets $h_n(t)$ are defined as,

(2.1) $h_n(t) = \psi(2^j t - k),\ j \geq 0,\ \ 0 \leq k < 2^j, n = 2^j + k, n, j, k \in \mathbb{Z},$

where

(2.2) $h_0(t) = 1, 0 \leq t < 1,\ \ \psi(t) = \begin{cases} 1, & 0 \leq t < \frac{1}{2}, \\ -1, & \frac{1}{2} \leq t < 1. \end{cases}$

Every Haar wavelet $h_n(t)$ has the support $\left[\frac{k}{2^j}, \frac{k+1}{2^j}\right)$ and is elsewhere zero in the interval $[0,1)$.

Function Approximation: Any square integrable function $g(t)$ can be expressed with respect to Haar wavelets as

$$g(t) = g_0 h_0(t) + \sum_{i=1}^{\infty} g_i h_i(t),\ \ i = 2^j + k,\ \ j \geq 0,\ \ 0 \leq k < 2^j,\ \ j,k \in \mathbb{N},$$

where g_i is given by

$$g_i = \int_0^1 g(t) h_i(t) dt,\ \ i = 0, 2^j + k,\ \ j \geq 0,\ \ 0 \leq k < 2^j,\ \ j,k \in \mathbb{N}.$$

The above infinite series can be truncated after 2^J terms(J is the level of resolution) as

$$g(t) = \sum_{i=1}^{2^J - 1} g(t) h_i(t),\ \ i = 2^j + k,\ \ 0 \leq j \leq J-1,\ \ 0 \leq k < 2^j,\ \ j,k \in \mathbb{N}.$$

Rewriting this equation in the vector form as

$$g(t) \simeq G^T H(t) = G H^T(t),$$

where G and $H(t)$ are Haar wavelet coefficients given as

$$G = [g_0, g_1, ..., g_{2^J - 1}],\ \ H(t) = [h_0(t), h_1(t), ..., h_{2^J - 1}(t)].$$

Similarly, any two dimensional function $k(s,t) \in L^2([0,1) \times ([0,1))$ can be written in terms of Haar wavelets as

$$k_{ij} = \int_0^1 \int_0^1 k(s,t) h_i(t) h_j(s) dt ds,\ \ i,j = 1,2,...N\ (N = 2^J).$$

For example, from equations (2.1) and (2.2), we can write

$$h_1(t) = \begin{cases} 1, & 0 \leq t < \frac{1}{2}, \\ -1, & \frac{1}{2} \leq t < 1, \end{cases}$$

$$h_2(t) = \begin{cases} 1, & 0 \leq t < \frac{1}{4}, \\ -1, & \frac{1}{4} \leq t < \frac{1}{2}, \end{cases}$$

$$h_3(t) = \begin{cases} 1, & \frac{1}{2} \leq t < \frac{3}{4}, \\ -1, & \frac{3}{4} \leq t < 1, \end{cases}$$

and so on.

2.3 Haar wavelets operational matrix of integration

HWOMI is computed as follows. Integrating equation (2.1), we get

(2.3) $\int_0^t H(s)ds \simeq PH(t)$,

where P is a matrix of order $N \times N$ and is called operational matrix of Haar wavelets.For example, for $N = 4$, we have

$$H(t) = [h_0(t), h_1(t), h_2(t), h_3(t)],$$

(2.4) $\int_0^t h_0(s)ds = t, \quad 0 \le t < 1,$

(2.5) $\int_0^t h_1(s)ds = \begin{cases} t, & 0 \le t < \frac{1}{2}, \\ 1 - t, & \frac{1}{2} \le t < 1, \end{cases}$

(2.6) $\int_0^t h_2(s)ds = \begin{cases} t, & 0 \le t < \frac{1}{4}, \\ (1/2) - t, & \frac{1}{4} \le t < \frac{1}{2}, \end{cases}$

(2.7) $\int_0^t h_3(s)ds = \begin{cases} t - \frac{1}{2}, & \frac{1}{2} \le t < \frac{3}{4}, \\ 1 - t, & \frac{3}{4} < t < 1. \end{cases}$

Thus, seeing equations (2.4), (2.5), (2.6) and (2.7), we can write P in general as,

(2.8) $P = \begin{cases} t - \frac{k}{m}, & \mathrm{t} \in \left[\frac{k}{2^j}, \frac{k+0.5}{2^j}\right), \\ \frac{k+1}{m} - t, & \mathrm{t} \in \left[\frac{k+0.5}{2^j}, \frac{k+1}{2^j}\right), \\ 0, & \text{elsewhere}. \end{cases}$

2.4 Haar wavelets stochastic operational matrix of integration

HWSOMI is written as follows,

(2.9) $\int_0^t H(s)dB(s) \simeq P_s H(t)$,

where P_s is a matrix of order $N \times N$ and is called stochastic operational matrix of Haar wavelets. For example, we obtain

(2.10) $\int_0^t h_0(s)dB(s) = B(t), \quad 0 \le t < 1,$

(2.11) $\int_0^t h_1(s)dB(s) = \begin{cases} B(t), & 0 \le t < \frac{1}{2}, \\ 2B(\frac{1}{2}) - B(t), & \frac{1}{2} \le t < 1, \end{cases}$

(2.12) $\int_0^t h_2(s)dB(s) = \begin{cases} B(t), & 0 \le t < \frac{1}{4}, \\ 2B(\frac{1}{4}) - B(t), & \frac{1}{4} \le t < \frac{1}{2}, \end{cases}$

(2.13) $\int_0^t h_3(s)dB(s) = \begin{cases} B(t) - B\left(\frac{1}{2}\right), & \frac{1}{2} \le t < \frac{3}{4}, \\ 2B\left(\frac{3}{4}\right) - B\left(\frac{1}{2}\right) - B(t), & \frac{3}{4} \le t < 1. \end{cases}$

Thus, seeing equations (2.10), (2.11), (2.12), and (2.13), we write the stochastic operational matrix of integration of Haar wavelets P_s in general as

(2.14) $P_s = \begin{cases} B(t) - B(\frac{k}{2^j}), & \mathrm{t} \in \left[\frac{k}{2^j}, \frac{k+0.5}{2^j}\right), \\ B(\frac{k+0.5}{2^j}) - B(\frac{k}{2^j}) - B(t), & \mathrm{t} \in \left[\frac{k+0.5}{2^j}, \frac{k+1}{2^j}\right), \\ 0, & \text{elsewhere}. \end{cases}$

***Remark* 2.1** *Using equation* (2.1) *for a N-vector G, we have*

(2.15) $H(t)H^T(t)G = \tilde{G}H(t)$,

where, $H(t)$ is the Haar wavelet coefficient matrix and $\tilde{G}$ is an $N \times N$ matrix given by

(2.16) $\tilde{G} = H\bar{G}H^{-1}$,

where $\bar{G} = diag(H^{-1}G)$. Also, for a $N \times N$ matrix X, we have

(2.17) $H^t XH(t) = \tilde{X^T} H(t)$,

where, $\tilde{X^T} = VH^{-1}$ and $V = diag(H^T XH)$ is a N-vector.

3 Method of solution

Consider the stochastic integral equation given in (1.1). Approximating the functions $U(t)$, $g(t)$, $k_1(x,t)$, and $k_2(x,t)$ using Haar wavelets, we get

(3.1) $U(t) \simeq U^T H(t) = UH^T(t)$,

(3.2) $g(t) \simeq G^T H(t) = GH^T(t)$,

(3.3) $k_1(s,t) \simeq H^T(s)K_1 H(t) = H^T(t)K_1^T H(s)$,

(3.4) $k_2(s,t) \simeq H^T(s)K_2 H(t) = H^T(t)K_2^T H(s)$,

where U and G are Haar wavelet coefficient vectors and K_1 and K_2 are Haar wavelet matrices. Substituting (3.1),(3.2),(3.3) and (3.4) in (1.1), we get

(3.5) $U^T H(t) \simeq G^T H(t) + H^T(t)K_1(\int_0^t H(s)H^T(s)Uds) + H^T(t)K_2(\int_0^t H(s)H^T(s)UdB(s))$.

By the use of *HWOMI*, *HWSOMI* and ***Remark* 2.1**, we have

(3.6) $U^T H(t) \simeq G^T H(t) + H^T(t)K_1\tilde{U}PH(t) + H^T(t)K_2\tilde{U}P_s H(t)$.

Using $\tilde{U_1} = K_1\tilde{U}P$ and $U_2 = K_2\tilde{U}P_s$ and using ***Remark* 2.1**, we get

(3.7) $U^T H(t) \simeq G^T H(t) + \tilde{U_1}H(t) + \tilde{U_2}H(t)$.

This gives,

(3.8) $U^T - \tilde{U_1} - \tilde{U_2} \simeq G^T$,

where $\tilde{U_1}$ and $\tilde{U_2}$ are functions of U and (3.8) is a system of linear equations. Solving this system of linear equations and substituting the obtained unknown vector U (3.1), we get the solution of (1.1).

4 Convergence and error analysis

The convergence and error analysis of the method presented for solving stochastic integral equations is studied.

***Theorem* 4.1** *Let $g(t) \in L^2[0,1)$ be any arbitrary function such that $|g'(t)| < \epsilon$, and $e_N(t) = g(t) - \sum_{i=0}^{N-1} g_i h_i(t)$, then*

(4.1) $\|e_N(t)\|_2 \le \dfrac{\epsilon}{\sqrt{3}N}$.

Proof. By the definition,

(4.2) $\|e_N(t)\|_2^2 = \int_0^1 \left(\sum_{i=N}^{\infty} g_i h_i(t)dt\right)^2 dt = \sum_{i=N}^{\infty} g_i^2$.

In equation (4.2), $i = 2^j + k$ and

$$g_i = \int_0^1 h_i(t)g(t)dt = 2^{\frac{j}{2}}\left(\int_{k2^{-j}}^{(k+\frac{1}{2})2^{-j}} g(t)dt - \int_{(k+\frac{1}{2})2^{-j}}^{(k+1)2^{-j}} g(t)dt\right).$$

Using the mean value theorem for integrals, there exist $\eta_{1j} \in \left(k2^{-j}, \left(\frac{k+1}{2}\right)2^{-j}\right)$ and $\eta_{2j} \in \left(\left(\frac{k+1}{2}\right)2^{-j}, (k+1)\,2^{-j}\right)$ such that

(4.3)
$$\begin{aligned}
g_i = \int_0^1 h_i(t)g(t)dt &= 2^{\frac{j}{2}}\left(g(\eta_{1j})\int_{k2^{-j}}^{(k+\frac{1}{2})2^{-j}} dt - g(\eta_{2j})\int_{(k+\frac{1}{2})2^{-j}}^{(k+1)2^{-j}} dt\right)\\
&= 2^{\frac{j}{2}}\left(g(\eta_{1j})\left[\left(k+\frac{1}{2}\right)2^{-j} - k2^{-j}\right] - g(\eta_{2j})\left[(k+1)\,2^{-j} - \left(k+\frac{1}{2}\right)2^{-j}\right]\right)\\
&= 2^{-\frac{-j}{2}-1}\left(g(\eta_{1j}) - g(\eta_{2j})\right)\\
&= 2^{-\frac{-j}{2}-1}\left(\eta_{1j} - \eta_{2j}\right)g'(\eta_j),\ \ \eta_{1j} < \eta_j < \eta_{2j}.
\end{aligned}$$

Equation (4.3) gives

(4.4)
$$\begin{aligned}
\|e_N(t)\|_2^2 = \sum_{i=N}^{\infty} g_i^2 &= \sum_{j=J}^{\infty} 2^{-j-2}(\eta_{1j} - \eta_{2j})^2\\
&\le \sum_{j=J}^{\infty} 2^{-j-2}2^{-2j}\epsilon^2\\
&= \frac{\epsilon^2}{4}\sum_{j=J}^{\infty} 2^{-3j}\\
&= \frac{\epsilon^2}{3}2^{-2J}.
\end{aligned}$$

Therefore,

(4.5) $\|e_N(t)\|_2 \le \dfrac{\epsilon}{\sqrt{3}N}$.

***Theorem* 4.2** *Let $g(s,t) \in L^2([0,1)\times[0,1))$ be any arbitrary function such that $|\frac{\partial^2 g}{\partial s \partial t}| < \epsilon$, and $e_N(s,t) = g(s,t) - \sum_{i=0}^{N-1}\sum_{j=0}^{N-1} g_{ij}h_i(s)h_j(t)$, then*

(4.6) $\|e_N(s,t)\|_2 \leq \dfrac{\epsilon}{3N^2}$.

Proof. By the definition,

(4.7) $\|e_N(s,t)\|_2^2 = \int_0^1 \left(\sum_{i=N}^{\infty}\sum_{j=N}^{\infty} g_{ij}h_i(s)h_j(t)\right)^2 dt = \sum_{i=N}^{\infty}\sum_{j=N}^{\infty} g_{ij}^2$.

In equation (4.7), $i = 2^j + k$, $j = 2^{j'} + k$, and

$$g_{ij} = \int_0^1\int_0^1 h_i(s)h_j(t)g(s,t)dsdt.$$

Using the mean value theorem for integrals, there exist η_j, η_{1j}, η_{2j}, $\eta_{j'}$, $\eta_{1j'}$, and $\eta_{2j'}$ such that

$$
\begin{aligned}
(4.8)\quad g_{ij} &= \int_0^1 h_i(s)(\int_0^1 h_i(t)g(s,t)dt)ds \\
&= \int_0^1 h_i(s)\left[2^{\frac{-j'}{1}-1}(\eta_{1j'}-\eta_{2j'})\tfrac{\partial g(s,\eta_{j'})}{\partial t}\right]ds \\
&= 2^{\frac{-j'}{1}-1}(\eta_{1j'}-\eta_{2j'})\int_0^1 \tfrac{\partial g(s,\eta_{j'})}{\partial t}h_i(s)ds \\
&= 2^{\frac{-j}{2}-\frac{j'}{2}-2}(\eta_{1j'}-\eta_{2j'})(\eta_{1j}-\eta_{2j})\frac{\partial^2 g(\eta_j,\eta_{j'})}{\partial t\partial s}.
\end{aligned}
$$

Equation (4.8) gives

$$
\begin{aligned}
(4.9)\quad \|e_N(s,t)\|_2^2 = \sum_{i=N}^{\infty}\sum_{j=N}^{\infty} g_{ij}^2 &= \sum_{j=J}^{\infty}\sum_{j'=J}^{\infty} 2^{-j-j'-4}(\eta_{1j'}-\eta_{2j'})^2(\eta_{1j}-\eta_{2j})^2\left|\tfrac{\partial^2 g(\eta_j,\eta_{j'})}{\partial t\partial s}\right|^2 \\
&\leq \sum_{j=J}^{\infty}\sum_{j'=J}^{\infty} \epsilon^2 2^{-3j-3j'-4}.
\end{aligned}
$$

From equation (4.4), we get

(4.10) $\|e_N(s,t)\|_2^2 \leq N^2\sum_{j=J}^{\infty}2^{-3j-2}\sum_{j'=J}^{\infty}2^{-3j'-2} = \frac{\epsilon^2}{(3N^2)^2}$.

Therefore

$$\|e_N(s,t)\|_2 \leq \frac{\epsilon}{3N^2}.$$

***Theorem* 4.3** *Let $U(t)$ and $U_N(t)$ be the exact and approximate solution of* (1.1). *Let us assume that*

1. $\|U(t)\| \leq \delta,\ t \in [0,1)$,
2. $\|k_i(s,t)\| \leq D_i,\ i = 1,2$,
3. $(D_1+\xi_1) + \|B(t)\|_\infty(D_2+\xi_2)$,

then,

$$\|U(t) - U_N(t)\|_2 \leq \frac{\mu_N + \xi_{1N} + \|B(t)\|_\infty \xi_{2N}}{1 - (D_1+\xi_1) - \|B(t)\|_\infty(D_2+\xi_{2N})},$$

where

$$\mu_N = \sup_{t\in[0,1)} \frac{g'(t)}{\sqrt{3}N},$$

$$\xi_i = \frac{1}{3N^2}\sup_{s,t\in[0,1)}\left|\frac{\partial^2 k_i(s,t)}{\partial s\partial t}\right|,\ i = 1,2.$$

Proof. *From equation* (1.1)*, we have*

$$
\begin{aligned}
U(t) - U_N(t) = g(t) - g_N(t) &+ \int_0^t (k_1(s,t)U(s) - k_{1N}(s,t)U_N(s)ds) \\
&+ \int_0^t (k_2(s,t)U(s) - k_{2N}(s,t)U_N(s))\,dB(s).
\end{aligned}
$$

By using mean value theorem we have,

(4.11)

$$
\begin{aligned}
\|U(t) - U_N(t)\| \leq \|g(t) - g_N(t)\| &+ t\|k_1(s,t)U(s) - k_{1N}(s,t)U_N(s)\| \\
&+ B(t)\|k_2(s,t)U(s) - k_{2N}(s,t)U_N(s)\|.
\end{aligned}
$$

Using ***Theorem*** **4.1** *and* ***Theorem*** **4.2***, we have*

(4.12)
$$\begin{aligned}\|k_i(s,t)U(s) - k_{iN}(s,t)U_N(s)\| \le{}& \|k_i(s,t)\|\,\|\|U(t) - U_N(t)\| \\ &+ \|k_i(s,t) - k_{iN}(s,t)\|\,\|\|U(t)\| \\ &+ \|k_i(s,t) - k_{iN}(s,t)\|\,\|\|U(t) - U_N(t)\|.\end{aligned}$$

Substituting (4.12) *in* (4.11)*, we get*

(4.13)
$$\begin{aligned}\|U(t) - U_N(t)\| \le{}& \mu_N + t\left[(D_1 + \xi_{1N})\|U(t) - U_N(t)\| + \delta\xi_{1N}\right] \\ &+ B(t)\left[(D_2 + \xi_{2N})\|U(t) - U_N(t)\| + \delta\xi_{2N}\right].\end{aligned}$$

Using the assumption (3)*, we get*

$$\|U(t) - U_N(t)\|_2 \le \frac{\mu_N + \xi_{1N} + \|B(t)\|_\infty \xi_{2N}}{1 - (D_1 + \xi_1) - \|B(t)\|_\infty (D_2 + \xi_{2N})}.$$

5 Numerical Experiments

Here some examples are presented in order to show the efficiency of the method presented.

Test Problem **5.1** *Consider the stochastic integral equation,*

(5.1) $U(t) = 1 + \int_0^t sin(s)U(s)dB(s),$

*where $U(t)$ is the unknown stochastic process defined on the probability space (Ω, F, P), and $B(t)$ is the Brownian motion process. Exact solution of equation (*5.1*) is*

(5.2) $U(t) = exp\left[\frac{-1}{4}(t - cos(t)sin(t)) + \int_0^t sin(s)dB(s)\right].$

Method of Implementation

For $N = 4$.

Comparing (5.1)with equation (1.1), we get

(5.3) $g(t) = 1,$

(5.4) $k_1(s,t) = 0,$

and

(5.5) $k_2(s,t) = sin(s).$

Approximating equations (5.3), (5.4), and (5.5) using Haar wavelets, we obtain

$$G = [\ 1 \quad 0 \quad 0 \quad 0\ ],$$

$$K_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } K_2 = \begin{pmatrix} 0.4609 & 0 & 0 & 0 \\ -0.2154 & 0 & 0 & 0 \\ -0.1208 & 0 & 0 & 0 \\ -0.0912 & 0 & 0 & 0 \end{pmatrix}.$$

Let our assumed solution be U and approximating this using Haar wavelets, we get

$$U(t) \simeq U^T H(t) = UH^T(t).$$

Substituting the obtained vector G, matrices K_1 and K_2 and the approximated unknown solution U in equation (5.1)and by the use of operational matrix of integration of Haar wavelets and the stochastic operational matrix of integration Haar wavelets, we obtain the unknown vector U as

$$U = [\ 0.90697 \quad 0.043482 \quad 0.024383 \quad 0.018413\ ].$$

Substituting this in $U(t) \simeq U^T H(t) = UH^T(t)$,we obtain the solution as

$$U(t) = [\ 0.9748 \quad 0.9261 \quad 0.8819 \quad 0.8451\ ].$$

The exact and approximate solutions of ***Test Problem*** **5.1** for $N = 4$ and $N = 8$ are shown in ***Table*** **5.1**, maximum absolute error (E_{max}) for different values of N are shown in ***Table*** **5.2** and the graphs of absolute errors for different values of N are shown in ***Figure*** **5.1**.

***Table* 5.1:** Exact solution, approximate solution and absolute errors for ***Test Problem* 5.1**.

	$N=4$			$N=8$		
t	Exact Solution	Approximate Solution	Absolute error	Exact Solution	Approximate Solution	Absolute error
0	1.0000	0.9256	0.0744	1.0000	1.0440	0.0440
0.1	0.9996	0.9799	0.0197	1.0024	1.0107	0.0083
0.2	0.9965	0.9602	0.0363	1.0070	1.0213	0.0143
0.3	0.9925	0.9407	0.0518	1.0142	1.0318	0.0176
0.4	0.9864	0.9217	0.0647	1.0225	1.0418	0.0193
0.5	0.9737	0.9040	0.0697	1.0310	1.0515	0.0204
0.6	0.9610	0.8863	0.0747	1.0390	1.0606	0.0216
0.7	0.9403	0.8709	0.0695	1.0459	1.0692	0.0233
0.8	0.9169	0.8561	0.0608	1.0510	1.0771	0.0261
0.9	1.4098	1.3190	0.0907	1.0536	1.0841	0.0306

***Table* 5.2:** Absolute errors for different values of N of ***Test Problem* 5.1**.

N	E_{max}
4	0.0907
8	0.0440
16	0.0194

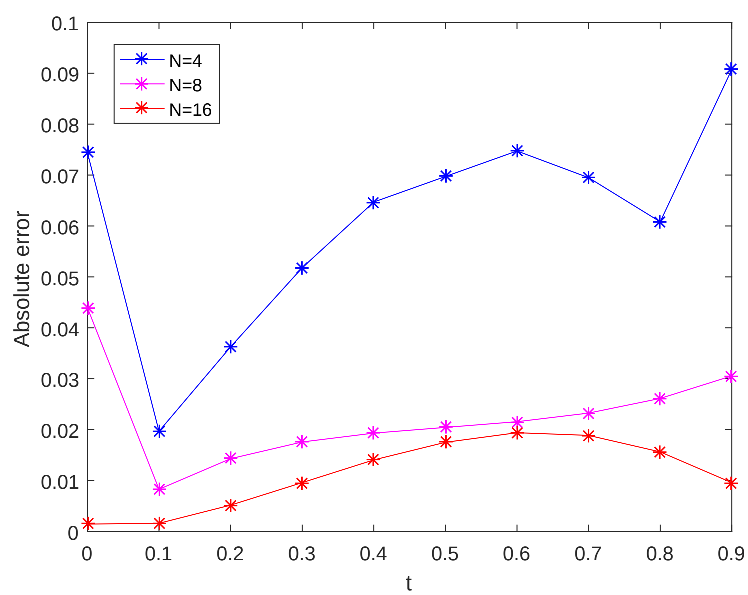

***Figure* 5.1:** Absolute errors for different values of N of ***Test Problem* 5.1**.

***Test Problem* 5.2** *Consider the stochastic integral equation*

$$(5.6)\quad U(t) = \frac{1}{12} + \int_0^t cos(s)U(s)ds + \int_0^t sin(s)U(s)dB(s),$$

where $U(t)$ *is the unknown stochastic process defined on the probability space* (Ω, F, P), *and* $B(t)$ *is the Brownian motion process. Exact solution of* (5.6) *is*

$$(5.7)\quad U(t) = \frac{1}{12} exp\left[\frac{-t}{4} + sin(t) + \frac{sin(2t)}{8} + \int_0^t sin(s)dB(s)\right].$$

Implementation is shown in ***Test Problem* 5.1**. The exact as well as approximate solutions of ***Test Problem* 5.2** for $N = 4$ and $N = 8$ are shown in ***Table* 5.3**, maximum absolute error (E_{max}) for different values of N are shown in ***Table* 5.4** and the graphs of absolute errors for different values of N are shown in ***Figure* 5.2**.

Table 5.3: Exact solution, approximate solution and absolute errors for ***Test Problem* 5.2**.

	$N = 4$			$N = 8$		
t	Exact Solution	Approximate Solution	Absolute error	Exact Solution	Approximate Solution	Absolute error
0	0.0833	0.0759	0.0075	0.0833	0.0729	0.0105
0.1	0.0921	0.0636	0.0285	0.0918	0.0729	0.0190
0.2	0.1017	0.0655	0.0362	0.1004	0.0782	0.0223
0.3	0.1116	0.0747	0.0368	0.1089	0.0710	0.0379
0.4	0.1214	0.0801	0.0413	0.1167	0.0775	0.0391
0.5	0.1311	0.0739	0.0572	0.1235	0.0710	0.0525
0.6	0.1409	0.0677	0.0732	0.1291	0.0647	0.0644
0.7	0.1487	0.0732	0.0756	0.1331	0.0720	0.0611
0.8	0.1560	0.0826	0.0734	0.1350	0.0649	0.0701
0.9	0.2133	0.1174	0.0959	0.1346	0.0711	0.0635

Table 5.4: Absolute errors for different values of N of test problem ***Test Problem* 5.2**.

N	E_{max}
4	0.0959
8	0.0701
16	0.0411

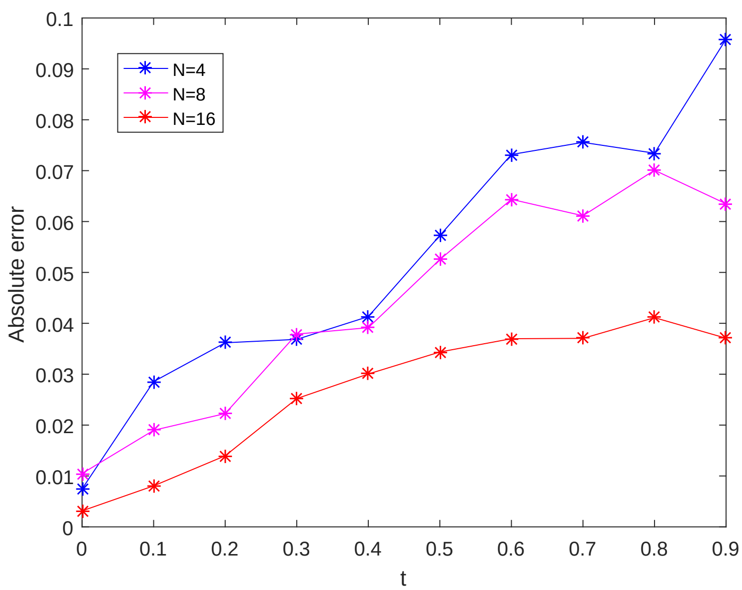

Figure 5.2: Absolute errors for different values of N of ***Test Problem* 5.2**.

***Test Problem* 5.3** *Consider the stochastic integral equation*

$$(5.8)\quad U(t) = \frac{1}{10} + \int_0^t ln(1+s)U(s)ds + \int_0^t sU(s)dB(s).$$

where $U(t)$ *is the unknown stochastic process defined on the probability space* (Ω, F, P)*, and* $B(t)$ *is the Brownian motion process. Exact solution of* (5.8) *is*

$$(5.9)\quad U(t) = \frac{1}{10} exp\left[(1+t)ln(1+t) - t - \frac{t^3}{6} + \int_0^t sdB(s)\right].$$

Implementation is shown in ***Test Problem* 5.1**. The exact as well as approximate solutions of ***Test Problem* 5.3** for $N = 4$ and $N = 8$ are shown in ***Table* 5.5**, maximum absolute error (E_{max}) for different values of N are shown in ***Table* 5.6** and the graphs of absolute errors (E_{max}) *for different values of* N are shown in ***Figure* 5.3**.

Table **5.5:** Exact solution, approximate solution and absolute errors for ***Test Problem* 5.3**.

	$N = 4$			$N = 8$		
t	Exact Solution	Approximate Solution	Absolute error	Exact Solution	Approximate Solution	Absolute error
0	0.0833	0.0750	0.0083	0.1000	0.0806	0.0194
0.1	0.0997	0.0907	0.0090	0.0999	0.0949	0.0051
0.2	0.0985	0.0847	0.0137	0.0997	0.0907	0.0090
0.3	0.0969	0.0798	0.0171	0.0992	0.0845	0.0147
0.4	0.0948	0.0736	0.0211	0.0981	0.0836	0.0145
0.5	0.0909	0.0636	0.0274	0.0963	0.0759	0.0205
0.6	0.0871	0.0535	0.0336	0.0939	0.0694	0.0246
0.7	0.0816	0.0536	0.0281	0.0907	0.0709	0.0198
0.8	0.0756	0.0571	0.0186	0.0867	0.0615	0.0251
0.9	0.1237	0.0932	0.0305	0.0817	0.0670	0.0147

Table **5.6:** Absolute errors for different values of N of ***Test Problem* 5.3**.

N	E_{max}
4	0.0336
8	0.0251
16	0.0074

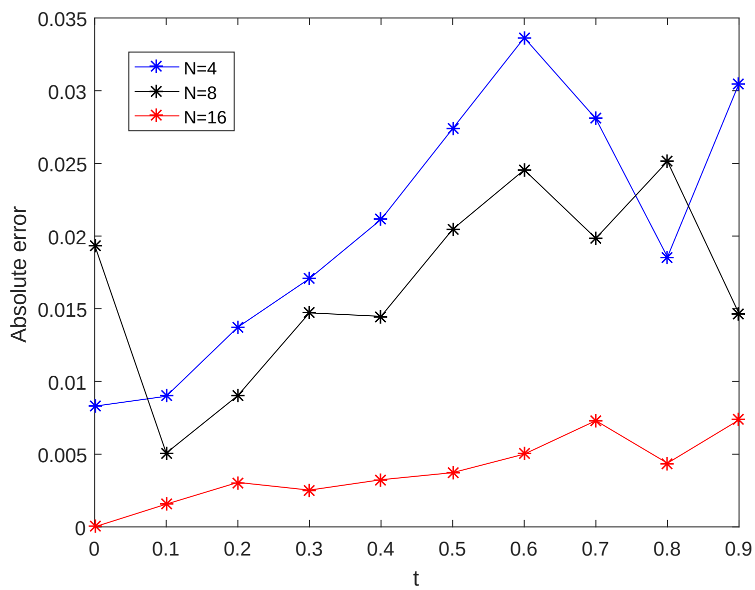

Figure **5.3:** Absolute errors for different values of N of ***Test Problem* 5.3**.

6 Conclusion

In this article, using Haar wavelets a new *HWSOMI* is developed to solve stochastic integral equations. From tables and figures we can see that the solutions obtained by proposed method are in good agreement with that of exact solutions. Hence, the investigated method is efficient and convenient for solving stochastic integral equations.

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(Dedicated to Honor Dr. R. C. Singh Chandel on His 75th Birth Anniversary Celebrations)

ON INTEGRALS INVOLVING A PRODUCT OF EXTENDED BESSEL MAITLAND FUNCTION AND I*-FUNCTION

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Abstract

The object of this paper is to establish some interesting integrals involving the product of extended Bessel-Maitland function and I*- function and then, express them in terms of extended I*- function. Further some special cases of our results are also deduced.
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Keywords and phrases: I*-function, Wright's generalized Bessel-Maitland function, I-function, definite integrals.

1 Introduction

In the last decade, many authors (for example, see [1], [4] - [9]) have developed numerous integral formulas involving a variety of special functions. Such integrals play a very important role in many diverse fields of engineering and sciences ([5], [11], [14]).

Recently, Ghayasuddin and Khan [4], Khan et al. ([6] - [9]) and Ali et al. [1] obtained certain interesting new class of integrals involving the generalized Bessel-Maitland function and expressed in terms of generalized (Wright) hypergeometric function.

In order to derive our main results, we are required to express following definitions and formulae of some well known special functions:

The I- function [13] defined in terms of following Mellin - Barnes type integral is given by

$$(1.1)\quad I^{m,n}_{p_i,q_i;r}\left[z\left|\begin{matrix}(a_j,\alpha_j)_{1,n}; & (a_{ji},\alpha_{ji})_{n+1,p_i}\\ (b_j,\beta_j)_{1,m}; & (b_{ji},\beta_{ji})_{m+1,q_i}\end{matrix}\right.\right]=\frac{1}{2\pi\omega}\int_L\varphi(\xi)z^{\xi}d\xi,\omega=\sqrt{(-1)},$$

where

$$\varphi(\xi)=\frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j\xi)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j\xi)}{\sum_{i=1}^{r}\left[\prod_{j=m+1}^{q_i}\Gamma(1-b_{ji})(+\beta_{ji}\xi)\prod_{j=n+1}^{p_i}\Gamma(a_{ji})(-\alpha_{ji}\xi)\right]}$$

$p_i,q_i(i=1,2,\dots,r),m$ and n are integers satisfying $0\le n\le p_i, 0\le m\le q_i(i=1,2,\dots,r)$, r is finite,$\alpha_j,\beta_j,\alpha_{ji},\beta_{ji}$ are real and positive and a_j,b_j,a_{ji},b_{ji} are numbers such that $\alpha_k(b_h+v)\neq\beta_h(a_k-1-k)$ for $k,v=0,1,2\dots;h=1,2,\dots,m;i=1,2,..,r$.

L is contour running form $\sigma-i\infty$ to $\sigma+i\infty$, where σ is real in the complex ξ-plane such that the poles

$$\xi=\frac{(a_j-1-v)}{\alpha_j}, j=1,2,\dots n; v=0,1,2,\cdots$$

$$\xi=\frac{(b_j+v)}{\beta_j}, j=1,2\cdots,m; v=0,1,2,\cdots$$

to the left and right hand sides of L, respectively.

The I*-function [5], related to the I-function [13], is introduced as a contour integral in complex ξ-plane given by

$$(1.2)\quad I^{m,n}_{p_i,q_i;r}\left[z\left|\begin{matrix}(a_j,\alpha_j)_{1,n}:(a_{ji},\alpha_{ji})_{1,pi}\\ (b_j,\beta_j)_{1,m}:(a_{ji},\beta_{ji})_{1,qi}\end{matrix}\right.\right]=\frac{1}{2\pi\omega}\int_L\varphi(\xi)z^{\xi}d\xi,$$

in which

$$\varphi(\xi)=\frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j\xi)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j\xi)}{\sum_{i=1}^{r}\left[\prod_{j=1}^{q_i}\Gamma(1-b_{ji}+\beta_{ji}\xi)\prod_{j=1}^{p_i}\Gamma(a_{ji}-\alpha_{ji}\xi)\right]}$$

Here, for finite value of r, all $p_i, q_i (i = 1, 2, \ldots r), m$ and n are positive integers, satisfying the inequalities; $0 \le n \le p, p_i \ge n \ge 1 (i = 1, 2, \ldots ..r), 1 \le m \le q, q_i \ge m \ge 1$ $(i = 1, 2, \ldots ..r), \alpha_j (j = 1, \ldots, n), \beta_j (j = 1, \ldots, m)$, $\alpha_{ji} (1 \le j \le p_i, i = 1, 2, \ldots r)$, $\beta_{ji} (1 \le j \le q_i, i = 1, 2, \ldots r)$ are real and positive and $a_j (j = 1, \ldots, n), b_j (j = 1, \ldots, m), a_{ji} (1 \le j \le p_i, i = 1, 2, \ldots r), b_{ji} (1 \le j \le q_i, i = 1, 2, \ldots r)$ are complex numbers such that $\alpha_k (b_h + \nu) \ne \beta_h (a_k - 1 - l) forl, \nu = 0, 1, 2, \ldots; h = 1, 2, \ldots, m$.

L is contour running from $\sigma - i\infty$ to $\sigma + i\infty$, where σ is real in the complex ξ- plane such that the poles $\xi = \frac{(a_j - 1 - l)}{\alpha_j}$, j=1,2,...,n; l=0,1,2,...,$\xi = \frac{(b_j + \nu)}{\beta_j}$, $j = 1, 2, \ldots, m$; $\nu = 0, 1, 2, \ldots$ lie to the left and right hand sides of the contour L respectively, the empty product is represented as 1.

The I^*- function converges absolutely in the ξ- plane if $|\arg z| < \frac{\pi}{2} A$.

(1.3) where, $A = \sum_{j=1}^{n} \alpha_j + \sum_{j=1}^{m} \beta_j - \max_{1 \le i \le r} \left[\sum_{j=1}^{p_i} \alpha_{ji} + \sum_{j=1}^{q_i} \beta_{ji} \right] > 0$

***Property* 1.1** *The I^*- function* (1.2) - (1.3) *is most probably identical to I-function given in* (1.1).

***Property* 1.2** *For $r = 1$, the I^*-function defined in* (1.2) - (1.3) *has a relation with Fox's H- function* [3] *as*

$$I^{m,n}_{p_1,q_1;1}\left[z\left|\begin{matrix}(a_j,\alpha_j)_{1,n} : (a_{j1},\alpha_{j1})_{1,p_1} \\ (b_j,\beta_j)_{1,m} : (a_{j1},\beta_{j1})_{1,q_1}\end{matrix}\right.\right] = H^{m,n}_{p_1+n,q_1+m}\left[z\left|\begin{matrix}(a_j,\alpha_j)_{1,n} : (a_{j1},\alpha_{j1})_{n+1,p_1}, (a_{j1},\alpha_{j1})_{1,n} \\ (b_j,\beta_j)_{1,m} : (b_{j1},\beta_{j1})_{m+1,q_1}, (b_{j1},\beta_{j1})_{1,m}\end{matrix}\right.\right]$$

where, $|\arg z| < \frac{\pi}{2} A$ *and* $A = \sum_{j=1}^{n} \alpha_j + \sum_{j=1}^{m} \beta_j - \left[\sum_{j=1}^{p_1} \alpha_{j1} + \sum_{j=1}^{q_1} \beta_{j1} \right] > 0.$

A new extension of Bessel-Maitland function is introduced by Khan et al. [9] *as*

(1.4) $J^{\beta,\delta,\sigma,t}_{\alpha,\gamma,\mu,\rho,v,s}(z) = \sum_{h=0}^{\infty} \frac{(\beta)_{\delta h} (\sigma)_{th} (-z)^h}{\Gamma(\gamma h + \alpha + 1)(v)_{sh} (\mu)_{\rho h}}$,

where $\alpha, \gamma, \mu, \rho, v, \beta, \delta, \sigma \in C; (\beta) > 0, (\gamma) > 0, (\delta) > 0, (\mu) > 0, (\rho) > 0, (\sigma) > 0, (v) > 0, (\alpha) \ge -1; s, t > 0$ *and* $t < (\gamma) + s$ *and* $(\beta)_0 = 1, (\beta)_{\delta h} = \frac{\Gamma(\beta + \delta h)}{\Gamma(\beta)}$ *denotes the generalized Pochhammer symbol.*

Some formulae of definite integrals are found by the authors:

By Edward [2] *as*

(1.5) $\int_0^1 \int_0^1 y^k (1 - x)^{k-1} (1 - y)^{l-1} (1 - xy)^{1-k-l} dxdy = \frac{\Gamma(k)\Gamma(l)}{\Gamma(k+l)}$,

provided $0 < (l) < (k)$.

By Oberhettinger [12] *given as*

(1.6) $\int_0^{\infty} x^{\alpha-1} (x + c + \sqrt{x^2 + 2cx})^{-\beta} dx = 2\beta c^{-\beta} (\frac{c}{2})^{\alpha} \frac{\Gamma(2\alpha)\Gamma(\beta-\alpha)}{\Gamma(1+\alpha+\beta)}$,

provided $0 < (\alpha) < (\beta)$.

MacRobert [10] *presented as*

(1.7) $\int_0^1 x^{\alpha-1} (1 - x)^{\beta-1} [cx + d(1 - x)]^{-\alpha-\beta} dx = \frac{1}{c^{\alpha} d^{\beta}} \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$,

provided $(\alpha) > 0, (\beta) > 0$ *and c, d are non zero constants and the expression* $cx + d(1 - x)$*, where* $0 \le x \le 1$ *is non zero.*

2 Main Results

In this section we derive following theorems on extended Bessel-Maitland function and I^*- function.

***Theorem* 2.1** *If* $\alpha, \gamma, \mu, \rho, v, \beta, \delta, \sigma \in C; (\beta) > 0, (\gamma) > 0, (\delta) > 0, (\mu) > 0, (\rho) > 0, (\sigma) > 0, (v) > 0, (\alpha) \ge -1; s, t > 0$ *and* $t < (\gamma) + s$ *and* $0 < (l) < (k)$*. Then following integral exists*

(2.1) $$\int_0^1 \int_0^1 y^k (1 - x)^{k-1} (1 - y)^{l-1} (1 - xy)^{1-k-l} J^{\beta,\delta,\sigma,t}_{\alpha,\gamma,\mu,\rho,v,s}\left[\frac{y(1-x)(1-y)}{(1-xy)^2}\right] I^{m,n}_{p_i,q_i;r}\left[\frac{zy(1-x)(1-y)}{(1-xy)^2}\left|\begin{matrix}(a_j, A_j)_{1,n} : (a_{ji}, A_{ji})_{1,pi} \\ (b_j, B_j)_{1,m} : (a_{ji}, B_{ji})_{1,qi}\end{matrix}\right.\right] dxdy$$

$$= \sum_{h=0}^{\infty} \frac{(-1)^h (\beta)_{\delta h} (\sigma)_{th}}{\Gamma(\gamma h + \alpha + 1)(v)_{sh} (\mu)_{\rho h}} I^{m,n+2}_{p_i,q_i+1;r}\left[z\left|\begin{matrix}(a_j, A_j)_{1,n}, (1 - k - h, 1), (1 - l - h, 1) : (a_{ji}, A_{ji})_{1,pi} \\ (b_j, B_j)_{1,m} : (a_{ji}, B_{ji})_{1,qi}, (1 - k - l - 2h, 2)\end{matrix}\right.\right],$$

provided that $|\arg z| < \frac{\pi}{2} A', A' = \sum_{j=1}^{n} A_j + \sum_{j=1}^{m} B_j - \max_{1 \le i \le r} [\sum_{j=1}^{p_i} A_{ji} + \sum_{j=1}^{q_i} B_{ji}] > 0.$

Proof. In left hand side of (2.1), expand $J^{\beta,\delta,\sigma,t}_{\alpha,\gamma,\mu,\rho,v,s}(.)$, in the series by (1.4) and the function $I^{m,n}_{p_i,q_i;r}[.]$, in the Mellin integral (1.2) and then, interchange the order of summation and integration, we get

(2.2) $\sum_{h=0}^{\infty} \frac{(-1)^h (\beta)_{\delta h} (\sigma)_{th}}{\Gamma(\gamma h + \alpha + 1)(v)_{sh} (\mu)_{\rho h}} \frac{1}{2\pi\omega} \int_L \varphi(\xi) z^{\xi} \int_0^1 \int_0^1 y^{k+h+\xi} (1 - x)^{k+h+\xi-1} (1 - y)^{l+h+\xi-1} (1 - xy)^{1-k-l-2h-2\xi} dxdyd\xi.$

Now, in (2.2) apply the formula (1.5) to achieve

(2.3) $\sum_{h=0}^{\infty} \frac{(-1)^h (\beta)_{\delta h} (\sigma)_{th}}{\Gamma(\gamma h + \alpha + 1)(v)_{sh} (\mu)_{\rho h}} \frac{1}{2\pi\omega} \int_L \varphi(\xi) \frac{\Gamma(k+h+\xi)\Gamma(l+h+\xi)}{\Gamma(k+l+2h+2\xi)} z^{\xi} d\xi,$

By definition of (1.2), the expression (2.3) immediately gives the result (2.1).

***Theorem* 2.2** *If $\alpha,\gamma,\mu,\rho,v,\beta,\delta,\sigma\in C;(\beta)>0,(\gamma)>0,(\delta)>0,(\mu)>0,(\rho)>0,(\sigma)>0,(v)>0,(\alpha)\geq -1;s,t>0,t<(\gamma)+s,0<(\tau)+1<(),|\arg z|<\frac{\pi}{2}A'$ and $A'=\sum_{j=1}^{n}A_j+\sum_{j=1}^{m}B_j-\max_{1\leq i\leq r}[\sum_{j=1}^{p_i}A_{ji}+\sum_{j=1}^{q_i}B_{ji}]>0,$ then following integral exists*

$$(2.4)\quad \int_0^\infty x^\tau(x+c+\sqrt{x^2+2cx})^{-}J^{\beta,\delta,\sigma,t}_{\alpha,\gamma,\mu,\rho,v,s}\left(\frac{x}{x+c+\sqrt{x^2+2cx}}\right)I^{m,n}_{p_i,q_i;r}\left[\frac{zx}{(x+c+\sqrt{x^2+2cx})}\middle|\begin{matrix}(a_j,A_j)_{1,n}:(a_{ji},A_{ji})_{1,pi}\\(b_j,B_j)_{1,m}:(a_{ji},B_{ji})_{1,qi}\end{matrix}\right]dx$$

$$=(c)^{1+\tau-\lambda}(\frac{1}{2})^{1+\tau}(-\tau)\sum_{h=0}^{\infty}\frac{(-\frac{1}{2})^h(\beta)_{\delta h}(\sigma)_{th}}{\Gamma(\gamma h+\alpha+1)(v)_{sh}(\mu)_{\rho h}}I^{m,n+1}_{p_i,q_i+1;r}\left[\frac{z}{2}\middle|\begin{matrix}(-1-2\tau-2h,2),(a_j,A_j)_{1,n}:(a_{ji},A_{ji})_{1,pi}\\(b_j,B_j)_{1,m}:(a_{ji},B_{ji})_{1,qi},(-1--\tau-2h,2)\end{matrix}\right]$$

$$+(c)^{1+\tau-\lambda}(\frac{1}{2})^{1+\tau}(-\tau-1)\sum_{h=0}^{\infty}\frac{(-\frac{1}{2})^h(\beta)_{\delta h}(\sigma)_{th}}{\Gamma(\gamma h+\alpha+1)(v)_{sh}(\mu)_{\rho h}}I^{m,n+1}_{p_i,q_i+1;r}\left[\frac{z}{2}\middle|\begin{matrix}(-1-2\tau-2h,2),(a_j,A_j)_{1,n}:(a_{ji},A_{ji})_{1,pi}\\(b_j,B_j)_{1,m}:(a_{ji},B_{ji})_{1,qi},(--\tau-2h,2)\end{matrix}\right],$$

Proof. In the integrand of (2.4) define by the definitions given in (1.2) and (1.4) and then interchanging the order of integration and summation to get that

$$(2.5)\quad \sum_{h=0}^{\infty}\frac{(-1)^h(\beta)_{\delta h}(\sigma)_{th}}{\Gamma(\gamma h+\alpha+1)(v)_{sh}(\mu)_{\rho h}}\frac{1}{2\pi\omega}\int_L\varphi(\xi)z^\xi\int_0^\infty x^{\tau+h+\xi}(x+c+\sqrt{x^2+2cx})^{--h-\xi}dxd\xi.$$

In the inner integral of (2.5), use the result (1.6) to find that

$$(2.6)\quad (c)^{1+\tau-\lambda}(\frac{1}{2})^{1+\tau}(-\tau)\sum_{h=0}^{\infty}\frac{(-\frac{1}{2})^h(\beta)_{\delta h}(\sigma)_{th}}{\Gamma(\gamma h+\alpha+1)(v)_{sh}(\mu)_{\rho h}}\frac{1}{2\pi\omega}\int_L\varphi(\xi)\frac{(2+2\tau+2h+2\xi)}{(2+\lambda+\tau+2h+2\xi)}(\frac{z}{2})^\xi d\xi$$

$$+(c)^{1+\tau-\lambda}(\frac{1}{2})^{1+\tau}(-\tau-1)\sum_{h=0}^{\infty}\frac{(-\frac{1}{2})^h(\beta)_{\delta h}(\sigma)_{th}}{\Gamma(\gamma h+\alpha+1)(v)_{sh}(\mu)_{\rho h}}\frac{1}{2\pi\omega}\int_L\varphi(\xi)\frac{(2+2\tau+2h+2\xi)}{(1+\lambda+\tau+2h+2\xi)}(\frac{z}{2})^\xi d\xi.$$

Finally, use the definition (1.2) in the contour integrals of (2.6), we obtain the right hand side of (2.4) as

$$(2.7)\quad (c)^{1+\tau-\lambda}(\frac{1}{2})^{1+\tau}(-\tau)\sum_{h=0}^{\infty}\frac{(-\frac{1}{2})^h(\beta)_{\delta h}(\sigma)_{th}}{\Gamma(\gamma h+\alpha+1)(v)_{sh}(\mu)_{\rho h}}I^{m,n+1}_{p_i,q_i+1;r}\left[\frac{z}{2}\middle|\begin{matrix}(-1-2\tau-2h,2),(a_j,A_j)_{1,n}:(a_{ji},A_{ji})_{1,pi}\\(b_j,B_j)_{1,m}:(a_{ji},B_{ji})_{1,qi},(-1--\tau-2h,2)\end{matrix}\right]$$

$$+(c)^{1+\tau-\lambda}(\frac{1}{2})^{1+\tau}(-\tau-1)\sum_{h=0}^{\infty}\frac{(-\frac{1}{2})^h(\beta)_{\delta h}(\sigma)_{th}}{\Gamma(\gamma h+\alpha+1)(v)_{sh}(\mu)_{\rho h}}I^{m,n+1}_{p_i,q_i+1;r}\left[\frac{z}{2}\middle|\begin{matrix}(-1-2\tau-2h,2),(a_j,A_j)_{1,n}:(a_{ji},A_{ji})_{1,pi}\\(b_j,B_j)_{1,m}:(a_{ji},B_{ji})_{1,qi},(--\tau-2h,2)\end{matrix}\right].$$

In the result (2.7), apply the conditions of the results (1.3), (1.4) and (1.6), we get the conditions given in the ***Theorem* 2.2**. Hence, the ***Theorem* 2.2** is followed.

***Theorem* 2.3** *If $\alpha,\gamma,\mu,\rho,v,\beta,\delta,\sigma\in C;(\beta)>0,(\gamma)>0,(\delta)>0,(\mu)>0,(\rho)>0,(\sigma)>0,(v)>0,(\alpha)\geq -1;s,t>0,t<(\gamma)+s,(k)>0,(l)>0,|\arg z|<\frac{\pi}{2}A'$ and $A'=\sum_{j=1}^{n}A_j+\sum_{j=1}^{m}B_j-\max_{1\leq i\leq r}[\sum_{j=1}^{p_i}A_{ji}+\sum_{j=1}^{q_i}B_{ji}]>0,$ then following integral holds good*

$$(2.8)\quad \int_0^1 x^{k-1}(1-x)^{l-1}[cx+d(1-x)]^{-k-l}J^{\beta,\delta,\sigma,t}_{\alpha,\gamma,\mu,\rho,v,s}\left[\frac{cdx(1-x)}{[cx+d(1-x)]^2}\right]I^{m,n}_{p_i,q_i;r}\left[\frac{zx(1-x)}{[cx+d(1-x)]^2}\middle|\begin{matrix}(a_j,A_j)_{1,n}:(a_{ji},A_{ji})_{1,pi}\\(b_j,B_j)_{1,m}:(a_{ji},B_{ji})_{1,qi}\end{matrix}\right]dx$$

$$=\frac{1}{c^kd^l}\sum_{h=0}^{\infty}\frac{(-1)^h(\beta)_{\delta h}(\sigma)_{th}}{\Gamma(\gamma h+\alpha+1)(v)_{sh}(\mu)_{\rho h}}I^{m,n+2}_{p_i,q_i+1;r}\left[\frac{z}{cd}\middle|\begin{matrix}(1-k-h,1),(1-l-h,1),(a_j,A_j)_{1,n}:(a_{ji},A_{ji})_{1,pi}\\(b_j,B_j)_{1,m}:(a_{ji},B_{ji})_{1,qi},(1-k-l-2h,2)\end{matrix}\right],$$

Proof. In the similar manner of the ***Theorems* 2.1** and **2.2**, and using the result of Eqn. (1.7), we obtain the result (2.7) of the ***Theorem* 2.3**.

3 Special cases

Here, among numerous special cases of the results in ***Section* 2**, only three of which are presented.

1. On replacing α by $\alpha-1$ in the ***Theorem* 2.1**,we get

$$(3.1)\quad \int_0^1\int_0^1 y^k(1-x)^{k-1}(1-y)^{l-1}(1-xy)^{1-k-l}E^{\beta,\delta,\sigma,t}_{\alpha,\gamma,\mu,\rho,v,s}\left[\frac{y(1-x)(1-y)}{(1-xy)^2}\right]I^{m,n}_{p_i,q_i;r}\left[\frac{zy(1-x)(1-y)}{(1-xy)^2}\middle|\begin{matrix}(a_j,A_j)_{1,n}:(a_{ji},A_{ji})_{1,pi}\\(b_j,B_j)_{1,m}:(a_{ji},B_{ji})_{1,qi}\end{matrix}\right]dxdy$$

$$=\sum_{h=0}^{\infty}\frac{(-1)^h(\beta)_{\delta h}(\sigma)_{th}}{\Gamma(\gamma h+\alpha)(v)_{sh}(\mu)_{\rho h}}I^{m,n+2}_{p_i,q_i+1;r}\left[z\middle|\begin{matrix}(a_j,A_j)_{1,n},(1-k-h,1),(1-l-h,1):(a_{ji},A_{ji})_{1,pi}\\(b_j,B_j)_{1,m}:(a_{ji},B_{ji})_{1,qi},(1-k-l-2h,2)\end{matrix}\right],$$

provided that $|\arg z|<\frac{\pi}{2}A',A'=\sum_{j=1}^{n}A_j+\sum_{j=1}^{m}B_j-\max_{1\leq i\leq r}[\sum_{j=1}^{p_i}A_{ji}+\sum_{j=1}^{q_i}B_{ji}]>0$, and $\gamma,\mu,\rho,v,\beta,\delta,\sigma\in C;(\beta)>0,(\gamma)>0,(\delta)>0,(\mu)>0,(\rho)>0,(\sigma)>0,(v)>0,(\alpha)>0;s,t>0 andt<(\gamma)+s,0<(l)<(k)$.

Here, $E^{\beta,\delta,\sigma,t}_{\alpha,\gamma,\mu,\rho,v,s}(z)$, a generalized Mittag - Leffler function, is defined by Khan and Ahmed [6].

2. On setting $r=1$ in ***Theorem* 2.1**, we obtain

$$(3.2)\quad \int_0^1\int_0^1 y^k(1-x)^{k-1}(1-y)^{l-1}(1-xy)^{1-k-l}J^{\beta,\delta,\sigma,t}_{\alpha,\gamma,\mu,\rho,v,s}\left[\frac{y(1-x)(1-y)}{(1-xy)^2}\right]I^{m,n}_{p_1,q_1;1}\left[\frac{zy(1-x)(1-y)}{(1-xy)^2}\middle|\begin{matrix}(a_j,A_j)_{1,n}:(a_{j1},A_{j1})_{1,p_1}\\(b_j,B_j)_{1,m}:(a_{j1},B_{j1})_{1,q_1}\end{matrix}\right]dxdy$$

$$= \sum_{h=0}^{\infty} \frac{(-1)^h (\beta)_{\delta h} (\sigma)_{th}}{\Gamma(\gamma h+\alpha+1)(v)_{sh}(\mu)_{\rho h}} H^{m,n+2}_{p_1+n+2,q_1+m+1}\left[z\left|\begin{matrix}(a_j,\alpha_j)_{1,n},(1-k-h,1),(1-l-h,1):(a_{j1},\alpha_{j1})_{n+1,p_1},(a_{j1},\alpha_{j1})_{1,n}\\(b_j,\beta_j)_{1,m}:(b_{j1},\beta_{j1})_{m+1,q_1},(b_{j1},\beta_{j1})_{1,m},(1-k-l-2h,2)\end{matrix}\right.\right],$$

provided that $\alpha,\gamma,\mu,\rho,v,\beta,\delta,\sigma \in C; (\beta) > 0, (\gamma) > 0, (\delta) > 0, (\mu) > 0, (\rho) > 0, (\sigma) > 0, (v) > 0, (\alpha) \geq -1; s,t > 0, t < (\gamma) + s, 0 < (l) < (k), |\arg z| < \frac{\pi}{2}A'$ and $A' = \sum_{j=1}^{n} A_j + \sum_{j=1}^{m} B_j - [\sum_{j=1}^{p} A_j + \sum_{j=1}^{q} B_j] > 0$.

3. On setting $\rho = s = t = \delta = 0$ in ***Theorem* 2.1**, we obtain

$$(3.3)\quad \int_0^1\int_0^1 y^k(1-x)^{k-1}(1-y)^{l-1}(1-xy)^{1-k-l} J^{\gamma}_{\alpha}\left[\frac{y(1-x)(1-y)}{(1-xy)^2}\right] I^{m,n}_{p_i,q_i;r}\left[\frac{zy(1-x)(1-y)}{(1-xy)^2}\left|\begin{matrix}(a_j,A_j)_{1,n}:(a_{ji},A_{ji})_{1,pi}\\(b_j,B_j)_{1,m}:(a_{ji},B_{ji})_{1,qi}\end{matrix}\right.\right]dxdy$$

$$= \sum_{h=0}^{\infty} \frac{(-1)^h}{h!\Gamma(\gamma h+\alpha+1)} I^{m,n+2}_{p_i,q_i+1;r}\left[z\left|\begin{matrix}(a_j,A_j)_{1,n},(1-k-h,1),(1-l-h,1):(a_{ji},A_{ji})_{1,pi}\\(b_j,B_j)_{1,m}:(a_{ji},B_{ji})_{1,qi},(1-k-l-2h,2)\end{matrix}\right.\right],$$

provided that $z \in C, (\beta) > 0, (\alpha) > -1, 0 < (l) < (k), |\arg z| < \frac{\pi}{2}A$ and $A = \sum_{j=1}^{n} A_j + \sum_{j=1}^{m} B_j - \max_{1\leq i\leq r}[\sum_{j=1}^{p_i} A_{ji} + \sum_{j=1}^{q_i} B_{ji}] > 0$, and $J^{\gamma}_{\alpha}(z)$ is Bessel-Maitland function [9, Eq.(8.3)].

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(Dedicated to Honor Dr. R. C. Singh Chandel on His 75^{th} Birth Anniversary Celebrations)

SOME APPLICATIONS OF MULTINOMINAL THEOREM IN SOLVING ALGEBRAIC EXPRESSIONS UNDER LINEAR MODELS

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Abstract

In this paper, an attempt has been made to provide the step-wise detailed algebraic expression for evaluation of expected values of different orders of error term (u) by employing multinominal theorem. The expectation of even order moments of error term (u) has been provided in case error term is not necessarily following normal distribution.

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1 Introduction

The literature review of linear regression has witnessed the extensive utilization of algebraic expression for providing proof of various problems. The several research articles have employed algebraic methods to provide important results but avoided the step-wise derivation for the sake of continuity. However, it take sufficient time for the young researchers to break down these expression by applying knowledge of Algebra. Therefore, the detailed derivations of expressions pertaining to expected value of disturbance term u for different distributional conditions have been obtained by employing well-known results of Multinominal theorem. Also, the detailed derivations of results pertaining to inversion of characteristic function have been obtained by using special methods such as integration by parts.

2 Model and measures of goodness of fit

Let us postulate the linear regression model as under

(2.1) $y = \alpha e + X\beta + u,$

where, y is a vector of $n \times 1$ observations on the variable to be explained, α is a scalar representing the intercept term, e is a $n \times 1$ vector with all elements of unity, X is a $n \times p$ full column rank matrix of n-observations on p-explanatory variables, β is a $p \times 1$ vector of the coefficients associated with them and u is a $n \times 1$ vector of disturbances. Further, it is assumed that the elements of disturbance vector (u) are distributed independently and identically.

The multinomial theorem has been employed to solve the expressions in this paper and statement of multinomial theorem is stated as here-under

Theorem 2.1: *It is stated that for real numbers $a_1, a_2, a_3, ..., a_m$ and non-negative integers $n, s_1, s_2, s_3, ..., s_m$, the following proposition holds*

(2.2) $(a_1 + a_2 + a_3 + ... + a_m)^n = \sum \frac{n!}{s_1!s_2!s_3!...s_m!} a_1^{s_1} a_2^{s_2} a_3^{s_3} ... a_m^{s_m},$

where, $\sum$ denotes the sum of all possible combinations of $(s_1, s_2, s_3, ..., s_m)$, such that

$$s_1 + s_2 + s_3 + ... + s_m = n.$$

Proof. The standard proof of aforesaid theorem can be seen in any related book of mathematics.

3 Derivation of results

Lets us assume that expressions pertaining to real numbers (a_i's) are given by

(3.1) $[a_1 + a_2 + ... + a_n]$

and expressions for different orders of (3.1) can be obtained by employing multinomial theorem, as under

$$(3.2)\quad (a_1 + a_2 + a_3 + ... + a_n)^1 = \frac{1!}{1!}\sum_i a_i = \binom{n}{1} a_i = na_i$$

$$(3.3)\quad (a_1 + a_2 + a_3 + ... + a_n)^2 = \frac{2!}{2!}\sum_i a_i^2 + \tfrac{2!}{1!1!}\sum_{i\neq j} a_i a_j = \sum_i a_i^2 + 2\sum_{i\neq j} a_i a_j$$
$$= \binom{n}{1} a_i^2 + \binom{n}{2}\binom{2}{1} a_i a_j = na_i^2 + n(n-1)a_i a_j$$
$$= na_i^2 + (n^2 - n)a_i a_j = na_i^2 + (n^2 - n)a_i a_j,$$

$$(3.4)\quad (a_1 + a_2 + a_3 + ... + a_n)^3 = \frac{3!}{3!}\sum_i a_i^3 + \tfrac{3!}{2!1!}\sum_{i\neq j} a_i^2 a_j + \tfrac{3!}{1!1!1!}\sum_{i\neq j\neq k} a_i a_j a_k$$
$$= \sum_i a_i^3 + 3\sum_{i\neq j} a_i^2 a_j + 6\sum_{i\neq j\neq k} a_i a_j a_k$$
$$= \binom{n}{1} a_i^3 + \binom{n}{2}\binom{2}{1} a_i^2 a_j + \binom{n}{3}\binom{3}{0} a_i a_j a_k$$
$$= na_i^3 + 3n(n-1)a_i^2 a_j + n(n-1)(n-2)a_i a_j a_k$$
$$= na_i^3 + (3n^2 - 3n)a_i^2 a_j + (n^3 - 3n^2 + 2n)a_i a_j a_k,$$

$$(3.5)\quad (a_1 + a_2 + a_3 + ... + a_n)^4 = \frac{4!}{4!}\sum_i a_i^4 + \tfrac{4!}{3!1!}\sum_{i\neq j} a_i^3 a_j^2 + \tfrac{4!}{2!2!}\sum_{i\neq j} a_i^2 a_j^2$$
$$+ \frac{4!}{2!}\sum_{i\neq j\neq k} a_i^2 a_j a_k + \tfrac{4!}{1!1!1!1!}\sum_{i\neq j\neq k\neq l} a_i a_j a_k a_l$$
$$= \sum_i a_i^4 + 4\sum_{i\neq j} a_i^3 a_j^2 + 6\sum_{i\neq j} a_i^2 a_j^2 + 12\sum_{i\neq j\neq k} a_i^2 a_j a_k$$
$$+ 24\sum_{i\neq j\neq k\neq l} a_i a_j a_k a_l$$
$$= \binom{n}{1}\binom{1}{1} a_i^4 + 4\binom{n}{2}\binom{2}{1} a_i^3 a_j^2 + 6\binom{n}{2}\binom{2}{2} a_i^2 a_j^2$$
$$+ 12\binom{n}{3}\binom{3}{1} a_i^2 a_j a_k + 24\binom{n}{4}\binom{4}{0} a_i a_j a_k a_l$$
$$= na_i^4 + \frac{8n(n-1)}{2} a_i^3 a_j^2 + \frac{6n(n-1)}{2} a_i^2 a_j^2$$
$$+ \frac{36n(n^2 - 3n + 2)}{6} a_i^2 a_j a_k + \frac{24(n^3 - 3n^2 + 2n)(n-3)}{24} a_i a_j a_k a_l$$
$$= na_i^4 + 4n(n-1)a_i^3 a_j^2 + 3n(n-1)a_i^2 a_j^2 + 6n(n-1)(n-2)$$
$$a_i^2 a_j a_k + n(n-1)(n-2)(n-3)a_i a_j a_k a_l$$
$$= na_i^4 + (4n^2 - 4n)a_i^3 a_j^2 + (3n^2 - 3n)a_i^2 a_j^2 + (6n^3 - 18n^2 + 12n)$$
$$a_i^2 a_j a_k + (n^4 - 6n^3 + 11n^2 - 6n)a_i a_j a_k a_l$$
$$= na_i^4 + (4n^2 - 4n)a_i^3 a_j^2 + (3n^2 - 3n)a_i^2 a_j^2 + (6n^3 - 18n^2 + 12n)$$
$$a_i^2 a_j a_k + (n^4 - 6n^3 + 11n^2 - 6n)a_i a_j a_k a_l,$$

$$(3.6)\quad (a_1 + a_2 + a_3 + ... + a_n)^5 = \frac{5!}{5!}\sum_i a_i^5 + \tfrac{5!}{4!1!}\sum_{i\neq j} a_i^4 a_j + \tfrac{5!}{3!2!}\sum_{i\neq j} a_i^3 a_j^2$$
$$+ \frac{5!}{3!1!1!}\sum_{i\neq j\neq k} a_i^3 a_j a_k + \tfrac{5!}{2!2!1!}\sum_{i\neq j\neq k} a_i^2 a_j^2 a_k$$
$$+ \frac{5!}{2!1!1!1!}\sum_{i\neq j\neq k\neq l} a_i^2 a_j a_k a_l + \tfrac{5!}{1!1!1!1!1!}\sum_{i\neq j\neq k\neq l\neq m} a_i a_j a_k a_l a_m$$
$$= na_i^5 + 5\frac{n(n-1)}{2} 2a_i^4 a_j + 10\frac{n(n-1)}{2} 2a_i^3 a_j^2$$
$$+ 20\frac{n(n-1)(n-2)}{6} 3a_i^3 a_j a_k + 30\frac{n(n-1)(n-2)}{6} 3a_i^2 a_j^2 a_k$$
$$+ 60\frac{n(n-1)(n-2)(n-3)}{24} 4a_i^2 a_j a_k a_l$$
$$+ 120\frac{n(n-1)(n-2)(n-3)(n-4)}{120} a_i a_j a_k a_l a_m$$

$$
\begin{aligned}
&= na_i^5 + 5n(n-1)a_i^4 a_j + 10n(n-1)a_i^3 a_j^2 \\
&\quad + 10n(n-1)(n-2)a_i^3 a_j a_k + 15n(n-1)(n-2)a_i^2 a_j^2 a_k \\
&\quad + 10n(n-1)(n-2)(n-3)a_i^2 a_j a_k a_l \\
&\quad + n(n-1)(n-2)(n-3)(n-4)a_i a_j a_k a_l a_m \\
&= na_i^5 + 5n(n-1)a_i^4 a_j + 10n(n-1)a_i^3 a_j^2 \\
&\quad + 10n(n-1)(n-2)a_i^3 a_j a_k + 15n(n-1)(n-2)a_i^2 a_j^2 a_k \\
&\quad + 10n(n-1)(n-2)(n-3)a_i^2 a_j a_k a_l \\
&\quad + n(n-1)(n-2)(n-3)(n-4)a_i a_j a_k a_l a_m \\
&= na_i^5 + (5n^2-5n)a_i^4 a_j + (10n^2-10n)a_i^3 a_j^2 \\
&\quad + (10n^3-30n^2+20n)a_i^3 a_j a_k \\
&\quad + (15n^3-45n^2+30n)a_i^2 a_j^2 a_k \\
&\quad + (10n^4-60n^3+110n^2-60n)a_i^2 a_j a_k a_l \\
&\quad + (n^5-10n^4+35n^3-50n^2+24n)a_i a_j a_k a_l a_m,
\end{aligned}
$$

$$
\begin{aligned}
(a_1+a_2+\dots+a_n)^6 &= \tfrac{6!}{6!}\textstyle\sum_i a_i^6 + \tfrac{6!}{5!1!}\sum_{i\neq j} a_i^5 a_j + \tfrac{6!}{4!2!}\sum_{i\neq j} a_i^4 a_j^2 + \tfrac{6!}{4!1!1!}\sum_{i\neq j\neq k} a_i^4 a_j a_k \\
&\quad + \tfrac{6!}{3!3!}\textstyle\sum_{i\neq j} a_i^3 a_j^3 + \tfrac{6!}{3!2!1!}\sum_{i\neq j\neq k} a_i^3 a_j^2 a_k + \tfrac{6!}{3!1!1!1!}\sum_{i\neq j\neq k\neq l} a_i^3 a_j a_k a_l \\
&\quad + \tfrac{6!}{2!2!2!}\textstyle\sum_{i\neq j\neq k} a_i^2 a_j^2 a_k^2 + \tfrac{6!}{2!2!1!1!}\sum_{i\neq j\neq k\neq l} a_i^2 a_j^2 a_k a_l \\
&\quad + \tfrac{6!}{2!1!1!1!1!}\textstyle\sum_{i\neq j\neq k\neq l\neq m} a_i^2 a_j a_k a_l a_m \\
&\quad + \tfrac{6!}{1!1!1!1!1!1!}\textstyle\sum_{i\neq j\neq k\neq l\neq m\neq n} a_i a_j a_k a_l a_m a_n \\
&= \textstyle\sum_i a_i^6 + 6\sum_{i\neq j} a_i^5 a_j + 15\sum_{i\neq j} a_i^4 a_j^2 + 30\sum_{i\neq j\neq k} a_i^4 a_j a_k + 20\sum_{i\neq j} a_i^3 a_j^3 \\
&\quad + 60\textstyle\sum_{i\neq j\neq k} a_i^3 a_j^2 a_k + 120\sum_{i\neq j\neq k\neq l} a_i^3 a_j a_k a_l + 90\sum_{i\neq j\neq k} a_i^2 a_j^2 a_k^2 \\
&\quad + 180\textstyle\sum_{i\neq j\neq k\neq l} a_i^2 a_j^2 a_k a_l + 360\sum_{i\neq j\neq k\neq l\neq m} a_i^2 a_j a_k a_l a_m \\
&\quad + 720\textstyle\sum_{i\neq j\neq k\neq l\neq m\neq n} a_i a_j a_k a_l a_m a_n \\
&= \tbinom{n}{1}\tbinom{1}{1}a_i^6 + 6\tbinom{n}{2}\tbinom{2}{1}a_i^5 a_j + 15\tbinom{n}{2}\tbinom{2}{2}a_i^4 a_j^2 \\
&\quad + 30\tbinom{n}{3}\tbinom{3}{1}a_i^4 a_j a_k + 20\tbinom{n}{2}\tbinom{2}{2}a_i^3 a_j^3 \\
&\quad + 60\tbinom{n}{3}\tbinom{3}{1}\tbinom{2}{1}a_i^3 a_j^2 a_k + 120\tbinom{n}{4}\tbinom{4}{1}a_i^3 a_j a_k a_l \\
&\quad + 90\tbinom{n}{3}\tbinom{3}{3}a_i^2 a_j^2 a_k^2 + 180\tbinom{n}{4}\tbinom{4}{2}a_i^2 a_j^2 a_k a_l \\
&\quad + 360\tbinom{n}{5}\tbinom{5}{1}a_i^2 a_j a_k a_l a_m + 720\tbinom{n}{6}\tbinom{6}{0}a_i a_j a_k a_l a_m a_n \\
&= na_i^6 + 6\tfrac{n(n-1)}{2}2a_i^5 a_j + 15\tfrac{n(n-1)}{2}2a_i^4 a_j^2 \\
&\quad + 30\tfrac{n(n-1)(n-2)}{6}3a_i^4 a_j a_k + 20\tfrac{n(n-1)}{2}a_i^3 a_j^3 \\
&\quad + 60\tfrac{n(n-1)(n-2)}{6}6a_i^3 a_j^2 a_k + 120\tfrac{n(n-1)(n-2)(n-3)}{24}4a_i^3 a_j a_k a_l \\
&\quad + 90\tfrac{n(n-1)(n-2)}{6}a_i^2 a_j^2 a_k^2 + 180\tfrac{n(n-1)(n-2)(n-3)}{24}6a_i^2 a_j^2 a_k a_l \\
&\quad + 360\tfrac{n(n-1)(n-2)(n-3)(n-4)}{120}5a_i^2 a_j a_k a_l a_m \\
&\quad + 720\tfrac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{720}a_i a_j a_k a_l a_m a_n \\
&= na_i^6 + 6n(n-1)a_i^5 a_j + 15n(n-1)a_i^4 a_j^2 + 15n(n-1)(n-2)a_i^4 a_j a_k \\
&\quad + 10n(n-1)a_i^3 a_j^3 + 60n(n-1)(n-2)a_i^3 a_j^2 a_k \\
&\quad + 20n(n-1)(n-2)(n-3)a_i^3 a_j a_k a_l \\
&\quad + 15n(n-1)(n-2)a_i^2 a_j^2 a_k^2 + 45n(n-1)(n-2)(n-3)a_i^2 a_j^2 a_k a_l \\
&\quad + 15n(n-1)(n-2)(n-3)(n-4)a_i^2 a_j a_k a_l a_m \\
&\quad + n(n-1)(n-2)(n-3)(n-4)(n-5)a_i a_j a_k a_l a_m a_n
\end{aligned}
\tag{3.7}
$$

$$
\begin{aligned}
&= na_i^6 + 6n(n-1)a_i^5 a_j + 15n(n-1)a_i^4 a_j^2 \\
&\quad + 15n(n-1)(n-2)a_i^4 a_j a_k + 10n(n-1)a_i^3 a_j^3 \\
&\quad + 60n(n-1)(n-2)a_i^3 a_j^2 a_k + 20n(n-1)(n-2)(n-3)a_i^3 a_j a_k a_l \\
&\quad + 15n(n-1)(n-2)a_i^2 a_j^2 a_k^2 + 45n(n-1)(n-2)(n-3)a_i^2 a_j^2 a_k a_l \\
&\quad + 15n(n-1)(n-2)(n-3)(n-4)a_i^2 a_j a_k a_l a_m \\
&\quad + n(n-1)(n-2)(n-3)(n-4)(n-5)a_i a_j a_k a_l a_m a_n \\
&= na_i^6 + (6n^2 - 6n)a_i^5 a_j + (15n^2 - 15n)a_i^4 a_j^2 \\
&\quad + (15n^3 - 45n^2 + 30n)a_i^4 a_j a_k + (10n^2 - 10n)a_i^3 a_j^3 \\
&\quad + (60n^3 - 180n^2 + 120n)a_i^3 a_j^2 a_k \\
&\quad + (20n^4 - 120n^3 + 220n^2 - 120n)a_i^3 a_j a_k a_l \\
&\quad + (15n^3 - 45n^2 + 30n)a_i^2 a_j^2 a_k^2 \\
&\quad + (45n^4 - 270n^3 + 495n^2 - 270n)a_i^2 a_j^2 a_k a_l \\
&\quad + (15n^5 - 150n^4 + 525n^3 - 750n^2 + 360n)a_i^2 a_j a_k a_l a_m \\
&\quad + (n^6 - 15n^5 + 85n^4 - 155n^3 + 274n^2 - 120)a_i a_j a_k a_l a_m a_n.
\end{aligned}
$$

From equations (3.2) to (3.7) we get respectively

(3.8)
$$
\begin{aligned}
(a_1 + a_2 + a_3 + \dots + a_n)^1 &= n[a_i], \\
(a_1 + a_2 + a_3 + \dots + a_n)^2 &= n\left[a_i^2\right] + (n^2 - n)\left[a_i a_j\right], \\
(a_1 + a_2 + a_3 + \dots + a_n)^3 &= n\left[a_i^3\right] + (3n^2 - 3n)\left[a_i^2 a_j\right] \\
&\quad + (n^3 - 3n^2 + 2n)\left[a_i a_j a_k\right], \\
(a_1 + a_2 + a_3 + \dots + a_n)^4 &= n\left[a_i^4\right] + (4n^2 - 4n)\left[a_i^3 a_j^2\right] \\
&\quad + (3n^2 - 3n)\left[a_i^2 a_j^2\right] \\
&\quad + (6n^3 - 18n^2 + 12n)\left[a_i^2 a_j a_k\right] \\
&\quad + (n^4 - 6n^3 + 11n^2 - 6n)\left[a_i a_j a_k a_l\right], \\
(a_1 + a_2 + a_3 + \dots + a_n)^5 &= n\left[a_i^5\right] + (5n^2 - 5n)\left[a_i^4 a_j\right] \\
&\quad + (10n^2 - 10n)\left[a_i^3 a_j^2\right] \\
&\quad + (10n^3 - 30n^2 + 20n)\left[a_i^3 a_j a_k\right] \\
&\quad + (15n^3 - 45n^2 + 30n)\left[a_i^2 a_j^2 a_k\right] \\
&\quad + (10n^4 - 60n^3 + 110n^2 - 60n)\left[a_i^2 a_j a_k a_l\right] \\
&\quad + (n^5 - 10n^4 + 35n^3 - 50n^2 + 24n)\left[a_i a_j a_k a_l a_m\right], \\
(a_1 + a_2 + a_3 + \dots + a_n)^6 &= n\left[a_i^6\right] + (6n^2 - 6n)\left[a_i^5 a_j\right] + (15n^2 - 15n)\left[a_i^4 a_j^2\right] \\
&\quad + (15n^3 - 45n^2 + 30n)\left[a_i^4 a_j a_k\right] \\
&\quad + (10n^2 - 10n)\left[a_i^3 a_j^3\right] \\
&\quad + (60n^3 - 180n^2 + 120n)\left[a_i^3 a_j^2 a_k\right] \\
&\quad + (20n^4 - 120n^3 + 220n^2 - 120n)\left[a_i^3 a_j a_k a_l\right] \\
&\quad + (15n^3 - 45n^2 + 30n)\left[a_i^2 a_j^2 a_k^2\right] \\
&\quad + (45n^4 - 270n^3 + 495n^2 - 270n)\left[a_i^2 a_j^2 a_k a_l\right] \\
&\quad + (15n^5 - 150n^4 + 525n^3 - 750n^2 + 360n)\left[a_i^2 a_j a_k a_l a_m\right]
\end{aligned}
$$

$$+ (n^6 - 15n^5 + 85n^4 - 155n^3 + 274n^2 - 120)\Big[a_i a_j a_k a_l a_m a_n\Big].$$

In case, we required the expectations of $(u'u)$, $(u'u)^2$, $(u'u)^3$, $(u'u)^4$, $(u'u)^5$ and $(u'u)^6$, the same can be obtained by using (3.8) as stated in ***Theorem* 3.1**.

***Theorem* 3.1** *The expected value of different orders of disturbance term (u), when u follows normal distribution with mean zero and variance σ^2 is given by*

(3.9) $E(u'u) = n\sigma^2$,

$E(u'uu'u) = \sigma^4(n^2 - 2n)$,

$E(u'uu'uu'u) = \sigma^6(n^3 + 6n^2 + 8n)$,

$E(u'uu'uu'uu'u) = \sigma^8(n^4 + 12n^3 + 44n^2 + 48)$,

$E(u'uu'uu'uu'uu'u) = \sigma^{10}(n^5 + 20n^4 + 140n^3 + 400n^2 + 384n)$,

$E(u'uu'uu'uu'uu'uu'u) = \sigma^{12}(n^6 + 30n^5 + 340n^4 + 1870n^3 + 4384n^2 + 3840n)$.

Proof. As disturbances are following normal distribution, the generalized form of central moments (with mean zero and variance σ^2) is given by

(3.10) $m_{2n} = \sigma^{2n}(2n-1)!!$ for even moments,

$m_{2n+1} = 0$ for odd moments.

Using (3.9) one can see that all odd order moments of normal distribution are zero and even order central moments for normal distribution with mean zero and variance σ^2 as

(3.11) $m_2 = \sigma^2(2-1)!! = \sigma^2$,

$m_4 = \sigma^4(4-1)!! = \sigma^4(4-1)(4-3) = 3\sigma^4$,

$m_6 = \sigma^6(6-1)!! = \sigma^6(6-1)(6-3)(6-1) = 15\sigma^6$,

$m_8 = \sigma^8(8-1)!! = \sigma^8(8-1)(8-3)(8-5)(8-7) = 105\sigma^8$,

$m_{10} = \sigma^{10}(10-1)!! = \sigma^1 0(10-1)(10-3)(10-5)(10-7)(10-9) = 945\sigma^8$,

$m_{12} = \sigma^{12}(12-1)!! = \sigma^{12}(12-1)(12-3)(12-5)(12-7)(12-9)(12-11) = 10395\sigma^{12}$.

By utilizing equations (3.1), (3.8) and (3.11) the expected value of disturbance term u following normal distribution with mean zero and variance σ^2 can be derived as

(3.12) $E(u_1^2 + u_2^2 + ... + u_n^2)^1 = E[a_1 + a_2 + a_3 + ... + a_n]^1 = nE[a_i] = nE[u_i^2] = n\sigma^2$,

(3.13) $E(u_1^2 + u_2^2 + ... + u_n^2)^2 = E[a_1 + a_2 + a_3 + ... + a_n]^2 = nE\Big[a_i^2\Big] + (n^2 - n)E\Big[a_i a_j\Big]$

$$= nE\Big[u_i^4\Big] + (n^2 - n)E\Big[u_i^2 u_j^2\Big] = nE\Big[u_i^4\Big] + (n^2 - n)E\Big[u_i^2\Big]\Big[u_j^2\Big]$$

$$= n[3\sigma^4] + (n^2 - n)[\sigma^2\sigma^2] = \sigma^4[n^2 - 2n],$$

(3.14) $E(u_1^2 + u_2^2 + ... + u_n^2)^3 = E[a_1 + a_2 + a_3 + ... + a_n]^3$

$$= nE\Big[a_i^3\Big] + (3n^2 - 3n)E\Big[a_i^2 a_j\Big] + (n^3 - 3n^2 + 2n)E\Big[a_i a_j a_k\Big]$$

$$= nE\Big[u_i^6\Big] + (3n^2 - 3n)E\Big[u_i^4 u_j^2\Big] + (n^3 - 3n^2 + 2n)E\Big[u_i^2 u_j^2 u_k^2\Big]$$

$$= nE\Big[u_i^6\Big] + (3n^2 - 3n)E\Big[u_i^4\Big]E\Big[u_j^2\Big] + (n^3 - 3n^2 + 2n)E\Big[u_i^2\Big]E\Big[u_j^2\Big]E\Big[u_k^2\Big]$$

$$= n[15\sigma^6] + (3n^2 - 3n)[3\sigma^4\sigma^2] + (n^3 - 3n^2 + 2n)[\sigma^6]$$

$$= \sigma^6[15n + 9n^2 - 9n + n^3 - 3n^2 + 2n]$$

$$= \sigma^6[n^3 + 6n^2 + 8n],$$

(3.15) $E(u_1^2 + u_2^2 + ... + u_n^2)^4 = E[a_1 + a_2 + a_3 + ... + a_n]^4$

$$=nE\Big[a_i^4\Big] + (4n^2 - 4n)E\Big[a_i^3 a_j^2\Big] + (3n^2 - 3n)E\Big[a_i^2 a_j^2\Big]$$

$$
\begin{aligned}
&+ (6n^3 - 18n^2 + 12n)E\left[a_i^2 a_j a_k\right] \\
&+ (n^4 - 6n^3 + 11n^2 - 6n)E\left[a_i a_j a_k a_l\right] \\
=&nE\left[u_i^8\right] + (4n^2 - 4n)E\left[u_i^6 u_j^4\right] + (3n^2 - 3n)E\left[u_i^4 u_j^4\right] \\
&+ (6n^3 - 18n^2 + 12n)E\left[u_i^4 u_j^2 u_k^2\right] \\
&+ (n^4 - 6n^3 + 11n^2 - 6n)E\left[u_i^2 u_j^2 u_k^2 u_l^2\right] \\
=&nE\left[u_i^8\right] + (4n^2 - 4n)E\left[u_i^6\right]E\left[u_j^4\right] \\
&+ (3n^2 - 3n)E\left[u_i^4\right]E\left[u_j^4\right] \\
&+ (6n^3 - 18n^2 + 12n)E\left[u_i^4\right]E\left[u_j^2\right]E\left[u_k^2\right] \\
&+ (n^4 - 6n^3 + 11n^2 - 6n)E\left[u_i^2\right]E\left[u_j^2\right]E\left[u_k^2\right]E\left[u_l^2\right] \\
=&n[105\sigma^8] + (4n^2 - 4n)[15\sigma^6 3\sigma^4] + (3n^2 - 3n)[3\sigma^4 3\sigma^4] \\
&+ (6n^3 - 18n^2 + 12n)[3\sigma^4\sigma^2\sigma^2] \\
&+ (n^4 - 6n^3 + 11n^2 - 6n)[\sigma^2\sigma^2\sigma^2\sigma^2] \\
=&\sigma^8[105n + 45(4n^2 - 4n) + 9(3n^2 - 3n) \\
&+ 3(6n^3 - 18n^2 + 12n) \\
&+ (n^4 - 6n^3 + 11n^2 - 6n)] \\
=&\sigma^8[n^4 + 12n^3 + 44n^2 + 48],
\end{aligned}
$$

(3.16) $E(u_1^2 + u_2^2 + ... + u_n^2)^5 = E[a_1 + a_2 + a_3 + ... + a_n]^5$

$$
\begin{aligned}
=&nE\left[a_i^5\right] + (5n^2 - 5n)E\left[a_i^4 a_j\right] + (10n^2 - 10n)E\left[a_i^3 a_j^2\right] \\
&+ (10n^3 - 30n^2 + 20n)E\left[a_i^3 a_j a_k\right] \\
&+ (15n^3 - 45n^2 + 30n)E\left[a_i^2 a_j^2 a_k\right] \\
&+ (10n^4 - 60n^3 + 110n^2 - 60n)E\left[a_i^2 a_j a_k a_l\right] \\
&+ (n^5 - 10n^4 + 35n^3 - 50n^2 + 24n)E\left[a_i a_j a_k a_l a_m\right] \\
=&nE\left[u_i^{10}\right] + (5n^2 - 5n)E\left[u_i^8 u_j^2\right] + (10n^2 - 10n)E\left[u_i^6 u_j^4\right] \\
&+ (10n^3 - 30n^2 + 20n)E\left[u_i^6 u_j^2 u_k^2\right] \\
&+ (15n^3 - 45n^2 + 30n)E\left[u_i^4 u_j^4 u_k^2\right] \\
&+ (10n^4 - 60n^3 + 110n^2 - 60n)E\left[u_i^4 u_j^2 u_k^2 u_l^2\right] \\
&+ (n^5 - 10n^4 + 35n^3 - 50n^2 + 24n)E\left[u_i^2 u_j^2 u_k^2 u_l^2 u_m^2\right] \\
=&nE\left[u_i^{10}\right] + (5n^2 - 5n)E\left[u_i^8\right]E\left[u_j^2\right] + (10n^2 - 10n)E\left[u_i^6\right]E\left[u_j^4\right] \\
&+ (10n^3 - 30n^2 + 20n)E\left[u_i^6\right]E\left[u_j^2\right]E\left[u_k^2\right] \\
&+ (15n^3 - 45n^2 + 30n)E\left[u_i^4\right]E\left[u_j^4\right]E\left[u_k^2\right] \\
&+ (10n^4 - 60n^3 + 110n^2 - 60n)E\left[u_i^4\right]E\left[u_j^2\right]E\left[u_k^2\right]E\left[u_l^2\right] \\
&+ (n^5 - 10n^4 + 35n^3 - 50n^2 + 24n) \\
&\quad E\left[u_i^2\right]E\left[u_j^2\right]E\left[u_k^2\right]E\left[u_l^2\right]E\left[u_m^2\right] \\
=&n\left[945\sigma^{10}\right] + (5n^2 - 5n)E\left[105\sigma^8\sigma^2\right] \\
&+ (10n^2 - 10n)E\left[15\sigma^6 3\sigma^4\right] + (10n^3 - 30n^2 + 20n)
\end{aligned}
$$

$$E\left[15\sigma^6\sigma^2\sigma^2\right] + (15n^3 - 45n^2 + 30n)E\left[3\sigma^4 3\sigma^4\sigma^2\right]$$
$$+ (10n^4 - 60n^3 + 110n^2 - 60n)E\left[3\sigma^4\sigma^2\sigma^2\sigma^2\right]$$
$$+ (n^5 - 10n^4 + 35n^3 - 50n^2 + 24n)E\left[\sigma^2\sigma^2\sigma^2\sigma^2\sigma^2\right]$$
$$= \sigma^{10}\left[n^5 + 20n^4 + 140n^3 + 400n^2 + 384n\right],$$

(3.17) $$E(u_1^2 + u_2^2 + ... + u_n^2)^6 = E\left[a_1 + a_2 + a_3 + ... + a_n\right]^6$$
$$= nE\left[u_i^{12}\right] + (6n^2 - 6n)E\left[u_i^{10}u_j^2\right]$$
$$+ (15n^2 - 15n)E\left[u_i^8 u_j^4\right] + (10n^2 - 10n)E\left[u_i^6 u_j^6\right]$$
$$+ (15n^3 - 45n^2 + 30n)E\left[u_i^8 u_j^2 u_k^2\right]$$
$$+ (60n^3 - 180n^2 + 120n)E\left[u_i^6 u_j^4 u_k^2\right]$$
$$+ (20n^4 - 120n^3 + 220n^2 - 120n)E\left[u_i^6 u_j^2 u_k^2 u_l^2\right]$$
$$+ (15n^3 - 45n^2 + 30n)E\left[u_i^4 u_j^4 u_k^4\right]$$
$$+ (45n^4 - 270n^3 + 495n^2 - 270n)E\left[u_i^4 u_j^4 u_k^2 u_l^2\right]$$
$$+ (15n^5 - 150n^4 + 525n^3 - 750n^2 + 360n)E\left[u_i^4 u_j^2 u_k^2 u_l^2 u_m^2\right]$$
$$+ (n^6 - 15n^5 + 85n^4 - 155n^3 + 274n^2 - 120)$$
$$E\left[u_i^2 u_j^2 u_k^2 u_l^2 u_m^2 u_n^2\right]$$
$$= nE\left[u_i^{12}\right] + (6n^2 - 6n)E\left[u_i^{10}\right]E\left[u_j^2\right]$$
$$+ (15n^2 - 15n)E\left[u_i^8\right]E\left[u_j^4\right] + (10n^2 - 10n)E\left[u_i^6\right]E\left[u_j^6\right]$$
$$+ (15n^3 - 45n^2 + 30n)E\left[u_i^8\right]E\left[u_j^2\right]\left[u_k^2\right]$$
$$+ (60n^3 - 180n^2 + 120n)E\left[u_i^6\right]E\left[u_j^4\right]E\left[u_k^2\right]$$
$$+ (20n^4 - 120n^3 + 220n^2 - 120n)$$
$$E\left[u_i^6\right]E\left[u_j^2\right]E\left[u_k^2\right]E\left[u_l^2\right]$$
$$+ (15n^3 - 45n^2 + 30n)E\left[u_i^4\right]E\left[u_j^4\right]E\left[u_k^4\right]$$
$$+ (45n^4 - 270n^3 + 495n^2 - 270n)$$
$$E\left[u_i^4\right]E\left[u_j^4\right]E\left[u_k^2\right]E\left[u_l^2\right]$$
$$+ (15n^5 - 150n^4 + 525n^3 - 750n^2 + 360n)$$
$$E\left[u_i^4\right]E\left[u_j^2\right]E\left[u_k^2\right]E\left[u_l^2\right]E\left[u_m^2\right]$$
$$+ (n^6 - 15n^5 + 85n^4 - 155n^3 + 274n^2 - 120)$$
$$E\left[u_i^2\right]E\left[u_j^2\right]E\left[u_k^2\right]E\left[u_l^2\right]E\left[u_m^2\right]E\left[u_n^2\right]$$
$$= nE\left[10395\sigma^{12}\right] + (6n^2 - 6n)E\left[945\sigma^{10}\right]E\left[\sigma^2\right]$$
$$+ (15n^2 - 15n)E\left[105\sigma^8\right]E\left[3\sigma^4\right]$$
$$+ (10n^2 - 10n)E\left[15\sigma^6\right]E\left[15\sigma^6\right]$$
$$+ (15n^3 - 45n^2 + 30n)E\left[105\sigma^8\right]E\left[\sigma^2\right]E\left[\sigma^2\right]$$
$$+ (60n^3 - 180n^2 + 120n)E\left[15\sigma^6\right]E\left[3\sigma^4\right]E\left[\sigma^2\right]$$
$$+ (15n^3 - 45n^2 + 30n)E\left[3\sigma^4\right]E\left[3\sigma^4\right]E\left[3\sigma^4\right]$$
$$+ (20n^4 - 120n^3 + 220n^2 - 120n)E\left[15\sigma^6\right]E\left[\sigma^2\right]$$

$$E\left[\sigma^2\right]E\left[\sigma^2\right] + (45n^4 - 270n^3 + 495n^2 - 270n)$$
$$E\left[3\sigma^4\right]E\left[3\sigma^4\right]E\left[\sigma^2\right]E\left[\sigma^2\right]$$
$$+ (15n^5 - 150n^4 + 525n^3 - 750n^2 + 360n)$$
$$E\left[3\sigma^4\right]\left[\sigma^2\right]\left[\sigma^2\right]\left[\sigma^2\right]\left[\sigma^2\right]$$
$$+ (n^6 - 15n^5 + 85n^4 - 155n^3 + 274n^2 - 120)$$
$$E\left[\sigma^2\right]\left[\sigma^2\right]\left[\sigma^2\right]\left[\sigma^2\right]\left[\sigma^2\right]\left[\sigma^2\right]$$
$$=\sigma^{12}\left[n^6 + 30n^5 + 340n^4 + 1870n^3 + 4384n^2 + 3840n\right].$$

Thus, using (3.12) to (3.17) we obtain the results (3.9) of the ***Theorem* 3.1**.

***Theorem* 3.2** *The even order moments of disturbance term (u), when u is distributed with non-normal distribution with finite moments as*

(3.18) $m_2 = \sigma^2$,

$m_4 = \sigma^4(\gamma_2 + 3)$,

$m_6 = \sigma^6(\gamma_4 + 15\gamma_2 + 10\gamma_1^2 + 15)$,

$m_8 = \sigma^8(\gamma_6 + 28\gamma_4 + 56\gamma_3\gamma_1 + 35\gamma_2^2 + 210\gamma_2 + 280\gamma_1^2 + 105)$.

Further, the expected value of different orders of $u'u$ are given by

(3.19) $E(u'u) = \sigma^2\left[n\right]$,

$E(u'u)^2 = \sigma^4\left[\gamma_2 n + n^2 + 2n\right]$,

$E(u'u)^3 = \sigma^6\left[\gamma_4 n + \gamma_2(3n^2 + 12n) + 10\gamma_1^2 n + 6n^2 + 8n\right]$,

$$E(u'u)^4 = \sigma^8\Big[n^4 + 12n^3 + 44n^2 + 48n + \gamma_2(6n^3 + 60n^2 + 144n)$$
$$+\gamma_4(4n^2 + 24n) + \gamma_6 n + \gamma_1^2(40n^2 + 240n) + \gamma_2^2(3n^2 + 32n) + 56\gamma_1\gamma_2 n\Big],$$

where, Pearson's measure of skewness and kurtosis are termed as γ_1 & γ_2 and γ_4, γ_6 may be treated as measures of deviation from the normality. Also, disturbance term is distributed with mean zero and variance σ^2 and elements of error term are i.i.d.

Proof. Using equations (3.1), (3.8) and (3.18) the expression for expectation of u under given conditions of non-normality as derived here-under

(3.20) $E(u'u) = nE(a_i) = nE(u_i^2) = \sigma^2[n]$,

(3.21) $E(u'u)^2 = nE(a_i^2) + (n^2 - n)E(a_i a_j) = nE(a_i^2) + (n^2 - n)E(a_i)E(a_j)$

$$= n\sigma^4(\gamma_2 + 3) + (n^2 - n)\sigma^2\sigma^2 = \sigma^4\left[n(\gamma_2 + 3) + n^2 - n\right]$$
$$= \sigma^4\left[n\gamma_2 + n^2 + 2n\right],$$

(3.22) $E(u'u)^3 = nE(a_i^3) + (3n^2 - 3n)E(a_i^2 a_j) + (n^3 - 3n^2 + 2n)E(a_i a_j a_k)$

$$= nE(a_i^3) + (3n^2 - 3n)E(a_i^2)E(a_j^2) + (n^3 - 3n^2 + 2n)E(a_i)E(a_j)E(a_k)$$
$$= n\sigma^6(\gamma_4 + 15\gamma_2 + 10\gamma_1^2 + 15) + \sigma^6(3n^2 - 3n)(\gamma_2 + 3)$$
$$+\sigma^6(n^3 - 3n^2 + 2n)$$
$$= \sigma^6\left[\gamma_4 n + \gamma_2(3n^2 + 12n) + 10\gamma_1^2 n + 6n^2 + 8n\right],$$

(3.23) $E(u_1^2 + u_2^2 + ... + u_n^2)^4 = E\left[a_1 + a_2 + a_3 + ... + a_n\right]^4$

$$= nE\left[a_i^4\right] + (4n^2 - 4n)E\left[a_i^3 a_j^2\right] + (3n^2 - 3n)E\left[a_i^2 a_j^2\right]$$

$$+(6n^3 - 18n^2 + 12n)E\Big[a_i^2 a_j a_k\Big]$$
$$+(n^4 - 6n^3 + 11n^2 - 6n)E\Big[a_i a_j a_k a_l\Big]$$
$$= nE\Big[u_i^8\Big] + (4n^2 - 4n)E\Big[u_i^6 u_j^4\Big] + (3n^2 - 3n)E\Big[u_i^4 u_j^4\Big]$$
$$+(6n^3 - 18n^2 + 12n)E\Big[u_i^4 u_j^2 u_k^2\Big]$$
$$+(n^4 - 6n^3 + 11n^2 - 6n)E\Big[u_i^2 u_j^2 u_k^2 u_l^2\Big]$$
$$= nE\Big[u_i^8\Big] + (4n^2 - 4n)E\Big[u_i^6\Big]E\Big[u_j^4\Big] + (3n^2 - 3n)E\Big[u_i^4\Big]E\Big[u_j^4\Big]$$
$$+(6n^3 - 18n^2 + 12n)E\Big[u_i^4\Big]E\Big[u_j^2\Big]E\Big[u_k^2\Big]$$
$$+(n^4 - 6n^3 + 11n^2 - 6n)E\Big[u_i^2\Big]E\Big[u_j^2\Big]E\Big[u_k^2\Big]E\Big[u_l^2\Big]$$
$$= n\sigma^8\Big[n^4 + 12n^3 + 44n^2 + 48n + \gamma_2(6n^3 + 60n^2 + 144n)$$
$$+\gamma_4(4n^2 + 24n) + \gamma_6 n + \gamma_1^2(40n^2 + 240n) + \gamma_2^2(3n^2 + 32n)$$
$$+56\gamma_1\gamma_2 n\Big] + (4n^2 - 4n)[\sigma^6(\gamma_4 + 15\gamma_2 + 10\gamma_1^2 + 15)\sigma^2]$$
$$+(3n^2 - 3n)[\sigma^4(\gamma_2 + 3)]^2 + (6n^3 - 18n^2 + 12n)[\sigma^8(\gamma_2 + 3)]$$
$$+(n^4 - 6n^3 + 11n^2 - 6n)[\sigma^2\sigma^2\sigma^2\sigma^2]$$
$$= \sigma^8[n^4 - 6n^3 + 11n^2 - 6n + 105n + \gamma_2(6n^3 - 18n^2 + 12n)$$
$$+18n^3 - 54n^2 + 36n + \gamma_2^2(3n^2 - 3n) + 27n^2 - 27n$$
$$+\gamma_2(18n^2 - 18n) + \gamma_4(4n^2 - 4n) + \gamma_2(60n^2 - 60n)$$
$$+\gamma_1^2(40n^2 - 40n) + 60n^2 - 60n + \gamma_6 n + 28\gamma_4 n + 56\gamma_2\gamma_1 n$$
$$+35\gamma_2^2 n + 210\gamma_2 n + 280\gamma_1^2 n + 105n$$
$$= \sigma^8\Big[n^4 + 12n^3 + 44n^2 + 48n + \gamma_2(6n^3 + 60n^2 + 144n)$$
$$+\gamma_4(4n^2 + 24n) + \gamma_6 n + \gamma_1^2(40n^2 + 240n) + \gamma_2^2(3n^2 + 32n)$$
$$+56\gamma_1\gamma_2 n\Big].$$

Thus, we obtain the results of the ***Theorem* 3.2**.

4 Derivation of expression pertaining to integrals

The derivation of integral utilized in solving expressions pertaining to inversion of characteristic function are provided in this section. Further, to carry out the detailed derivations of integrals expression by utilizing integration by parts method, let us state following theorem:

***Theorem* 4.1** *The integration of I_1, I_2 and I_3 as follows*

(4.1) $$I_1 = \frac{1}{2\Pi}\int_{-\infty}^{\infty} te^{-at^2/2}e^{-\iota tW_1}dt = -\Big[\frac{\iota W_1}{a}\Big]f(W_1),$$
$$I_2 = \frac{1}{2\Pi}\int_{-\infty}^{\infty} t^2 e^{-at^2/2}e^{-\iota tW_1}dt = \Big[\frac{\iota^2 W_1^2}{a^2}\Big]f(W_1),$$
$$I_3 = \frac{1}{2\Pi}\int_{-\infty}^{\infty} t^3 e^{-at^2/2}e^{-\iota tW_1}dt == -\iota\Big[\frac{3W_1}{a^2} - \frac{W_1^3}{a^3}\Big]f(W_1),$$

where,

(4.2) $$\frac{1}{2\Pi}\int_{-\infty}^{\infty} e^{-at^2/2}e^{-\iota tW_1}dt = f(W_1),$$

given that W_1 follows normal distribution with mean zero and variance (σ^2) equals to a, such that $a > 0$.

Proof. In order to proof results at (4.1), let us write $f(x_1)$, $f(x_2)$ and $f(x_3)$ as

(4.3) $f(x_1) = \int te^{-at^2/2},$

$f(x_2) = \int t^2 e^{-at^2/2},$

$f(x_3) = \int t^3 e^{-at^2/2},$

where, integration of $f(x_1)$, $f(x_2)$ and $f(x_3)$ can be derived as given by

$$\begin{aligned}
(4.4) \qquad \int f(x_1) &= \int te^{-at^2/2} \\
&= \int e^{-au/2}\tfrac{du}{2} \qquad (Let \quad t^2 = u \implies 2tdt = du) \\
&= \frac{1}{2}\Big[\frac{e^{-au/2}}{-a/2}\Big] \\
&= -\frac{e^{-au/2}}{a} \\
&= -\frac{1}{a}e^{-at^2/2} + C,
\end{aligned}$$

$$\begin{aligned}
(4.5) \quad 3cmf(x_2) &= \int t^2 e^{-at^2/2}dt \\
&= t^2\Big[\frac{e^{-at^2/2}}{-at}\Big] - 2\int \tfrac{e^{-at^2/2}}{-a}dt \\
&= -\frac{t}{a}e^{-at^2/2} - \frac{2}{a^2}\frac{e^{-at^2/2}}{t} + C,
\end{aligned}$$

$$\begin{aligned}
(4.6) \qquad f(x_3) &= \int t^3 e^{-at^2/2}dt \\
&= t^3\Big[\frac{e^{-at^2/2}}{-at}\Big] - \int 3t^2 \tfrac{e^{-at^2/2}}{-at}dt \\
&= -\frac{t^2}{a}e^{-at^2/2} - \frac{3}{a^2}e^{-at^2/2} + C.
\end{aligned}$$

Further, let $g(x) = \cos(tW_1) - \iota\sin(tW_1)$ and derivative of $g(x)$ is given by

$$\begin{aligned}
(4.7) \quad g'(x) &= \frac{d}{dx}[(\cos(tW_1) - \iota\sin(tW_1))]dt \\
&= -W_1\sin(tW_1) - \iota W_1\cos(tW_1) \\
&= -\iota W_1 g(x).
\end{aligned}$$

We may re-write I_1 as

$$\begin{aligned}
(4.8) \quad I_1 &= \frac{1}{2\Pi}\int_{-\infty}^{\infty} te^{-at^2/2}e^{-\iota tW_1}dt \\
&= \frac{1}{2\Pi}\int_{-\infty}^{\infty} te^{-at^2/2}[\cos(tW_1) - \iota\sin(tW_1)]dt \\
&= \frac{1}{2\Pi}\int_{-\infty}^{\infty} f(x_1)g(x),
\end{aligned}$$

where, $f(x_1) = \int te^{-at^2/2}$ and $g(x) = \cos(tW_1) - \iota\sin(tW_1)$ as defined in equations (4.3)) and (4.7). Further, employing integration by parts and utilizing results (4.4) and (4.7), we derive

$$\begin{aligned}
(4.9) \quad I_1 &= \frac{1}{2\Pi}\int_{-\infty}^{\infty} g(x)f(x_1)dt \\
&= \frac{1}{2\Pi}\Big[g(x)\int_{-\infty}^{\infty} f(x_1)dt - \int_{-\infty}^{\infty} g'(x)\int_{-\infty}^{\infty} f(x_1)dtdt\Big] \\
&= \frac{1}{2\Pi}\Big[-g(x)\frac{1}{a}e^{-at^2/2}\Big|_{-\infty}^{+\infty} - \frac{\iota W_1}{2\Pi}\frac{1}{a}\int_{-\infty}^{\infty} g(x)e^{-at^2/2}dt\Big] \\
&= \frac{1}{2\Pi}\Big[-\frac{1}{a}g(x)e^{-at^2/2}\Big|_{-\infty}^{+\infty} - \frac{\iota W_1}{2\Pi}\frac{1}{a}\int_{-\infty}^{\infty} g(x)e^{-at^2/2}dt\Big] \\
&= -\Big[\frac{\iota W_1}{a}\Big]f(W_1).
\end{aligned}$$

Similarly

(4.10)

$$
\begin{aligned}
I_2 =& \frac{1}{2\Pi}\int_{-\infty}^{\infty} t^2 e^{-at^2/2} e^{-\iota t W_1} dt\\
=& \frac{1}{2\Pi}\int_{-\infty}^{\infty} g(x) f(x_2) dt\\
=& \frac{1}{2\Pi}\Big[g(x)\int_{-\infty}^{\infty} f(x_2)dt + \iota W_1 \int_{-\infty}^{\infty} g(x)\Big(-\tfrac{t}{a}e^{-at^2/2} - \tfrac{2}{a^2}\tfrac{e^{-at^2/2}}{t}\Big)\Big]dt\\
=& \frac{1}{2\Pi}\Big[-\frac{\iota W_1}{a}\int_{-\infty}^{\infty} g(x) t e^{-at^2/2} dt - \tfrac{2\iota W_1}{a^2}\int_{-\infty}^{\infty} g(x)\tfrac{e^{-at^2/2}}{t}\Big]\\
=& \frac{1}{2\Pi}\Big[-\frac{-\iota W_1}{a}\Big(2\Pi\frac{-\iota W_1}{a} f(W_1)\Big)\Big]\\
=& \Big[\frac{\iota^2 W_1^2}{a^2}\Big] f(W_1).
\end{aligned}
$$

Using (4.7) and (4.8), we obtain

(4.11)

$$
\begin{aligned}
I_3 =& \frac{1}{2\Pi}\int_{-\infty}^{\infty} t^3 e^{-at^2/2} e^{-\iota t W_1} dt\\
=& \frac{1}{2\Pi}\int_{-\infty}^{\infty} g(x) f(x_3) dt\\
=& \frac{1}{2\Pi}\Big[g(x)\int_{-\infty}^{\infty} f(x_3)dt - \int_{-\infty}^{\infty} g'(x)\int_{-\infty}^{\infty} f(x_3)dtdt\Big]\\
=& + \iota W_1 \int_{-\infty}^{\infty} g(x)\Big[\tfrac{-t^2}{a}e^{-at^2/2} - \tfrac{3}{a^2}e^{-at^2/2}\Big]dt\\
=& -\Big[\frac{\iota W_1}{2\Pi a}\Big]\int_{-\infty}^{\infty} t^2 e^{-\iota t W_1} e^{-at^2/2} dt - \Big[\tfrac{3\iota W_1}{2\Pi a^2}\Big]\int_{-\infty}^{\infty} e^{-\iota t W_1} e^{-at^2/2} dt\\
=& -\frac{iW_1}{a}\Big[\frac{\iota^2 W_1^2}{a^2}\Big] f(W_1) - \Big[\frac{3\iota W_1}{a^2}\Big] f(W_1)\\
=& -\iota\Big[\frac{3W_1}{a^2} - \frac{W_1^3}{a^3}\Big] f(W_1).
\end{aligned}
$$

Thus, we obtain the results of ***Theorem*** **4.1**.

5 Conclusion

The present work provide insights to the algebraic expression, utilized for the deducing the several important results available in literature, by providing step-wise derivation.

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(Dedicated to Honor Dr. R. C. Singh Chandel on His 75^{th} Birth Anniversary Celebrations)

DYNAMICS OF AN SIRS EPIDEMIC MODEL WITH PARTICULAR NON-LINEAR INCIDENCE RATE AND MEDIA EFFECTS

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Abstract

This paper deals with the nonlinear dynamics of Susceptible, Infected, Recovered epidemic model with a particular non linear incidence rate and the consequences of media awareness program. The model analysis shows that the spread of an infectious disease can be controlled by using awareness programs. Different equilibrium points and their stability are discussed. The basic reproduction number R_0 is obtained. We also apply Lasalle's invariance principle to show that the disease-free equilibrium is globally asymptotically stable if $R_0 < 1$ and we use a geometric approach to find out the global stability of the endemic equilibrium. In addition, to our analytical results, several numerical simulations are also illustrated.
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1 Introduction

Mathematical models of infectious diseases are proven to be very important in better understanding of epidemiological patterns and disease control in human populations. In the study of the epidemiological models, incidence rate play a very important role while controlling the transmission of infectious diseases. Epidemic models with different types of incidence rate have been studied and developed by many authors. In order to model this disease transmission process many authors use the incidence functions: The earliest one is the bilinear incidence rate λSI used by Kermack and Mckendrick [7] in 1927. In 1978, Capasso and Serio [3] introduced a saturated incidence rate $\frac{\lambda I}{1+\alpha I}$ by research of the Cholera epidemic spread in Bari. Also in 1978, May and Anderson [1] proposed the saturated incidence rate $\frac{\lambda SI}{1+\alpha S}$. The general incidence rate $\frac{\lambda I^p S}{1+\alpha I^q}$ was proposed by Liu et. al. [10, 11] in 1986-87, Derick and Ven Den Driessche [4] in 1993, etc. Ruan and Wang [17] studied an epidemic model with a specific nonlinear incidence rate $\frac{\lambda I^2 S}{1+\alpha I^2}$ and presented a detailed quantitative analysis and bifurcation analysis and Bogdanov-Takens bifurcation for the model in 2003. Xiao and Zhou [20] considered the non-monotone incidence rate $\frac{\lambda IS}{1+\beta I+\alpha I^2}$ in 2006. To model the effects of psychological factor, protection measures and intervention policies when a serious disease emerges, Xiao and Ruan [21] proposed the specific incidence rate $\frac{\lambda IS}{1+\alpha I^2}$ in 2007.

Controlling infectious diseases has been an increasingly more and more complex issue in recent years. In the field of epidemiology, treatment, vaccination, isolation, media awareness program and many more play a crucial role in controlling the disease spread. Media is can be helpful to develop the awareness among in common people regarding the rich nature of the disease, make people knowledgeable about the disease to take precautions such as social distancing, wearing protective masks, vaccination etc., to reduce their probabilities of being infected and some other impacts. It is observed that, media coverage gives rise to healthy behaviour among the population. Few research works on media coverage can be found in [8, 9, 12, 16, 18]. However, mathematical models to study the disease transmission dynamics together with media effect is still largely remain unexplored.

In the present research, we intend to study the influence of media coverage to control and eradicate the disease with a particular non-linear incidence function $U(S,I) = \frac{\lambda SI}{1+\alpha_1 S+\alpha_2 I^2}$ used by [2], which has the property of being saturated with infectives as well as with susceptible individuals. We can see that

$$\frac{\partial U(S,I)}{\partial I} = \frac{\lambda S(1+\alpha_1 S-\alpha_2 I^2)}{(1+\alpha_1 S+\alpha_2 I^2)^2},$$

which is positive when $I^2 < \frac{(1+\alpha_1 S)}{\alpha_2}$ and negative when $I^2 > \frac{(1+\alpha_1 S)}{\alpha_2}$. Hence, $U(S,I)$ is a non-monotonic function with respect to I, since it increases when the number of infectives is relatively small but decreases as the number of infectives becomes larger. On the other hand

$$\frac{\partial U(S,I)}{\partial S} = \frac{\lambda I(1+\alpha_2 I^2)}{(1+\alpha_1 S+\alpha_2 I^2)^2} > 0,$$

so $U(S,I)$ grows monotonically with respect to susceptibles.

This kind of non-linear and non-monotonic incidence function models the idea that, at the beginning of the infection, the population has little awareness of preventive measures, so the contact rate increases rapidly. As time advances, media reporting on early stage symptoms of the disease, the population becomes more aware of the risk and takes measures to control or eradicate the disease, so the number of infectious contacts decreases.

This manuscript is organized as follows: In ***Sect.*2**, $SIRS$ model is presented. In ***Sect.*3**, basic properties of solutions are discussed. In ***Sect.*4**, we calculate the basic reproduction number then in ***Sect.*5**, we determine all possible equilibria of model. In ***Sect.*6**, we discuss and analyze the local stability of the equilibriums. In ***Sect.*7**, we discuss and analyze the global stability of the equilibriums. We present in ***Sect.*8**, some numerical examples of the dynamics of the model. Finally, in ***Sect.*9**, swe discussed the conclusion.

2 Model Formulation

In this section, deterministic nonlinear $SIRS$ model is considered by taking media awareness and particular incidence rate into account. The variables and parameters of the model are described in ***Table* 2.1** and ***Table* 2.2** respectively.

***Table* 2.1:** Description of the model state variables.

State variables	Description
$S(t)$	Number of susceptible individuals at time t
$I(t)$	Number of infected individuals at time t
$R(t)$	Number of recovered individuals at time t
$N(t)$	The total population size at time t

To model the situation considered a region with total population $N(t)$ at any instant of time t. By taking into account the aforementioned considerations, the system of equations that capture the dynamics of the infectious disease is designed and the ordinary differential equations of the system (2.1) is as follows.

$$(2.1)\quad \begin{cases} \dfrac{dS}{dt} = a - dS - \dfrac{\lambda SI}{1+\alpha_1 S+\alpha_2 I^2} + \beta R - pSM \\ \dfrac{dI}{dt} = \dfrac{\lambda SI}{1+\alpha_1 S+\alpha_2 I^2} - (d+\delta)I \\ \dfrac{dR}{dt} = \delta I - (d+\beta)R + pSM, \end{cases}$$

whose state space is the first quadrant $R_3^+ = \{(S,\ I,\ R) : S \geq 0,\ I \geq 0,\ R \geq 0\}$ and subject to the initial conditions $S(0) = S_0 \geq 0$, $I(0) = I_0 \geq 0$, $R(0) = R_0 \geq 0$. It is assumed that all the parameters are positive.

***Table* 2.2:** Description of the model parameters.

State parameters	Description
a	Recruitment rate of the population
d	The natural death rate of the population
λ	The effective contact rate
$\alpha_1 \& \alpha_2$	The parameter measures of the psychological or inhibitory effect
β	The rate at which recovered individuals lose immunity and return to susceptible class
δ	The natural recovery rate of infection
p	The dissemination rate of awareness among unaware susceptible due to which they form a different class
M	The media control parameter (fixed)

3 Basic Properties of the Model

Summing up the four equations of model (2.1) and denoting

$$N(t) = S(t) + I(t) + R(t),$$

having

$$N'(t) = a - dN.$$

If disease is not present, then $N'(t) = a - dN$. This shows that population size $N \to \frac{a}{d}$ as $t \to \infty$. It follows that the solutions of model (2.1) exists in the region defined by

$$\Omega = \{(S,\ I,\ R) \in R_4^+ : S,\ I,\ R \geq 0,\ S + I + R \leq a/d\}. \tag{3.1}$$

This gives the following lemma which shows that the solutions of model (2.1) are bounded, continuous for all positive time and lie in a compact set.

***Lemma* 3.1** *The set* Ω *defined in* (3.1) *is a positively invariant region for model* (2.1)*. Moreover, every trajectory of model* (2.1) *is eventually staying in a compact subset of* Ω*.*

4 Basic Reproductive Number

The basic reproduction number sometimes called basic reproductive rate or basic reproductive ratio is one of the most useful threshold parameters which characterize mathematical problems concerning infectious diseases. This metric is useful because it helps determine whether or not an infectious disease will spread through a population. In this section, we will calculate the basic reproduction number R_0 of system (2.1) by using the next-generation matrix method described in [19]. For that, we rewrite model (2.1) as

$$\frac{dx}{dt} = F(x) - \mathfrak{V}(x),$$

where $x = (I, R, S)$,

$$F(x) = \begin{pmatrix} \frac{\lambda S I}{1+\alpha_1 S+\alpha_2 I^2} \\ 0 \\ 0 \end{pmatrix} \text{ and } \mathfrak{V}(x) = \begin{pmatrix} (\gamma + \delta + d + d_1)I \\ -\delta I + (\mu + d + d_2)Q \\ -\gamma I - \mu Q + dR \\ -A + dS + \frac{\beta S I}{(1+\alpha_1 S)(1+\alpha_2 I)} \end{pmatrix}.$$

We calculate the Jacobian matrices for $F(x)$ and $\mathfrak{V}(x)$ at the disease-free equilibrium $x_0 = (0, 0, a/d + pM)$.

$$F = \begin{pmatrix} \frac{\lambda a}{\alpha_1 a+d+pM} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, V = \begin{pmatrix} d+\delta & 0 & 0 \\ -\delta & d+\beta & -pM \\ \frac{\lambda a}{d+pM+\alpha_1 a} & -\beta & d+pM \end{pmatrix}.$$

FV^{-1} is the next generation matrix for model (2.1). It then follows that the spectral radius of matrix FV^{-1} is $\rho(FV^{-1}) = \frac{\lambda a}{(a\alpha_1+d+pM)(d+\delta)}$. Thus, the basic reproduction number of model (2.1) is

$$R_0 = \frac{\lambda a}{(a\alpha_1 + d + pM)(d + \delta)}.$$

5 Existence of Equilibria

In this section, we obtain the existence of the disease-free equilibrium E_0 and the endemic equilibrium E^* of model (2.1).

Set the right sides of model (2.1) equal zero, that is,

$$(5.1)\quad \begin{cases} a - dS - \dfrac{\lambda SI}{1+\alpha_1 S+\alpha_2 I^2} + \beta R - pSM = 0 \\ \dfrac{\lambda SI}{1+\alpha_1 S+\alpha_2 I^2} - (d+\delta)I = 0 \\ \delta I - (d+\beta)R + pSM = 0. \end{cases}$$

The model (2.1) always has the disease-free equilibrium point $E_0(\frac{a}{d+pM}, 0, 0)$. Solving (5.1) we also get a unique positive, endemic equilibrium point $E^*(S^*, I^*, R^*)$ of the model (2.1), where

$$S^* = \frac{(d+\delta)(1+\alpha_2 I^2)}{\lambda - \alpha_1(d+\delta)},$$
$$R^* = \frac{\delta I + pSM}{(d+\beta)},$$

and I^* is given as a root of the quadratic equation $\Omega_1 I^2 + \Omega_2 I + \Omega_3 = 0$,
where

$$\Omega_1 = -\alpha_2 d(d+\delta)(d+\beta+pM),$$
$$\Omega_2 = -(d^2+d\beta+\delta d)(\lambda - \alpha_1 d - \alpha_1\delta),$$
$$\Omega_3 = a(d+\beta)[\lambda-(d+\delta)\alpha_1] + (d+\delta)[-pMd - d(d+\beta)].$$

Now,

$$I^* = \frac{(d^2+d\beta+\delta d)(\lambda-\alpha_1 d-\alpha_1\delta) + \sqrt{\Delta}}{-2\alpha_2 d(d+\delta)(d+\beta+pM)},$$

where,

$$\Delta^2 = [-(d^2+d\beta+\delta d)(\lambda-\alpha_1 d-\alpha_1\delta)]^2 - 4[\{-\alpha_2 d(d+\delta)(d+\beta+pM\}a(d+\beta)\times$$
$$\{\lambda-(d+\delta)\alpha_1\} + (d+\delta)\{-pMd - d(d+\beta)\}].$$

6 Local Stability Analysis

In this section, we study the local stability of the disease-free equilibrium E_0 and the endemic equilibrium E^* of model (2.1).

***Theorem* 6.1** *If $R_0 < 1$, the disease-free equilibrium E_0 of model* (2.1) *is locally asymptotically stable. If $R_0 > 1$, the disease-free equilibrium E_0 is unstable.*

Proof. The Jacobian matrix of model (2.1) at the disease-free equilibrium E_0 is

$$J(E_0) = \begin{pmatrix} -d-pM & \dfrac{-\lambda a}{d+pM+\alpha_1 a} & \beta \\ 0 & \dfrac{\lambda a}{d+pM+\alpha_1 a} - (d+\delta) & 0 \\ pM & \delta & -(d+\beta) \end{pmatrix}.$$

The characteristic equation of $J(E_0)$ is

$$\left\{\{\frac{\lambda a}{d+pM+\alpha_1 a} - d - \delta - \mu\}\{\mu^2 + (2d+pM+\beta)\mu + (d^2+d\beta+pMd)\}\right\} = 0.$$

Clearly, the one eigenvalue $\mu_1 = \frac{\lambda a}{d+pM+\alpha_1 a} - (d+\delta)$ and other two eigenvalues are given by the quadratic equation

$$\mu^2 + (2d+pM+\beta)\mu + (d^2+d\beta+pMd) = 0$$

or

$$\mu^2 + \mu\psi_1 + \psi_2 = 0, A_0 \neq 0,$$

where $\psi_1 = 2d + pM + \beta$, $\psi_2 = d^2 + d\beta + pMd$.

By Routh-Hurwitz criteria, we know that the model is stable if $\psi_1 > 0$ and $\psi_2 > 0$, while $\mu_1 < 0$ for $R_0 < 1$ and $\mu_1 > 0$ for $R_0 > 1$.

Hence E_0 is locally asymptotically stable for $R_0 < 1$, while it is unstable for $R_0 > 1$.

***Theorem* 6.2** *If* $R_0 > 1$*, the endemic equilibrium* E^* *of model* (2.1) *is locally asymptotically stable.*

Proof. Consider

$$J(E^*) = \begin{pmatrix} -V_1 - d - pM & -V_2 & \beta \\ V_1 & V_2 - (d+\delta) & 0 \\ pM & \delta & -(d+\beta) \end{pmatrix},$$

where $V_1 = \frac{(1+\alpha_1 S^* + \alpha_2 I^{*2})\lambda I^* - \lambda S^* I^* \alpha_1}{(1+\alpha_1 S^* + \alpha_2 I^{*2})^2}$, $V_2 = \frac{(1+\alpha_1 S^* + \alpha_2 I^{*2})\lambda S^* - 2\lambda S^* I^{*2}\alpha_2}{(1+\alpha_1 S^* + \alpha_2 I^{*2})^2}$.

The characteristic equation of $J(E^*)$ is

$$\mu^3 + \mu^2 A_1 + \mu A_2 + A_3 = 0, A_0 \neq 0,$$

where

$$A_1 = (3d + \delta + \beta + pM + v_1 - v_2),$$
$$A_2 = (d^2 + \beta v_1 + pMv_2 + 2\beta pM + 2d\beta + \beta\delta - \delta v_1 - \beta v_2 - \delta pM),$$
$$A_3 = (d+\beta)(dv_2 + pMv_2 - d^2 - dpM - dv_1 - d\delta - \delta pM - \delta v_1) + \beta v_1 \delta + \beta pM(d + \delta - v_2).$$

We know that $A_1 > 0$ if $3d+\delta+\beta+pM+v_1 > v_2$ and $A_2 > 0$ if $(d^2+\beta v_1+pMv_2+2\beta pM+2d\beta+\beta\delta) > (\delta v_1+\beta v_2+\delta pM)$. By Routh-Hurwitz criteria, endemic equilibrium E^* of model (2.1) is locally asymptotically stable if and only if $A_1 > 0$, $A_2 > 0$ and $A_1 A_2 > A_0 A_3$.

7 Global Stability Analysis

In this section, we study the global stability of the disease-free equilibrium E_0 and the endemic equilibrium E^* of model (2.1).

***Theorem* 7.1** *If* $R_0 < 1$*, the disease-free equilibrium* E_0 *of model* (2.1) *is globally asymptotically stable.*

Proof. We prove the global stability of the model (2.1) at the equilibrium E_0 when $R_0 < 1$. Taking the Lyapunov function

$$V(S, I, R) = I(t).$$

Calculating the derivative of $V(t)$ along the positive solution of model (2.1), it follows that

$$\frac{dV}{dt} = \frac{dI}{dt} = \frac{\lambda SI}{(1 + \alpha_1 S + \alpha_2 I^2)} - (d+\delta)I.$$

Since the incidence function

$$\frac{\lambda SI}{(1 + \alpha_1 S + \alpha_2 I^2)} \leq \frac{\frac{\lambda Ia}{d+pM}}{1 + \frac{\alpha_1 a}{d+pM} + \alpha_2 I^2}$$

for $0 \leq S \leq \frac{a}{d+pM}$,

$$\dot{V}(t) \leq \left[\frac{\lambda a}{(d + pM + \alpha_1 a)} - (d+\delta)\right] I$$
$$= (d+\delta)[R_0 - 1] I \leq 0.$$

Furthermore, $\dot{V} = 0$ only if $I = 0$, so the largest invariant set contained $\{(S,\ I,\ R) \in \Omega : \dot{V} = 0\}$ is the plane $I = 0$. By Lassalle's invariance principle [13], this implies that all solution in Ω approach the plane $I = 0$ as $t \to \infty$. On the other hand, solutions of (2.1) contained in such plane satisfy $\frac{dS}{dt} = a - dS + \beta R - pSM$, $\frac{dR}{dt} = -(d+\beta)R + pSM$, which implies that $S \to \frac{a}{d+pM}$ and $R \to 0$ as $t \to \infty$, that is, all of these solutions approach E_0 is globally asymptotically stable in Ω.

Next, we analysis the global stability of an endemic equilibrium E^* by using geometric approach method described by Li and Muldowney in [14]. For that, we need to consider a parameter

$$w = \max\left\{ \begin{array}{c} -pM - \frac{\lambda I}{1+\alpha_1 S+\alpha_2 I^2}\left(1 - \frac{S(\alpha_1 - 2\alpha_2 I)}{1+\alpha_1 S+\alpha_2 I^2}\right) + \beta, \delta(2-p) - \beta - \\ \frac{\lambda}{1+\alpha_1 S+\alpha_2 I^2}\left(S + I - \frac{SI\alpha_1}{1+\alpha_1 S+\alpha_2 I^2}\right), -\beta + \delta - \frac{-2\lambda S\alpha_2 I^2}{1+\alpha_1 S+\alpha_2 I^2} \end{array} \right\},$$

and we will make use of the following ***Theorem***.

***Theorem* 7.2** *(**Li Muldowney** [14]). Suppose that the system $x' = f(x)$, with $f : D \subset \mathbb{R}^n \to \mathbb{R}^n$, satisfies the following:*
($H1$) D is a simply connected open set,
($H2$) there is a compact absorbing set $K \subset D$,
($H3$) x^* is the only equilibrium in D.
Then the equilibrium x^* is globally stable in D if there exists a Lozinskiĭ measure ηsuch that

(7.1) $\lim_{t\to\infty} \sup \sup_{x_0\in K} \frac{1}{t}\int_0^t \eta(B(x(s,x_0)))ds < 0,$

(7.2) $B = P_f P^{-1} + PJ^{[2]}P^{-1}$

and $Q \to Q(x)$ is an $\binom{n}{2}\times\binom{n}{2}$ matrix valued function.

In our case, model (2.1) can be written as $x' = f(x)$ with $f : D \subset \mathbb{R}^n \to \mathbb{R}^n$ and D being the interior of the feasible region Ω. The existence of a compact absorbing set $K \subset D$ is equivalent to proving that (2.1) is uniformly persistent (see [14, 5]) and the proof for this in the case when $R_0 > 1$ is similar to that of proposition 4.2 of [14]. Hence, ($H1$) and ($H2$) hold for system (2.1), and by assuming the uniqueness of the endemic equilibrium in D, we can prove its global stability with the aid of ***Theorem* 7.2**.

***Theorem* 7.3** *If $R_0 > 1$, $d < w$ and the endemic equilibrium E^* of system* (2.1) *is unique, then E^* is globally asymptotically stable in the feasible region* Ω.

Proof. Let J be the Jacobian matrix of the system (2.1). Then the second additive compound matrix [15] of J is given by

$$J^{[2]} = \begin{pmatrix} J_{11}+J_{22} & J_{23} & -J_{13} \\ J_{32} & J_{11}+J_{33} & J_{12} \\ -J_{31} & J_{21} & J_{22}+J_{33} \end{pmatrix},$$

$$J^{[2]} = \begin{pmatrix} -d-pM-v_1+v_2-(d+\delta) & 0 & -\beta \\ \delta & -d-pM-v_1+(-d-\beta) & -v_2 \\ -pM & v_1 & v_2-(d+\delta)-(d+\beta) \end{pmatrix},$$

where, $v_1 = \frac{(1+\alpha_1 S+\alpha_2 I^2)\lambda I-\lambda S I\alpha_1}{(1+\alpha_1 S+\alpha_2 I^2)^2}$, $v_2 = \frac{(1+\alpha_1 S+\alpha_2 I^2)\lambda S-2\lambda S I^2\alpha_2}{(1+\alpha_1 S+\alpha_2 I^2)^2}$.

Let P be the matrix-valued function defined by $P = P(S,I,R) = \mathrm{diag}(\frac{S}{I},\frac{S}{I},\frac{S}{I})$; then P is C^1 and non-singular in the interior of Ω, $P_f = \mathrm{diag}(\frac{S'I-SI'}{I^2},\frac{S'I-SI'}{I^2},\frac{S'I-SI'}{I^2})$ and $P^{-1} = \mathrm{diag}(\frac{I}{S},\frac{I}{S},\frac{I}{S})$, $P_fP^{-1} = \mathrm{diag}(\frac{S'}{S}-\frac{I'}{I},\frac{S'}{S}-\frac{I'}{I},\frac{S'}{S}-\frac{I'}{I})$ and $B = P_fP^{-1} + PJ^{[2]}P^{-1}$. Then B can be written in the block form

$$B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix},$$

with $B_{11} = \frac{S'}{S} - \frac{I'}{I} - d - pM - v_1 + v_2 - (d+\delta)$, $B_{12} = (0, -\beta)$, $B_{21} = \begin{pmatrix} \delta \\ -pM \end{pmatrix}$ and

$$B_{22} = \begin{pmatrix} \frac{S'}{S}-\frac{I'}{I}-d-pM-v_1+(-d-\beta) & -v_2 \\ v_1 & \frac{S'}{S}-\frac{I'}{I}+v_2-(d+\delta)-(d+\beta) \end{pmatrix}.$$

Consider the vector norm in R^3 defined by $\|(u,v,w)\| = \max\{|u|, |v|+|w|\} \in R^3$ and let $\eta_1(B)$ be the *Lozinskiĭ* measure with respect to this norm. Then

(7.3) $\eta_1(B) \le \sup\{g_1, g_2\}$,

where, $g_1 = (B_{11}) + |B_{12}|$, $g_2 = \mu(B_{22}) + |B_{21}|$, $|B_{12}|$ and $|B_{21}|$ denote the matrix norm with respect to l_1 vector norm in norm R^2 and η_1 is the *Lozinskiĭ* measure of B_{22} with respect to l_1 vector norm in R^2. We have $|B_{12}| = 0$, $|B_{21}| = \delta$, $\mu(B_{22}) = \frac{S'}{S} - \frac{I'}{I} - d + \max\{-pM - v_1 - d - \beta, v_2 - d - \delta - \beta\}$. From the second equation in the system (2.1), we have

$$\frac{I'}{I} = \frac{\beta S}{(1+\alpha_1 S)(1+\alpha_2 I)} - (\gamma+\delta+d+d_1+q).$$

Therefore,

$$\mu(B_{22}) = g_2 = S'/S - d + \max\left\{\delta - pM - \beta - \frac{\lambda}{(1+\alpha_1 S+\alpha_2 I^2)}(S + I - \frac{SI\alpha_1}{(1+\alpha_1 S+\alpha_2 I^2)})\right.$$

$$-\beta-\frac{2\lambda SI^2\alpha_2}{(1+\alpha_1 S+\alpha_2 I^2)^2}\Big\}.$$

Then

$$g_1=S'/S-(d+pM)-\frac{\lambda I}{(1+\alpha_1 S+\alpha_2 I^2)}(1-\frac{S[(\alpha_1-2\alpha_2 I)]}{[(1+\alpha_1 S+\alpha_2 I^2)]}+\beta,)$$

$$g_2=S'/S-d+\max\Big\{\delta-pM-\beta-\frac{\lambda}{(1+\alpha_1 S+\alpha_2 I^2)}(S+I-\frac{SI\alpha_1}{(1+\alpha_1 S+\alpha_2 I^2)})$$

$$-\beta-\frac{2\lambda SI^2\alpha_2}{(1+\alpha_1 S+\alpha_2 I^2)^2}\Big\}+\delta.$$

By (7.3), this implies that

$$\eta_1(B)\le S'/S-d+\max\left\{\begin{array}{c}-pM-\frac{\lambda I}{1+\alpha_1 S+\alpha_2 I^2}(1-\frac{S(\alpha_1-2\alpha_2 I)}{1+\alpha_1 S+\alpha_2 I^2})+\beta,\delta(2-p)-\beta-\\ \frac{\lambda}{1+\alpha_1 S+\alpha_2 I^2}(S+I-\frac{SI\alpha_1}{1+\alpha_1 S+\alpha_2 I^2}),-\beta+\delta-\frac{-2\lambda S\alpha_2 I^2}{1+\alpha_1 S+\alpha_2 I^2}\end{array}\right\}$$

$$=S'/S-(d-w).$$

By integrating both sides at the same time, we obtain

$$\frac{1}{t}\int_0^t\eta_1(B)\mathrm{ds}\le\tfrac{1}{t}\mathrm{In}\tfrac{S(t)}{S(0)}-(d-w).$$

Thus

$$\lim_{t\to\infty}\sup\sup\frac{1}{t}\int_0^t\eta_1(B)\mathrm{ds}\le-(d-w)$$

and therefore,

$$\lim_{t\to\infty}\sup\sup\frac{1}{t}\int_0^t\eta_1(B)\mathrm{ds}<0,$$

provided $d>w$. Hence, E^* is globally asymptotically stable in Ω.

8 Numerical Simulations

In this section, we will give some numerical examples to illustrate our main results by using Milstein's Higher Order Method [6]. All simulations are done using the function ode45, which is *MATLAB's* standard solver for ordinary differential equations (*ODEs*).

As the present study is not a case study, no real data are available. Hence, the choice of parametric values is hypothetical with appropriate units and does not base on data. They are chosen only for illustrative purpose. Because the parametric values are not related to a specific disease, system (2.1) can be considered to be dimensionless. The interval of time is supposed to be [0, 50], while the various set of initial size of population are assumed to be $(S(0), I(0), R(0)) = (50, 40, 30)$. Here we present some numerical examples to discuss the effect of the choice of the parameters, nonlinear incidence rate and media awareness effect on the basic reproduction number R_0. For simulations, we take the set of parameters as shown in ***Table* 8.1** and ***Table* 8.2**.

***Table* 8.1:** Parameters used for simulation purpose when $R_0 = 0.09375 < 1$.

Symbol	a	β	d	δ	λ	M	p	α_1	α_2
Value	15	0.5	0.01	0.9	0.5	0.8	20	0.7	0.1

***Table* 8.2:** Parameters used for simulation purpose when $R_0 = 3.024194 > 1$.

Symbol	a	β	d	δ	λ	M	p	α_1	α_2
Value	1.0	0.1	0.1	0.4	2.0	0.8	0.78	0.1	0.7

For this simulation, we take the set of parameters as shown in ***Table* 8.1**. In this case, $S(t)$ approaches to its steady state value while $I(t)$, $Q(t)$ and $R(t)$ approaches to zero as $t\to\infty$. Hence the disease disappears and dies out. (***Fig.* 8.1**).

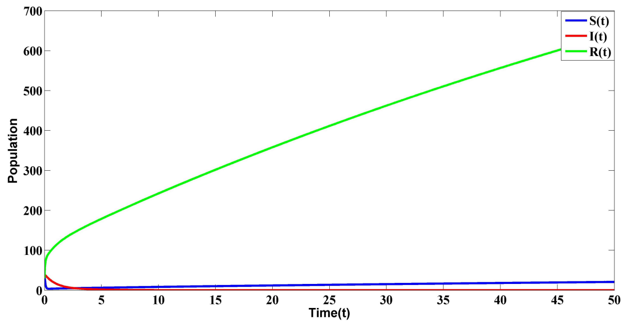

***Figure* 8.1:** The figure represents that the disease dies out.

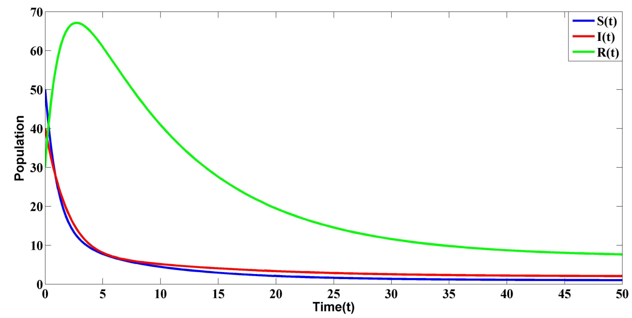

***Figure* 8.2:** The figure represents that the disease endemic.

For these simulations, we take the set of parameters as shown in ***Table* 8.2**. Here, ***Fig.* 8.2** present $S(t)$, $I(t)$ and $R(t)$ all approaches to their steady state values as $t \to \infty$. Hence the disease becomes endemic.

Fig.* 8.3** we present the variation of susceptible class when media effect is applied. The main importance of applying media effect can be observed in ***Fig.* 8.4**, where we draw the variations of infected individuals. It is observed that when media effect is applied optimally, the infected class population remains the least.Fig.* 8.5** represents the variation of recovered class of population. Thus the ***Figs.* 8.3-8.5** represent the behavioral change of all classes of population as time evolves. ***Fig.* 8.6**, represents the phase portrait in $SIRS$-space with different initial conditions. This phase diagram shows that $\lim_{t\to\infty}(S(t), I(t), R(t)) = (S^*, I^*, R^*)$ for $R_0 > 1$.

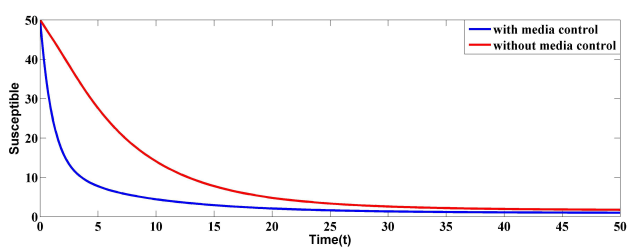

***Figure* 8.3:** Variation of the susceptible population for media control.

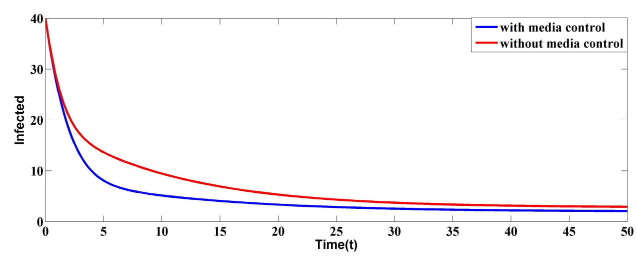

***Figure* 8.4:** Variation of the infected population for media control.

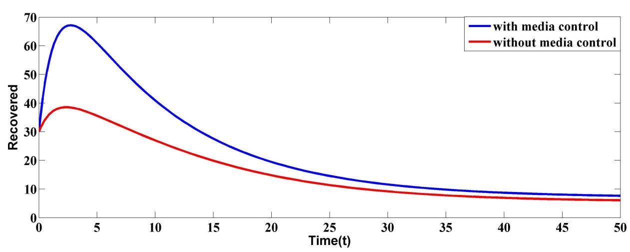

***Figure* 8.5:** Variation of the recovered population for media control.

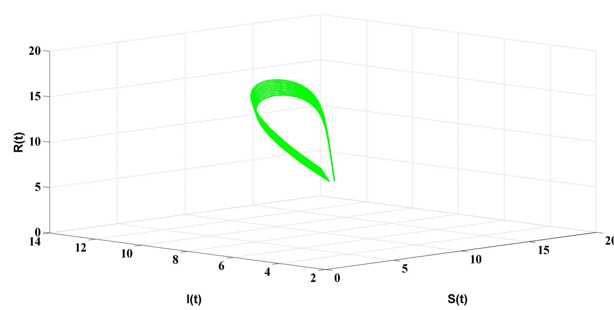

***Figure* 8.6:** The phase diagram at different initial values endemic equilibrium.

9 Discussions and Conclusions

This paper presented a mathematical study of $SIRS$ epidemiological model with a non-monotonic incidence rate and effect of awareness program through media coverage is considered as measure of disease control. The mathematical analysis shows that the basic reproduction number R_0 plays an important role to control the disease, we see that the basic reproduction number R_0 of our model contains the term $a\alpha_1$ in the denominator. Hence the saturation factor of epidemic control (α_1) can contribute to reducing R_0, whereas the inhibition factor with respect to infective (α_2) does not influence that value. We also show that the disease-free equilibrium E_0 is locally and globally asymptotically stable if $R_0 < 1$ and unstable if $R_0 > 1$ under certain conditions. Similarly, for the endemic equilibrium E^*, it has been obtained under certain conditions for locally as well as globally asymptotically stable. The phase diagram is demonstrated in ***Fig.* 8.6**, at different initial values to validate the global stability. The model analysis further shows that awareness programs through the media campaigning are helpful in decreasing the spread of infectious diseases.

This kind of models can be particularly enlightening for the planning of public health policies for the control of diseases such as influenza, malaria, salmonella, cholera, whooping cough, and measles, COVID-19 since we can discover the many different behaviours the model can have as the parameters are varied.

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(Dedicated to Honor Dr. R. C. Singh Chandel on His 75^{th} Birth Anniversary Celebrations)

APPLICATION IN INITIAL VALUE PROBLEMS VIA OPERATIONAL TECHNIQUES ON A CONTOUR INTEGRAL FOR SRIVASTAVA - DAOUST FUNCTION OF TWO VARIABLES

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Abstract

In this paper, we introduce the contour integrals for two variables functions namely as Srivastava - Daoust and generalized Kampé de Fériet functions and then, by the fractional and partial derivatives operational techniques, obtain their many results and relations for various special functions useful in quantum mechanical fields. Again then, apply them to solve the fractional calculus problems involving the initial values with the Caputo fractional derivatives and Riemann - Liouville fractional integrals.
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1 Introduction

In this section, we introduce some preliminaries and formulae to be used in our investigation:

Mittag - Leffler [22], in his a series of five notes (from 1901 to 1905), defined a function

(1.1) $E_\nu(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\nu k+1)}, \nu, z \in \mathbb{C}, \Re(\nu) > 0,$ $\mathbb{C}$ being the set of complex numbers,

and studied various properties of his named Mittag - Leffler function (1.1) (see also in, [2] and [32]) to propose the generalization of the Laplace-Abel integral in the form $\int_0^{+\infty} e^{-t} E_\nu(zt^\nu)dt$.

The properties and generalization of (1.1) are introduced as namely as generalized Mittag - Leffler function (see, Dattoli et al. [2], Humbert and Agrawal [11], Wiman [32])

(1.2) $E_{\nu,\rho}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\nu k+\rho)}, z, \nu, \rho \in \mathbb{C}, \Re(\nu) > 0, \Re(\rho) > 0.$

The functions (1.1) and (1.2) have used in various scientific and physical problems found in the literature of the authors (for example [4], [8] - [10], [12], [13] - [19], [21]). The Mittag - Leffler function (1.2) has a relation to the function (1.1) as $E_{\nu,1}(z) = E_\nu(z)$ and an integral formula of (1.2) is used in quantum mechanical problems by Borel transforms (see in Dattoli and Licciardi [3]) as

(1.3) $\int_{-\infty}^{\infty} E_{1,\beta+1}(-x^2)dx = \frac{1}{\Gamma(\beta+\frac{1}{2})}, \Re(\beta) > -\frac{1}{2}.$

Recently, the contour integral has studied for Kummer confluent hypergeometric function by (see [1], [7], [24])

(1.4) $M(\alpha, \beta, y) = \dfrac{y^{1-\beta}\Gamma(\beta)}{2\pi i} \int_{-\infty}^{(0^+,1^+)} e^{yt} t^{-\beta} \left(1 - \frac{1}{t}\right)^{-\alpha} dt,\ \alpha, \beta \in \mathbb{C},\ |\arg(y)| < \frac{\pi}{2}, i = \sqrt{(-1)}.$

The function $M(\alpha, \beta, y)$ in the series form ([1], [5], [30, p.36]) is written by

$$M(\alpha, \beta, y) = {}_1F_1(\alpha; \beta; y) = \frac{\Gamma(\beta)}{\Gamma(\alpha)} \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+k)}{\Gamma(\beta+k)} \frac{y^k}{k!}.$$

Here in (1.4), the relation between Gamma function and Pochhammer symbol (a generalized factorial function) is known as

$$\frac{\Gamma(\nu + k)}{\Gamma(\nu)} = (\nu)_k = \begin{cases} 1, k = 0; \\ \nu(\nu + 1) \ldots (\nu + k - 1), \forall k \in \mathbb{N}; \end{cases}$$

where, $\mathbb{N}$ being the set of natural numbers.

In the present investigation, we introduce a contour integral for two variables Srivastava and Daoust function ([20], [26]-[28]) defined by

(1.5) $S^{A\,:\,B;B'}_{C\,:\,D;D'}\begin{pmatrix} z \\ w \end{pmatrix} = S^{A\,:\,B;B'}_{C\,:\,D;D'}\left(\begin{matrix} [(a) : \theta, \vartheta] : [(b) : \psi]; [(b') : \psi']; \\ [(c) : \delta, \kappa] : [(d) : \varphi]; [(d') : \varphi']; \end{matrix} z, w \right) = \sum_{m,n=0}^{\infty} H^{A\,:\,B;B'}_{C\,:\,D;D'}(m, n) \frac{z^m}{m!} \frac{w^n}{n!},$

where,

$$H^{A:B;B'}_{C:D;D'}(m,n)=\frac{\prod_{j=1}^{A}\Gamma(a_j+\theta_j m+\vartheta_j n)}{\prod_{j=1}^{C}\Gamma(c_j+\delta_j m+\kappa_j n)}\frac{\prod_{j=1}^{B}\Gamma(b_j+\psi_j m)}{\prod_{j=1}^{D}\Gamma(d_j+\varphi_j m)}\frac{\prod_{j=1}^{B'}\Gamma(b'_j+\psi'_j n)}{\prod_{j=1}^{D'}\Gamma(d'_j+\varphi'_j n)}.$$

The series in (1.5) is convergent under the conditions given by

$$\sum_{j=1}^{C}\delta_j+\sum_{j=1}^{D}\varphi_j-\sum_{j=1}^{A}\theta_j-\sum_{j=1}^{B}\psi_j+1>0;\sum_{j=1}^{C}\kappa_j+\sum_{j=1}^{D'}\varphi'_j-\sum_{j=1}^{A}\vartheta_j-\sum_{j=1}^{B'}\psi'_j+1>0.$$

Again, by the formula (1.5), on setting $\theta_j=\vartheta_j=1$ with $(j=1,2,\ldots,A)$; $\psi_j=1$ with $(j=1,2,\ldots,B)$; $\psi'_j=1$ with $(j=1,2,\ldots,B')$; $\delta_j=\kappa_j=1$ with $(j=1,2,\ldots,C)$; $\varphi_j=1$ with $(j=1,2,\ldots,D)$; $\varphi'_j=1$ with $(j=1,2,\ldots,D')$; a relation between the Srivastava and Daoust function $S^{A:B;B'}_{C:D;D'}\begin{pmatrix}z\\w\end{pmatrix}$ function (1.5) and the generalized Kampé de Fériet function [31], denoted by $F^{A:B;B'}_{C:D;D'}\begin{bmatrix}z\\w\end{bmatrix}$, is found in the form

$$(1.6)\quad \frac{\prod_{j=1}^{C}\Gamma(c_j)}{\prod_{j=1}^{A}\Gamma(a_j)}\frac{\prod_{j=1}^{D}\Gamma(d_j)}{\prod_{j=1}^{B}\Gamma(b_j)}\frac{\prod_{j=1}^{D'}\Gamma(d'_j)}{\prod_{j=1}^{B'}\Gamma(b'_j)}S^{A:B;B'}_{C:D;D'}\left(\begin{matrix}[(a):1,1]:[(b):1];[(b'):1];\\ [(c):1,1]:[(d):1];[(d'):1];\end{matrix}z,w\right)$$

$$=F^{A:B;B'}_{C:D;D'}\left[\begin{matrix}(a_j)_{1,A}:(b_j)_{1,B};(b'_j)_{1,B'};\\ (c_j)_{1,C}:(d_j)_{1,D};(d'_j)_{1,D'};\end{matrix}z,w\right].$$

The generalized Kampé de Fériet function, given in right hand side of (1.6), has the relations with various one and two variables functions of Appell's and Lauricella's functions used in various fields of science and technologies (see in [6], [29], [30]). Recently in [25], Pathan and Kumar presented a representation of multi-parametric Mittag - Leffler function in terms of Srivstava and Daoust function (1.5) and used in analysis of multivariable Cauchy residue theorem. On the other hand, currently, Chandel and Kumar [1] established two contour integral representations involving Mittag - Leffler functions (i) for a two variable generalized hypergeometric function of Srivastava and Daoust function (1.5) and (ii) a sum of the Kummer's confluent hypergeometric functions (1.4). Motivated by above researches, we will introduce a new contour integral in (2.3) in the complex t- plane for Srivastava and Daoust function (1.5) and then obtain various results and relations through operational techniques. Finally, we use these results in solving of some of the initial value problems consisting of Caputo fractional derivatives and Riemann - Liouville fractional integrals.

2 The contour integral representation for Srivastava and Daoust function and related special cases

***Lemma* 2.1** *If in the complex t - plane, $\alpha,\beta,z,y\in\mathbb{C}$, $|\arg(y)|<\frac{\pi}{2}$, and for $c>\Re(t)$, $\min\{\Re(t),\Re(z),\Re(\beta)\}>0$, then, there exists a contour integral for Kummer's confluent hypergeometric function as*

$$(2.1)\quad \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}e^{yt}t^{-\beta}\left(1-\tfrac{z}{t}\right)^{-\alpha}dt=\tfrac{y^{\beta-1}}{\Gamma(\beta)}{}_1F_1(\alpha;\beta;yz).$$

Then,

$$(2.2)\quad \frac{1}{\Gamma(\beta)}\int_0^{\infty}e^{-ty}y^{\beta-1}{}_1F_1(\alpha;\beta;yz)dy=\tfrac{1}{t^{\beta}}\left(1-\tfrac{z}{t}\right)^{-\alpha}$$

Again then,

$$\int_0^{\infty}e^{-ty}\left[\tfrac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}e^{y\xi}\xi^{-\beta}\left(1-\tfrac{z}{\xi}\right)^{-\alpha}d\xi\right]dy=y^{-\beta}\left(1-\tfrac{z}{y}\right)^{-\alpha}\delta_{ty};$$

$$\delta_{mn}=\begin{cases}0, m\neq n,\\ 1, m=n;\end{cases}\text{ being the Dirac - delta function.}$$

Proof. In the left hand side of first integral in (2.2), expand ${}_1F_1(\alpha;\beta;yz)$ by the series (1.4) and thus find that

$$\sum_{k=0}^{\infty}\tfrac{(\alpha)_k z^k}{\Gamma(\beta+k)}\int_0^{\infty}e^{-ty}y^{\beta+k-1}dy=t^{-\beta}\left(1-\tfrac{z}{t}\right)^{-\alpha},\left(\text{ since, }\int_0^{\infty}e^{-ty}y^{\beta+k-1}dy=\tfrac{\Gamma(\beta+k)}{t^{\beta+k}}\right).$$

Further, apply the formula (2.1) in the left hand side of equation (2.2) to get

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\xi^{-\beta}\left(1-\tfrac{z}{\xi}\right)^{-\alpha}\left\{\int_0^{\infty}e^{-y(t-\xi)}dy\right\}d\xi=\tfrac{-1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\tfrac{y^{-\beta}\left(1-\frac{z}{y}\right)^{-\alpha}}{y-t}dy=y^{-\beta}\left(1-\tfrac{z}{y}\right)^{-\alpha}\delta_{ty}.$$

Hence, by the statement of the ***Lemma* 2.1**, we get the equalities of (2.2). (Also, see Erdélyi et al. [5, p. 217], Srivastava and Manocha [30, p.219]).

Particularly, for $z=1$, the contour integral relation in (2.1) becomes the integral relation in (1.4).

***Theorem* 2.1** *In the complex t- plane, if* $\nu,\mu\in\mathbb{R}$ *such that* $\nu>0,\mu>0$ *and* $(\nu+\mu)>0$, *and also* $\alpha,\beta,\rho,w,z,y,\lambda\in\mathbb{C}$, $|\arg(y)|<\frac{\pi}{2}$, *where,* $\lambda\neq 0,\Re(\rho)>0$, *then, for* $c>\Re(t)$, *and* $\min\{\Re(t),\Re(z),\Re(\beta)\}>0$, *there exists a contour integral* $\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}e^{yt}E_{\nu,\rho}(\lambda w^{\nu}(t-z)^{-\mu})t^{-\beta}\left(1-\frac{z}{t}\right)^{-\alpha}dt$, *and thus it has the equality for Srivastava - Daoust function as*

$$(2.3)\quad \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}e^{yt}E_{\nu,\rho}(\lambda w^{\nu}(t-z)^{-\mu})t^{-\beta}\left(1-\tfrac{z}{t}\right)^{-\alpha}dt \quad = y^{\beta-1}S^{1:1;0}_{1:2;0}\left(\begin{matrix}[\alpha:\mu,1]:[1:1];[-:-];\\ [\beta:\mu,1]:[\rho:\nu],[\alpha:\mu];[-:-];\end{matrix}\lambda w^{\nu}y^{\mu},yz\right),$$

provided that $(\nu+\mu)>0$.

Proof. In left hand side of (2.3), define the Mittag - Leffler function (1.2), to get the equality

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}e^{yt}E_{\nu,\rho}(\lambda w^{\nu}(t-z)^{-\mu})t^{-\beta}\left(1-\tfrac{z}{t}\right)^{-\alpha}dt=\sum_{k=0}^{\infty}\frac{(\lambda w^{\nu})^{k}}{\Gamma(\nu k+\rho)}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}e^{yt}t^{-\beta-\mu k}\left(1-\tfrac{z}{t}\right)^{-\alpha-\mu k}dt,$$

and then, in right hand side of this equality apply the formula (2.1), to find the double series

$$y^{\beta-1}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{\Gamma(\alpha+\mu k+m)\Gamma(1+k)}{\Gamma(\beta+\mu k+m)\Gamma(\rho+\nu k)\Gamma(\alpha+\mu k)}\frac{(\lambda w^{\nu}y^{\mu})^{k}}{k!}\frac{(yz)^{m}}{m!}.$$

Again, with the help of convergence conditions of (1.5), this double series converges for $(\nu+\mu)>0$. Hence, by the definition of Srivastava and Daoust function (1.5), the following formula holds

$$(2.4)\quad \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}e^{yt}E_{\nu,\rho}(\lambda w^{\nu}(t-z)^{-\mu})t^{-\beta}\left(1-\tfrac{z}{t}\right)^{-\alpha}dt=y^{\beta-1}S^{1:1;0}_{1:2;0}\left(\begin{matrix}[\alpha:\mu,1]:[1:1];[-:-];\\ [\beta:\mu,1]:[\rho:\nu],[\alpha:\mu];[-:-];\end{matrix}\lambda w^{\nu}y^{\mu},yz\right),$$

provided that $(\nu+\mu)>0$.

Hence, the equality (2.3) is followed.

***Corollary* 2.1** *For all conditions of the* ***Theorem* 2.2** *and on specialization of the parameters, with* $\nu=\mu=1$, *of the formula* (2.3) *following equality holds for the Kampé de Fériet function* (1.6) *as*

$$(2.5)\quad \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}e^{yt}E_{1,\rho}(\lambda w(t-z)^{-1})t^{-\beta}\left(1-\tfrac{z}{t}\right)^{-\alpha}dt=\frac{y^{\beta-1}}{\Gamma(\beta)\Gamma(\rho)}F^{1:1;0}_{1:2;0}\left(\begin{matrix}[\alpha:1,1]:[1:1];[-:-];\\ [\beta:1,1]:[\rho:1],[\alpha:1];[-:-];\end{matrix}\lambda wy,yz\right).$$

***Remark* 2.1** *Again, on specialization of some other parameters in the formula* (2.3) *following special cases are discussed:*
***Special case* 2.1.1** *In the formula* (2.3), *set* $\rho=1,\alpha,w\to 0,\beta=k+1$, *then, it becomes Cauchy integral formula as* [25]

$$(2.6)\quad \lim_{\alpha,w\to 0}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}e^{yt}E_{\nu,1}(\lambda w^{\nu}(t-z)^{-\mu})t^{-k-1}\left(1-\tfrac{z}{t}\right)^{-\alpha}dt=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}e^{yt}t^{-k-1}dt=\frac{y^{k}}{k!}.$$

***Special case* 2.1.2** *Again, set* $\rho=1,z=1$, *in the formula* (2.3), *then, by use of the formula* (1.4), *and for the conditions given in the* ***Theorem* 2.2**, *its limiting case for* $w\to 0$, *gives us the equalities*

$$(2.7)\quad \lim_{w\to 0}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}e^{yt}E_{\nu,1}(\lambda w^{\nu}(t-1)^{-\mu})t^{-\beta}\left(1-\tfrac{1}{t}\right)^{-\alpha}dt=\frac{z^{1-\beta}\Gamma(\beta)}{2\pi i}\int_{-\infty}^{(0^{+},1^{+})}e^{yt}t^{-\beta}\left(1-\tfrac{1}{t}\right)^{-\alpha}dt=M(\alpha,\beta,y)$$

***Special case* 2.1.3** *Further, in the Eqn.* (2.3), *set* $\nu=\mu,w=t,y=-1,z=\frac{-x}{(-t)^{\kappa-1}},\beta=1,|\kappa|<1$, *and consider that* $f\left(\frac{x}{(-t)^{\kappa}}\right)=-\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}(\alpha+\mu k)_{m}\frac{(\lambda)^{k}}{\Gamma(\mu k+\rho)}\left(\frac{x}{(-t)^{\kappa}}\right)^{m}$, *then, it becomes the inverse Borel transformation formula of that function* $f\left(\frac{x}{(-t)^{\kappa}}\right)$ *as (see in* [3]*)*

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}e^{-t}E_{\mu,\rho}(\lambda\left(1-\tfrac{x}{(-t)^{\kappa}}\right)^{-\mu})t^{-1}(1-\tfrac{x}{(-t)^{\kappa}})^{-\alpha}dt=\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}(\alpha+\mu k)_{m}\frac{(\lambda)^{k}}{\Gamma(\mu k+\rho)}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{e^{-t}}{t}\left(\frac{x}{(-t)^{\kappa}}\right)^{m}dt$$
$$=\frac{i}{2\pi}\int_{c-i\infty}^{c+i\infty}\frac{e^{-t}}{t}f(\tfrac{x}{(-t)^{\kappa}})dt,|\kappa|<1.$$

Here,

$$(2.8)\quad f\left(\frac{x}{(-t)^{\kappa}}\right)=-\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}(\alpha+\mu k)_{m}\frac{(\lambda)^{k}}{\Gamma(\mu k+\rho)}\left(\frac{x}{(-t)^{\kappa}}\right)^{m}.$$

3 Operational techniques by fractional and partial derivatives on the contour integrals

In this section, we operate the contour integral defined in the ***Theorem* 2.1** by the Caputo fractional and partial derivatives and then obtain various results and relations.

The Caputo fractional derivative of the function $f(t)$, denoted by ${}_t^C D_{0^+}^{\alpha} f(t)$ where, $m-1<\alpha\leq m, \forall m\in\mathbb{N}$, is defined by ([4], [12], [21])

(3.1) $({}_t^C D_{0^+}^{\alpha} f)(t) = (I^{m-\alpha} f^{(m)})(t)$,

where, $f^{(m)}(t) = D_t^m f(t), \left\{D_t^m \equiv \frac{d^m}{dt^m} = \frac{d}{dt}\left(\frac{d^{m-1}}{dt^{m-1}}\right)\right\}, I^{m-\alpha}$ being the Riemann - Liouville fractional integral ([4], [12], [23])

(3.2) $(I^{m-\alpha} f)(t) = \begin{cases} \frac{1}{\Gamma(m-\alpha)}\int_0^t (t-\tau)^{m-\alpha-1} f(\tau)d\tau, t>0, m-1<\alpha\leq m, \\ f(t), \alpha = m, \forall m\in\mathbb{N}. \end{cases}$

The operation of Caputo derivative (3.1); $\forall m\in\mathbb{N}$, $m-1<\nu\leq m$; on the Mittag - Leffler function (1.1) is found as ([4], [12])

(3.3) ${}_w^C D_{0^+}^{\nu} E_\nu(\lambda w^\nu) = \lambda E_\nu(\lambda w^\nu)$.

***Theorem* 3.1** *In the complex t- plane, if $\nu,\mu\in\mathbb{R}$ such that $m-1<\nu\leq m, \forall m\in\mathbb{N}, \mu>0$ and $(\nu+\mu)>0$, and $\alpha,\beta,y,w,z,\rho,\lambda\in\mathbb{C}, |\arg(y)|<\frac{\pi}{2}$, where, $\lambda\neq 0, \Re(\rho)>0$, then, for $c>\Re(t)$ and $\min\{\Re(t),\Re(z),\Re(\beta)\}>0$, there exists a contour integral* (2.3) *in the form*

(3.4) $\dfrac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{yt} E_{\nu,\rho}(\lambda w^\nu (t-z)^{-\mu}) t^{-\beta}\left(1-\frac{z}{t}\right)^{-\alpha} dt = I_\lambda^{\alpha,\beta,\nu,\rho,\mu}(w,z;y)$, *(let).*

Thus, following operational formulae hold

(3.5) ${}_w^C D_{0^+}^{\nu}\left\{I_\lambda^{\alpha,\beta,\nu,1,\mu}(w,z;y)\right\} = \lambda I_\lambda^{\alpha+\mu,\beta+\mu,\nu,1,\mu}(w,z;y)$.

and

(3.6) $\nu\dfrac{\partial}{\partial y}\left\{I_\lambda^{\alpha,\beta,\nu,\rho,\mu}(w,z;y)\right\} = \nu I_\lambda^{\alpha,\beta-1,\nu,\rho,\mu}(w,z;y)$.

Proof. Operate (3.4) by the Caputo derivative (3.1) with respect to w to get

$$\begin{aligned} {}_w^C D_{0^+}^{\nu}\left\{I_\lambda^{\alpha,\beta,\nu,1,\mu}(w,z;y)\right\} &= {}_w^C D_{0^+}^{\nu}\left\{\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{yt} E_{\nu,1}(\lambda w^\nu (t-z)^{-\mu}) t^{-\beta}\left(1-\frac{z}{t}\right)^{-\alpha} dt\right\} \\ &= \left\{\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{yt}\, {}_w^C D_{0^+}^{\nu}\left\{E_\nu(\lambda w^\nu (t-z)^{-\mu})\right\} t^{-\beta}\left(1-\frac{z}{t}\right)^{-\alpha} dt\right\}, \end{aligned}$$

since then, in the first and last relations, applying the relations of Mittag - Leffler function, given in (1.3) and (3.3) to get the identities

$${}_w^C D_{0^+}^{\nu}\left\{I_\lambda^{\alpha,\beta,\nu,1,\mu}(w,z;y)\right\} = \left\{\frac{\lambda}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{yt}\left\{E_{\nu,1}(\lambda w^\nu (t-z)^{-\mu})\right\} t^{-\beta-\mu}\left(1-\frac{z}{t}\right)^{-\alpha-\mu} dt\right\} = \lambda I_\lambda^{\alpha+\mu,\beta+\mu,\nu,1,\mu}(w,z;y).$$

In the similar manner to find

$$\nu\frac{\partial}{\partial y}\left\{I_\lambda^{\alpha,\beta,\nu,\rho,\mu}(w,z;y)\right\} = \left\{\frac{\nu}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{yt} E_{\nu,\rho}(\lambda w^\nu (t-z)^{-\mu}) t^{1-\beta}\left(1-\frac{z}{t}\right)^{-\alpha} dt\right\} = \nu I_\lambda^{\alpha,\beta-1,\nu,\rho,\mu}(w,z;y).$$

Hence, the results (3.5) and (3.6) hold good.

***Theorem* 3.2** *If in the complex t- plane, $\nu,\mu\in\mathbb{R}$ such that $m-1<\nu\leq m, \forall m\in\mathbb{N}, \mu>0$ and $(\nu+\mu)>0$, and α,β, $w,y,z,\lambda\in\mathbb{C}$, $|\arg(y)|<\frac{\pi}{2}$, where, $\lambda\neq 0$, then, for $c>\Re(t)$ and $\min\{\Re(t),\Re(z),\Re(\beta)\}>0$, there exists a contour integral*

(3.7) $\dfrac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{yt} E_\nu(\lambda w^\nu (t-z)^{-\mu}) t^{-\beta}\left(1-\frac{z}{t}\right)^{-\alpha} dt = I_\lambda^{\alpha,\beta,\nu,\mu}(w,z;y)$.

Again then, following operational formulae hold

(3.8) $\int_0^\infty\left[\exp(-ty + {}_w^C D_{0^+}^{\nu})\left\{I_\lambda^{\alpha,\beta,\nu,\mu}(w,z;y)\right\}\right]dy = \exp\left\{\frac{\lambda}{(t-z)^\mu}\right\} E_\nu(\lambda w^\nu (t-z)^{-\mu}) t^{-\beta}\left(1-\frac{z}{t}\right)^{-\alpha}$.

and

(3.9) $\int_0^\infty\left[\exp\left(-ty + \nu\frac{\partial}{\partial y}\right)\left\{I_\lambda^{\alpha,\beta,\nu,\rho,\mu}(w,z;y)\right\}\right]dy = \exp\{\nu t\} E_\nu(\lambda w^\nu (t-z)^{-\mu}) t^{-\beta}\left(1-\frac{z}{t}\right)^{-\alpha}$.

Proof. On application of the ***Theorem* 3.1**, and by the properties of Mittag - Leffler function given in (1.3) and (3.3), we obtain

(3.10) $$({}^C_wD^{\nu}_{0^+})^n\Big\{I^{\alpha,\beta,\nu,\mu}_{\lambda}(w,z;y)\Big\} = \frac{\lambda^n}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{yt}E_{\nu}(\lambda w^{\nu}(t-z)^{-\mu})t^{-\beta-\mu n}\left(1-\tfrac{z}{t}\right)^{-\alpha-\mu n}dt = \lambda^n I^{\alpha+\mu n,\beta+\mu n,\nu,\mu}_{\lambda}(w,z;y).$$

and

(3.11) $$\left(\nu\frac{\partial}{\partial y}\right)^n\Big\{I^{\alpha,\beta,\nu,\mu}_{\lambda}(w,z;y)\Big\} = \frac{\nu^n}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{yt}E_{\nu}(\lambda w^{\nu}(t-z)^{-\mu})t^{n-\beta}\left(1-\tfrac{z}{t}\right)^{-\alpha}dt = \nu^n I^{\alpha,\beta-n,\nu,\mu}_{\lambda}(w,z;y).$$

Now, multiply by $\frac{1}{n!}$ in both of the sides of Eqns. (3.10) and (3.11) and then summing up n from $n=0$ to $n=\infty$ to get

(3.12) $$\exp({}^C_wD^{\nu}_{0^+})\Big\{I^{\alpha,\beta,\nu,\mu}_{\lambda}(w,z;y)\Big\} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{yt}E_{\nu}(\lambda w^{\nu}(t-z)^{-\mu})t^{-\beta}\left(1-\tfrac{z}{t}\right)^{-\alpha}\exp\Big\{\tfrac{\lambda}{(t-z)^{\mu}}\Big\}dt$$

and

(3.13) $$\exp\left(\nu\frac{\partial}{\partial y}\right)\Big\{I^{\alpha,\beta,\nu,\rho,\mu}_{\lambda}(w,z;y)\Big\} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{yt}E_{\nu}(\lambda w^{\nu}(t-z)^{-\mu})t^{-\beta}\left(1-\tfrac{z}{t}\right)^{-\alpha}\exp\Big\{\nu t\Big\}dt$$

Now, multiply by e^{-ty} both the sides of (3.12) and (3.13) and then integrate to them with respect to y from $y=0$ to $y=\infty$, and with the help of the ***Lemma* 2.1**, we find the operational relations (3.8) and (3.9).

***Theorem* 3.3** *If in the complex t- plane,*$\alpha,\beta,y,z,w\in\mathbb{C}$, *and* $|\arg(y)|<\frac{\pi}{2}$, *then, for* $c>\Re(t)$ *and* $\min\Big\{\Re(w),\Re(t),\Re(\beta)\Big\}>0$, *there exists a contour integral for* $K\in\mathbb{N}^*=\Big\{2,3,4,\ldots,L\Big\}, L<\infty$,

(3.14) $$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{yt}E_{\frac{1}{K}}((zt)^{\frac{1}{K}})t^{-\beta}\left(1-\tfrac{w}{t}\right)^{-\alpha}dt = I^{\alpha,\beta,K}_1(z;y,w),$$

then for $z,z_0\in\mathbb{C}$, *a function* $F(z)$ *is defined as such that* $\int_{z_0}^{z}F(z)dz<\infty$, *and thus, following equalities hold*

(3.15) $$\int_{z_0}^{z}F(z)d\Big\{I^{\alpha,\beta,K}_1(z;y,w)\Big\} = \sum_{r=0}^{K-1}\frac{\Big\{I_1^{\alpha,\beta+\frac{r}{K}-1,K}(0;y,w)\Big\}}{\Gamma(1-\frac{r}{K})}\int_{z_0}^{z}F(z)z^{-\frac{r}{K}}dz$$
$$= \sum_{r=0}^{K-1}\Big\{{}_1F_1(\alpha;\beta+\tfrac{r}{K}-1;wy)\Big\}\frac{y^{\beta+\frac{r}{K}-2}}{\Gamma(\beta+\frac{r}{K}-1)\Gamma(1-\frac{r}{K})}\int_{z_0}^{z}F(z)z^{-\frac{r}{K}}dz.$$

Proof. By the relation (3.14), we write

$$\frac{d}{dz}\Big\{I^{\alpha,\beta,K}_1(z;y,w)\Big\} = \frac{d}{dz}\Big\{\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{(y+z)t}\Big\{e^{-zt}E_{\frac{1}{K}}((zt)^{\frac{1}{K}})\Big\}t^{-\beta}\left(1-\tfrac{w}{t}\right)^{-\alpha}dt\Big\}$$
$$= \Big\{\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{(y+z)t}\Big\{e^{-zt}E_{\frac{1}{K}}((zt)^{\frac{1}{K}})\Big\}t^{1-\beta}\left(1-\tfrac{w}{t}\right)^{-\alpha}dt\Big\}$$
$$+\Big\{\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{(y+z)t}\frac{d}{d\phi}\Big\{e^{-\phi}E_{\frac{1}{K}}((\phi)^{\frac{1}{K}})\Big\}t^{1-\beta}\left(1-\tfrac{w}{t}\right)^{-\alpha}dt\Big\},\Big\{\text{ on setting }\phi=zt\Big\},$$

and since then, applying the result by Mathai and Haubold [21, p. 84], we have a relation

$$\frac{d}{d\phi}\Big\{e^{-\phi}E_{\frac{1}{K}}((\phi)^{\frac{1}{K}})\Big\} = e^{-\phi}\sum_{r=1}^{K-1}\frac{\phi^{-\frac{r}{K}}}{\Gamma(1-\frac{r}{K})}$$

and thus to find

$$\frac{d}{dz}\Big\{\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{yt}E_{\frac{1}{K}}((zt)^{\frac{1}{K}})t^{-\beta}\left(1-\tfrac{w}{t}\right)^{-\alpha}dt\Big\} = \Big\{\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{yt}E_{\frac{1}{K}}((zt)^{\frac{1}{K}})t^{1-\beta}\left(1-\tfrac{w}{t}\right)^{-\alpha}dt\Big\}$$
$$+\sum_{r=1}^{K-1}\frac{z^{-\frac{r}{K}}}{\Gamma(1-\frac{r}{K})}\Big\{\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{yt}t^{1-\frac{r}{K}-\beta}\left(1-\tfrac{w}{t}\right)^{-\alpha}dt\Big\} = \sum_{r=0}^{K-1}\frac{z^{-\frac{r}{K}}}{\Gamma(1-\frac{r}{K})}\Big\{\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{yt}t^{1-\frac{r}{K}-\beta}\left(1-\tfrac{w}{t}\right)^{-\alpha}dt\Big\}$$

It becomes

(3.16) $$\int_{z_0}^{z}F(z)d\Big\{\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{yt}E_{\frac{1}{K}}((zt)^{\frac{1}{K}})t^{-\beta}\left(1-\tfrac{w}{t}\right)^{-\alpha}dt\Big\} = \sum_{r=0}^{K-1}\int_{z_0}^{z}F(z)\frac{z^{-\frac{r}{K}}}{\Gamma(1-\frac{r}{K})}dz\Big\{\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{yt}t^{1-\frac{r}{K}-\beta}\left(1-\tfrac{w}{t}\right)^{-\alpha}dt\Big\}.$$

Again, by (3.14), we again write

$$\lim_{z\to 0}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{yt}E_{\frac{1}{K}}((zt)^{\frac{1}{K}})t^{-\beta}\left(1-\tfrac{w}{t}\right)^{-\alpha}dt = \lim_{z\to 0}I^{\alpha,\beta,K}_1(z;y,w)$$

and since by the definition (1.1) , there exists a result $\lim_{z\to 0}E_{\frac{1}{K}}((zt)^{\frac{1}{K}})=1$, and hence by ***Lemma* 2.1**, there implies that

(3.17) $$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{yt}t^{-\beta}\left(1-\tfrac{w}{t}\right)^{-\alpha}dt = I^{\alpha,\beta,K}_1(0;y,w) = \frac{y^{\beta-1}}{\Gamma(\beta)}{}_1F_1(\alpha;\beta;yw).$$

Finally, using the definition of (3.14) and the relation of (3.17) in the result (3.16), we obtain the equalities of (3.15).

***Theorem* 3.4** *If in the* ***Theorem* 3.2** *, set* $\nu = 1$, $w = x^2$, $x,\mu \in \mathbb{R}$, $\mu > 0$ *and* $\alpha,\beta,\rho,z,y \in \mathbb{C}$, $|\arg(y)| < \frac{\pi}{2}$, $\lambda = -1$, *replace* ρ *by* $\rho + 1$, *then, for* $c > \Re(t)$ *and* $\min\{\Re(t),\Re(z),\Re(\beta)\} > 0$, *there exists a contour integral*

$$(3.18)\quad \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{yt}E_{1,\rho+1}(-x^2(t-z)^{-\mu})t^{-\beta}\left(1-\tfrac{z}{t}\right)^{-\alpha}dt = I_{-1}^{\alpha,\beta,1,\rho,\mu}(x^2,z;y).$$

Again then, following formula holds

$$(3.19)\quad \int_{-\infty}^{\infty} I_{-1}^{\alpha,\beta,1,\rho,\mu}(x^2,z;y)dx = \tfrac{1}{\Gamma(\rho+\frac{1}{2})}\tfrac{y^{\beta-\frac{\mu}{2}-1}}{\Gamma(\beta-\frac{\mu}{2})}{}_1F_1(\alpha-\tfrac{\mu}{2};\beta-\tfrac{\mu}{2};yz),$$

provided that $\rho > -\frac{1}{2}$ *and* $\beta - \frac{\mu}{2} \neq 0,-1,-2,\ldots, |yz| < \infty$.

Proof. Integrate both sides of (3.18) with respect to x from $x = -\infty$ to $x = \infty$, to get

$$\int_{-\infty}^{\infty} I_{-1}^{\alpha,\beta,1,\rho,\mu}(x^2,z;y)dx = \tfrac{1}{2\pi i}\int_{-\infty}^{\infty}\int_{c-i\infty}^{c+i\infty} e^{yt}E_{1,\rho+1}(-x^2(t-z)^{-\mu})t^{-\beta}\left(1-\tfrac{z}{t}\right)^{-\alpha}dtdx$$

Now, in the right hand side of the double integral of above equality, change the order of integration and set $x^2(t-z)^{-\mu} = y^2$, to find that

$$\int_{-\infty}^{\infty} I_{-1}^{\alpha,\beta,1,\rho,\mu}(x^2,z;y)dx = \tfrac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{yt}t^{-\beta+\frac{\mu}{2}}\left(1-\tfrac{z}{t}\right)^{-\alpha+\frac{\mu}{2}}\int_{-\infty}^{\infty}E_{1,\rho+1}(-y^2)dydt,$$

where, in right hand side, in the inner integral use the Borel transformation formula (1.3), we get

$$\int_{-\infty}^{\infty} I_{-1}^{\alpha,\beta,1,\rho,\mu}(x^2,z;y)dx = \tfrac{1}{2\pi i}\tfrac{1}{\Gamma(\rho+\frac{1}{2})}\int_{c-i\infty}^{c+i\infty} e^{yt}t^{-\beta+\frac{\mu}{2}}\left(1-\tfrac{z}{t}\right)^{-\alpha+\frac{\mu}{2}}dt,$$

which on defining by the formula given in ***Lemma* 2.1**, we obtain

$$(3.20)\quad \int_{-\infty}^{\infty} I_{-1}^{\alpha,\beta,1,\rho,\mu}(x^2,z;y)dx = \tfrac{1}{\Gamma(\rho+\frac{1}{2})}\tfrac{y^{\beta-\frac{\mu}{2}-1}}{\Gamma(\beta-\frac{\mu}{2})}{}_1F_1(\alpha-\tfrac{\mu}{2};\beta-\tfrac{\mu}{2};yz),$$

which is valid for $\rho + \frac{1}{2} > 0$ and $\beta - \frac{\mu}{2} \neq 0,-1,-2,\ldots$ and $|yz| < \infty$.

Hence, by the result (3.20), the ***Theorem* 3.4** has followed.

4 Application with numerical examples

In this section, we apply our above results in following problems:

***Problem* 4.1** *If* $I_{\lambda}^{\alpha,\beta,\nu,\rho,\mu}(w,z;y) = ({}_{y}^{C}D_{0^+}^{\eta}F)(y)$, *where the* $I_{\lambda}^{\alpha,\beta,\nu,\rho,\mu}(w,z;y)$ *is given in (3.4) and* ${}_{t}^{C}D_{0^+}^{\eta}$ *being the Caputo fractional derivative* (3.1)*, for* $m-1 < \eta \leq m, \forall m \in \mathbb{N}$, *and with the initial conditions* $\frac{d^{m-1}}{dy^{m-1}}F(y)\Big|_{y\,=\,0^+} = 0\ \forall m = 1,2,3,\ldots.$

Then, following solution exists

$$(4.1)\quad F(y) = I_{\lambda}^{\alpha,\beta+\eta,\nu,\rho,\mu}(w,z;y) = y^{\beta+\eta-1}S\,{}^{1\,:\,1;0}_{1\,:\,2;0}\left(\begin{matrix}[\alpha:\mu,1]:[1:1];[-:-];\\ [\beta+\eta:\mu,1]:[\rho:\nu],[\alpha:\mu];[-:-];\end{matrix}\lambda w^{\nu}y^{\mu},yz\right),$$

provided that $(\nu + \mu) > 0$.

***Solution*.** On applying (3.4), the equation, $I_{\lambda}^{\alpha,\beta,\nu,\rho,\mu}(w,z;y) = ({}_{y}^{C}D_{0^+}^{\eta}F)(y)$, of the ***Problem* 4.1**, is written as

$$(4.2)\quad \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{yt}E_{\nu,\rho}(\lambda w^{\nu}(t-z)^{-\mu})t^{-\beta}\left(1-\tfrac{z}{t}\right)^{-\alpha}dt = ({}_{y}^{C}D_{0^+}^{\eta}F)(y)$$

Now, multiply by $e^{-yt}, t > 0$, in both of the sides of (4.2) , and integrate it with respect to y, (from $y = 0$ to $y = \infty$) to get as

$$(4.3)\quad \int_{0}^{\infty}\tfrac{1}{2\pi i}e^{-yt}\int_{c-i\infty}^{c+i\infty} e^{yt}E_{\nu,\rho}(\lambda w^{\nu}(t-z)^{-\mu})t^{-\beta}\left(1-\tfrac{z}{t}\right)^{-\alpha}dtdy = \int_{0}^{\infty}e^{-yt}\left({}_{y}^{C}D_{0^+}^{\eta}F\right)(y)dy,$$

Then, in right hand side of (4.3), apply the result by [12] as

$$\int_{0}^{\infty}e^{-yt}\left({}_{y}^{C}D_{0^+}^{\eta}F\right)(y)dy = t^{\eta}f(t) - \textstyle\sum_{k=0}^{m-1}t^{\eta-1-k}F^{(k)}(0^+)\forall m-1 < \eta \leq m,$$

where, $f(t) = \int_{0}^{\infty}e^{-yt}F(y)dy$,and inversely, $F(y) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}e^{yt}f(t)dt$ and thus use the ***Lemma* 2.1**, with the initial conditions given in the ***Problem* 4.1**, we find

$$(4.4)\quad f(t) = E_{\nu,\rho}(\lambda w^{\nu}(t-z)^{-\mu})t^{-\beta-\eta}\left(1-\frac{z}{t}\right)^{-\alpha}$$

Finally, on taking contour integration (see, inverse Laplace transformation before (4.4)) of both sides of (4.4) and using by (3.4) to obtain equality in first and second results of (4.1) as

$$(4.5)\quad F(y) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}e^{yt}f(t)dt = \tfrac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}e^{yt}E_{\nu,\rho}(\lambda w^{\nu}(t-z)^{-\mu})t^{-\beta-\eta}\left(1-\tfrac{z}{t}\right)^{-\alpha}dt = I_{\lambda}^{\alpha,\beta+\eta,\nu,\rho,\mu}(w,z;y).$$

In right hand side of (4.5), use the ***Theorem* 2.2**, we find equality in first and third results of (4.1).

***Problem* 4.2** *If* $I_{\lambda}^{\alpha,\beta,\nu,\rho,\mu}(w,z;y) = (I^{\eta}F)(y)$, *where the* $I_{\lambda}^{\alpha,\beta,\nu,\rho,\mu}(w,z;y)$ *is given in (3.4) and* I^{η} *being the Riemann - Liouville fractional integral of order* η *defined in (3.2) for* $\eta > 0$, *then*

(4.6) $F(y) = I_{\lambda}^{\alpha,\beta-\eta,\nu,\rho,\mu}(w,z;y)$.

***Solution*.** On using formula (3.4), the equation in ***Problem* 4.2** is written by

(4.7) $$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{yt}E_{\nu,\rho}(\lambda w^{\nu}(t-z)^{-\mu})t^{-\beta}\left(1-\tfrac{z}{t}\right)^{-\alpha}dt = (I^{\eta}F)(y).$$

Now, in both sides of (4.7), multiply $e^{-yt}, t > 0$, and then integrate them with respect to y from $y = 0$ to $y = \infty$, to get that

$$\int_0^{\infty} e^{-yt}\tfrac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{yt}E_{\nu,\rho}(\lambda w^{\nu}(t-z)^{-\mu})t^{-\beta}\left(1-\tfrac{z}{t}\right)^{-\alpha}dtdy = \int_0^{\infty} e^{-yt}\left\{\tfrac{1}{\Gamma(\eta)}\int_0^{y}(y-x)^{\eta-1}F(x)dx\right\}dy$$

Since, by Laplace convolution theorem, there exists a relation

$$\frac{1}{\Gamma(\eta)}\int_0^{y}(y-x)^{\eta-1}F(x)dx = \tfrac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{yt}t^{-\eta}f(t)dt,$$

and hence right hand side becomes

$$= \int_0^{\infty} e^{-yt}\left\{\tfrac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{yt}t^{-\eta}f(t)dt\right\}dy.$$

Then, in both of the sides, apply the ***Lemma* 2.1**, to obtain

$$E_{\nu,\rho}(\lambda w^{\nu}(t-z)^{-\mu})t^{-\beta}\left(1-\frac{z}{t}\right)^{-\alpha} = t^{-\eta}f(t)$$

or to find

$$f(t) = E_{\nu,\rho}(\lambda w^{\nu}(t-z)^{-\mu})t^{-(\beta-\eta)}\left(1-\frac{z}{t}\right)^{-\alpha},$$

again, use the formula given in (4.4), we get

$$F(y) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{yt}E_{\nu,\rho}(\lambda w^{\nu}(t-z)^{-\mu})t^{-(\beta-\eta)}\left(1-\tfrac{z}{t}\right)^{-\alpha}dt = I_{\lambda}^{\alpha,\beta-\eta,\nu,\rho,\mu}(w,z;y).$$

Finally, we obtain the result (4.6).

***Problem* 4.3** *If* $I_{\lambda}^{\alpha,\beta,\nu,\rho,\mu}(w,z;y) = \frac{d}{dy}F(y) + \left({}_{y}^{C}D_{0^+}^{\eta}F\right)(y)$, *where the* $I_{\lambda}^{\alpha,\beta,\nu,\rho,\mu}(w,z;y)$ *is given in* (3.4) *and* ${}_{t}^{C}D_{0^+}^{\eta}$ *being the Caputo fractional derivative* (3.1), *for* $m-1 < \eta \leq m, \forall m \in \mathbb{N}$, *and with the initial conditions* , $\frac{d^{m-1}}{dy^{m-1}}F(y)\Big|_{y\,=\,0^+} = 0 \forall m = 1,2,3,\ldots$.

Then

(4.8) $F(y) = \sum_{k=0}^{\infty}(-1)^{k}I_{\lambda}^{\alpha,\beta+\eta+(\eta-1)k,\nu,\rho,\mu}(w,z;y)$.

***Solution*.** In the similar manner of the ***Problem* 4.1**, for $m-1 < \eta \leq m, \forall m \in \mathbb{N}$, by ***Problem* 4.3**, we have

(4.9) $$(t+t^{\eta})f(t) = E_{\nu,\rho}(\lambda w^{\nu}(t-z)^{-\mu})t^{-\beta}\left(1-\frac{z}{t}\right)^{-\alpha}.$$

Then, by Eqn. (4.9), it implies that

$$f(t) = (1+t^{1-\eta})^{-1}E_{\nu,\rho}(\lambda w^{\nu}(t-z)^{-\mu})t^{-\beta-\eta}\left(1-\frac{z}{t}\right)^{-\alpha},$$

and then we have

$$f(t) = \sum_{k=0}^{\infty}(-1)^{k}E_{\nu,\rho}(\lambda w^{\nu}(t-z)^{-\mu})t^{-\beta-\eta-(\eta-1)k}\left(1-\tfrac{z}{t}\right)^{-\alpha}$$

and then inversing it by formula (4.5), finally, we get the solution (4.8) as

(4.10) $F(y) = \sum_{k=0}^{\infty}(-1)^{k}I_{\lambda}^{\alpha,\beta+\eta+(\eta-1)k,\nu,\rho,\mu}(w,z;y)$.

Concluding remarks

This research work centralizes about a contour integral defined in the ***Theorem* 2.1** for the Srivastava and Daoust function of two variables (1.5). Specially, it gives the formula for the generalized Kampé de Fériet function (1.6) in the ***Corollary* 2.1.** Other applicable special cases are also discussed in the ***Remark* 2.1**. In the ***Section* 3**, by partial and Caputo fractional derivative operation techniques, various results and relations are obtained, in which ***Theorem* 3.2** may generate many generating functions and relations for various multiple special functions through Lie group theoretic techniques. By the Borel transforms [3], (see in the ***Theorem* 3.4**) obtained results may use in quantum mechanical problems. Next, we use and discuss our results in some numerical problems consisting of Caputo fractional derivative and Riemann Liouville integrals. These fractional operators are helpful in construction and solving of many diffusion and wave problems (see for example [8] - [10], [12], [13], [19], [21], [23] and others). It is also remarked that the Laplace operator techniques play an important role in solving the given problems consisting of contour integral in (3.4) and has a relation with two variables Srivastava - Daoust function (1.5) in Eqn. (2.3) and another relation with Kampé de Fériet function in (2.5). Also other results and relations by different contour integral formulae have carried out in the ***Sections* 2** and **3**. Hence, the contour integral in (2.3) becomes very much useful in doing of further researches in the area of fractional calculus along with the fractional diffusion and wave problems occurring in the modern science and technology.

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(Dedicated to Honor Dr. R. C. Singh Chandel on His 75th Birth Anniversary Celebrations)

COMMON FIXED POINT THEOREMS IN BICOMPLEX VALUED b-METRIC SPACES FOR RATIONAL CONTRACTIONS

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Abstract

In this paper we prove some common fixed point theorems in a complete bicomplex valued b-metric spaces for rational contractions.

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1 Introduction, Definitions and Notations.

Segre's [41] paper, published in 1892 made a pioneering attempt in the development of special algebras. He conceptualized commutative generalization of complex numbers as bicomplex numbers, tricomplex numbers, etc. as elements of an infinite set of algebras. Unfortunately this significant work of Segre failed to earn the attention of the mathematicians for almost a century. However, recently a renewed interest in this subject contributes a lot in the different fields of mathematical sciences and other branches of science and technology.

Price [36] developed the bicomplex algebra and function theory. In this field an impressive body of work has been developed by different researchers during the last few years. One can see some of the attempts in (cf.[3]-[6], [15], [16], [24]-[33], [39], [40], [42], [43]). Azam et al. [1] introduced a concept of complex valued metric space and established a common fixed point theorem for a pair of self contracting mappings. Rouzkard & Imdad [37] generalized the result obtained by Azam et al. [1] and proved another common fixed point theorem satisfying some rational inequality in complex valued metric space. The Banach contraction principle (cf. [12]) is a very popular and effective tool to solve the existence problems in many branches of mathematical analysis and it is an active area of research since 1922. The famous Banach theorem (cf. [12]) states that "Let (X, d) be a metric space and T be a mapping of X into itself satisfying $d(Tx, Ty) \leq kd(x, y), \forall x, y \in X$, where k is a constant in $(0, 1)$. Then T has a unique fixed point $x^* \in X$". In this connection Choudhury et al. ([13]&[14]) proved some fixed point results in partially ordered complex valued metric spaces for rational type expressions. Datta & Ali [7] proved common fixed point theorems for four mappings in complex valued metric space. Also one can see the attempts in (cf. [2], [8], [9], [44], [46], 47).

The concept of complex-valued b-metric spaces introduced by Rao et al.[38] proved a common fixed point theorem in complex valued b-metric spaces. Mukheimer [34] proved some common fixed point theorems in complex-valued b-metric spaces. Also Dubey et al.[18] proved some common fixed point theorems for contractive mappings in complex-valued b-metric spaces and Singh et al.[45] common fixed point theorems in complex-valued b-metric spaces. In this connection Mitra[35] proved a common fixed point theorem in complex valued b-metric spaces and Kumar et al.[19] proved common fixed point theorem in complex valued b-metric space for rational contraction. We write the set of real, complex and bicomplex numbers respectively as $\mathbb{C}_0, \mathbb{C}_1$ and $\mathbb{C}_2$. In this paper we are going to prove some common fixed point theorem in bicomplex valued b-metric space for rational contraction.

Let $z_1, z_2 \in \mathbb{C}_1$ be any two complex numbers, then the partial order relation $\precsim$ on $\mathbb{C}_1$ is defined as follows:
$z_1 \precsim z_2$ if and only if $Re(z_1) \leq Re(z_2)$ and $Im(z_1) \leq Im(z_2)$,
i.e., $z_1 \precsim z_2$ if one of the following conditions is satisfied:

(1) $Re(z_1) = Re(z_2),\ Im(z_1) = Im(z_2)$,

(2) $Re(z_1) < Re(z_2),\ Im(z_1) = Im(z_2)$,

(3) $Re(z_1) = Re(z_2),\ Im(z_1) < Im(z_2)$ and

(4) $Re(z_1) < Re(z_2),\ Im(z_1) < Im(z_2)$.

In particular, we can say $z_1 \lnsim z_2$ if $z_1 \precsim z_2$ and $z_1 \neq z_2$ i.e. one of (2), (3) and (4) is satisfied and $z_1 \prec z_2$ if only (4) is satisfied. We can easily check the following fundamental properties of partial order relation $\precsim$ on $\mathbb{C}_1$:

1. If $0 \precsim z_1 \lnsim z_2$, then $|z_1| < |z_2|$,
2. If $z_1 \precsim z_2, z_2 \prec z_3$ then $z_1 \prec z_3$ and
3. If $z_1 \precsim z_2$ and $\lambda > 0$ is a real number then $\lambda z_1 \precsim \lambda z_2$.

1.1 Complex valued metric space.

Azam et al.[1] defined the complex valued metric spaces as

***Definition* 1.1** *Let X be a nonempty set. Suppose the mapping $d : X \times X \to \mathbb{C}_1$ satisfies the following conditions:*

1. $0 \precsim d(x, y)$ *for all* $x, y \in X$,
2. $d(x, y) = 0$ *if and only if* $x = y$,
3. $d(x, y) = d(y, x)$ *for all* $x, y \in X$ *and*
4. $d(x, y) \precsim d(x, z) + d(z, y)$ *for all* $x, y, z \in X$.

Then d is called a complex valued metric on X and (X, d) is called the complex valued metric space.

***Definition* 1.2** *Let X be a nonempty set and let $s \geq 1$. Suppose the mapping $d : X \times X \to \mathbb{C}_1$ satisfies the following conditions:*

1. $0 \precsim d(x, y)$ *for all* $x, y \in X$,
2. $d(x, y) = 0$ *if and only if* $x = y$,
3. $d(x, y) = d(y, x)$ *for all* $x, y \in X$ *and*
4. $d(x, y) \precsim s\left[d(x, z) + d(z, y)\right]$ *for all* $x, y, z \in X$.

Then d is called a complex valued b-metric on X and (X, d) is called the complex valued b-metric space.

1.2 Bicomplex Number.

Segre [41] defined the bicomplex number as:

$$\xi = a_1 + a_2 i_1 + a_3 i_2 + a_4 i_1 i_2,$$

where $a_1, a_2, a_3, a_4 \in \mathbb{C}_0$ and the independent units i_1, i_2 are such that $i_1^2 = i_2^2 = -1$ and $i_1 i_2 = i_2 i_1$. We denote $i_1 i_2 = j$, which is known as the hyperbolic unit and such that $j^2 = 1$, $i_1 j = j i_1 = -i_2$, $i_2 j = j i_2 = -i_1$. Also $\mathbb{C}_2$ is defined as:

$$\mathbb{C}_2 = \{\xi : \xi = a_1 + a_2 i_1 + a_3 i_2 + a_4 i_1 i_2, a_1, a_2, a_3, a_4 \in \mathbb{C}_0\}$$

i.e.,

$$\mathbb{C}_2 = \{\xi : \xi = z_1 + i_2 z_2, z_1, z_2 \in \mathbb{C}_1\},$$

where $z_1 = a_1 + a_2 i_1 \in \mathbb{C}_1$ and $z_2 = a_3 + a_4 i_1 \in \mathbb{C}_1$.

If $\xi = z_1 + i_2 z_2$ and $\eta = w_1 + i_2 w_2$ be any two bicomplex numbers then the sum is $\xi \pm \eta = (z_1 + i_2 z_2) \pm (w_1 + i_2 w_2) = (z_1 \pm w_1) + i_2 (z_2 \pm w_2)$ and the product is $\xi.\eta = (z_1 + i_2 z_2).(w_1 + i_2 w_2) = (z_1 w_1 - z_2 w_2) + i_2 (z_1 w_2 + z_2 w_1)$.

1.2.1 Idempotent representation of bicomplex number.

There are four idempotent elements in $\mathbb{C}_2$, they are $0, 1, e_1 = \frac{1+i_1 i_2}{2}$, and $e_2 = \frac{1-i_1 i_2}{2}$ out of which e_1 and e_2 are nontrivial such that $e_1 + e_2 = 1$ and $e_1 e_2 = 0$. Every bicomplex number $z_1 + i_2 z_2$ can uniquely be expressed as the combination of e_1 and e_2, namely

$$\xi = z_1 + i_2 z_2 = (z_1 - i_1 z_2) e_1 + (z_1 + i_1 z_2) e_2.$$

This representation of ξ is known as the idempotent representation of bicomplex number and the complex coefficients $\xi_1 = (z_1 - i_1 z_2)$ and $\xi_2 = (z_1 + i_1 z_2)$ are known as idempotent components of the bicomplex number ξ.

1.2.2 Non-Singular and Singular elements.

An element $\xi = z_1 + i_2 z_2 \in \mathbb{C}_2$ is said to be invertible if there exists another element η in $\mathbb{C}_2$ such that $\xi\eta = 1$ and η is said to be the inverse (multiplicative) of ξ. Consequently ξ is said to be the inverse (multiplicative) of η. An element which has an inverse in $\mathbb{C}_2$ is said to be the nonsingular element of $\mathbb{C}_2$ and an element which does not have an inverse in $\mathbb{C}_2$ is said to be the singular element of $\mathbb{C}_2$.

An element $\xi = z_1 + i_2 z_2 \in \mathbb{C}_2$ is nonsingular if and only if $\left|z_1^2 + z_2^2\right| \neq 0$ and singular if and only if $\left|z_1^2 + z_2^2\right| = 0$ and the inverse of ξ is defined as

$$\xi^{-1} = \eta = \frac{z_1 - i_2 z_2}{z_1^2 + z_2^2}.$$

Zero is the only one element in $\mathbb{R}$ which does not have multiplicative inverse and in $\mathbb{C}, 0 = 0 + i0$ is the only one element which does not have multiplicative inverse. We denote the set of singular elements of $\mathbb{R}$ and $\mathbb{C}$ by O_0 and O_1 respectively. But there are more than one element in $\mathbb{C}_2$ which do not have multiplicative inverse; we denote this set by O_2 and clearly $O_0 = O_1 \subset O_2$.

1.2.3 Norm of a bicomplex number.

The norm $\|\cdot\|$ of $\mathbb{C}_2$ is a positive real valued function and $\|\cdot\| : \mathbb{C}_2 \to \mathbb{R}^+$ is defined by

$$\begin{aligned}\|\xi\| = \|z_1 + i_2 z_2\| &= \left\{|z_1|^2 + |z_2|^2\right\}^{\frac{1}{2}} \\ &= \left[\frac{|(z_1 - i_1 z_2)|^2 + |(z_1 + i_1 z_2)|^2}{2}\right]^{\frac{1}{2}} = \left(a_1^2 + a_2^2 + a_3^2 + a_4^2\right)^{\frac{1}{2}},\end{aligned}$$

where $\xi = a_1 + a_2 i_1 + a_3 i_2 + a_4 i_1 i_2 = z_1 + i_2 z_2 \in \mathbb{C}_2$.

The linear space $\mathbb{C}_2$ with respect to defined norm is a norm linear space, also $\mathbb{C}_2$ is complete; therefore $\mathbb{C}_2$ is the Banach space. If $\xi, \eta \in \mathbb{C}_2$ then $\|\xi\eta\| \leq \sqrt{2}\,\|\xi\|\,\|\eta\|$ holds instead of $\|\xi\eta\| \leq \|\xi\|\,\|\eta\|$, therefore $\mathbb{C}_2$ is not the Banach algebra.

Now we define the partial order relation $\precsim_{i_2}$ on $\mathbb{C}_2$ as follows:

Let $\mathbb{C}_2$ be the set of bicomplex numbers and $\xi = z_1 + i_2 z_2, \eta = w_1 + i_2 w_2 \in \mathbb{C}_2$ then $\xi \precsim_{i_2} \eta$ if and only if $z_1 \precsim w_1$. and $z_2 \precsim w_2$ i.e., $\xi \precsim_{i_2} \eta$ if one of the following conditions is satisfied:

(1) $z_1 = w_1,\ z_2 = w_2$,
(2) $z_1 \prec w_1,\ z_2 = w_2$,
(3) $z_1 = w_1,\ z_2 \prec w_2$ and
(4) $z_1 \prec w_1,\ z_2 \prec w_2$.

In particular we can write $\xi \lnsim_{i_2} \eta$ if $\xi \precsim_{i_2} \eta$ and $\xi \neq \eta$ i.e. one of (2), (3) and (4) is satisfied and we will write $\xi \prec_{i_2} \eta$ if only (4) is satisfied.

For any two bicomplex numbers $\xi, \eta \in \mathbb{C}_2$ we can verify the followings:

(i) $\xi \precsim_{i_2} \eta \to \|\xi\| \leq \|\eta\|$,
(ii) $\|\xi + \eta\| \leq \|\xi\| + \|\eta\|$,
(iii) $\|a\xi\| = a\,\|\xi\|$ if a is a non negative real number,
(iv) $\|\xi\eta\| \leq \sqrt{2}\,\|\xi\|\,\|\eta\|$ and the equality holds only when at least one of ξ and η is equal to zero,
(v) $\left\|\xi^{-1}\right\| = \|\xi\|^{-1}$ if ξ is a nonsingular bicomplex number with $0 \prec \xi$,
(vi) $\left\|\frac{\xi}{\eta}\right\| = \frac{\|\xi\|}{\|\eta\|}$, if η is a nonsingular bicomplex number.

1.3 Bicomplex valued metric space.

Choi et al.[17] defined the bicomplex valued metric space as follows:

***Definition* 1.3** *Let X be a nonempty set. Suppose the mapping $d : X \times X \to \mathbb{C}_2$ satisfies the following conditions:*

1. $0 \precsim_{i_2} d(x, y)$ *for all* $x, y \in X$,
2. $d(x, y) = 0$ *if and only if* $x = y$,
3. $d(x, y) = d(y, x)$ *for all* $x, y \in X$ *and*
4. $d(x, y) \precsim_{i_2} d(x, z) + d(z, y)$ *for all* $x, y, z \in X$.

Then d is called a bicomplex valued metric on X and (X, d) is called the bicomplex valued metric space.

***Definition* 1.4** *Let X be a nonempty set and let $s \geq 1$. Suppose the mapping $d : X \times X \to \mathbb{C}_2$ satisfies the following conditions:*

1. $0 \precsim_{i_2} d(x, y)$ *for all* $x, y \in X$,

2. $d(x,y) = 0$ *if and only if* $x = y$,
3. $d(x,y) = d(y,x)$ *for all* $x, y \in X$ *and*
4. $d(x,y) \precsim_{i_2} s\left[d(x,z) + d(z,y)\right]$ *for all* $x, y, z \in X$.

Then d is called a bicomplex valued b-metric on X and (X,d) *is called the bicomplex valued b-metric space.*

***Example* 1.1** *Let* $X = [0,1]$, *and consider the mapping* $d : X \times X \to \mathbb{C}_2$ *as defined by* $d(x,y) = (1 + i_1 + i_2 + i_1 i_2)\,|x-y|^2$.

Then for all $x, y, z \in X$,

$$\begin{aligned}
d(x,y) &= (1 + i_1 + i_2 + i_1 i_2)\,|x-y|^2 \\
&= (1 + i_1 + i_2 + i_1 i_2)\,|x-z+z-y|^2 \\
&= (1 + i_1 + i_2 + i_1 i_2)\left(|x-z|^2 + |z-y|^2 + 2\,|x-z|\,|z-y|\right) \\
&\precsim_{i_2} (1 + i_1 + i_2 + i_1 i_2)\left(|x-z|^2 + |z-y|^2 + |x-z|^2 + |z-y|^2\right) \\
&\precsim_{i_2} 2\left[d(x,z) + d(z,y)\right]
\end{aligned}$$

therefore (X,d) is a bicomplex valued b-metric space as $s = 2$.

***Definition* 1.5** (i). *Let* $A \subseteq X$ *and* $a \in A$ *is said to be an interior point of A if there exists a* $0 \prec_{i_2} r \in \mathbb{C}_2$ *such that*

$$B(a,r) = \{x \in X : d(a,x) \prec_{i_2} r\} \subseteq A$$

and the subset $A \subseteq X$ *is said to be an open set if each point of A is an interior point of A.*
(ii). *A point* $a \in X$ *is said to be a limit point of A if for all* $0 \prec_{i_2} r \in \mathbb{C}_2$ *such that*

$$B(a,r) \cap \{A - \{a\}\} \neq \phi$$

and the subset $A \subseteq X$ *is said to be a closed set if all the limit points of A belong to A.*
(iii). *The family*

$$F = \{B(a,r) : a \in X, 0 \prec_{i_2} r \in \mathbb{C}_2\}$$

is a sub-basis for a Hausdorff topology τ *on X.*

***Definition* 1.6** *For a bicomplex valued metric space* (X,d)

(i). *A sequence* $\{x_n\}$ *in X is said to be a convergent sequence and converges to a point x if for any* $0 \prec_{i_2} r \in \mathbb{C}_2$ *there is a natural number* $n_0 \in \mathbb{N}$ *such that* $d(x_n, x) \prec_{i_2} r$, *for all* $n > n_0$ *and we write* $\lim_{n\to\infty} x_n = x$ *or* $x_n \to x$ *as* $n \to \infty$.

(ii). *A sequence* $\{x_n\}$ *in X is said to be a Cauchy sequence in* (X,d) *if for any* $0 \prec_{i_2} r \in \mathbb{C}_2$ *there is a natural number* $n_0 \in \mathbb{N}$ *such that* $d(x_n, x_{n+m}) \prec_{i_2} r$, *for all* $m, n \in \mathbb{N}$ *and* $n > n_0$.

(iii). *If every cauchy sequence in X is convergent in X then* (X,d) *is said to be a complete bicomplex valued metric space.*

***Definition* 1.7** *Let* (X,d) *be a bicomplex valued metric space and* $S, T : X \to X$ *be two self-mappings then S and T are said to be compatible if* $\lim_{n\to\infty} d(STx, TSx_n) = 0$, *whenever* $\{x_n\}$ *is a sequence in X such that* $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = u$ *for some* $u \in X$.

***Definition* 1.8** *Let* $S, T : X \to X$ *be two self-mappings then S and T are said to be weakly compatible if* $STx = TSx$ *whenever* $Sx = Tx$*for all* $x \in X$.

***Definition* 1.9** *Let* $S, T : X \to X$ *be two self-mappings then, S and T are said to be commuting if* $TSx = STx$ *for all* $x \in X$.

***Definition* 1.10** *Let* (X,d) *be a bicomplex valued metric space and* $S, T : X \to X$ *be two Self-mappings then S and T are said to be weakly commuting if* $d(STx, TSx) \precsim_{i_2} d(Sx, Tx)$ *for all* $x \in X$.

***Definition* 1.11** *Let* (X,d) *be a cone metric space then the self-mapping* $T : X \to X$ *is said to be almost Jaggi contraction if it satisfies the following condition:*

$$(1.1)\quad d(Tx,Ty) \leq \alpha \frac{d(x,Tx)\,d(y,Ty)}{d(x,y)} + \beta d(x,y) + L \min\{d(x,Ty), d(y,Tx)\}$$

for all $x, y \in X$,*where* $L \geq 0$ *and* α, β *are non-negative real numbers with* $\alpha + \beta < 1$.

***Definition* 1.12** *Let (X, d) be a cone metric space then the self-mapping $T : X \to X$ is said to be Jaggi contraction if it satisfies the following condition:*

$$d(Tx, Ty) \leq \alpha \frac{d(x, Tx)\, d(y, Ty)}{d(x, y)} + \beta d(x, y)$$

for all $x, y \in X$, where $L \geq 0$ and α and β are non-negative real numbers with $\alpha + \beta < 1$.

***Definition* 1.13** *Let (X, d) be a complete complex valued $b-$metric space then the self-mapping $T : X \to X$ is said to be Jaggi contraction if it satisfies the following condition:*

$$d(Tx, Ty) \precsim \alpha \frac{d(x, Tx)\, d(y, Ty)}{d(x, y)} + \beta d(x, y)$$

for all $x, y \in X$, where α and β are non-negative real numbers with $s(\alpha + \beta) < 1$.

***Definition* 1.14** *Let (X, d) be a cone metric space then the self-mapping $T : X \to X$ is said to be Dass-Gupta contraction if it satisfies the following condition:*

$$d(Tx, Ty) \leq \alpha \frac{d(y, Ty)\,[1 + d(x, Tx)]}{1 + d(x, y)} + \beta d(x, y) + L \min\{d(x, Tx), d(x, Ty), d(y, Tx)\}$$

for all $x, y \in X$, where $L \geq 0$ and α, β are non-negative real numbers with $\alpha + \beta < 1$.

***Definition* 1.15** *Let (X, d) be a complete complex valued $b-$metric space with coefficient $s \geq 1$, then the self-mapping $T : X \to X$ is said to be Dass-Gupta contraction if it satisfies the following condition:*

$$d(Tx, Ty) \precsim \alpha \frac{d(y, Ty)\,[1 + d(x, Tx)]}{1 + d(x, y)} + \beta d(x, y) + L \min\{d(x, Tx), d(x, Ty), d(y, Tx)\}$$

for all $x, y \in X$, where $L \geq 0$ and α, β are non-negative real numbers with $s(\alpha + \beta) < 1$.

2 Lemmas.

In this section we present some lemmas which will be needed in the sequel.

***Lemma* 2.1** *[20] Let (X, d) be a bicomplex valued metric space and a sequence $\{x_n\}$ in X is said to be convergent to a point x if and only if $\lim_{n\to\infty} \|d(x_n, x)\| = 0$.*

***Lemma* 2.2** *[20] Let (X, d) be a bicomplex valued metric space and a sequence $\{x_n\}$ in X is said to be a Cauchy sequence in X if and only if $\lim_{n\to\infty} \|d(x_n, x_{n+m})\| = 0$.*

3 Main Theorems.

In this section we prove some fixed point theorems on bicomplex valued b-metric space for rational contraction.

***Theorem* 3.1** *Let (X, d) be a complete bicomplex valued b-metric space with the coefficient $s \geq 1$. Let the self-mapping $T : X \to X$ be almost Jaggi contraction satisfying the condition*

$$(3.1) \quad d(Tx, Ty) \precsim_{i_2} \alpha \frac{d(x, Tx)\, d(y, Ty)}{d(x, y)} + \beta d(x, y) + L \min\{d(x, Ty), d(y, Tx)\}.$$

for all $x, y \in X$ and $d(x, y)$ is nonsingular where $L \geq 0$ and α, β are non-negative real numbers with $s\left(\sqrt{2}\alpha + \beta\right) < 1$. Then T has a unique fixed point in X.

Proof. Let $\{x_n\}$ be a sequence in X such that

$$x_n = Tx_{n-1}, \text{ for all } n = 1, 2, ...$$

where x_0 is an arbitrary fixed point in X.
Therefore by using (3.1) we obtain that

$$\begin{aligned}
d(x_n, x_{n+1}) =& d(Tx_{n-1}, Tx_n) \\
\precsim_{i_2}& \alpha \frac{d(x_{n-1}, Tx_{n-1})\, d(x_n, Tx_n)}{d(x_{n-1}, x_n)} + \beta d(x_{n-1}, x_n) + \\
& L \min\{d(x_{n-1}, Tx_n), d(x_n, Tx_{n-1})\} \\
\precsim_{i_2}& \alpha \frac{d(x_{n-1}, x_n)\, d(x_n, x_{n+1})}{d(x_{n-1}, x_n)} + \beta d(x_{n-1}, x_n) +
\end{aligned}$$

$$L\min\{d(x_{n-1},x_{n+1}),d(x_n,x_n)\}$$
$$\precsim_{i_2}\alpha\frac{d(x_{n-1},x_n)\,d(x_n,x_{n+1})}{d(x_{n-1},x_n)}+\beta d(x_{n-1},x_n)$$

Hence
$$\|d(x_n,x_{n+1})\|\le\sqrt{2}\alpha\frac{\|d(x_{n-1},x_n)\|\,\|d(x_n,x_{n+1})\|}{\|d(x_{n-1},x_n)\|}+\beta\|d(x_{n-1},x_n)\|$$
$$\le\sqrt{2}\alpha\|d(x_n,x_{n+1})\|+\beta\|d(x_{n-1},x_n)\|,$$

i.e.,$\|d(x_n,x_{n+1})\|\le\frac{\beta}{1-\sqrt{2}\alpha}\|d(x_{n-1},x_n)\|,$

i.e.,$\|d(x_n,x_{n+1})\|\le h\|d(x_{n-1},x_n)\|,$

where $h=\frac{\beta}{1-\sqrt{2}\alpha}$ and $0\le h<1$, as $s\left(\sqrt{2}\alpha+\beta\right)<1$ and $s\ge1$. Therefore for all $n=1,2,3,...$
$$\|d(x_n,x_{n+1})\|\le h\|d(x_{n-1},x_n)\|\le h^2\|d(x_{n-2},x_{n-1})\|\le...\le h^n\|d(x_0,x_1)\|.$$

Thus

(3.2) $\|d(x_{n+1},x_{n+2})\|\le h^{n+1}\|d(x_0,x_1)\|.$

Since $s\left(\sqrt{2}\alpha+\beta\right)<1$ and $s\ge1\rightarrow sh=\frac{s\beta}{1-\sqrt{2}\alpha}<1.$

Then for any two positive integers m,n with $m>n$ we get that
$$d(x_n,x_m)\precsim_{i_2}s[d(x_n,x_{n+1})+d(x_{n+1},x_m)].$$

Therefore,
$$\|d(x_n,x_m)\|$$
$$\le s\|d(x_n,x_{n+1})\|+s\|d(x_{n+1},x_m)\|$$
$$\le s\|d(x_n,x_{n+1})\|+s^2\|d(x_{n+1},x_{n+2})\|+s^2\|d(x_{n+2},x_m)\|$$
$$\le s\|d(x_n,x_{n+1})\|+s^2\|d(x_{n+1},x_{n+2})\|+s^3\|d(x_{n+2},x_{n+3})\|+s^3\|d(x_{n+3},x_m)\|$$
$$\cdots\qquad\cdots\qquad\cdots$$
$$\cdots\qquad\cdots\qquad\cdots$$

i.e.,$\|d(x_n,x_m)\|\le s\|d(x_n,x_{n+1})\|+s^2\|d(x_{n+1},x_{n+2})\|+s^3\|d(x_{n+2},x_{n+3})\|+...+s^{m-n-1}\|d(x_{m-1},x_m)\|$
$$\le\left\{\begin{array}{c}\|d(x_n,x_{n+1})\|+s^2\|d(x_{n+1},x_{n+2})\|+\\ s^3\|d(x_{n+2},x_{n+3})\|+...+s^{m-n}\|d(x_{m-1},x_m)\|\end{array}\right\},\text{ as }s\ge1.$$

Therefore by using(3.2) we get that
$$\|d(x_n,x_m)\|\le sh^n\|d(x_0,x_1)\|+s^2h^{n+1}\|d(x_0,x_1)\|$$
$$+s^3h^{n+2}\|d(x_0,x_1)\|+...+s^{m-n}h^{m-1}\|d(x_0,x_1)\|$$

i.e., $\|d(x_n,x_m)\|\sum_{i=1}^{m-n}s^ih^{i+n-1}\|d(x_0,x_1)\|,$

i.e.,$\|d(x_n,x_m)\|\sum_{i=1}^{m-n}s^{i+n-1}h^{i+n-1}\|d(x_0,x_1)\|,$ as $s\ge1,$

i.e.,$\|d(x_n,x_m)\|\sum_{j=n}^{m-1}s^jh^j\|d(x_0,x_1)\|,$

i.e.,$\|d(x_n,x_m)\|\sum_{j=n}^{\infty}(sh)^j\|d(x_0,x_1)\|,$

i.e.,$\|d(x_n,x_m)\|\le\frac{(sh)^n}{1-sh}\|d(x_0,x_1)\|.$

Since $\frac{(sh)^n}{1-sh}\to0$ as $n\to\infty$, therefore for any $\varepsilon>0$ there exists a positive integer n_0 such that $\|d(x_n,x_m)\|<\varepsilon$, for all $m,n>n_0$. Hence $\{x_n\}$ is Cauchy in X. Since X is a complete bicomplex valued b-metric space, then there exists $u\in X$ such that $\lim\limits_{n\to\infty}x_n=u$.

Now we show that $u=Tu$, if not then there exists $0\prec_{i_2}\xi\in\mathbb{C}_2$ such that $d(u,Tu)=\xi$.

Therefore,
$$\xi=d(u,Tu)$$
$$\precsim_{i_2}sd(u,x_{n+1})+sd(x_{n+1},Tu)$$
$$\precsim_{i_2}sd(u,x_{n+1})+sd(Tx_n,Tu)$$
$$\precsim_{i_2}sd(u,x_{n+1})+s\alpha\frac{d(x_n,Tx_n)\,d(u,Tu)}{d(x_n,u)}+s\beta d(x_n,u)+sL\min\{d(x_n,Tu),d(u,Tx_n)\}$$
$$\precsim_{i_2}sd(u,x_{n+1})+s\alpha\frac{d(x_n,x_{n+1})\,\xi}{d(x_n,u)}+s\beta d(x_n,u)+sL\min\{d(x_n,Tu),d(u,x_{n+1})\}.$$

Therefore,

$$\|\xi\| \leq s\,\|d\,(u, x_{n+1})\| + s\,\sqrt{2}\alpha\frac{\|d\,(x_n, x_{n+1})\|\,\|\xi\|}{\|d\,(x_n, u)\|} + s\beta\,\|d\,(x_n, u)\| + s\,\|L \min\{d\,(x_n, Tu)\,, d\,(u, x_{n+1})\}\|\,.$$

Since $\lim_{n\to\infty} x_n = u$, taking limit both sides as $n \to \infty$ we get that $\|\xi\| \leq 0$, which is a contradiction, hence $\|\xi\| = 0 \to \|d\,(u, Tu)\| = 0 \to u = Tu$. Therefore u is a fixed point of T.

Now we show that T has a unique fixed point. If possible let $u^* \in X$ be another fixed point of T.

Then

$$\begin{aligned} d\,(u, u^*) &= d\,(Tu, Tu^*) \\ &\precsim_{i_2} \alpha\frac{d\,(u, Tu)\,d\,(u^*, Tu^*)}{d\,(u, u^*)} + \beta d\,(u, u^*) + L \min\{d\,(u, Tu^*)\,, d\,(u^*, Tu)\} \\ &\precsim_{i_2} \alpha\frac{d\,(u, u)\,d\,(u^*, u^*)}{d\,(u, u^*)} + \beta d\,(u, u^*) + L \min\{d\,(u, u^*)\,, d\,(u^*, u)\} \\ &\precsim_{i_2} (\beta + L)\,d\,(u, u^*) \end{aligned}$$

i.e., $\|d(u, u^*)\| \leq (\beta + L)\|d(u, u^*)\|$

i.e., $\|d(u, u^*)\| = 0$

i.e., $u = u^*$

This completes the proof of the ***Theorem*** **3.1.**

***Example* 3.1** *Let $X = [0, 1]$ and consider the mapping $d : X \times X \to \mathbb{C}_2$ defined by $d\,(x, y) = (1 + i_1 + i_2 + i_1 i_2)\,|x - y|^2$*

Then for all $x, y, z \in X$,we can easily show that

$$d(x, y) \precsim_{i_2} 2\,[d(x, z) + d(z, y)]$$

therefore (X, d) is a bicomplex valued b-metric space with $s = 2$.

Let us consider the mapping $T : X \to X$ by $Tx = \frac{x}{2}$, then

$$\begin{aligned} d\,(Tx, Ty) &= d\left(\frac{x}{2}, \frac{y}{2}\right) \\ &= (1 + i_1 + i_2 + i_1 i_2)\left|\frac{x}{2} - \frac{y}{2}\right|^2 \\ &= \frac{1}{4}\,(1 + i_1 + i_2 + i_1 i_2)\,|x - y|^2 \\ &= \frac{1}{4} d\,(x, y)\,. \end{aligned}$$

if we choose $\alpha = \frac{3}{16\sqrt{2}}$ and $\beta = \frac{1}{4}$ then $s\left(\sqrt{2}\alpha + \beta\right) = 2\left(\sqrt{2}\frac{3}{16\sqrt{2}} + \frac{1}{4}\right) = \frac{7}{8} < 1$ and for all $L \geq 0$ then all conditions of the ***Theorem*** **3.1** is satisfied. And clearly 0 is the unique fixed point of T.

***Corollary* 3.1** *Let (X, d) be a complete bicomplex valued b-metric space with the coefficient $s \geq 1$. Let the self-mapping $T : X \to X$ be Jaggi contraction satisfying the condition*

$$d\,(Tx, Ty) \precsim_{i_2} \alpha\frac{d\,(x, Tx)\,d\,(y, Ty)}{d\,(x, y)} + \beta d\,(x, y)$$

for all $x, y \in X$ and $d\,(x, y)$ is nonsingular where α, β are non-negative real numbers with $s\left(\sqrt{2}\alpha + \beta\right) < 1$. Then T has a unique fixed point in X.

Proof. This can be proved by taking $L = 0$ in ***Theorem*** **3.1.**

***Theorem* 3.2** *Let (X, d) be a complete bicomplex valued b-metric space with the coefficient $s \geq 1$. Let the mappings $S, T : X \to X$ be almost Jaggi contraction satisfying the condition*

(3.3) $$d\,(Sx, Ty) \precsim_{i_2} \alpha\frac{d\,(x, Sx)\,d\,(y, Ty)}{d\,(x, y)} + \beta d\,(x, y) + L \min\{d\,(x, Ty)\,, d\,(y, Sx)\}$$

for all $x, y \in X$ and $d\,(x, y)$ is nonsingular where $L \geq 0$ and α, β are non-negative real numbers with $s\left(\sqrt{2}\alpha + \beta\right) < 1$. Then the mappings S and T have a unique common fixed point in X.

Proof. Let $\{x_n\}$ be any sequence in X and x_0 be an arbitrary point in X. We define

$$x_{2n+1} = Sx_{2n}, \quad x_{2n+2} = Tx_{2n+1}, \quad n = 0, 1, 2, ...$$

Therefore by using (3.3) we obtain that

$$\begin{aligned}
d(x_{2n+1}, x_{2n+2}) &= d(Sx_{2n}, Tx_{2n+1}) \\
&\precsim_{i_2} \alpha \frac{d(x_{2n}, Sx_{2n})\, d(x_{2n+1}, Tx_{2n+1})}{d(x_{2n}, x_{2n+1})} + \beta d(x_{2n}, x_{2n+1}) + L \min\{d(x_{2n}, Tx_{2n+1}), d(x_{2n+1}, Sx_{2n})\} \\
&\precsim_{i_2} \alpha \frac{d(x_{2n}, x_{2n+1})\, d(x_{2n+1}, x_{2n+2})}{d(x_{2n}, x_{2n+1})} + \beta d(x_{2n}, x_{2n+1}) + L \min\{d(x_{2n}, x_{2n+2}), d(x_{2n+1}, x_{2n+1})\} \\
&\precsim_{i_2} \alpha \frac{d(x_{2n}, x_{2n+1})\, d(x_{2n+1}, x_{2n+2})}{d(x_{2n}, x_{2n+1})} + \beta d(x_{2n}, x_{2n+1}).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|d(x_{2n+1}, x_{2n+2})\| &\leq \sqrt{2}\alpha \frac{\|d(x_{2n}, x_{2n+1})\|\, \|d(x_{2n+1}, x_{2n+2})\|}{\|d(x_{2n}, x_{2n+1})\|} + \beta \|d(x_{2n}, x_{2n+1})\| \\
&\leq \sqrt{2}\alpha \|d(x_{2n+1}, x_{2n+2})\| + \beta \|d(x_{2n}, x_{2n+1})\|
\end{aligned}$$

(3.4) i.e., $\|d(x_{2n+1}, x_{2n+2})\| \leq \dfrac{\beta}{1 - \sqrt{2}\alpha} \|d(x_{2n}, x_{2n+1})\|$

Similarly we get that

(3.5) $\|d(x_{2n+2}, x_{2n+3})\| \leq \dfrac{\beta}{1 - \sqrt{2}\alpha} \|d(x_{2n+1}, x_{2n+2})\|$

From (3.4) and (3.5) we can say that

$$\|d(x_{n+1}, x_{n+2})\| \leq \frac{\beta}{1 - \sqrt{2}\alpha} \|d(x_n, x_{n+1})\|$$

Let $h = \frac{\beta}{1-\sqrt{2}\alpha}$. Then $0 \leq h < 1$, as $s\left(\sqrt{2}\alpha + \beta\right) < 1$ and $s \geq 1$. Therefore for all $n = 0, 1, 2, ...$

(3.6) $\|d(x_{n+1}, x_{n+2})\| \leq h\|d(x_n, x_{n+1})\| \leq h^2 \|d(x_{n-1}, x_n)\| \leq ... \leq h^{n+1} \|d(x_0, x_1)\|.$

Since $s\left(\sqrt{2}\alpha + \beta\right) < 1$ and $s \geq 1$, therefore $sh = \frac{s\beta}{1-\sqrt{2}\alpha} < 1$

Then for any two positive integers m, n with $m > n$, we obtain that

$$d(x_n, x_m) \precsim_{i_2} s[d(x_n, x_{n+1}) + d(x_{n+1}, x_m)].$$

Therefore,

$$\begin{aligned}
&\|d(x_n, x_m)\| \\
\leq& s\|d(x_n, x_{n+1})\| + s\|d(x_{n+1}, x_m)\| \\
\leq& s\|d(x_n, x_{n+1})\| + s^2 \|d(x_{n+1}, x_{n+2})\| + s^2 \|d(x_{n+2}, x_m)\| \\
\leq& s\|d(x_n, x_{n+1})\| + s^2 \|d(x_{n+1}, x_{n+2})\| + s^3 \|d(x_{n+2}, x_{n+3})\| + s^3 \|d(x_{n+3}, x_m)\| \\
&... \qquad ... \qquad ... \\
&... \qquad ... \qquad ...
\end{aligned}$$

$$\begin{aligned}
i.e., \|d(x_n, x_m)\| \leq& s\|d(x_n, x_{n+1})\| + s^2 \|d(x_{n+1}, x_{n+2})\| \\
&+ s^3 \|d(x_{n+2}, x_{n+3})\| + ... + s^{m-n-1} \|d(x_{m-1}, x_m)\| \\
\leq& \left\{ \begin{array}{c} \|d(x_n, x_{n+1})\| + s^2 \|d(x_{n+1}, x_{n+2})\| + \\ s^3 \|d(x_{n+2}, x_{n+3})\| + ... + s^{m-n} \|d(x_{m-1}, x_m)\| \end{array} \right\}, \text{as} \quad s \geq 1.
\end{aligned}$$

Therefore by using(3.6) we obtain that

$$\begin{aligned}
\|d(x_n, x_m)\| \leq& sh^n \|d(x_0, x_1)\| + s^2 h^{n+1} \|d(x_0, x_1)\| \\
&+ s^3 h^{n+2} \|d(x_0, x_1)\| + ... + s^{m-n} h^{m-1} \|d(x_0, x_1)\|
\end{aligned}$$

i.e., $\|d(x_n, x_m)\| \sum_{i=1}^{m-n} s^i h^{i+n-1} \|d(x_0, x_1)\|$

i.e.,$\|d(x_n, x_m)\| \sum_{i=1}^{m-n} s^{i+n-1} h^{i+n-1} \|d(x_0, x_1)\|$, as $\quad s \geq 1.$

i.e.,$\|d(x_n, x_m)\| \sum_{j=n}^{m-1} s^j h^j \|d(x_0, x_1)\|$

i.e., $\|d(x_n, x_m)\| \sum_{j=n}^{\infty} (sh)^j \|d(x_0, x_1)\|$

i.e.,$\|d(x_n, x_m)\| \leq \frac{(sh)^n}{1-sh} \|d(x_0, x_1)\|$.

Since $\frac{(sh)^n}{1-sh} \to 0$ as $n \to \infty$, therefore for any $\varepsilon > 0$ there exists a positive integer n_0 such that $\|d(x_n, x_m)\| < \varepsilon$, for all $m, n > n_0$. Hence $\{x_n\}$ is a Cauchy in X. Again since X is a complete bicomplex valued b-metric space, there exists a $u \in X$ such that $\lim_{n\to\infty} x_n = u$.

Now we show that $u = Su$, if not then there exists $0 \prec_{i_2} \xi \in \mathbb{C}_2$ such that $d(u, Su) = \xi$.

Therefore,

$$\begin{aligned}
\xi &= d(u, Su) \\
&\precsim_{i_2} sd(u, x_{2n+2}) + sd(x_{2n+2}, Su) \\
&\precsim_{i_2} sd(u, x_{2n+2}) + sd(Su, Tx_{2n+1}) \\
&\precsim_{i_2} sd(u, x_{2n+2}) + s\alpha \frac{d(u, Su)\, d(x_{2n+1}, Tx_{2n+1})}{d(u, x_{2n+1})} + s\beta d(u, x_{2n+1}) + sL \min\{d(u, Tx_{2n+1}), d(x_{2n+1}, Su)\} \\
&\precsim_{i_2} sd(u, x_{2n+2}) + s\alpha \frac{\xi d(x_{2n+1}, x_{2n+2})}{d(u, x_{2n+1})} + s\beta d(u, x_{2n+1}) + sL \min\{d(u, x_{2n+2}), d(x_{2n+1}, Su)\}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|\xi\| \leq & s\|d(u, x_{2n+2})\| + s\alpha \frac{\|\xi\|\, \|d(x_{2n+1}, x_{2n+2})\|}{\|d(u, x_{2n+1})\|} + \\
& s\beta \|d(u, x_{2n+1})\| + s\|L \min\{d(u, x_{2n+2}), d(x_{2n+1}, Su)\}\|.
\end{aligned}$$

Since $\lim_{n\to\infty} x_n = u$, taking limit both sides as $n \to \infty$ we get that $\|\xi\| \leq 0$, which is a contradiction, hence $\|\xi\| = 0 \to \|d(u, Su)\| = 0 \to u = Su$. Therefore u is a fixed point of S. Similarly we can show that $Tu = u$.

Now we show that S and T have a unique common fixed point. For this let $u^* \in X$ be another common fixed point of S and T, i.e. $Su^* = Tu^* = u^*$.

Then

$$\begin{aligned}
d(u, u^*) &= d(Tu, Tu^*) \\
&\precsim_{i_2} \alpha \frac{d(u^*, Tu^*)[1 + d(u, Tu)]}{1 + d(u, u^*)} + \beta d(u, u^*) + L \min\{d(u, Tu), d(u, Tu^*), d(u^*, Tu)\} \\
&\precsim_{i_2} \alpha \frac{d(u^*, u^*)[1 + d(u, u)]}{d(u, u^*)} + \beta d(u, u^*) + L \min\{d(u, u), d(u, u^*), d(u^*, u)\} \\
&\precsim_{i_2} \beta d(u, u^*)
\end{aligned}$$

i.e., $\|d(u, u^*)\| \leq \beta \|d(u, u^*)\|$

i.e., $\|d(u, u^*)\| = 0$

i.e., $u = u^*$

Hence the proof of the ***Theorem* 3.2**. is established.

***Theorem* 3.3** *Let (X, d) be a complete bicomplex valued $b-$metric space with coefficient $s \geq 1$. Let the self-mapping $T : X \to X$ be a Dass-Gupta contraction satisfying the condition*

$$(3.7)\quad d(Tx, Ty) \precsim_{i_2} \alpha \frac{d(y, Ty)[1 + d(x, Tx)]}{1 + d(x, y)} + \beta d(x, y) + L \min\{d(x, Tx), d(x, Ty), d(y, Tx)\}$$

for all $x, y \in X$ and $1 + d(x, y)$ be nonsingular, where $L \geq 0$ and α, β are non-negative real numbers with $s\left(\sqrt{2}\alpha + \beta\right) < 1$. Then T has a unique fixed point in X.

Proof. Let $\{x_n\}$ be a sequence in X such that

$$x_n = Tx_{n-1}, \text{ for all } n = 1, 2, ...$$

where x_0 is an arbitrary fixed point in X.

Therefore by using (3.7) we obtain that

$$\begin{aligned}
d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \\
&\precsim_{i_2} \alpha \frac{d(x_n, Tx_n)[1 + d(x_{n-1}, Tx_{n-1})]}{1 + d(x_{n-1}, x_n)} + \beta d(x_{n-1}, x_n) \\
&\quad + L \min\{d(x_{n-1}, Tx_{n-1}), d(x_{n-1}, Tx_n), d(x_n, Tx_{n-1})\}
\end{aligned}$$

$$\precsim_{i_2} \alpha \frac{d(x_n, x_{n+1})[1 + d(x_{n-1}, x_n)]}{1 + d(x_{n-1}, x_n)} + \beta d(x_{n-1}, x_n) +$$
$$L \min\{d(x_{n-1}, x_n), d(x_{n-1}, x_{n+1}), d(x_n, x_n)\}$$
$$\precsim_{i_2} \alpha \frac{d(x_n, x_{n+1})[1 + d(x_{n-1}, x_n)]}{1 + d(x_{n-1}, x_n)} + \beta d(x_{n-1}, x_n).$$

Therefore,

$$\|d(x_n, x_{n+1})\| \le \sqrt{2}\alpha \frac{\|d(x_n, x_{n+1})\| \|1 + d(x_{n-1}, x_n)\|}{\|1 + d(x_{n-1}, x_n)\|} + \beta \|d(x_{n-1}, x_n)\|$$
$$\le \sqrt{2}\alpha \|d(x_n, x_{n+1})\| + \beta \|d(x_{n-1}, x_n)\|,$$

i.e.,$\|d(x_n, x_{n+1})\| \le \frac{\beta}{1-\sqrt{2}\alpha} \|d(x_{n-1}, x_n)\|,$
i.e.,$\|d(x_n, x_{n+1})\| \le h \|d(x_{n-1}, x_n)\|,$
where $h = \frac{\beta}{1-\sqrt{2}\alpha}$, then $0 \le h < 1$, since $s\left(\sqrt{2}\alpha + \beta\right) < 1$ and $s \ge 1$. Therefore for all $n = 1, 2, 3, ...$

$$\|d(x_n, x_{n+1})\| \le h \|d(x_{n-1}, x_n)\| \le h^2 \|d(x_{n-2}, x_{n-1})\| \le ... \le h^n \|d(x_0, x_1)\|.$$

Therefore,

$$\|d(x_{n+1}, x_{n+2})\| \le h^{n+1} \|d(x_0, x_1)\|$$

Since $s(\alpha + \beta) < 1$ and $s \ge 1 \to sh = \frac{s\beta}{1-\sqrt{2}\alpha} < 1$

Then for any two positive integers m, n with $m > n$

$$d(x_n, x_m) \precsim_{i_2} s[d(x_n, x_{n+1}) + d(x_{n+1}, x_m)].$$

Therefore,

$$\|d(x_n, x_m)\|$$
$$\le s\|d(x_n, x_{n+1})\| + s\|d(x_{n+1}, x_m)\|$$
$$\le s\|d(x_n, x_{n+1})\| + s^2\|d(x_{n+1}, x_{n+2})\| + s^2\|d(x_{n+2}, x_m)\|$$
$$\le s\|d(x_n, x_{n+1})\| + s^2\|d(x_{n+1}, x_{n+2})\| + s^3\|d(x_{n+2}, x_{n+3})\| + s^3\|d(x_{n+3}, x_m)\|$$

... ...

... ...

$$i.e., \|d(x_n, x_m)\| \le s\|d(x_n, x_{n+1})\| + s^2\|d(x_{n+1}, x_{n+2})\| + s^3\|d(x_{n+2}, x_{n+3})\| + ... + s^{m-n-1}\|d(x_{m-1}, x_m)\|$$
$$\le \left\{ \begin{array}{c} \|d(x_n, x_{n+1})\| + s^2\|d(x_{n+1}, x_{n+2})\| + \\ s^3\|d(x_{n+2}, x_{n+3})\| + ... + s^{m-n}\|d(x_{m-1}, x_m)\| \end{array} \right\} \text{ as } s \ge 1.$$

Therefore by using (3.2) we get that

$$\|d(x_n, x_m)\| \le sh^n\|d(x_0, x_1)\| + s^2h^{n+1}\|d(x_0, x_1)\| + s^3h^{n+2}\|d(x_0, x_1)\| + ... + s^{m-n}h^{m-1}\|d(x_0, x_1)\|$$

i.e.,$\|d(x_n, x_m)\| \sum_{i=1}^{m-n} s^i h^{i+n-1} \|d(x_0, x_1)\|$
i.e.,$\|d(x_n, x_m)\| \sum_{i=1}^{m-n} s^{i+n-1} h^{i+n-1} \|d(x_0, x_1)\|$, since $s \ge 1$.
i.e., $\|d(x_n, x_m)\| \sum_{j=n}^{m-1} s^j h^j \|d(x_0, x_1)\|$
i.e., $\|d(x_n, x_m)\| \sum_{j=n}^{\infty} (sh)^j \|d(x_0, x_1)\|$
i.e., $\|d(x_n, x_m)\| \le \frac{(sh)^n}{1-sh} \|d(x_0, x_1)\|$.

Since $\frac{(sh)^n}{1-sh} \to 0$ as $n \to \infty$, therefore for any $\varepsilon > 0$ there exists a positive integer n_0 such that $\|d(x_n, x_m)\| < \varepsilon$, for all $m, n > n_0$. Hence $\{x_n\}$ is Cauchy in X. Since X is a complete bicomplex valued b-metric space, then there exists $u \in X$ such that $\lim_{n\to\infty} x_n = u$.

Now we show that $u = Tu$, if not then there exists $0 \prec_{i_2} \xi \in \mathbb{C}_2$ such that $d(u, Tu) = \xi$.

Therefore,

$$\xi = d(u, Tu)$$
$$\precsim_{i_2} sd(u, x_{n+1}) + sd(x_{n+1}, Tu)$$
$$\precsim_{i_2} sd(u, x_{n+1}) + sd(Tx_n, Tu)$$
$$\precsim_{i_2} sd(u, x_{n+1}) + s\alpha \frac{d(u, Tu)[1 + d(x_n, Tx_n)]}{1 + d(x_n, u)} +$$

$$s\beta d(x_n,u) + sL\min\{d(x_n,Tx_n), d(x_n,Tu), d(u,Tx_n)\}$$
$$\precsim_{i_2} sd(u,x_{n+1}) + s\alpha\frac{\xi[1+d(x_n,x_{n+1})]}{d(x_n,u)} + s\beta d(x_n,u) +$$
$$sL\min\{d(x_n,x_{n+1}), d(x_n,Tu), d(u,x_{n+1})\}.$$

Thus

$$\|\xi\| \leq s\|d(u,x_{n+1})\| + s\sqrt{2}\alpha\frac{\|\xi\|\,\|1+d(x_n,x_{n+1})\|}{\|1+d(x_n,u)\|} + s\beta\|d(x_n,u)\| +$$
$$s\|L\min\{d(x_n,x_{n+1}), d(x_n,Tu), d(u,x_{n+1})\}\|.$$

Since $\lim_{n\to\infty} x_n = u$, taking limit both sides as $n\to\infty$ we get that $\|\xi\| \leq s\alpha\|\xi\|$, which is a contradiction. Hence $\|\xi\| = 0 \to \|d(u,Tu)\| = 0 \to u = Tu$. Therefore u is a fixed point of T.

Now we show that T has a unique fixed point. For this let $u^* \in X$ be another fixed point of T.

Then

$$d(u,u^*) = d(Tu,Tu^*)$$
$$\precsim_{i_2} \alpha\frac{d(u^*,Tu^*)[1+d(u,Tu)]}{1+d(u,u^*)} + \beta d(u,u^*) +$$
$$L\min\{d(u,Tu), d(u,Tu^*), d(u^*,Tu)\}$$
$$\precsim_{i_2} \alpha\frac{d(u^*,u^*)[1+d(u,u)]}{d(u,u^*)} + \beta d(u,u^*) +$$
$$L\min\{d(u,u), d(u,u^*), d(u^*,u)\}$$
$$\precsim_{i_2} \beta d(u,u^*)$$

i.e., $\|d(u,u^*)\| \leq \beta\|d(u,u^*)\|$

i.e., $\|d(u,u^*)\| = 0$

i.e., $u = u^*$.

This completes the proof of the ***Theorem*** **3.3**.

Corollary **3.2** *Let (X,d) be a complete bicomplex valued $b-$metric space with coefficient $s \geq 1$. Let the self-mapping $T: X \to X$ be a Dass-Gupta rational contraction satisfying the condition*

$$(3.8)\quad d(Tx,Ty) \precsim_{i_2} \alpha\frac{d(y,Ty)[1+d(x,Tx)]}{1+d(x,y)} + \beta d(x,y)$$

for all $x,y \in X$ and $1+d(x,y)$ be non degenerated, where α,β are non-negative real numbers with $s\left(\sqrt{2}\alpha+\beta\right) < 1$. Then T has a unique fixed point in X.

Proof. This can be proved by taking $L = 0$ in ***Theorem*** **3.3**.

Corollary **3.3** *Let (X,d) be a complete bicomplex valued $b-$metric space with coefficient $s \geq 1$. Let the self-mapping $T: X \to X$ be a Dass-Gupta rational contraction satisfying the condition*

$$(3.9)\quad d(T^n x,T^n y) \precsim_{i_2} \alpha\frac{d(y,T^n y)[1+d(x,T^n x)]}{1+d(x,y)} + \beta d(x,y)$$

for all $x,y \in X$ and $1+d(x,y)$ be nonsingular, where α,β are non-negative real numbers with $s\left(\sqrt{2}\alpha+\beta\right) < 1$. Then T has a unique fixed point in X.

By ***Corollary*** **3.2** there exists a unique point $u \in X$ such that

$$T^n u = u.$$

Therefore,

$$d(Tu,u) = d(TT^n u,T^n u) = d(T^n Tu,T^n u) \precsim_{i_2} \alpha\frac{d(u,T^n u)[1+d(Tu,T^n Tu)]}{1+d(Tu,u)} + \beta d(Tu,u)$$

i.e., $d(Tu,u) \precsim_{i_2} \alpha\frac{d(u,u)[1+d(Tu,T^n Tu)]}{1+d(Tu,u)} + \beta d(Tu,u)$

i.e., $d(Tu,u) \precsim_{i_2} \alpha d(Tu,u)$

i.e., $\|d(Tu,u)\| \leq \alpha\|d(Tu,u)\|$

i.e., $\|d(Tu,u)\| = 0$

i.e., $Tu = u$.

This completes the proof of the ***Corollary*** **3.3**.

4 Future prospect.

In the line of the works as carried out in the paper one may think of the deduction of fixed point theorems using fuzzy metric, quasi metric, partial metric, probabilistic metric, p-adic metric (where p is a prime number), cone metric, quasi semi metric, convex metric, D-metric and other different types of metrics under the flavour of bicomplex analysis. This may be regarded as an active area of research to the future workers in this branch.

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(Dedicated to Honor Dr. R. C. Singh Chandel on His 75^{th} Birth Anniversary Celebrations)

COMPARATIVE STUDY OF WAVELET METHODS FOR SOLVING BERNOULLI'S EQUATIONS

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Abstract

A comparative study of two numerical techniques is presented for solving nonlinear differential equations of the Bernoulli's type. Proposed techniques are based on the conversion of nonlinear differential equations into linear differential equations by substituting particular factor and utilization of Haar wavelet collocation method *(HWCM)* and Hermite wavelet collocation method *(HeWCM)* to these linear equations. Searching for numerical solutions of such equations has attracted a considerable amount of research work where computer symbolic systems facilitate the computational work.

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1 Introduction

As nonlinear differential equations have many applications in real life problems, several numerical methods have been developed like Adomian decomposition method *(ADM)*, Homotopy Perturbation method *(HPM)*, Homotopy Analysis method *(HAM)*, Laplace transform method, B-splines methods and finite difference methods *(FDM)*. All these methods have huge procedure for solving nonlinear differential equations. Wavelets are more powerful tools for solving differential as well as integral equations in comparison to pre-existing classical methods. Numerical solutions of inverse euler-Bernoulli problem with integral overdetermination and periodic boundary conditions have been presented in [1]. In [2], Chebyshev collocation method has been presented for solving Voltra-Fredholm integro-differential equations. New *ICI* self-cancellation technique has been investigated to mitigate the effect of *ICI* in *FFT-OFDM* and compared with *DCT* based *OFDM* system in terms of bit error rate (*BER*) and carrier to interference ratio (*CIR*) in [3]. Problem of determining the time-dependent leading coefficient to the time derivative of heat equation with nonlocal boundary and integral conditions has been discussed in [4]. In [5], a constructive approach has been developed for solving system of linear and nonlinear fractional differential equations with the help of modified differential transform method and Adomian polynomials. Haar wavelet is simplest and more reliable in comparison to other members of wavelet family. Haar wavelets are not applied directly for solving differential equations due to some shortcomings and these shortcomings were removed by regularizing the piecewise constant Haar functions with interpolation splines [8, 9] or by expanding the highest derivative appearing in the differential equation into the Haar series and other derivatives are obtained through integrations [10, 11]. The first possibility was discarded because by using this technique, it is difficult to find the solution easily and simplicity of Haar wavelets gets lost. Haar wavelet based numerical schemes have been discussed in [6, 14, 15, 18, 19, 20] for solving differential and integral equations. In [7, 12, 13, 16, 17, 21], numerical techniques based on Hermite wavelet and collocation points have been discussed for solving various variety of differential and integral equations.

The main objective of this research, is to compare the two wavelets based numerical techniques for solving differential equations of the Bernoulli type. For this purpose Haar and Hermite wavelets are utilized. The mathematical formulation of such differential equation is:

$$\frac{dy}{dx} + P(x)y = Q(x).F(y), \tag{1.1}$$

where $F(y)$ is nonlinear function in y. The initial conditions is $y(0) = a$, a is constant.

2 Haar wavelets and their operational matrices

Haar functions are an orthogonal family of switched rectangular waveforms where amplitudes can differ from one function to another. Haar wavelet is a sequence of rescaled square shaped functions which together forms a wavelet family or basis. The Haar wavelet function $h_i(x)$ is defined in the interval $[\alpha, \gamma]$ as

$$(2.1)\quad h_i(x) = \begin{cases} 1, & \alpha \le x < \beta, \\ -1, & \beta \le x < \gamma, \\ 0, & \text{elsewhere}, \end{cases}$$

where $\alpha = \frac{k}{m}$, $\beta = \frac{k+0.5}{m}$, $\gamma = \frac{k+1}{m}$, $m = 2^j$ and $j = 0, 1, 2, ..., J$. J denotes the level of resolution. The integer $k = 0, 1, 2, ..., m-1$ is the translation parameter. The index i is calculated as: $i = m + k + 1$. The minimal value of $i = 2$ and the maximal value of $i = 2^{j+1}$.

The collocation points are calculated as

$$(2.2)\quad x_l = \frac{l - 0.5}{2M}, \quad l = 1, 2, 3, ..., 2M.$$

The operational matrix P, which is $2M \times 2M$, is calculated as below

$$(2.3)\quad P_{1,i}(x) = \int_0^x h_i(x)dx,$$

and

$$(2.4)\quad P_{n+1,i}(x) = \int_0^x P_{n,i}(x)dx, \quad n = 1, 2, 3, ...$$

From (2.3), we obtain:

$$(2.5)\quad P_{1,i}(x) = \begin{cases} x - \alpha, & \alpha \le x < \beta, \\ \gamma - x, & \beta \le x < \gamma, \\ 0, & \text{elsewhere}, \end{cases}$$

3 Hermite wavelets and their operational matrices

Wavelets constitute a family of functions from dilation and translation of a single function known as mother wavelet. The continuous variation of dilation parameter α and translation parameter β, form a family of continuous wavelets as:

$$(3.1)\quad \psi_{\alpha,\beta}(x) = |\,\alpha\,|^{-\frac{1}{2}}\psi\Big(\frac{x-\beta}{\alpha}\Big), \quad \alpha, \beta \in R, \quad \alpha \neq 0,$$

if the dilation and translation parameters are restricted to discrete values by setting $\alpha = {\alpha_0}^{-k}, \beta = n\beta_0{\alpha_0}^{-k}$, $\alpha_0 > 1$, $\beta_0 > 0$, we obtain the following family of discrete wavelets:

$$(3.2)\quad \psi_{k,n}(x) = |\,\alpha\,|^{-\frac{1}{2}}\psi({\alpha_0}^k x - n\beta_0), \quad \alpha, \beta \in R, \quad \alpha \neq 0,$$

where $\psi_{k,n}$, form a wavelet basis for $L^2(R)$. For special case, if $\alpha_0 = 2$ and $\beta_0 = 1$, then $\psi_{k,n}(x)$ forms an orthonormal basis. Hermite wavelets are defined as:

$$(3.3)\quad \psi_{n,m}(x) = \begin{cases} \frac{2^{\frac{k+1}{2}}}{\sqrt{\pi}} H_m(2^k m - 2n + 1), & \frac{n-1}{2^{k-1}} \le x < \frac{n}{2^{k-1}}, \\ 0, & \text{Otherwise}, \end{cases}$$

where $m = 0, 1, 2, 3, ..., M-1$ and $n = 1, 2, 3, ..., 2^{k-1}$ and k is assumed any positive integer. Also, H_m are Hermite polynomials of degree m with respect to weight function $W(x) = \sqrt{1-x^2}$ on the real line R and satisfies the following recurrence relation

$$(3.4)\quad H_{m+2}(x) = 2xH_{m+1}(x) - 2(m+1)H_m(x),$$

where $m = 0, 1, 2, ...,$ $H_0(x) = 1$ and $H_1(x) = 2x$.

3.1 Operational matrices of integration[21]

For $k = 1$ and $M = 6$, Assume the six basis functions on $[0, 1]$ as:

$$(3.5)\quad \begin{cases} \psi_{1,0}(x) = \frac{2}{\sqrt{\pi}}, \\ \psi_{1,1}(x) = \frac{2}{\sqrt{\pi}}(4x - 2), \\ \psi_{1,2}(x) = \frac{2}{\sqrt{\pi}}(16x^2 - 16x + 2), \\ \psi_{1,3}(x) = \frac{2}{\sqrt{\pi}}(64x^3 - 96x^2 + 36x - 2), \\ \psi_{1,4}(x) = \frac{2}{\sqrt{\pi}}(256x^4 - 512x^3 + 320x^2 - 64x + 2), \\ \psi_{1,5}(x) = \frac{2}{\sqrt{\pi}}(1024x^5 - 2560x^4 + 2240x^3 - 800x^2 + 100x - 2). \end{cases}$$

Let $\psi_6(x) = [\psi_{1,0}(x), \psi_{1,1}(x), \psi_{1,2}(x), \psi_{1,3}(x), \psi_{1,4}(x), \psi_{1,5}(x)]^T$. Integrating the above equations with respect to x, from 0 to x and after expressing in the matrix form, we obtain

$$(3.6)\quad \begin{cases} \int_0^x \psi_{1,0}(x)dx = \frac{2}{\sqrt{\pi}}x = \Big[\frac{1}{2}, \frac{1}{4}, 0, 0, 0, 0\Big]\psi_6(x), \\ \int_0^x \psi_{1,1}(x)dx = \frac{2}{\sqrt{\pi}}(2x^2 - 2x) = \Big[-\frac{1}{4}, 0, \frac{1}{8}, 0, 0, 0\Big]\psi_6(x), \\ \int_0^x \psi_{1,2}(x)dx = \frac{2}{\sqrt{\pi}}(\frac{16}{3}x^3 - 8x^2 + 2x) = \Big[-\frac{1}{3}, -\frac{1}{4}, 0, \frac{1}{12}, 0, 0\Big]\psi_6(x), \\ \int_0^x \psi_{1,3}(x)dx = \frac{2}{\sqrt{\pi}}(16x^4 - 32x^3 + 18x^2 - 2x) = \Big[\frac{1}{8}, 0, -\frac{1}{8}, 0, \frac{1}{16}, 0\Big]\psi_6(x), \\ \int_0^x \psi_{1,4}(x)dx = \frac{2}{\sqrt{\pi}}(\frac{256}{5}x^5 - \frac{512}{4}x^4 + \frac{320}{3}x^3 - 32x^2 + 2x) = \Big[-\frac{1}{15}, 0, 0, -\frac{1}{12}, 0, \frac{1}{20}\Big]\psi_6(x), \\ \int_0^x \psi_{1,5}(x)dx = \frac{2}{\sqrt{\pi}}(\frac{512}{3}x^6 - \frac{2560}{5}x^5 + \frac{1120}{2}x^4 - \frac{800}{3}x^3 + 50x^2 - 2x) = \Big[\frac{1}{24}, 0, 0, 0, -\frac{1}{16}, 0\Big]\psi_6(x). \end{cases}$$

Therefore,

$$(3.7)\quad \int_0^x \psi_6(x)dx = P_{6\times6}\psi_6(x) + \overline{\psi_6}(x),$$

where

$$(3.8)\quad P_{6\times6} = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & 0 & 0 & 0 & 0 \\ -\frac{1}{4} & 0 & \frac{1}{8} & 0 & 0 & 0 \\ -\frac{1}{3} & -\frac{1}{4} & 0 & \frac{1}{12} & 0 & 0 \\ \frac{1}{8} & 0 & -\frac{1}{8} & 0 & \frac{1}{16} & 0 \\ -\frac{1}{15} & 0 & 0 & -\frac{1}{12} & 0 & \frac{1}{20} \\ \frac{1}{24} & 0 & 0 & 0 & -\frac{1}{16} & 0 \end{pmatrix}$$

and

$$(3.9)\quad \bar{\psi_6}(x) = \Big(0, 0, 0, 0, 0, \frac{1}{24}\psi_{1,6}(x)\Big)^T.$$

Similarly integrating (3.7) with respect to x, from 0 to x, we obtain

$$(3.10)\quad \int_0^x \int_0^x \psi_6(x)dxdx = Q_{6\times6}\psi_6(x) + \bar{\bar{\psi_6}}(x),$$

where

$$(3.11)\quad Q_{6\times6} = \begin{pmatrix} \frac{3}{16} & \frac{1}{8} & \frac{1}{32} & 0 & 0 & 0 \\ -\frac{1}{6} & -\frac{3}{32} & 0 & \frac{1}{96} & 0 & 0 \\ -\frac{3}{32} & -\frac{1}{12} & -\frac{1}{24} & 0 & \frac{1}{192} & 0 \\ \frac{1}{10} & \frac{1}{16} & 0 & -\frac{1}{64} & 0 & \frac{1}{320} \\ -\frac{1}{24} & -\frac{1}{60} & \frac{1}{96} & 0 & -\frac{1}{120} & 0 \\ \frac{1}{42} & \frac{1}{96} & 0 & \frac{1}{192} & 0 & -\frac{1}{192} \end{pmatrix}$$

and

$$(3.12)\quad \bar{\bar{\psi_6}}(x) = \Big(0, 0, 0, 0, \frac{1}{480}\psi_{1,6}(x), \frac{1}{672}\psi_{1,7}(x)\Big)^T.$$

4 Function Approximation

4.1 Haar wavelet method

Consider any square integrable function $y(x)$ can be expanded in terms of infinite series of Haar basis functions as:

$$(4.1)\quad y(x) = \sum_{i=1}^{\infty} a_i h_i(x),$$

where a_i are constants of this infinite series, known as Haar wavelet coefficients. For numerical approximation the above infinite series is truncated upto finite terms as:

$$(4.2)\quad y(x) = \sum_{i=1}^{2M} a_i h_i(x) = A^T h(x),$$

where A and $h(x)$ are $2M \times 1$ matrices and are given by

$$(4.3)\quad A^T = [a_1, a_2, ..., a_{2M}],$$

and

$$(4.4)\quad h(x) = [h_1(x), h_2(x), ..., h_{2M}(x)]^T.$$

4.2 Hermite wavelet method

Consider any square integrable function $u(x)$ can be expanded in terms of infinite series of Hermite basis functions as:

$$(4.5)\quad u(x) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} C_{n,m}\psi_{n,m}(x),$$

where $C_{n,m}$ are constants of this infinite series, known as Hermite wavelet coefficients. For numerical approximation the above infinite series is truncated upto finite terms as:

$$(4.6)\quad u(x) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} C_{n,m}\psi_{n,m}(x) = C^T\Psi(x),$$

where C and Ψ are $2^{k-1}M \times 1$ matrices and are given by

$$(4.7)\quad C^T = [C_{1,0}, ..., C_{1,M-1}, ..., C_{2^{k-1},0}, ..., C_{2^{k-1},M-1}]$$

and

$$(4.8)\quad \Psi = [\psi_{1,0}, ..., \psi_{1,M-1}, ..., \psi_{2^{k-1},0}, ..., \psi_{2^{k-1},M-1}]^T.$$

5 Proposed methods for solving Bernoulli's equation

Consider the Bernoulli's equation

(5.1) $\dfrac{dy}{dx} + P(x)y = Q(x).F(y),$

where $F(y)$ is a nonlinear term. Above equation (5.1) is nonlinear differential equation. Divide (5.1) with function $F(y)$. The transformed equation is

(5.2) $F_1(y)\dfrac{dy}{dx} + P(x)F_2(y) = Q(x),$

where F_1, F_2 are functions of y. Putting $F_2(y) = z$ in (5.2), we get

(5.3) $\dfrac{dz}{dx} + P_1(x)z = Q_1(x),$

where P_1 and Q_1 are new functions of x. Equation (5.3) is linear differential equation with initial condition $z(0) = b$, where b is constant.

5.1 Haar wavelet collocation method

(*HWCM*) Consider the wavelet approximation

(5.4) $\dfrac{dz}{dx} = \sum_{i=1}^{2M} a_i h_i(x).$

Integrating (5.4) with respect to x, from 0 to x, we get

(5.5) $z(x) = z(0) + \sum_{i=1}^{2M} a_i P_{1,i}(x).$

Substituting (5.4) and (5.5) in (5.3), and applying initial conditions, we get

(5.6) $\sum_{i=1}^{2M} a_i\Big[h_i(x) + P_1(x)P_{1,i}(x)\Big] = Q_1(x) - b.P_1(x).$

From (5.6), we get Haar wavelet coefficient. The Haar wavelet solution $z(x)$ is obtained by substituting the values of wavelet coefficients into (5.5). The solution of (5.1) is obtained from the relation $y = {F_2}^{-1}(z)$.

5.2 Hermite wavelet collocation method

(HeWCM) Consider the wavelet approximation

(5.7) $\dfrac{dz}{dx} = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} C_{n,m}\psi_{n,m}(x) = C^T\Psi(x),$

Integrating (5.7) with respect to x, from 0 to x, we get

(5.8) $z(x) = z(0) + \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} C_{n,m}\int_0^x \psi_{n,m}(x)dx.$

Substituting (5.7) and (5.8) in (5.3), and applying initial conditions, we get

(5.9) $\sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} C_{n,m}\Big[\psi_{n,m}(x) + P_1(x)\int_0^x \psi_{n,m}(x)dx\Big] = Q_1(x) - b.P_1(x).$

From (5.9), we get Hermite wavelet coefficient. The Hermite wavelet solution $z(x)$ is obtained by substituting the values of wavelet coefficients into (5.8). The solution of (5.1) is obtained from the relation $y = {F_2}^{-1}(z)$.

6 Numerical Observations

We present here, numerical examples for solving some nonlinear differential equations, to illustrate the accuracy of the proposed method with the aid of two efficient techniques such as Haar wavelet method and Hermite wavelet method.

Example 6.1: Consider the nonlinear differential equation

(6.1) $\dfrac{dy}{dx} + x\sin 2y = x^3\cos^2 y,$

with initial condition $y(0) = 0$. The exact solution of the equation is

(6.2) $\tan y = \dfrac{1}{2}(x^2 - 1) + \dfrac{1}{2}e^{-x^2}.$

Table **6.1:** Numerical solutions of ***Example*** 6.1 for $2M = 8$.

x	Exact solution	Haar wavelet solution	Hermite wavelet solution
1/16	$3.8097e-006$	$1.5141e-005$	$4.5026e-006$
3/16	$3.0540e-004$	$4.3214e-004$	$3.0590e-004$
5/16	$2.3084e-003$	$2.6383e-003$	$2.3089e-003$
7/16	$8.6014e-003$	$9.1744e-003$	$8.6019e-003$
9/16	$2.2581e-002$	$2.3382e-002$	$2.2581e-002$
11/16	$4.7963e-002$	$4.8928e-002$	$4.7964e-002$
13/16	$8.8234e-002$	$8.9264e-002$	$8.8234e-002$
15/16	$1.4602e-001$	$1.4701e-001$	$1.4602e-001$

Table* 6.2:** Comparison of absolute errors of ***Example 6.1 for $2M = 8$.

x	Absolute errors for Haar wavelet	Absolute errors for Hermite wavelet
1/16	$1.1331e-005$	$6.9287e-007$
3/16	$1.2674e-004$	$4.9578e-007$
5/16	$3.2985e-004$	$5.0176e-007$
7/16	$5.7292e-004$	$4.4160e-007$
9/16	$8.0075e-004$	$4.0076e-007$
11/16	$9.6413e-004$	$3.2832e-007$
13/16	$1.0300e-003$	$3.0043e-007$
15/16	$9.8735e-004$	$1.0734e-007$

***Table* 6.1** represents the comparison of numerical solutions obtained by Haar and Hermite wavelet methods with exact solution of ***Example* 6.1**. ***Table* 6.2** represents the comparison of absolute errors obtained by Haar wavelet method and Hermite wavelet method. ***Figure* 6.1** and ***Figure* 6.2** show the absolute errors of ***Example* 6.1** for $2M = 8$.

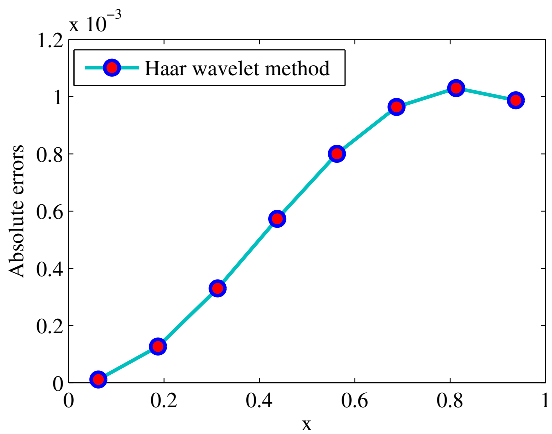

***Figure* 6.1:** Absolute errors of ***Example* 6.1** for $2M = 8$.

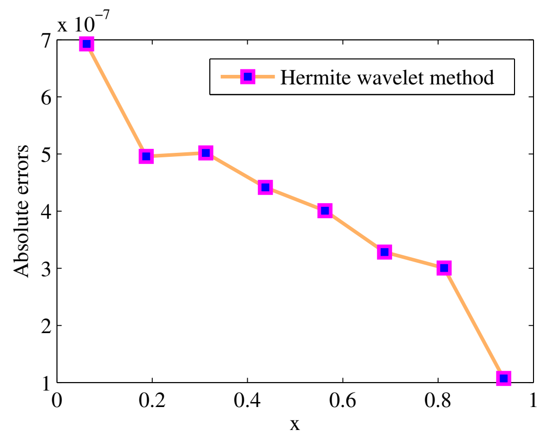

***Figure* 6.2:** Absolute errors of ***Example* 6.1** for $2M = 8$.

***Example* 6.2**: Consider the nonlinear differential equation

$$e^y\left(\frac{dy}{dx} + 1\right) = e^x, \tag{6.3}$$

with initial condition $y(0) = 0$. The exact solution of the equation is

$$e^y = \frac{1}{2}(e^x + e^{-x}). \tag{6.4}$$

***Table* 6.3:** Numerical solutions of ***Example* 6.2** for $2M = 8$.

x	Exact solution	Haar wavelet solution	Hermite wavelet solution
1/16	$1.9519e-003$	$3.7866e-003$	$1.9519e-003$
3/16	$1.7476e-002$	$1.9089e-002$	$1.7476e-002$
5/16	$4.8053e-002$	$4.9471e-002$	$4.8053e-002$
7/16	$9.2797e-002$	$9.4047e-002$	$9.2797e-002$
9/16	$1.5050e-001$	$1.5161e-001$	$1.5050e-001$
11/16	$2.1977e-001$	$2.2076e-001$	$2.1977e-001$
13/16	$2.9910e-001$	$3.0000e-001$	$2.9910e-001$
15/16	$3.8703e-001$	$3.8786e-001$	$3.8703e-001$

***Table* 6.4:** Comparison of absolute errors of ***Example* 6.2** for $2M = 8$.

x	Absolute errors for Haar wavelet	Absolute errors for Hermite wavelet
1/16	$1.8348e-003$	$1.1023e-010$
3/16	$1.6130e-003$	$6.6898e-011$
5/16	$1.4174e-003$	$6.3810e-011$
7/16	$1.2496e-003$	$5.1481e-011$
9/16	$1.1095e-003$	$4.5878e-011$
11/16	$9.9544e-004$	$3.6280e-011$
13/16	$9.0478e-004$	$3.7286e-011$
15/16	$8.3425e-004$	$5.4857e-012$

***Table* 6.3** represents the comparison of numerical solutions obtained by Haar and Hermite wavelet methods with exact solution of ***Example* 6.2**. ***Table* 6.4** represents the comparison of absolute errors obtained by Haar wavelet method and Hermite wavelet method. ***Figure* 6.3** and ***Figure* 6.4** show the absolute errors of ***Example* 6.2** for $2M = 8$.

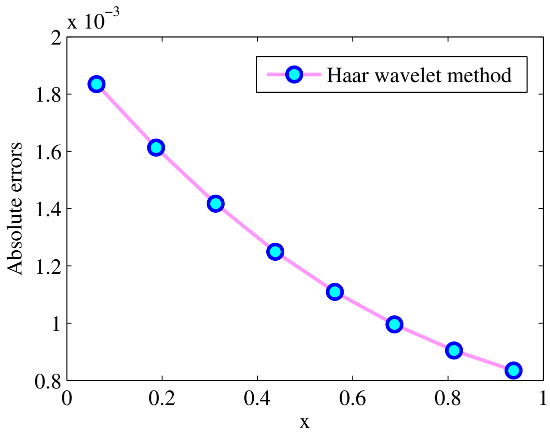

***Figure* 6.3:** Absolute errors of ***Example* 6.2** for $2M = 8$.

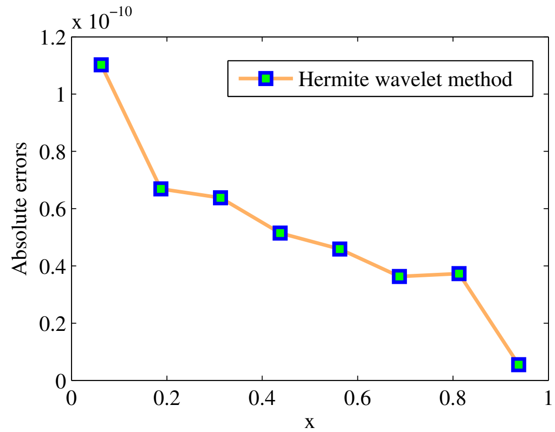

***Figure* 6.4:** Absolute errors of ***Example* 6.2** for $2M = 8$.

***Example* 6.3**: Consider the nonlinear differential equation

(6.5) $$xy(1 + xy^2)\frac{dy}{dx} = 1,$$

with initial condition $x(0) = 1$. The exact solution of the equation is

(6.6) $$\frac{1}{x} = (2 - y^2) - e^{-\frac{y^2}{2}}.$$

***Table* 6.5:** Numerical solutions of ***Example* 6.3** for $2M = 8$.

y	Exact solution	Haar wavelet solution	Hermite wavelet solution
1/16	1.0020	1.0039	1.0020
3/16	1.0181	1.0201	1.0181
5/16	1.0526	1.0549	1.0526
7/16	1.1113	1.1139	1.1113
9/16	1.2049	1.2081	1.2049
11/16	1.3553	1.3594	1.3553
13/16	1.6104	1.6162	1.6104
15/16	2.0977	2.1075	2.0977

Table **6.6**: Comparison of absolute errors of ***Example* 6.3** for $2M = 8$.

y	Absolute errors for Haar wavelet	Absolute errors for Hermite wavelet
1/16	$1.9627e-003$	$7.1279e-008$
3/16	$2.0568e-003$	$5.3094e-008$
5/16	$2.2577e-003$	$5.9452e-008$
7/16	$2.5976e-003$	$6.1041e-008$
9/16	$3.1475e-003$	$6.9398e-008$
11/16	$4.0705e-003$	$7.7959e-008$
13/16	$5.7923e-003$	$1.1056e-007$
15/16	$9.7275e-003$	$8.6224e-008$

***Table* 6.5** represents the comparison of numerical solutions obtained by Haar and Hermite wavelet methods with exact solution of ***Example* 6.3**. ***Table* 6.6** represents the comparison of absolute errors obtained by Haar wavelet method and Hermite wavelet method. ***Figure* 6.5** and ***Figure* 6.6** show the absolute errors of ***Example* 6.3** for $2M = 8$.

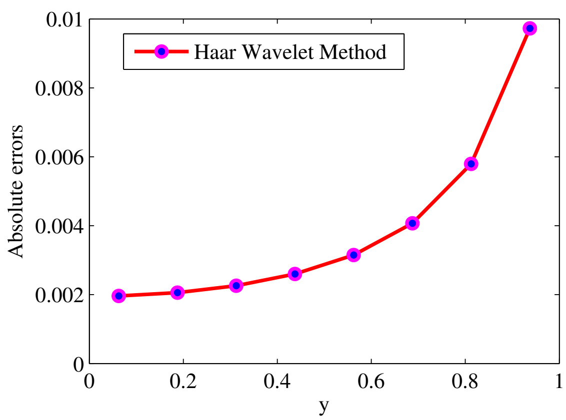

Figure **6.5**: Absolute errors of ***Example* 6.3** for $2M = 8$.

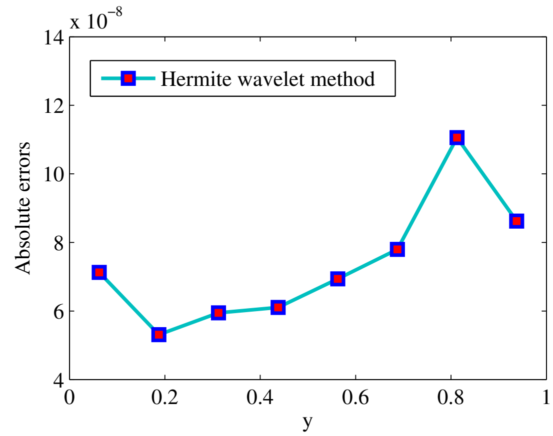

Figure **6.6**: Absolute errors of ***Example* 6.3** for $2M = 8$.

7 Conclusion

From above discussion, it is concluded that the Hermite wavelet based collocation is much better in comparison to Haar wavelet based collocation method for solving nonlinear differential equations of the Bernoulli's type. For getting the necessary accuracy the number of collocation points may be increased.

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(Dedicated to Honor Dr. R. C. Singh Chandel on His 75^{th} Birth Anniversary Celebrations)

APPLICATION OF INTERVAL VALUED FUZZY SET AND SOFT SET

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Abstract

Molodtsov was a father of soft set approach. We can't easily settle the membership degree in some practical application. So it must be much better to describe interval-valued data instead of explaining membership degree. In this paper, we introduce the latest approach of the interval-valued fuzzy soft set by combining the interval-valued fuzzy set and soft set models. This approach successfully follows distributive, associative and DeMorgans laws as well. In the end, a decision problem is solved by this approach.
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1 Introduction

In 1999 a great researcher Molodtsov was born in Russia who gave an approach which is known as a soft set approach. There are many approach which deal with uncertainty for example, probability, the theory of fuzzy set etc. But this approach is different from all these approach. Molodtsov applied this approach and got positive results.

Many researchers are working on it, because of this approach has many application in several directions. Many operations applied on soft sets by Maji et al. [21]. This approach has also solved many judgment constructing issue by adopting fuzzy mathematics [20]. The characterization devaluation of soft sets analyzed by Chen at al. [10]. The main objective of Kong at al. [16, 17, 18] is to show a devaluation of soft sets and fuzzy soft sets. They also explained algorithm of normal parameter reduction.

By above analysis, we observed that classical soft set [26, 28] approach is the root of all above work. We link this model with any mathematical model. Aktas et al. [1] obtained elementary properties of soft groups by this approach. Feng et al. [11] suggested approach of soft semirings, soft subsemirings, soft ideals, idealistic soft semirings, soft semiring homomorphisms and their characteristics. Approach of soft BCK/BCI-algebras and soft subalgebras are presented by Jun [13, 14, 15]. Maji et al. [20] suggested this approach. It is a amalgam of fuzzy set and soft set models. Yange et al. [29] applied some process on soft sets, these were negation, triangular norm etc. Zou et al. [31] presented soft set, fuzzy soft set into the insufficient situation.

In fuzzy mathematics, The utility of fuzzy set theory in handling uncertainty, arising from deficiencies of information available from a situation in pattern recognition problems have been proposed by many researchers. Controversy has surrounded the concept of fuzziness since its inception. Some maintains that probability theory can handle any kind of uncertainty; other think that fuzziness is probability in disguise or that probability is only sensible way to describe any kind of uncertainty. This theory provide an approximate, effective and more flexible means of classifying the patterns which are too complex or have to ill-defined features to be handled by the classical approach. Both fuzzy logic and probability theory are closely related, the key difference is their meaning. Probability is associated with events and not facts, and those events will either occur or not occur. There is nothing fuzzy about it. Fuzzy Logic is all about degree of truth.

The merging of interval-valued fuzzy set and soft set are shown here. So we get latest soft set model which is Interval-valued Fuzzy Soft Set. To make this work easy, Section 2 having detail about standard soft set and fuzzy soft set. Section 3 having approach about interval-valued fuzzy soft set. We then apply the concept of complement, *AND* and *OR* on interval-valued fuzzy soft sets. Section 4, show that interval-valued fuzzy soft set successfully works on judgment constructing issue. At last we then give summary and direction for new research.We observed these hotels by net and saw their websites. First we saw these attributes and got the rating of all these attributes from their websites.

Triangular norms and conorms are operations which generalize the logical conjunction and logical disjunction to fuzzy logic. They are a natural interpretation of the conjunction and disjunction in the semantics of mathematical fuzzy logics and they are used to combine criteria in multi-criteria decision making.

2 Preliminaries

A fuzzy soft relation is defined as soft set over the fuzzy power set of the Cartesian product of two crisp sets.

By this process Molodtsov [22] explained soft set. Suppose M and N indicate universal set, parameter set respectively. We show soft set by this pair (F, N). In this case F indicates a function of N towards all subsets of set M.

Now make all subsets of M and indicated by $P(M)$, F mapping shown below [10].

$$F : N \to P(M),$$

$\forall n \in N,\ F(n)$ indicates the family of n-approximate members of soft sets (F, N) [2, 7, 23].

Suppose assemblage of every subsets of M can denote by $P(M)$. So fuzzy soft set for $\mathcal{P}(M)$ as reported by $(\tilde{F}, N)$. Here $\tilde{F}$ transformation represented.

$$\tilde{F} \colon N \to \mathcal{P}(M).$$

It indicates the parameterized group of fuzzy subsets of M. It is mapping from parameters to a universe, so this indicates the particular process of soft set.

In general, fuzzy value set for n is indicated by $\tilde{F}(n)$ and it is the subset of M. We can differentiate it from the classic soft set, suppose $(\mu_{\tilde{F}(n)}(HO))$ indicate the membership degree of element HO, which holds the parameter n here. $HO \in M$,$n \in N$. $\tilde{F}$ given below.

$$\tilde{F}(n) = \left\{(HO, \mu_{\tilde{F}(n)}(HO)) \colon HO \in M\right\}.$$

***Definition* 2.1** *The AND operation on the two interval-valued fuzzy soft sets* $(\tilde{F}, A)$, $(\tilde{G}, B)$ *is defined by*

$$(\tilde{F}, A) \wedge (\tilde{G}, B) = (\tilde{H}, A \times B),$$

where $\tilde{H}(\alpha,\beta) = \tilde{F}(\alpha) \cap \tilde{G}(\beta), \forall(\alpha,\beta) \in (A \times B)$.

***Definition* 2.2** *The OR operation on the two interval-valued fuzzy soft sets* $(\tilde{F}, A)$, $(\tilde{G}, B)$ *is defined by*

$$(\tilde{F}, A) \vee (\tilde{G}, B) = (\tilde{H}, A \times B),$$

where $\tilde{H}(\alpha,\beta) = \tilde{F}(\alpha) \cup \tilde{G}(\beta), \forall(\alpha,\beta) \in (A \times B)$.

3 Interval-valued fuzzy soft set

3.1 Approach of interval-valued fuzzy soft set

It can obtained with the help of fuzzy set and soft set theory definitely. Note that we can't easily settle the membership degree in some practical application. So it must be much better to describe interval-valued data instead of explaining membership degree.Due to this approach, Zadeh described this approach. Interval-valued fuzzy soft set model can be described by joining interval-valued fuzzy set and soft set. Now, we shortly suggest this theory. Mapping of interval-valued fuzzy set [8, 9, 30] $\hat{S}$ on the universe M.

$$\hat{S} : M \to Int([0, 1]),$$

Here, collection of every closest subintervals of [0, 1] is indicated by Int([0, 1]). Group of all interval-valued fuzzy sets on M is denoted by $\mathcal{P}(M)$.

Let $\hat{S} \in \tilde{P}(M), \forall m \in M$.

The degree of membership is given by $\mu_{\hat{S}}(HO) = [\mu^-_{\hat{S}}(HO), \mu^+_{\hat{S}}(HO)]$ of element HO to $\hat{S}$.

$\mu^-_{\hat{S}}(HO), \mu^+_{\hat{S}}(HO)$ Indicate the lower and upper membership degree of HO to $\hat{S}$. Here

$$\mu^-_{\hat{S}}(HO), \mu^+_{\hat{S}}(HO).$$

We can describe complement, intersection and other operation of the interval-valued fuzzy set here.

Suppose $\hat{S}$, $\hat{T} \in \tilde{\mathcal{P}}$(M)

1. $\hat{S}^C$ indicates the complement of $\hat{S}$
   $\mu_{\hat{S}^C}(HO) = 1 - \mu_{\hat{S}}(HO) = [1 - \mu^+_{\hat{S}}(HO), 1 - \mu^-_{\hat{S}}(HO)]$;
2. $\hat{S} \cap \hat{T}$ Indicates the intersection of $\hat{S}$ and $\hat{T}$
   $\mu_{\hat{S}\cap\hat{T}}(HO) = inf[\mu_{\hat{S}}(HO), \mu_{\hat{T}}(HO)]$
   $= [inf(\mu^-_{\hat{S}}(HO), \mu^-_{\hat{T}}(HO)), inf(\mu^+_{\hat{S}}(HO), \mu^+_{\hat{T}}(HO))]$;

***Table* 3.1:** $(I\tilde{v}f_1, \Omega_1)$ serve as interval valued fuzzy soft set.

M	n_1	n_2	n_3	n_4
HO_1	[0.6 – 0.8]	[0.4 – 0.6]	[0.6 – 0.8]	[0.7 – 0.9]
HO_2	[0.6 – 0.8]	[0.5 – 0.7]	[0.6 – 0.8]	[0.4 – 0.6]
HO_3	[0.8 – 1.0]	[0.7 – 0.9]	[0.8 – 1.0]	[0.8 – 1.0]
HO_4	[0.8 – 1.0]	[0.6 – 0.8]	[0.8 – 1.0]	[0.8 – 1.0]
HO_5	[0.6 – 0.8]	[0.5 – 0.7]	[0.7 – 0.9]	[0.7 – 0.9]
HO_6	[0.6 – 0.8]	[0.5 – 0.7]	[0.6 – 0.8]	[0.7 – 0.9]

***Table* 3.2:** $(I\tilde{v}f_2, \Omega_2)$ serve as interval valued fuzzy soft set.

M	σ_1	σ_2	σ_3
HO_1	[0.6 – 0.8]	[0.4 – 0.6]	[0.6 – 0.8]
HO_2	[0.6 – 0.8]	[0.4 – 0.6]	[0.6 – 0.8]
HO_3	[0.8 – 1.0]	[0.7 – 0.9]	[0.8 – 1.0]
HO_4	[0.8 – 1.0]	[0.7 – 0.9]	[0.8 – 1.0]
HO_5	[0.7 – 0.9]	[0.6 – 0.8]	[0.7 – 0.9]
HO_6	[0.6 – 0.8]	[0.4 – 0.6]	[0.6 – 0.8]

3. $\hat{S} \cup \hat{T}$ indicates the union of $\hat{S}$ and $\hat{T}$.
   $\mu_{\hat{S}\cup\hat{T}}(HO) = sup[\mu_{\hat{S}}(HO), \mu_{\hat{T}}(HO)] = [sup(\mu^-_{\hat{S}}(HO), \mu^-_{\hat{T}}(HO)), sup(\mu^+_{\hat{S}}(HO), \mu^+_{\hat{T}}(HO))]$.

Suppose M and N indicate the universal set, parameters set respectively, an ordered pair $(I\tilde{v}f_1, N)$ indicates interval-valued fuzzy soft set for $\tilde{\mathcal{P}}(M)$, here $I\tilde{v}f_1$ defined below [3, 4, 5].

$$I\tilde{v}f_1 : N \to \tilde{\mathcal{P}}(M).$$

By parameterized family of subsets of M we prepare interval-valued fuzzy soft set. Hence, its universe is the set of all interval-valued fuzzy sets of M, i.e. $\tilde{\mathcal{P}}(M)$.
$\forall n \in N$, the interval fuzzy value set of guideline represented by $I\tilde{v}f_1(n)$, which indicates interval-valued fuzzy set of M where $HO \in M$ and $n \in N$, we write in the form:

$$I\tilde{v}f_1(n) = \{\langle HO, \mu_{I\tilde{v}f_1(n)}(HO)\rangle : HO \in M\},$$

here, $I\tilde{v}f_1(n)$ is the interval-valued fuzzy membership degree of object HO which holds on guideline $n \; \forall \; n \in N$, $\forall$ $HO \in M$, $\mu^-_{I\tilde{v}f_1(n)}(HO) = \mu^+_{I\tilde{v}f_1(n)}(HO)$, then $I\tilde{v}f_1(n)$ will degenerated to be a standard fuzzy set and then $(I\tilde{v}f_1, N)$ will be degenerated to be a traditional fuzzy soft set. Let us consider,

- M is the collection of hotel according to choice and M=$\{HO_1, HO_2, HO_3, HO_4, HO_5, HO_6\}$
- Ω_1 is the collection of attributes of hotel,
  Ω_1=$\{n_1, n_2, n_3, n_4\}$=\{Staff, Value for Money, Facilities, Location\}

An interval valued fuzzy soft set $(I\tilde{v}f_1, \Omega_1)$ has given in ***Table* 3.1**.

3.2 Source of Numerical Data

The lower and upper limits of an evaluation is given. For example, we cannot present the precise degree of how beautiful hotel HO_1 is, yet, hotel HO_1 is at least beautiful on the degree of 0.6 and it is at most beautiful on the degree of 0.8. According to these attributes. By questionnaire to 150 visiters who visites these hotels, we collected membership value of these attributes. For example: By net we obtained the staff rating of hotel Le Meridien is 8.7. So 90 percent visiters assign this [0.6-0.8] and 10 percent [0.7-0.9] then we give [0.6-0.8] vale to the staff attribute. likewise, we make interval valued fuzzy soft sets for all attributes of all hotels.

Let $(I\tilde{v}f_1, N)$ indicates interval-valued fuzzy soft set for $\tilde{\mathcal{P}}$(M), $I\tilde{v}f_1$(n) indicates interval fuzzy value set for n, then all interval fuzzy value sets in interval-valued fuzzy soft set $(I\tilde{v}f_1, N)$ are specify to as the interval fuzzy value class of $(I\tilde{v}f_1, N)$, $G_{(I\tilde{v}f_1,N)}$ symbolic representation of it and presented as follows [19, 24, 25].

$$G_{(I\tilde{v}f_1,N)} = \{I\tilde{v}f_1(n) \colon n \in N\}.$$

Table **3.1**, we got $G_{(I\tilde{v}f_1,\Omega_1)} = \{I\tilde{v}f_1(n_1), I\tilde{v}f_1(n_2), I\tilde{v}f_1(n_3), I\tilde{v}f_1(n_4)\}$here:
$I\tilde{v}f_1(n_1) = \{(HO_1, [0.6-0.8]), (HO_2, [0.6-0.8]), (HO_3, [0.8-1.0]), (HO_4, [0.8-1.0]), (HO_5, [0.6-0.8]), (HO_6, [0.6-0.8])\}$
$I\tilde{v}f_1(n_2) = \{(HO_1, [0.4-0.6]), (HO_2, [0.5-0.7]), (HO_3, [0.7-0.9]), (HO_4, [0.6-0.8]), (HO_5, [0.5-0.7]), (HO_6, [0.5-0.7])\}$
$I\tilde{v}f_1(n_3) = \{(HO_1, [0.6-0.8]), (HO_2, [0.6-0.8]), (HO_3, [0.8-1.0]), (HO_4, [0.8-1.0]), (HO_5, [0.7-0.9]), (HO_6, [0.6-0.8])\}$
$I\tilde{v}f_1(n_4) = \{(HO_1, [0.7-0.9]), (HO_2, [0.9-0.6]), (HO_3, [0.8-1.0]), (HO_4, [0.8-1.0]), (HO_5, [0.7-0.9]), (HO_6, [0.7-0.9])\}$

Suppose M and N indicates universal set and parameter set respectively, let $\Omega_1, \Omega_2 \subset N$, $(I\tilde{v}f_1, \Omega_1)$ and $(I\tilde{v}f_2, \Omega_2)$ indicates interval-valued fuzzy soft sets, $(I\tilde{v}f_1, \Omega_1)$ is an interval-valued fuzzy soft subset of $(I\tilde{v}f_2, \Omega_2)$ if

1. $\Omega_1 \subset \Omega_2$
2. $\forall\ n \in \Omega_1, I\tilde{v}f_1(n)$

shows interval-valued fuzzy subset of $I\tilde{v}f_2(n)$; shown as follows $(I\tilde{v}f_1, \Omega_1)\tilde{\subset}(I\tilde{v}f_2, \Omega_2)$.
$(I\tilde{v}f_1, \Omega_1)$ and $(I\tilde{v}f_2, \Omega_2)$ indicates two interval-valued fuzzy soft sets, $M = \{HO_1, HO_2, HO_3, HO_4, HO_5, HO_6\}$. M indicates the collection of hotels, $\Omega_1 = \{n_1, n_2\} = \{Staff, ValueforMoney\}$, Ω_2=$\{n_1, n_2, n_3\}$=$\{Staff, ValueforMoney, Facilities\}$, and
$I\tilde{v}f_1(n_1) = \{(HO_1, [0.6-0.8]), (HO_2, [0.6-0.8]), (HO_3, [0.8-1.0]), (HO_4, [0.8-1.0]), (HO_5, [0.6-0.8]), (HO_6, [0.6-0.8])\}$
$I\tilde{v}f_1(n_2) = \{(HO_1, [0.4-0.6]), (HO_2, [0.5-0.7]), (HO_3, [0.7-0.9]), (HO_4, [0.6-0.8]), (HO_5, [0.5-0.7]), (HO_6, [0.5-0.7])\}$
$I\tilde{v}f_2(n_1) = \{(HO_1, [0.8-1.0]), (HO_2, [0.6-0.8]), (HO_3, [0.9-1.0]), (HO_4, [0.8-1.0]), (HO_5, [0.8-1.0]), (HO_6, [0.8-1.0])\}$
$I\tilde{v}f_2(n_2) = \{(HO_1, [0.6-0.7]), (HO_2, [0.9-1.0]), (HO_3, [0.8-0.9]), (HO_4, [0.7-0.8]), (HO_5, [0.6-0.8]), (HO_6, [0.9-1.0])\}$
$I\tilde{v}f_2(n_3) = \{(HO_1, [0.6-0.8]), (HO_2, [0.6-0.8]), (HO_3, [0.8-1.0]), (HO_4, [0.8-1.0]), (HO_5, [0.6-0.8]), (HO_6, [0.6-0.8])\}$

Obviously, we can see $(I\tilde{v}f_1, \Omega_1)\ \tilde{\subset}\ (I\tilde{v}f_2, \Omega_2)$. Suppose $(I\tilde{v}f_1, \Omega_1)$ and $(I\tilde{v}f_2, \Omega_2)$ indicates two interval-valued fuzzy soft sets, These two sets will equal if

1. $(I\tilde{v}f_1, \Omega_1)$ is a subset of $(I\tilde{v}f_2, \Omega_2)$,
2. $(I\tilde{v}f_2, \Omega_2)$ is a subset of $(I\tilde{v}f_1, \Omega_1)$.

It is represented as $(I\tilde{v}f_1, \Omega_1) = (I\tilde{v}f_2, \Omega_2)$.

Operation on interval-valued fuzzy soft sets $(I\tilde{v}f_1, \Omega_1)^C$ is the complement of $(I\tilde{v}f_1, \Omega_1)$ and it is explained as $(I\tilde{v}f_1, \Omega_1)^C$=$(I\tilde{v}f_1{}^C, \neg\Omega_1)$,
Here $\forall\ \theta_1 \in \Omega_1$, $\neg\ \theta_1$=not θ_1, not belongs to the parameter θ_1, it means it is opposite of θ_1;

$$I\tilde{v}f_1{}^C : \neg\Omega_1 \to \tilde{\mathcal{P}}(M),$$

is the function given by $I\tilde{v}f_1{}^C(\theta_2) = (I\tilde{v}f_1(\neg\theta_2))^C, \forall\theta_2 \in \neg\Omega_1$.

Another interval-valued fuzzy soft set taken into consideration $(I\tilde{v}f_2, \Omega_2)$ and given in ***Table*** **3.2**. Universal set M is same in both table, i.e $M = \{HO_1, HO_2, HO_3, HO_4, HO_5, HO_6\}$ it is the collection of hotels; Ω_2= $\{\sigma_1, \sigma_2, \sigma_3\}$ = {Cleanliness, Free WiFi, Comfort} it is the collection of parameters.

So, with the help **1** of ***Section*** **3.1**, we get
$I\tilde{v}f_2{}^C(\neg\sigma_1) = \{(HO_1, [0.2, 0.4]), (HO_2, [0.2, 0.4]), (HO_3, [0.0, 0.2]), (HO_4, [0.0, 0.2]), (HO_5, [0.1, 0.3]), (HO_6, [0.2, 0.4])\}$
$I\tilde{v}f_2{}^C(\neg\sigma_2) = \{(HO_1, [0.4, 0.6]), (HO_2, [0.4, 0.6]), (HO_3, [0.1, 0.3]), (HO_4, [0.1, 0.3]), (HO_5, [0.2, 0.4]), (HO_6, [0.4, 0.6])\}$
$I\tilde{v}f_2{}^C(\neg\sigma_3) = \{(HO_1, [0.2, 0.4]), (HO_2, [0.2, 0.4]), (HO_3, [0.0, 0.2]), (HO_4, [0.0, 0.2]), (HO_5, [0.1, 0.3]), (HO_6, [0.2, 0.4])\}$

The "*AND*" procedure on $(I\tilde{v}f_1, \Omega_1)$ $(I\tilde{v}f_2, \Omega_2)$ is shown below

$$(I\tilde{v}f_1, \Omega_1) \wedge (I\tilde{v}f_2, \Omega_2) = (I\tilde{v}f_3, \Omega_1 \times \Omega_2).$$

Here

$$I\tilde{v}f_3(\theta_1, \theta_2) = I\tilde{v}f_1(\theta_1) \cap I\tilde{v}f_2(\theta_2), \forall(\theta_1, \theta_2) \in \Omega_1 \times \Omega_2.$$

We can apply "AND" procedure on $(I\tilde{v}f_1, \Omega_1)$ and $(I\tilde{v}f_2, \Omega_2)$ and can show in tabular form below.

***Table* 3.3:** "AND" procedure on ***Tables* 3.1** and **3.2**.

M	$n_1\sigma_1$	$n_1\sigma_2$	$n_1\sigma_3$	$n_2\sigma_1$	$n_2\sigma_2$	$n_2\sigma_3$
HO_1	[0.6, 0.8]	[0.4, 0.6]	[0.6, 0.8]	[0.4, 0.6]	[0.4, 0.6]	[0.4, 0.6]
HO_2	[0.6, 0.8]	[0.4, 0.6]	[0.6, 0.8]	[0.5, 0.7]	[0.4, 0.6]	[0.5, 0.7]
HO_3	[0.8, 1.0]	[0.7, 0.9]	[0.8, 1.0]	[0.7, 0.9]	[0.7, 0.9]	[0.7, 0.9]
HO_4	[0.8, 1.0]	[0.7, 0.9]	[0.8, 1.0]	[0.6, 0.8]	[0.6, 0.8]	[0.6, 0.8]
HO_5	[0.6, 0.8]	[0.6, 0.8]	[0.6, 0.8]	[0.5, 0.7]	[0.5, 0.7]	[0.5, 0.7]
HO_6	[0.6, 0.8]	[0.4, 0.6]	[0.6, 0.8]	[0.5, 0.7]	[0.4, 0.6]	[0.5, 0.7]
M	$n_3\sigma_1$	$n_3\sigma_2$	$n_3\sigma_3$	$n_4\sigma_1$	$n_4\sigma_2$	$n_4\sigma_3$
HO_1	[0.6, 0.8]	[0.4, 0.6]	[0.6, 0.8]	[0.6, 0.8]	[0.4, 0.6]	[0.6, 0.8]
HO_2	[0.6, 0.8]	[0.4, 0.6]	[0.6, 0.8]	[0.4, 0.6]	[0.4, 0.6]	[0.4, 0.6]
HO_3	[0.8, 1.0]	[0.7, 0.9]	[0.8, 1.0]	[0.8, 1.0]	[0.7, 0.9]	[0.8, 1.0]
HO_4	[0.8, 1.0]	[0.7, 0.9]	[0.8, 1.0]	[0.8, 1.0]	[0.7, 0.9]	[0.8, 1.0]
HO_5	[0.7, 0.9]	[0.6, 0.8]	[0.7, 0.9]	[0.7, 0.9]	[0.6, 0.8]	[0.7, 0.9]
HO_6	[0.6, 0.8]	[0.4, 0.6]	[0.6, 0.8]	[0.6, 0.8]	[0.4, 0.6]	[0.6, 0.8]

***Table* 3.4:** "OR" procedure on ***Tables* 3.1** and **3.2**.

M	$n_1\sigma_1$	$n_1\sigma_2$	$n_1\sigma_3$	$n_2\sigma_1$	$n_2\sigma_2$	$n_2\sigma_3$
HO_1	[0.6, 0.8]	[0.6, 0.8]	[0.6, 0.8]	[0.6, 0.8]	[0.4, 0.6]	[0.6, 0.8]
HO_2	[0.6, 0.8]	[0.6, 0.8]	[0.6, 0.8]	[0.6, 0.8]	[0.5, 0.7]	[0.6, 0.8]
HO_3	[0.8, 1.0]	[0.8, 1.0]	[0.8, 1.0]	[0.8, 1.0]	[0.7, 0.9]	[0.8, 1.0]
HO_4	[0.8, 1.0]	[0.8, 1.0]	[0.8, 1.0]	[0.8, 1.0]	[0.7, 0.9]	[0.8, 1.0]
HO_5	[0.7, 0.9]	[0.6, 0.8]	[0.7, 0.9]	[0.7, 0.9]	[0.6, 0.8]	[0.7, 0.9]
HO_6	[0.6, 0.8]	[0.6, 0.8]	[0.6, 0.8]	[0.6, 0.8]	[0.5, 0.7]	[0.6, 0.8]
M	$n_3\sigma_1$	$n_3\sigma_2$	$n_3\sigma_3$	$n_4\sigma_1$	$n_4\sigma_2$	$n_4\sigma_3$
HO_1	[0.6, 0.8]	[0.6, 0.8]	[0.6, 0.8]	[0.7, 0.9]	[0.7, 0.9]	[0.7, 0.9]
HO_2	[0.6, 0.8]	[0.6, 0.8]	[0.6, 0.8]	[0.6, 0.8]	[0.4, 0.6]	[0.6, 0.8]
HO_3	[0.8, 1.0]	[0.8, 1.0]	[0.8, 1.0]	[0.8, 1.0]	[0.8, 1.0]	[0.8, 1.0]
HO_4	[0.8, 1.0]	[0.8, 1.0]	[0.8, 1.0]	[0.8, 1.0]	[0.8, 1.0]	[0.8, 1.0]
HO_5	[0.7, 0.9]	[0.7, 0.9]	[0.7, 0.9]	[0.7, 0.9]	[0.7, 0.9]	[0.7, 0.9]
HO_6	[0.6, 0.8]	[0.6, 0.8]	[0.6, 0.8]	[0.7, 0.9]	[0.7, 0.9]	[0.7, 0.9]

By ***Definition* 2.1**, we obtain
$I\tilde{v}f_3(n_1, \sigma_2) = I\tilde{v}f_1(n_1) \cap I\tilde{v}f_2(\sigma_2)$
$=\{(HO_1, [0.4, 0.6]), (HO_2, [0.4, 0.6]), (HO_3, [0.7, 0.9]), (HO_4, [0.7, 0.9]), (HO_5, [0.6, 0.8]), (HO_6, [0.4, 0.6])\}$
By ***Table* 3.3** we obtain the value of $(I\tilde{v}f_1, \Omega_1) \wedge (I\tilde{v}f_2, \Omega_2)$

The "OR" procedure on $(I\tilde{v}f_1, \Omega_1)$ $(I\tilde{v}f_2, \Omega_2)$ is shown below [6, 12].

$$(I\tilde{v}f_1, \Omega_1) \vee (I\tilde{v}f_2, \Omega_2) = (I\tilde{v}f_3, \Omega_1 \times \Omega_2),$$

Here $I\tilde{v}f_3(\theta_1,\theta_2)=I\tilde{v}f_1(\theta_1) \cup I\tilde{v}f_2(\theta_2)$, $\forall$ $(\theta_1,\theta_2) \in \Omega_1 \times \Omega_2$.

The output of the "OR" procedure on $(I\tilde{v}f_1, \Omega_1)$ and $(I\tilde{v}f_2, \Omega_2)$ in ***Tables* 3.1** and **3.2** is given in ***Table* 3.4**.

Let us, we apply DeMorgan's Laws on two interval-valued fuzzy soft sets $(I\tilde{v}f_1, \Omega_1)$, $(I\tilde{v}f_2, \Omega_2)$ as shown below.
$((I\tilde{v}f_1, \Omega_1)\wedge(I\tilde{v}f_2, \Omega_2))^C = (I\tilde{v}f_1, \Omega_1)^C \vee (I\tilde{v}f_2, \Omega_2)^C$
$((I\tilde{v}f_1, \Omega_1)\vee(I\tilde{v}f_2, \Omega_2))^C = (I\tilde{v}f_1, \Omega_1)^C \wedge (I\tilde{v}f_2, \Omega_2)^C$
Proof. $(I\tilde{v}f_1, \Omega_1)^C \vee (I\tilde{v}f_2, \Omega_2)^C = (I\tilde{v}f_1{}^C, \neg\Omega_1) \vee (I\tilde{v}f_2{}^C, \neg\Omega_2) = (I\tilde{v}f_4, \neg\Omega_1 \times \neg\Omega_2)$
$where, I\tilde{v}f_4(\neg\theta_1, \neg\theta_2) = I\tilde{v}f_1{}^C(\neg\theta_1) \cup I\tilde{v}f_2{}^C(\neg\theta_2)$
$= (I\tilde{v}f_4, \neg(\Omega_1 \times \Omega_2))$.
Assume $(I\tilde{v}f_1, \Omega_1)\wedge(I\tilde{v}f_2, \Omega_2) = (I\tilde{v}f_3, \Omega_1 \times \Omega_2)$, then we have $((I\tilde{v}f_1, \Omega_1)\wedge(I\tilde{v}f_2, \Omega_2))^C = (I\tilde{v}f_3, \Omega_1 \times \Omega_2)^C = (I\tilde{v}f_3{}^C, \neg(\Omega_1 \times \Omega_2))$.

***Table* 3.5:** Consequence interval-valued fuzzy soft set $(I\tilde{v}f_3, \Omega_3)$

M	$n_1\sigma_2$	$n_2\sigma_1$	$n_3\sigma_2$	$n_3\sigma_3$	$n_4\sigma_1$
HO_1	[0.4 – 0.6]	[0.4 – 0.6]	[0.4 – 0.6]	[0.6 – 0.8]	[0.6 – 0.8]
HO_2	[0.4 – 0.6]	[0.5 – 0.7]	[0.4 – 0.6]	[0.6 – 0.8]	[0.4 – 0.6]
HO_3	[0.7 – 0.9]	[0.7 – 0.9]	[0.7 – 0.9]	[0.8 – 1.0]	[0.8 – 1.0]
HO_4	[0.7 – 0.9]	[0.6 – 0.8]	[0.7 – 0.9]	[0.8 – 1.0]	[0.8 – 1.0]
HO_5	[0.6 – 0.8]	[0.5 – 0.7]	[0.6 – 0.8]	[0.7 – 0.9]	[0.7 – 0.9]
HO_6	[0.4 – 0.6]	[0.5 – 0.7]	[0.4 – 0.6]	[0.6 – 0.8]	[0.6 – 0.8]

$\forall\ (\theta_1, \theta_2) \in \Omega_1 \times \Omega_2$, we know
$I\tilde{v}f_3{}^C(\neg\theta_1, \neg\theta_2) = (I\tilde{v}f_3(\theta_1, \theta_2))^C = (I\tilde{v}f_1(\theta_1) \cap I\tilde{v}f_2(\theta_2))^C = (I\tilde{v}f_1(\theta_1))^C \cup (I\tilde{v}f_2(\theta_2))^C = (I\tilde{v}f_1{}^C(\neg\theta_1)) \cup (I\tilde{v}f_2{}^C(\neg\theta_2))$

By above analysis,we got $((I\tilde{v}f_1, \Omega_1) \wedge (I\tilde{v}f_2, \Omega_2))^C = (I\tilde{v}f_1, \Omega_1)^C \vee (I\tilde{v}f_2, \Omega_2)^C$.

In the same way, we can prove that $((I\tilde{v}f_1, \Omega_1) \vee (I\tilde{v}f_2, \Omega_2))^C = (I\tilde{v}f_1, \Omega_1)^C \wedge (I\tilde{v}f_2, \Omega_2)^C$.

Suppose we have interval-valued fuzzy soft sets, $(I\tilde{v}f_1, \Omega_1), (I\tilde{v}f_2, \Omega_2) and (I\tilde{v}f_3, \Omega_3)$.

[**Associative law**]
$(I\tilde{v}f_1, \Omega_1) \wedge ((I\tilde{v}f_2, \Omega_2) \wedge (I\tilde{v}f_3, \Omega_3)) = ((I\tilde{v}f_1, \Omega_1) \wedge (I\tilde{v}f_2, \Omega_2)) \wedge (I\tilde{v}f_3, \Omega_3)$.
$(I\tilde{v}f_1, \Omega_1) \vee ((I\tilde{v}f_2, \Omega_2) \vee (I\tilde{v}f_3, \Omega_3)) = ((I\tilde{v}f_1, \Omega_1) \vee (I\tilde{v}f_2, \Omega_2)) \vee (I\tilde{v}f_3, \Omega_3)$.

[**Distributive law**]
$(I\tilde{v}f_1, \Omega_1) \wedge ((I\tilde{v}f_2, \Omega_2) \vee (I\tilde{v}f_3, \Omega_3)) = ((I\tilde{v}f_1, \Omega_1) \wedge (I\tilde{v}f_2, \Omega_2)) \vee ((I\tilde{v}f_1, \Omega_1) \wedge (I\tilde{v}f_3, \Omega_3))$.
$(I\tilde{v}f_1, \Omega_1) \vee ((I\tilde{v}f_2, \Omega_2) \wedge (I\tilde{v}f_3, \Omega_3)) = ((I\tilde{v}f_1, \Omega_1) \vee (I\tilde{v}f_2, \Omega_2)) \wedge ((I\tilde{v}f_1, \Omega_1) \vee (I\tilde{v}f_3, \Omega_3))$.

Proof. $\forall\ \theta_1 \in \Omega_1, \theta_2 \in \Omega_2$ and $\forall \theta_3 \in \Omega_3$, Now
$I\tilde{v}f_1(\theta_1) \cap (I\tilde{v}f_2(\theta_2) \cap I\tilde{v}f_3(\theta_3)) = (I\tilde{v}f_1(\theta_1) \cap I\tilde{v}f_2(\theta_2)) \cap I\tilde{v}f_3(\theta_3)$
from it, we can say $(I\tilde{v}f_1, \Omega_1) \wedge ((I\tilde{v}f_2, \Omega_2) \wedge (I\tilde{v}f_3, \Omega_3)) = ((I\tilde{v}f_1, \Omega_1) \wedge (I\tilde{v}f_2, \Omega_2)) \wedge (I\tilde{v}f_3, \Omega_3)$.

In the same way, we can say $(I\tilde{v}f_1, \Omega_1) \vee ((I\tilde{v}f_2, \Omega_2) \vee (I\tilde{v}f_3, \Omega_3)) = ((I\tilde{v}f_1, \Omega_1) \vee (I\tilde{v}f_2, \Omega_2)) \vee (I\tilde{v}f_3, \Omega_3)$.
$\forall\ \theta_1 \in \Omega_1, \theta_2 \in \Omega_2$ and $\forall \theta_3 \in \Omega_3$, we have $I\tilde{v}f_1(\theta_1) \cap (I\tilde{v}f_2(\theta_2) \cup I\tilde{v}f_3(\theta_3)) = (I\tilde{v}f_1(\theta_1) \cap I\tilde{v}f_2(\theta_2)) \cup (I\tilde{v}f_1(\theta_1) \cap I\tilde{v}f_3(\theta_3))$

In the same way, we can say $(I\tilde{v}f_1, \Omega_1) \wedge ((I\tilde{v}f_2, \Omega_2) \vee (I\tilde{v}f_3, \Omega_3)) = ((I\tilde{v}f_1, \Omega_1) \wedge (I\tilde{v}f_2, \Omega_2)) \vee ((I\tilde{v}f_1, \Omega_1) \wedge (I\tilde{v}f_3, \Omega_3))$.

Similarity, we also have $(I\tilde{v}f_1, \Omega_1) \vee ((I\tilde{v}f_2, \Omega_2) \wedge (I\tilde{v}f_3, \Omega_3)) = ((I\tilde{v}f_1, \Omega_1) \vee (I\tilde{v}f_2, \Omega_2)) \wedge ((I\tilde{v}f_1, \Omega_1) \vee (I\tilde{v}f_3, \Omega_3))$.

4 Application of interval-valued fuzzy soft set

Object can be identified by using algorithm, given by Roy et al. [27], different objects are compared in it. Kong et al. [18] said that Roy's innovation has error, Kong gave correct innovation, in it they compare choice values of different objects.

***Rule* 4.1**:Take the set of interval-valued fuzzy soft sets.
$(I\tilde{v}f_1, \Omega_1)$ and $(I\tilde{v}f_2, \Omega_2)$ demonstrate in ***Tables* 3.1** and **3.2** are under deliberation.
Here, we obtain the value of sets $(I\tilde{v}f_1, \Omega_1)$ and $(I\tilde{v}f_2, \Omega_2)$ from ***Tables* 3.1** and **3.2**.

***Rule* 4.2**:Input the guideline set X as detected by the viewer.
We take these two sets $(I\tilde{v}f_1, \Omega_1)$ and $(I\tilde{v}f_2, \Omega_2)$,***Table* 3.3** obtained by applying "*AND*" procedure on these sets. Suppose we are taking these values of parameters X such that X={{$n_1\sigma_2$},{$n_2\sigma_1$},
{$n_3\sigma_2$},{$n_3\sigma_3$},{$n_4\sigma_1$}}.

***Rule* 4.3**:We obtain resultant $(I\tilde{v}f_3, \Omega_3)$ set from $(I\tilde{v}f_1, \Omega_1)$ and $(I\tilde{v}f_2, \Omega_2)$.
***Table* 3.5** the resultant interval-valued fuzzy soft set is obtained from ***Table* 3.3** according to the set of parameters.

***Rule* 4.4**: $\forall HO_i \in M$, In this way, we calculate the choice value d for every hotel HO.

$$d_i = [d_i^-, d_i^+] = \left[\sum_{r\in R} \mu^-_{I\tilde{v}f_3(r)}(HO_i), \sum_{r\in R} \mu^+_{I\tilde{v}f_3(r)}(HO_i)\right].$$

The result is shown in ***Table* 4.1**.

Table **4.1:** Choice value.

M	$n_1\sigma_2$	$n_2\sigma_1$	$n_3\sigma_2$	$n_3\sigma_3$	$n_4\sigma_1$	d_i
HO_1	$[0.4, 0.6]$	$[0.4, 0.6]$	$[0.4, 0.6]$	$[0.6, 0.8]$	$[0.6, 0.8]$	$d_1 = [2.4, 3.4]$
HO_2	$[0.4, 0.6]$	$[0.5, 0.7]$	$[0.4, 0.6]$	$[0.6, 0.8]$	$[0.4, 0.6]$	$d_2 = [2.3, 3.3]$
HO_3	$[0.7, 0.9]$	$[0.7, 0.9]$	$[0.7, 0.9]$	$[0.8, 1.0]$	$[0.8, 1.0]$	$d_3 = [3.7, 4.7]$
HO_4	$[0.7, 0.9]$	$[0.6, 0.8]$	$[0.7, 0.9]$	$[0.8, 1.0]$	$[0.8, 1.0]$	$d_4 = [3.6, 4.6]$
HO_5	$[0.6, 0.8]$	$[0.5, 0.7]$	$[0.6, 0.8]$	$[0.7, 0.9]$	$[0.7, 0.9]$	$d_5 = [3.1, 4.1]$
HO_6	$[0.4, 0.6]$	$[0.5, 0.7]$	$[0.4, 0.6]$	$[0.6, 0.8]$	$[0.6, 0.8]$	$d_6 = [2.5, 3.5]$

***Rule* 4.5**:$\forall HO_i \in M$,calculate the value of t_i for each hotel HO_i in this way.

$$t_i = \sum_{HO_i \in M} \left((d_i^- - d_j^-) + (d_i^+ - d_j^+)\right).$$

By calculation, we obtained $t_1 = -6.4, t_2 = -7.4, t_3 = 9.2, t_4 = 8.0, t_5 = 2.0, t_6 = -5.2$.
***Rule* 4.6**:The solution is any one of the elements in Z here $Z = max_{HO_i \in M}(t_i)$
In this problem, hotel HO_3 is the best option because $max_{HO_i \in M}(t_i) = (HO_3)$. This result is reasonable because we can see that $d_3 > d_i$ here i=1,2,3,4,5,6 i.e. HO_3 has the highest choice value.

5 Conclusion

Soft set theory solves problems which contain uncertainty, fuzziness or vagueness. At last, an example shows that interval-valued fuzzy soft set works properly in judgment constructing issue. In previous work Yang at al. [29] took house problem which can be solved by Fuzzy Soft Maximum Minimum Decision Making Method. In the present work we took very important daily life problem which can only solved by this method. There are parameterizations reduction of interval-valued fuzzy soft set which is new topic for further research. In the end, this work also having the utilization in the field of automobile for buying car, buying laptop according to our desire parameter.

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(Dedicated to Honor Dr. R. C. Singh Chandel on His 75th Birth Anniversary Celebrations)

UPPER BOUND ON FOURTH HANKEL DETERMINANT FOR CERTAIN SUBCLASS OF MULTIVALENT FUNCTIONS

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Abstract

The present investigation is concerned with the estimation of the upper bound to the $H_4(p)$ Hankel determinant for a subclass of p-valent functions in the open unit disc $E = \{z :| z |< 1\}$. This work will motivate the researchers to work in the direction of investigation of fourth Hankel determinant for several other subclasses of univalent and multivalent functions.

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1 Introduction

Let P denote the class of analytic functions $p(z)$ of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n,$$

whose real parts are positive in E.

By A_p, we denote the class of functions of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k, (p \in N = \{1, 2, 3, ...\}), \tag{1.1}$$

which are analytic in the unit disc $E = \{z :| z |< 1\}$ and normalized by $f(0) = f'(0) - 1 = 0$.

Let S be the class $A_1 \equiv A$ consisting of functions of the form (1.1) and which are univalent in E.

Let R represent the class of functions $f \in A$, which satisfy the condition

$$Re(f'(z)) > 0.$$

The class R was introduced by MacGregor [12] and functions in this class are called bounded turning functions.

By R_1, we denote the class of functions $f \in A$, with the condition that

$$Re\left(\frac{f(z)}{z}\right) > 0.$$

R_1 is a subclass of close-to-star functions and was studied by MacGregor [13].

Further, Murugusundramurthi and Magesh [15] introduced the following class:

$$R(\alpha) = \left\{f : f \in A, Re\left\{(1-\alpha)\frac{f(z)}{z} + \alpha f'(z)\right\} > 0, 0 \leq \alpha \leq 1, z \in E\right\}.$$

In particular, $R(1) \equiv R$ and $R(0) \equiv R_1$.

Later on, Vamshee Krishna et al. [8] introduced a subclass of p-valent functions as follows:

$$RT_p = \left\{f : f \in A_p, Re\left(\frac{f'(z)}{pz^{p-1}}\right) > 0, z \in E\right\}.$$

For $p = 1$, $RT_1 \equiv R$.

Motivated by the above defined classes, Amourah et al. [2] defined the following subclass of p-valent functions:

$$R_p(\alpha) = \left\{f : f \in A_p, Re\left\{(1-\alpha)\frac{f(z)}{z^p} + \alpha\frac{f'(z)}{pz^{p-1}}\right\} > 0, 0 \leq \alpha \leq 1, z \in E\right\}.$$

The following observations are obvious:

(i) $R_1(\alpha) \equiv R(\alpha)$,
(ii) $R_p(1) \equiv RT_p$,
(iii) $R_1(1) \equiv R$,
(iv) $R_1(0) \equiv R_1$.

In 1976, Noonan and Thomas [16] stated the qth Hankel determinant for $q \geq 1$ and $n \geq 1$ as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \dots & a_{n+q+1} \\ a_{n+1} & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ a_{n+q+1} & \dots & \dots & a_{n+2q-2} \end{vmatrix}.$$

In the particular cases, $q = 2, n = p, a_1 = 1$ and $q = 2, n = p + 1$, the Hankel determinant simplifies respectively to $H_2(p) = |a_{p+2} - a_{p+1}^2|$ and $H_2(p+1) = |a_{p+1}a_{p+3} - a_{p+2}^2|$.

This paper is concerned with the Hankel determinant in the case $q = 3$ and $n = p$ as

$$H_3(p) = \begin{vmatrix} a_p & a_{p+1} & a_{p+2} \\ a_{p+1} & a_{p+2} & a_{p+3} \\ a_{p+2} & a_{p+3} & a_{p+4} \end{vmatrix},$$

which is known as Hankel determinant of order 3.

For $f \in A_p$ and $a_p = 1$, we have

$$H_3(p) = a_{p+2}(a_{p+1}a_{p+3} - a_{p+2}^2) - a_{p+3}(a_{p+3} - a_{p+1}a_{p+2}) + a_{p+4}(a_{p+2} - a_{p+1}^2),$$

and using the triangle inequality, it yields

$$(1.2)\quad |H_3(p)| \leq |a_{p+2}||a_{p+1}a_{p+3} - a_{p+2}^2| + |a_{p+3}||a_{p+3} - a_{p+1}a_{p+2}| + |a_{p+4}||a_{p+2} - a_{p+1}^2|.$$

For any $f \in A_p$ of the form (1.1), we can represent the fourth Hankel determinant as

$$(1.3)\quad H_{4,p}(f) = a_{p+6}H_3(p) - a_{p+5}D_1 + a_{p+4}D_2 - a_{p+3}D_3,$$

where D_1, D_2 and D_3 are determinants of order 3 given by

$$(1.4)\quad D_1 = (a_{p+2}a_{p+5} - a_{p+3}a_{p+4}) - a_{p+1}(a_{p+1}a_{p+5} - a_{p+2}a_{p+4}) + a_{p+3}(a_{p+1}a_{p+3} - a_{p+2}^2),$$

$$(1.5)\quad D_2 = (a_{p+3}a_{p+5} - a_{p+4}^2) - a_{p+1}(a_{p+2}a_{p+5} - a_{p+3}a_{p+4}) + a_{p+2}(a_{p+2}a_{p+4} - a_{p+3}^2),$$

$$(1.6)\quad D_3 = a_{p+1}(a_{p+3}a_{p+5} - a_{p+4}^2) - a_{p+2}(a_{p+2}a_{p+5} - a_{p+3}a_{p+4}) + a_{p+3}(a_{p+2}a_{p+4} - a_{p+3}^2).$$

Hankel determinant has been considered by several authors. For example, Noor [17] determined the rate of growth of $H_q(n)$ as $n \to \infty$ for the functions given by Eq.(1.1) with bounded boundary. Ehrenborg [5] studied the Hankel determinant of exponential polynomials and in [10], the Hankel transform of an integer sequence is defined and some of its properties have been discussed by Layman.

Second Hankel determinant for various classes has been extensively studied by various authors including Mehrok and Singh [14], Janteng et al.[7] and many others. Third Hankel determinants for various classes were studied by some of the researchers including Babalola [3], Shanmugam et al.[18], Altinkaya and Yalcin [1] and Singh and Singh [19]. Also the Hankel determinant for various subclasses of p-valent functions were studied by various authors including Krishna and Ramreddy [8] and Hayami and Owa [6].

In this paper, we seek upper bound for the functional $H_{4,p}(f)$ for the functions belonging to the class $R_p(\alpha)$. This paper will motivate the future researchers to investigate the fourth Hankel determinant for some other subclasses of univalent and multivalent functions.

2 Preliminary results

***Lemma* 2.1**[4,11] *If $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \in P$, then for $n, k \in N = \{1, 2, 3, ...\}$, we have the following inequalities*:

$$|c_{n+k} - \lambda c_n c_k| \leq 2, 0 \leq \lambda \leq 1,$$

and

$$|c_n| \leq 2.$$

***Lemma* 2.2** *If $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \in P$, then for $n, k \in N = \{1, 2, 3, ...\}$, we have*:

$$|c_{n+k} - \lambda c_n c_k| \leq 4\lambda - 2, \lambda \geq 1.$$

Proof. For $\lambda \geq 1$, we have

$$|c_{n+k} - \lambda c_n c_k| \leq |c_n c_k - c_{n+k}| + (\lambda - 1)|c_n c_k|.$$

Using ***Lemma* 2.1**, the above inequality yields

$$|c_{n+k} - \lambda c_n c_k| \leq 4\lambda - 2.$$

***Lemma* 2.3**[2] *If $f \in R_p(\alpha)$, then*

$$|a_{p+j}| \leq \frac{2p}{p + j\alpha}.$$

***Lemma* 2.4**[2] *If $f \in R_p(\alpha)$, then*

$$|a_{p+2} - a_{p+1}^2| \leq \frac{2p}{p + 2\alpha}.$$

***Lemma* 2.5**[9] *If $f \in R_p(\alpha)$, then*

$$|a_{p+1}a_{p+3} - a_{p+2}^2| \leq \frac{4p^2}{(p + 2\alpha)^2}.$$

***Lemma* 2.6**[2] *If $f \in R_p(\alpha)$, then*

$$|a_{p+1}a_{p+2} - a_{p+3}| \leq \begin{cases} 2 & \text{if } \alpha = 0, \\ \dfrac{2p(6\alpha^2 + 3p\alpha + p^2)^{\frac{3}{2}}}{3\sqrt{6}\alpha(p + \alpha)(p + 2\alpha)(p + 3\alpha)} & \text{if } 0 < \alpha \leq 1. \end{cases}$$

***Lemma* 2.7** *If $f \in R_p(\alpha)$, then*

$$|H_3(p)| \leq \begin{cases} 16 & \text{for } \alpha = 0, \\ \dfrac{4p^2}{p + 2\alpha}\left[\dfrac{2p}{(p + 2\alpha)^2} + \dfrac{1}{p + 4\alpha} + \dfrac{(6\alpha^2 + 3p\alpha + p^2)^{\frac{3}{2}}}{3\sqrt{6}\alpha(p + \alpha)(p + 3\alpha)^2}\right] & \text{for } 0 < \alpha \leq 1. \end{cases}$$

***Proof*.** From ***Lemma* 2.3**, we have

$$(2.1) \quad |a_{p+2}| \leq \frac{2p}{p + 2\alpha},$$

$$(2.2) \quad |a_{p+3}| \leq \frac{2p}{p + 3\alpha},$$

and

$$(2.3) \quad |a_{p+4}| \leq \frac{2p}{p + 4\alpha}.$$

Using equations (2.1),(2.2) and (2.3), ***Lemma* 2.4**, ***Lemma* 2.5** and ***Lemma* 2.6** in (1.2), the result is obvious.
For $p = 1$, ***Lemma* 2.7** yields the following result:

***Corollary* 2.1** *If $f \in R(\alpha)$, then*

$$|H_3(1)| \leq \begin{cases} 16 & \text{for } \alpha = 0, \\ \dfrac{4}{1 + 2\alpha}\left[\dfrac{2}{(1 + 2\alpha)^2} + \dfrac{1}{1 + 4\alpha} + \dfrac{(6\alpha^2 + 3\alpha + 1)^{3/2}}{3\sqrt{6}\alpha(1 + \alpha)(1 + 3\alpha)^2}\right] & \text{for } 0 < \alpha \leq 1. \end{cases}$$

For $p = 1, \alpha = 1$, ***Lemma* 2.7** gives the following result proved by Babalola [3]:

***Corollary* 2.2** *If $f \in R$, then*

$$|H_3(1)| \leq 0.7423.$$

3 Fourth Hankel determinant for the class $R_p(\alpha)$

***Theorem* 3.1** *If $f \in R_p(\alpha)$, then*

$$(3.1) \quad |H_4(p)| \leq \begin{cases} 152.0866 & \text{for } \alpha = 0, \\ \dfrac{8p^3}{(p + 2\alpha)(p + 6\alpha)}\left[\dfrac{2p}{(p + 2\alpha)^2} + \dfrac{1}{p + 4\alpha} + \dfrac{(6\alpha^2 + 3p\alpha + p^2)^{3/2}}{3\sqrt{6}\alpha(p + \alpha)(p + 3\alpha)^2}\right] & \\ + \dfrac{2p}{(p + 5\alpha)}u(p, \alpha) + \dfrac{2p}{(p + 4\alpha)}v(p, \alpha) + \dfrac{2p}{(p + 3\alpha)}w(p, \alpha) & \text{for } 0 < \alpha \leq 1, \end{cases}$$

where

(3.2) $$u(p,\alpha) = 2p^2(4p-2)\left[\frac{1}{(p+\alpha)^2(p+5\alpha)} + \frac{1}{(p+3\alpha)(p+2\alpha)^2} + \frac{1}{(p+\alpha)(p+3\alpha)^2}\right] + \frac{174p^2(4p-2)+4p^2}{48(p+\alpha)(p+2\alpha)(p+4\alpha)},$$

(3.3) $$v(p,\alpha) = \left[\frac{63p^2(4p-2)}{25(p+\alpha)(p+2\alpha)(p+5\alpha)} + \frac{18p^2(4p-2)}{5(p+4\alpha)(p+2\alpha)^2} + \frac{150p^2(4p-2)+4p^2}{75(p+2\alpha)(p+3\alpha)^2}\right]$$

and

(3.4) $$w(p,\alpha) = 2p^2(4p-2) \times \left[\frac{1}{(p+2\alpha)^2(p+5\alpha)} + \frac{1}{(p+\alpha)(p+3\alpha)(p+5\alpha)} + \frac{2}{(p+3\alpha)^3} + \frac{1}{(p+\alpha)(p+4\alpha)^2}\right] + \frac{34p^2(4p-2)}{16(p+2\alpha)(p+3\alpha)(p+4\alpha)} + \frac{p^2}{(p+\alpha)(p+2\alpha)^2(p+3\alpha)(p+4\alpha)^2(p+5\alpha)}.$$

Proof. Using ***Lemma* 2.3** in (1.4), (1.5) and (1.6), it gives

(3.5) $$D_1 = \frac{p^2c_2c_5}{(p+2\alpha)(p+5\alpha)} - \frac{p^2c_3c_4}{(p+3\alpha)(p+4\alpha)} - \frac{p^3c_1^2c_5}{(p+\alpha)^2(p+5\alpha)} + \frac{p^3c_1c_2c_4}{(p+\alpha)(p+2\alpha)(p+4\alpha)} + \frac{p^3c_1c_3^2}{(p+\alpha)(p+3\alpha)^2} - \frac{p^3c_3c_2^2}{(p+3\alpha)(p+2\alpha)^2},$$

(3.6) $$D_2 = \frac{p^2c_3c_5}{(p+3\alpha)(p+5\alpha)} - \frac{p^2c_4^2}{(p+4\alpha)^2} - \frac{p^3c_1c_2c_5}{(p+\alpha)(p+2\alpha)(p+5\alpha)} + \frac{p^3c_1c_3c_4}{(p+\alpha)(p+3\alpha)(p+4\alpha)} + \frac{p^3c_4c_2^2}{(p+2\alpha)^2(p+4\alpha)} - \frac{p^3c_2c_3^2}{(p+2\alpha)(p+3\alpha)^2}$$

and

(3.7) $$D_3 = \frac{p^3c_1c_3c_5}{(p+\alpha)(p+3\alpha)(p+5\alpha)} - \frac{p^3c_1c_4^2}{(p+\alpha)(p+4\alpha)^2} - \frac{p^3c_2^2c_5}{(p+2\alpha)^2(p+5\alpha)} + \frac{2p^3c_2c_3c_4}{(p+2\alpha)(p+3\alpha)(p+4\alpha)} - \frac{p^3c_3^3}{(p+3\alpha)^3}.$$

On rearranging the terms in (3.5), (3.6) and (3.7), it yields

(3.8) $$D_1 = \frac{p^2c_5(c_2-pc_1^2)}{(p+\alpha)^2(p+5\alpha)} + \frac{p^2c_3(c_4-pc_2^2)}{(p+3\alpha)(p+2\alpha)^2} - \frac{p^2c_3(c_4-pc_1c_3)}{(p+\alpha)(p+3\alpha)^2} - \frac{67p^2c_4(c_3-pc_1c_2)}{48(p+\alpha)(p+2\alpha)(p+4\alpha)} + \frac{19p^2c_2(c_5-pc_1c_4)}{48(p+\alpha)(p+2\alpha)(p+4\alpha)} + \frac{p^2c_2c_5}{48(p+\alpha)(p+2\alpha)(p+4\alpha)},$$

(3.9) $$D_2 = \frac{p^2c_5(c_3-pc_1c_2)}{(p+\alpha)(p+2\alpha)(p+5\alpha)} - \frac{p^2c_4(c_4-pc_2^2)}{(p+4\alpha)(p+2\alpha)^2} - \frac{p^2c_3(c_5-pc_2c_3)}{(p+2\alpha)(p+3\alpha)^2} - \frac{4p^2c_4(c_4-pc_1c_3)}{5(p+4\alpha)(p+2\alpha)^2} - \frac{13p^2c_3(c_5-pc_1c_4)}{50(p+\alpha)(p+2\alpha)(p+5\alpha)} + \frac{p^2c_3c_5}{75(p+2\alpha)(p+3\alpha)^2}$$

and

(3.10) $$D_3 = \frac{p^2c_5(c_4-pc_2^2)}{(p+2\alpha)^2(p+5\alpha)} - \frac{p^2c_5(c_4-pc_1c_3)}{(p+\alpha)(p+3\alpha)(p+5\alpha)} + \frac{p^2c_3(c_6-pc_3^2)}{(p+3\alpha)^3} - \frac{p^2c_3(c_6-pc_2c_4)}{(p+3\alpha)^3} + \frac{p^2c_4(c_5-pc_1c_4)}{(p+\alpha)(p+4\alpha)^2} - \frac{17p^2c_4(c_5-pc_2c_3)}{16(p+2\alpha)(p+3\alpha)(p+4\alpha)} + \frac{p^2c_4c_5}{4(p+\alpha)(p+2\alpha)^2(p+3\alpha)(p+4\alpha)^2(p+5\alpha)}.$$

Using ***Lemma* 2.2** and applying triangle inequality in (3.8), (3.9) and (3.10), we obtain

(3.11) $|D_1| \le u(p,\alpha)$,

(3.12) $|D_2| \le v(p,\alpha)$

and

(3.13) $|D_3| \leq w(p, \alpha)$,

where $u(p, \alpha), v(p, \alpha)$ and $w(p, \alpha)$ are defined in (3.2), (3.3) and (3.4) respectively.
Hence using ***Lemma* 2.3**, ***Lemma* 2.7** and equations (3.11), (3.12), (3.13) in equation (1.3) and applying triangle inequality, the result (3.1) is obvious.
On putting $p = 1$ in ***Theorem* 3.1**, we obtain the following result:
***Corollary* 3.1** *If $f \in R(\alpha)$, then*

$$|H_4(1)| \leq \begin{cases} 152.0866 & \text{for } \alpha = 0, \\ \dfrac{8}{(1+2\alpha)(1+6\alpha)}\left[\dfrac{2}{(1+2\alpha)^2} + \dfrac{1}{1+4\alpha} + \dfrac{(6\alpha^2+3\alpha+1)^{\frac{3}{2}}}{3\sqrt{6}\alpha(1+\alpha)(1+3\alpha)^2}\right] & \\ +\dfrac{2}{(1+5\alpha)}p(\alpha) + \dfrac{2}{(1+4\alpha)}q(\alpha) + \dfrac{2}{(1+3\alpha)}r(\alpha) & \text{for } 0 < \alpha \leq 1, \end{cases}$$

where

$$p(\alpha) = 4\left[\frac{1}{(1+\alpha)^2(1+5\alpha)} + \frac{1}{(1+3\alpha)(1+2\alpha)^2} + \frac{1}{(1+\alpha)(1+3\alpha)^2}\right] + \frac{29}{4(1+\alpha)(1+2\alpha)(1+4\alpha)},$$

$$q(\alpha) = 4\left[\frac{63}{50(1+\alpha)(1+2\alpha)(1+5\alpha)} + \frac{9}{5(1+4\alpha)(1+2\alpha)^2} + \frac{76}{75(1+2\alpha)(1+3\alpha)^2}\right]$$

and

$$r(\alpha) = 4\left[\frac{1}{(1+2\alpha)^2(1+5\alpha)} + \frac{1}{(1+\alpha)(1+3\alpha)(1+5\alpha)} + \frac{2}{(1+3\alpha)^3} + \frac{1}{(1+\alpha)(1+4\alpha)^2}\right] + \frac{68}{16(1+2\alpha)(1+3\alpha)(1+4\alpha)} + \frac{1}{(1+\alpha)(1+2\alpha)^2(1+3\alpha)(1+4\alpha)^2(1+5\alpha)}.$$

On putting $p = 1, \alpha = 1$ in ***Theorem* 3.1**, the following result is obvious:
***Corollary* 3.2** *If $f \in R$, then*

$$|H_{4,1}(f)| \leq 0.7973.$$

4 Conclusion.

In the present work, we estimated the bounds for the fourth Hankel determinant for a subclass of multivalent bounded turning functions. The estimation of fourth Hankel determinant for the various subclasses of analytic functions is a new concept in the field of geometric function theory. Till now much work has been done on the study of second and third Hankel determinants for various subclasses of univalent functions, so this paper will work as a milestone to the future researchers in this field.

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(Dedicated to Honor Dr. R. C. Singh Chandel on His 75^{th} *Birth Anniversary Celebrations)*

COMBINATORIAL OPTIMIZED TECHNIQUE FOR COMPUTATION OF TRADITIONAL COMBINATIONS

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Abstract

This paper presents a computing method and models for optimizing the combination defined in combinatorics. The optimized combination has been derived from the iterative computation of multiple geometric series and summability by specialized approach. The optimized combinatorial technique has applications in science, engineering and management. In this paper, several properties and consequences on the innovative optimized combination has been introduced that are useful for scientific researchers who are solving scientific problems and meeting today's challenges.

2010 Mathematics Subject Classifications: 05-xx, 05A10, 05A19.
Keywords and phrases: optimized combination, combinatorics, counting technique, binomial coefficient.

1 Introduction

Combinatorics is a collection of various counting techniques or methods and models and has many applications in science, technology, and management. In the research paper, optimized combination of combinatorics is introduced that are useful for scientific researchers who are solving scientific problems and meeting today's challenges.

2 Optimized Combination

The growing complexity of mathematical modelling and its application demands the simplicity of numerical equations and combinatorial techniques for solving the scientific problems facing today. In view of this idea, the optimized combination of combinatorics is introduced that is

$$V_r^n = \frac{(r+1)(r+2)(r+3)\cdots(r+n-1)(r+n)}{n!}, (n, r \in N, n \geq 1, r \geq 0)$$

where $N = \{0, 1, 2, 3, 4, 5, ...\}$ is the set of natural numbers including the element 0.

This optimized combination is derived from the iterative computations [1 - 4] of multi-geometric series and summability as follows

(A) $\sum_{i_1=0}^{n-1} \sum_{i_2=i_1}^{n-1} \sum_{i_3=i_2}^{n-1} \cdots \sum_{i_n=i_{n-1}}^{n-1} x^{i_n} = \sum_{i=0}^{n-1} V_i^p x^i, (p \in N,\ 1 \leq p \leq n-1),$

where V_i^p is a binomial coefficient and its mathematical expressions are given below:

$$V_i^p = \frac{(i+1)(i+2)(i+3)\ldots(i+p)}{p!}(1 \leq p \leq n-1).$$

$$V_{i-k}^p = \frac{(i-k+1)(i-k+2)(i-k+3)\ldots(i-k+p)}{p!}.$$

Let us prove the equation (A) using the multiple geometric series.

$$\sum_{i_1=0}^{n-1} \sum_{i_2=i_1}^{n-1} x^{i_2} = \sum_{i_2=0}^{n-1} x^{i_2} + \sum_{i_2=1}^{n-1} x^{i_2} + \sum_{i_2=2}^{n-1} x^{i_2} + \cdots + \sum_{i_2=n-1}^{n} x^{i_2} = \sum_{i=0}^{n-1} \frac{(i+1)}{1!} x^i = \sum_{i=0}^{n-1} V_i^1 x^i,$$

where

$$\sum_{i_2=0}^{n-1} x^{i_2} + \sum_{i_2=1}^{n-1} x^{i_2} + \sum_{i_2=2}^{n-1} x^{i_2} + \cdots + \sum_{i_2=n-1}^{n} x^{i_2} = 1 + 2x + 3x^2 + \cdots + \frac{n}{1!} x^{n-1}.$$

$$\sum_{i_1=0}^{n-1} \sum_{i_2=i_1}^{n-1} \sum_{i_3=i_2}^{n-1} x^{i_3} = \sum_{i_2=0}^{n-1} \sum_{i_3=i_2}^{n-1} x^{i_3} + \sum_{i_2=1}^{n-1} \sum_{i_3=i_2}^{n-1} x^{i_3} + \sum_{i_2=2}^{n-1} \sum_{i_3=i_2}^{n-1} x^{i_3} + \cdots + \sum_{i_2=n-1}^{n-1} \sum_{i_3=i_2}^{n-1} x^{i_3}$$

$$= (1 + 2x + 3x^2 + \cdots + nx^{n-1}) + (x + 2x^2 + 3x^3 \cdots + (n-1)x^{n-1}) + \cdots x^{n-1}$$

$$= 1 + 3x + 6x^2 + 10x^3 + \cdots + \frac{n(n+1)}{2!} x^{n-1} = \sum_{i=0}^{n-1} \frac{(i+1)(i+2)}{2!} x^i = \sum_{i=0}^{n-1} V_i^2 x^i,$$

where

$$\sum_{i_1=0}^{n-1} \sum_{i_2=i_1}^{n-1} \sum_{i_3=i_2}^{n-1} x^{i_3} = 1 + 3x + 6x^2 + 10x^3 + 15x^4 + 21x^5 + \cdots + \frac{n(n+1)}{2!} x^{n-1}.$$

$$\sum_{i_1=0}^{n-1} \sum_{i_2=i_1}^{n-1} \sum_{i_3=i_2}^{n-1} \sum_{i_4=i_3}^{n-1} x^{i_4} = \sum_{i=0}^{n-1} \frac{(i+1)(i+2)(x+3)}{3!} x^i = \sum_{i=0}^{n-1} V_i^3 x^i$$

where

$$\sum_{i_1=0}^{n-1} \sum_{i_2=i_1}^{n-1} \sum_{i_3=i_2}^{n-1} \sum_{i_4=i_3}^{n-1} x^{i_4} = 1 + 4x + 10x^2 + 20x^3 + 35x^4 + \cdots + \frac{n(n+1)(n+2)}{3!} x^{n-1}.$$

If we continue like this, the binomial coefficient of the multisereis is $V_i^p(1 \leq p \leq n-1)$.

3 To convert combinations

3.1 To convert the combination ${}^nC_r$ into the optimized combination

$${}^nC_r = \frac{n!}{r!(n-r)!} = (V_0^r)(V_r^{n-1}) = V_r^{n-r} \text{ where } V_0^r = 1.$$

Let us consider $n - r = k$ for easily understood.
Then,

$$V_r^{n-r} = V_r^k = \frac{(r+1)(r+2)(r+3)\cdots(r+k)}{k!}.$$

3.2 To convert the combination ${}^nC_n$ into the optimized combination

$${}^nC_n = \frac{n!}{n!} = V_0^n = 1.$$

3.3 To convert the combination ${}^{(n+r)}C_r$ into the optimized combination

$${}^{(n+r)}C_r = \frac{n!}{r!(n+r-r)!} = \frac{n!}{r!n!} = \frac{1.2.3\cdots r(r+1)(r+2)\cdots(r+n)}{r!n!} = (V_0^r)(V_r^n).$$

$(V_0^r)(V_r^n) = V_r^n$, where $V_0^r = 1$.

Now $V_r^n(n, r \in N, n \geq 1, r \geq 0)$ is considered as optimized combination.

4 Some results with proofs on the optimized combination [5,6]

***Result* 4.1** $V_0^1 = V_0^n = 1$.

Proof.

(4.1) $V_0^1 = \dfrac{(0+1)}{1!} = 1.$

(4.2) $V_0^n = \dfrac{(0+1)(0+2)(0+3)\cdots(0+n)}{n!} = \dfrac{n!}{n!} = 1.$

From (4.1) and (4.2), the ***Result* 4.1** is true.

***Result* 4.2** $V_r^{n+1} - V_r^n = V_{r-1}^n$.

Proof. $V_r^n = \frac{(r+1)(r+2)(r+3)\cdots(r+n)}{n!}$,

$$V_r^{n+1} = \frac{(r+1)(r+2)(r+3)\cdots(r+n)(r+n+1)}{(n+1)!},$$

$$V_r^{n+1} - V_r^n = \frac{(r+1)(r+2)(r+3)\cdots(r+n)}{n!}[\frac{r+n+1}{n+1} - 1],$$

(4.3) $V_r^{n+1} - V_r^n = \dfrac{r(r+1)(r+2(r+3)+\cdots+(r+n)}{n!} = V_{r-1}^n.$

It is understood from (4.3) that the ***Result* 4.2** is true.

***Result* 4.3** $1 + V_1^1 + V_1^2 + V_1^3 + \cdots + V_1^n = V_2^n$.

Proof.

(4.4) $V_2^n = \dfrac{(2+1)(2+2)(2+3)\cdots(2+n-1)(2+n)}{n!} = \dfrac{(n+1)(n+2)}{2!},$

(4.5) $1 + V_1^1 + V_1^2 + V_1^3 + \cdots + V_1^n = 1 + 2 + 3 + \cdots + n + 1 = \dfrac{(n+1)(n+2)}{2!}.$

From (4.4) and (4.5), the ***Result* 4.3** is true.

***Result* 4.4** $V_r^n = V_n^r(n, r \geq 1n, r \in N)$.

Proof.

$V_r^n = V_n^r$ implies $\dfrac{(r+1)(r+2)\cdots(r+n)}{n!} = \dfrac{(n+1)(n+2)\cdots(n+r)}{r!}$.

Assume that $r = n + m(m \in Nm \geq 1)$. Let us show that $V_{n+m}^n = V_n^{n+m}$.

$$(4.6)\quad V_{n+m}^n = \frac{(n+m+1)(n+m+2)\cdots(n+m+n)}{n!} = \frac{(n+1)(n+2)\cdots(n+m+n)}{(n+m)!}$$

$$(4.7)\quad V_n^{n+m} = \frac{(n+1)(n+2)\cdots(n+n)(n+n+1)(n+n+2)\cdots(n+n+m)}{(n+m)!}$$

From (4.6) and (4.7), $V_{n+m}^n = V_n^{n+m}$ is true.

Assume that $r = n - m(n > m)$. Let us show that $V_{n-m}^n = V_n^{n-m}$.

$$(4.8)\quad V_{n-m}^n = \frac{(n-m+1)(n-m+2)\cdots(n-m+n)}{n!} = \frac{(n+1)(n+2)\cdots(n+n-m)}{(n-m)!}.$$

$$(4.9)\quad V_n^{n-m} = \frac{(n+1)(n+2)\cdots(n+n-m)}{(n-m)!},$$

From (4.8) and (4.9), $V_{n-m}^n = V_n^{n-m}$ is true.

If $r = n$, $V_r^n = V_n^r$ is obivously true for $r = n$.

Hence, the ***Result* 4.4** is true.

***Result* 4.5** $V_n^n = 2V_{n-1}^n$.

Proof.

$$V_n^n = \frac{(n+1)(n+2)\cdots(n+n-1)2n}{(n-1)!n} = \frac{2(n+1)(n+2)\cdots(n+n-1)}{(n-1)!} = 2V_{n-1}^n.$$

Hence, the Result 4.5 is true.

***Result* 4.6** $V_0^n + V_1^n + V_2^n + V_3^n + \cdots + V_{r-1}^n + V_r^n = V_r^{n+1}$.

Proof. This result is proved by mathematical induction. Basis. Let $r = 1.V_0^n + V_1^n = V_1^{n+1}$ implies $n+2 = n+2$.

Inductive hypothesis.

Let us assume that $V_0^n + V_1^n + V_2^n + \cdots + V_{k-1}^n = V_{k-1}^{n+1}$ is true for $r = k-1$.

Inductive step. We must show that the inductive hypothesis is true for $r = k$.

$V_0^n + V_1^n + \cdots + V_{k-1}^n + V_k^n = V_k^{n+1}$ implies $V_0^n + V_1^n + \cdots + V_{k-1}^n = V_k^{n+1} - V_k^n = V_{k-1}^{n+1}$.

Hence, it is proved.

To convert the combination ${}^{(n+r)}C_r$ into the optimized combination:

$${}^{(n+r)}C_r = \frac{n!}{r!(n+r-r)!} = \frac{n!}{r!n!} = \frac{1.2.3\cdots r(r+1)(r+2)\cdots(r+n)}{r!n!} = (V_0^r)(V_r^n).$$

$(V_0^r)(V_r^n) = V_r^n$ where $V_0^r = 1$.

5 Conclusion

In the research paper, a computing method and models for optimizing the combination defined in combinatorics has been introduced that are useful for scientific researchers who are solving scientific problems and meeting today's challenges.

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A QUADRUPLE FIXED POINT THEOREM FOR A MULTIMAP IN A HAUSDORFF FUZZY METRIC SPACE

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Abstract

Rao and Rao[16] obtained a triple fixed point theorem for a multimap in Hausdorff fuzzy metric space. Extending this idea we generalize the concept of triple fixed point, we define quadruple fixed point. In this paper we have established a result regarding it in Hausdorff fuzzy metric space.
2010 Mathematics Subject Classifications: 47H10, 54H25
Keywords and phrases: Quadruple fixed point, Hausdorff fuzzy metric space, multimaps

1 Introduction and Preliminaries

Zadeh [23] introduced the concept of fuzzy sets in 1965. Since then, it was developed extensively by many authors. Fuzzy metric spaces have been defined by several researchers in several ways (e.g.[6,7]). The concepts of fuzzy metric space introduced by Kramosil and Michlek [12] have been modified by George and Veeramani [7] and also induced a Hausdro topology on such fuzzy metric space. The contraction principle in the setting of fuzzy metric spaces introduced in [7] was later proved by Grabiec[9]. Some interesting references for fixed point theorems in fuzzy metric spaces are given in [3,4,5,21].

Nadler [14] initiated the study of fixed points for multivalued contraction mappings using the Hausdor metric. In 2004, Rodrguez-Lpez and Romaguera [17] introduced Hausdors fuzzy metric on the set of the nonempty compact subsets of a given fuzzy metric space. Later some fixed point theorems for multivalued maps in fuzzy metric spaces (e.g., [1,11,20,22]) were proved by several authors. Many authors studied the existence of fixed points for various multivalued contractive mappings under dierent conditions, refer to [12-14] and the references therein.

In 2006 coupled fixed point in partially ordered metric spaces was introduced by Gnana Bhaskar and Lakshmikantham [8] and some problems of the uniqueness of a coupled fixed point was discussed and the results were applied to periodic boundary value problems. In 2011, Samet and Vetro [18] proved the coupled fixed point theorems for a multivalued mapping. Berinde and Borcut [2] also introduced the concept of triple fixed points and obtained a triple fixed point theorem for a single valued map in partially ordered metric spaces.

In this paper, we obtain a quadruple fixed point theorem for a multimap in a Hausdor fuzzy metric space and using it, we obtain a common quadruple fixed point for a multi- and single valued maps.

For this we need the following.

***Definition* 1.1** [19] *A binary operation* $*:[0,1]\times[0,1]\to[0,1]$ *is a continuous t-norm if it satisfies the following conditions:*

1. $*$ *is associative and commutative,*
2. $*$ *is continuous,*
3. $a*1=a$ *for all* $a\epsilon[0,1]$,
4. $a*b\leq c*d$ *whenever* $a\leq c$ *and* $b\leq d$, *for each* $a,b,c,d\in[0,1]$.

Two typical examples of continuous t-norm are $a*b=ab$ *and* $a*b=\min\{a,b\}$.

***Definition* 1.2** [7] *A 3-tuple $(X, M, *)$ is called a fuzzy metric space if X is an arbitrary (nonempty) set, $*$ is a continuous t-norm, and is a fuzzy set on $X^2 \times (0, \infty)$, satisfying the following conditions for each $x, y, z \in X$ and each t and $s > 0$,*
1. $M(x, y, t) > 0$,

2. $M(x, y, t) = 1$ *if and only if* $x = y$,

3. $M(x, y, t) = M(y, x, t)$,

4. $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,

5. $M(x, y,) : (0, \infty) \to [0, 1]$ *is continuous.*

Let $(X, M, *)$ be a fuzzy metric space. For > 0, the open ball $B(x, r, t)$ with centre $x \in X$ and radius $0 < r < 1$ is defined by $B(x, r, t) = \{y \in X : M(x, y, t) > 1 - r\}$.

A subset $A \subset X$ is called open if for each $x \in A$ there exist $t > 0$ and $0 < r < 1$ such that $B(x, r, t) \subset A$. Let τ denote the family of all open subsets of X. Then τ is called the topology on X induced by the fuzzy metric M. This topology is Hausdorff and first countable. A subset A of X is said to be F-bounded if there exist $t > 0$ and $0 < r < 1$ such that $M(x, y, t) > 1 - r$ for all $x, y \in A$.

***Lemma* 1.1** [9] *Let $(X, M, *)$ be a fuzzy metric space. Then $M(x, y, t)$ is nondecreasing with respect to t for all x, y in X.*

***Definition* 1.3** [17] *Let $(X, M, *)$ be a fuzzy metric space. M is said to be continuous on $X^2 \times (0, \infty)$ if*

(1.1) $\lim_{n\to\infty} M(x_n, y_n, t_n) = M(x, y, t)$,

whenever a sequence $\{(x_n, y_n, t_n)\}$ in $X^2 \times (0, \infty)$ converges to a point $(x, y, t) \in X^2 \times (0, \infty)$, that is, whenever

$$\lim_{n\to\infty} M(x_n, x, t) = \lim_{n\to\infty} M(y_n, y, t) = M(x, y, t) = 1,$$

(1.2) $\lim_{n\to\infty} (x, y, t_n) = M(x, y, t)$.

***Lemma* 1.2** [17] *Let $(X, M, *)$ be a fuzzy metric space. Then M is a continuous function on $X^2 \times (0, \infty)$. Also let us take the condition:*

(1.3) $\lim_{t\to\infty} M(x, y, t) = 1, \ \forall\, x, y \in X$.

***Lemma* 1.3** [13] *Let $\{y_n\}$ be a sequence in fuzzy metric space $(X, M, *)$ satisfying condition (3). If there exists a positive number $k < 1$ such that*

(1.4) $M(y_n, y_{n+1}, kt) \geq M(y_{n-1}, y_n, t), \quad t > 0, n = 1, 2....$

***Definition* 1.4** [17] *Let B be a nonempty subset of a fuzzy metric space $(X, M, *)$. For $a \in X$ and $t > 0$, define $M(a, B, t) = \sup\left\{\frac{a,b,t}{b} \in B\right\}$.*

In this paper let $K(X)$ denotes the class of all non empty compact subsets of X.

***Lemma* 1.4** [17] *Let $(X, M, *)$ be a fuzzy metric space. Then for each $a \in X$, $B \in K(X)$ and $t > 0$, there exists $b \in B$ such that $M(a, B, t) = M(a, b, t)$.*

***Definition* 1.5** [17] *Let $(X, M, *)$ be a fuzzy metric space. For each $A, B \in K(X)$ and $t > 0$, set*

(1.5) $H_M(A, B, t) = \min\left\{\inf_{x\in A} M(x, B, t),\ \inf_{y\in B} M(A, y, t)\right\}$.

The 3-tuple $\left(K(X), H_M, *\right)$ is called a Hausdorff fuzzy metric space.

***Lemma* 1.5** [10] *Let X be a nonempty set and $g : X \to X$ be a mapping. Then there exists a subset $E \subseteq X$ such that $g(E) = g(X)$ and $g : X \to X$ is one one.*

***Definition* 1.6** *Let X be a nonempty set, $T : X\times X\times X\times X \to 2^X$ (collection of all nonempty subset of X) and $f : X \to X$.*
(i) The point $(s, x, y, z) \in X \times X \times X \times X$ is called a quadruple fixed point of T if

(1.6) $s \in T(s, x, y, z)$,

$y \in T(y, z, s, x)$,

$z \in T(z, s, x, y)$.

(ii) The point $(s, x, y, z) \in X \times X \times X \times X$ is called a quadruple coincident point of T and f if

(1.7) $f_s \in T(s, x, y, z)$,

$f_x \in T(x,y,z,s)$,

$f_y \in T(y,z,s,x)$,

$f_z \in T(z,s,x,y)$.

(iii) The point $(s,x,y,z) \in X \times X \times X \times X$ is called a quadruple common fixed point of T and f if

(1.8) $s = f_s \in T(s,x,y,z)$,

$x = f_x \in T(x,y,z,s)$,

$y = f_y \in T(y,z,s,x)$,

$z = f_z \in T(z,s,x,y)$.

***Definition* 1.7** *Let $T : X \times X \times X \times X \to 2^X$ be a multivalued map and f be a self map on X. The Hybrid pair $\{T, f\}$ is called w-compatible if $f(T(s,x,y,z)) \subseteq T(fs, fx, fy, fz)$ whenever (s,x,y,z) is quadruple coincident point of T and f.*

2 Main Result

Let us prove a slightly different result from Lemma 1.3 which we will use to prove our main result.

***Lemma* 2.1** *Let $\{s_n\}, \{x_n\}, \{y_n\}$ and $\{z_n\}$ be sequences in fuzzy metric space $(X, M, *)$ satisfying condition (1.3). If there exists a positive number $k < 1$ such that*

(2.1) $$\min\left\{M(s_n, s_{n+1}, kt)(x_n, x_{n+1}, kt)(y_n, y_{n+1}, kt)(z_n, z_{n+1}, kt)\right\}$$
$$\geq \min\left\{M(s_{n-1}, s_n, t)(x_{n-1}, x_n, t)(y_{n-1}, y_n, t)(z_{n-1}, z_n, t)\right\},$$

for all $t > 0,\ n = 1, 2, ...,$ then $\{s_n\}, \{x_n\}, \{y_n\}$ and $\{z_n\}$ are Cauchy sequences in X.

Proof. We have

(2.2) $$\min\left\{M(s_n, s_{n+1}, kt), M(x_n, x_{n+1}, kt), M(y_n, y_{n+1}, kt), M(z_n, z_{n+1}, kt)\right\}$$
$$\geq \min\left\{M(s_{n-1}, s_n, t), M(x_{n-1}, x_n, t), M(y_{n-1}, y_n, t), M(z_{n-1}, z_n, t)\right\}$$
$$\geq \min\left\{M\left(s_{n-2}, s_{n-1}, \frac{t}{k^2}\right), M\left(x_{n-2}, x_{n-1}, \frac{t}{k^2}\right), M\left(y_{n-2}, y_{n-1}, \frac{t}{k^2}\right),\right.$$
$$\left.M\left(z_{n-2}, z_{n-1}, \frac{t}{k^2}\right)\right\}$$
$$\vdots$$
$$\geq \min\left\{M\left(s_0, s_1, \frac{t}{k^n}\right), M\left(x_0, x_1, \frac{t}{k^n}\right), M\left(y_0, y_1, \frac{t}{k^n}\right), M\left(z_0, z_1, \frac{t}{k^n}\right)\right\}$$

.

Hence,

(2.3) $$M(s_n, s_{n+1}, t) \geq \min\left\{M\left(s_0, s_1, \frac{t}{k^n}\right), M\left(x_0, x_1, \frac{t}{k^n}\right), M\left(y_0, y_1, \frac{t}{k^n}\right), M\left(z_0, z_1, \frac{t}{k^n}\right)\right\}.$$

Now, for any positive integer p,

(2.4) $$M\left(s_n, s_{n+p}, t\right) \geq M\left(s_n, s_{n+1}, \frac{t}{p}\right) * M\left(s_{n+1}, s_{n+2}, \frac{t}{p}\right) * ... * M\left(s_{n+p-1}, s_{n+p}, \frac{t}{p}\right)$$
$$\geq \min\left\{M\left(s_0, s_1, \frac{t}{pk^n}\right), M\left(x_0, x_1, \frac{t}{pk^n}\right), M\left(y_0, y_1, \frac{t}{pk^n}\right), M\left(z_0, z_1, \frac{t}{pk^n}\right)\right\}$$
$$* \min\left\{M\left(s_0, s_1, \frac{t}{pk^{n+1}}\right)\left(x_0, x_1, \frac{t}{pk^{n+1}}\right)\left(y_0, y_1, \frac{t}{pk^{n+1}}\right)\left(z_0, z_1, \frac{t}{pk^{n+1}}\right)\right\}$$
$$*...* \geq \min\left\{M\left(s_0, s_1, \frac{t}{pk^{n+p-1}}\right), M\left(x_0, x_1, \frac{t}{pk^{n+p-1}}\right), M\left(y_0, y_1, tpk^{n+p-1}\right), M(z_0, z_1, \frac{t}{pk^{n+p-1}})\right\}.$$

Letting $n \to \infty$ and using condition (3), we have

$$\lim_{n\to\infty} M\Big(x_n, x_{n+p}, t\Big) \geq 1 * 1 * ... * 1 = 1.$$

Hence,

$$(2.5)\quad \lim_{n\to\infty} M\Big(x_n, x_{n+p}, t\Big) = 1.$$

Thus $\{s_n\}$ is a Cauchy sequence in X. Similarly, we can show that $\{x_n\}, \{y_n\}$ and $\{z_n\}$ are Cauchy sequences in X.
Now, let us prove our first main result.

***Theorem* 2.1** *Let $(X, M, *)$ be a complete fuzzy metric space satisfying condition (1.3) and $F : X \times X \times X \times X \to K(X)$ be a set valued mapping satisfying*

$$(2.6)\quad H_M\Big(F(s, x, y, z), F(h, u, v, w), kt\Big) \geq \min\Big\{M(s, h, t), M(x, u, t)M(y, v, t), M(z, w, t)\Big\}$$

for each $s, x, y, z, h, u, v, w \in X,\ t > 0$, where $0 < k < 1$.

Then F has a quadruple fixed point.

Proof. Let $s_0, x_0, y_0, z_0 \in X$.

Choose $s_1 \in F(s_0, x_0, y_0, z_0), x_1 \in F(x_0, y_0, z_0, s_0), y_1 \in (y_0, z_0, s_0, x_0), z_1 \in F(z_0, s_0, x_0, y_0)$.

Since F is compact valued, by Lemma 1.4, there exists $s_2 \in F(s_1, x_1, y_1, z_1)$ such that

$$\begin{aligned}(2.7)\quad M(s_1, s_2, kt) &= \sup_{x\in F(s_1,x_1,y_1,z_1)} M(s_1, s, kt)\\ &\geq H_M\Big(F(s_0, x_0, y_0, z_0), F(s_1, x_1, y_1, z_1), kt\Big)\\ &\geq \min\Big\{M(s_0, s_1, t), M(x_0, x_1, t), M(y_0, y_1, t), M(z_0, z_1, t)\Big\}.\end{aligned}$$

Since F is compact valued, by Lemma 1.4, there exists $x_2 \in F(x_1, y_1, z_1, s_1)$ such that

$$\begin{aligned}(2.8)\quad M(x_1, x_2, kt) &= \sup_{x\in F(x_1,y_1,z_1,s_1)} M(x_1, x, kt)\\ &\geq H_M(F(x_0, y_0, z_0, s_0), F(x_1, y_1, z_1, s_1), kt)\\ &\geq \min\{M(x_0, x_1, t), M(y_0, y_1, t), M(z_0, z_1, t), M(s_0, s_1, t)\}\\ &\geq \min\{M(s_0, s_1, t), M(x_0, x_1, t), M(y_0, y_1, t), M(z_0, z_1, t)\}.\end{aligned}$$

Since F is compact valued, by Lemma 1.4, there exists $y_2 \in F(y_1, z_1, s_1, x_1)$ such that

$$\begin{aligned}(2.9)\quad M(y_1, y_2, kt) &= \sup_{x\in F(y_1,z_1,s_1,x_1)} M(y_1, y, kt)\\ &\geq H_M(F(y_0, z_0, s_0, x_0), F(y_1, z_1, s_1, x_1), kt)\\ &\geq \min\{M(y_0, y_1, t), M(z_0, z_1, t), M(s_0, s_1, t), M(x_0, x_1, t)\}.\\ &\geq \min\{M(s_0, s_1, t), M(x_0, x_1, t), M(y_0, y_1, t), M(z_0, z_1, t)\}.\end{aligned}$$

Since F is compact valued, by Lemma 1.4, there exists $z_2 \in F(z_1, s_1, x_1, y_1)$ such that

$$\begin{aligned}(2.10)\quad M(z_1, z_2, kt) &= \sup_{x\in F(z_1,s_1,x_1,y_1)} M(z_1, z, kt)\\ &\geq H_M(F(z_0, s_0, x_0, y_0), F(z_1, s_1, x_1, y_1), kt)\\ &\geq \min\{M(z_0, z_1, t), M(s_0, s_1, t), M(x_0, x_1, t), M(y_0, y_1, t)\}\\ &\geq \min\{M(s_0, s_1, t), M(x_0, x_1, t), M(y_0, y_1, t), M(z_0, z_1, t)\}.\end{aligned}$$

Thus,

$$\begin{aligned}(2.11)\quad &\min\{M(s_1, s_2, kt), M(x_1, x_2, kt), M(y_1, y_2, kt), M(z_1, z_2, kt)\}\\ &\geq \min\{M(s_0, s_1, t), M(x_0, x_1, t), M(y_0, y_1, t), M(z_0, z_1, t)\}.\end{aligned}$$

Continuing in this way we can find the sequences $\{s_n\}, \{x_n\}, \{y_n\}$ and $\{z_n\}$ in X such that

$$s_{n+1} \in F(s_n, x_n, y_n, z_n), x_{n+1} \in F(x_n, y_n, z_n, s_n), y_{n+1} \in F(y_n, z_n, s_n, x_n),$$
$$z_{n+1} \in F(z_n, s_n, x_n, y_n)$$

Such that

(2.12) $\min\{M(s_n, s_{n+1}, kt), M(x_n, x_{n+1}, kt), M(y_n, y_{n+1}, kt), M(z_n, z_{n+1}, kt)\}$
$$\geq \min\{M(s_{n-1}, s_n, t), M(x_{n-1}, x_n, t), M(y_{n-1}, y_n, t), M(z_{n-1}, z_n, t)\}.$$

Hence, by Lemma 2.1, $\{s_n\}, \{x_n\}, \{y_n\}$ and $\{z_n\}$ are Cauchy sequences in X.

Since X is complete, there exists $s, x, y, z \in X$ such that $\lim_{n\to\infty}\{s_n\} = s, \lim_{n\to\infty}\{x_n\} = x, \lim_{n\to\infty}\{y_n\} = y, \lim_{n\to\infty}\{z_n\} = z$.

Consider

(2.13) $H_M(F(s_n, x_n, y_n, z_n), F(s, x, y, z), kt)$
$$\geq \min\{M(s_n, s, t), M(x_n, x, t), M(y_n, y, t), M(z_n, z, t)\}.$$

Let $n \to \infty$, we get

(2.14) $\lim\limits_{n\to\infty} H_M(F(s_n, x_n, y_n, z_n), F(s, x, y, z), kt) = 1$ so that
$$\lim_{n\to\infty} H_M(F(s_n, x_n, y_n, z_n), F(s, x, y, z), t) = 1.$$

Similarly we can show that

(2.15) $\lim\limits_{n\to\infty} H_M(F(x_n, y_n, z_n, s_n), F(x, y, z, s), t) = 1,$
$$\lim_{n\to\infty} H_M(F(y_n, z_n, s_n, x_n), F(y, z, s, x), t) = 1,$$
$$\lim_{n\to\infty} H_M(F(z_n, s_n, x_n, y_n), F(z, s, x, y), t) = 1.$$

Since
$$s_{n+1} \in F(s_n, x_n, y_n, z_n), x_{n+1} \in F(x_n, y_n, z_n, s_n), y_{n+1} \in F(y_n, z_n, s_n, x_n),$$
$$z_{n+1} \in F(z_n, s_n, x_n, y_n),$$

From (2.12) and (2.13), we have

(2.16) $\lim\limits_{n\to\infty} \sup\limits_{a\in F(s,x,y,z)} M(s_{n+1}, a, t) = 1,$
$$\lim_{n\to\infty} \sup_{b\in F(x,y,z,s)} M(x_{n+1}, b, t) = 1,$$
$$\lim_{n\to\infty} \sup_{c\in F(y,z,s,x)} M(y_{n+1}, c, t) = 1,$$
$$\lim_{n\to\infty} \sup_{d\in F(z,s,x,y)} M(z_{n+1}, d, t) = 1.$$

Hence there exist sequences $l_n \in F(s, x, y, z), p_n \in F(x, y, z, s), q_n \in F(y, z, s, x)$ and $r_n \in F(z, s, x, y)$ such that

(2.17) $\lim\limits_{n\to\infty} M(s_{n+1}, l_n, t) = 1,$
$$\lim_{n\to\infty} M(x_{n+1}, p_n, t) = 1,$$
$$\lim_{n\to\infty} M(y_{n+1}, q_n, t) = 1,$$
$$\lim_{n\to\infty} M(z_{n+1}, r_n, t) = 1.$$

for each $t > 0$.

Now for each $n \in N$, we have

(2.18) $M(l_n, s, t) \geq M(l_n, s_{n+1}, t/2) * M(s_{n+1}, s, t/2).$

Letting $n \to \infty$, we obtain

(2.19) $\lim\limits_{n\to\infty} M(l_n, s, t) = 1$ so that $\lim\limits_{n\to\infty} l_n = s.$

Similarly, we can show that

(2.20) $\lim\limits_{n\to\infty} p_n = x, \quad \lim\limits_{n\to\infty} q_n = y, \quad \lim\limits_{n\to\infty} r_n = z.$

Since $F(s, x, y, z), F(x, y, z, s), F(y, z, s, x)$ and $F(z, s, x, y)$ are compact, we have
$s \in F(s, x, y, z), x \in F(x, y, z, s), y \in F(y, z, s, x)$ and $z \in F(z, s, x, y)$.

Thus (s, x, y, z) is a quadruple fixed point of F.

***Theorem* 2.2** *Let $(X, M, *)$ be a complete fuzzy metric space satisfying condition (1.3) and $F : X \times X \times X \times X \to K(X)$ and $gX \to X$ be a mappings satisfying*

$$(2.21)\quad H_M\Big(F(s,x,y,z), F(h,u,v,w), kt\Big) \geq \min\Big\{M(gs,gh,t), M(gx,gu,t), M(gy,gv,t), M(gz,gw,t)\Big\},$$

for all $s,x,y,z,h,u,v,w \in X$, $t > 0$ and $0 < k < 1$. further assume that $F(X \times X \times X \times X) \subseteq g(x)$, then F and g have a quadrupled coincidence point. Moreover, F and g have a quadrupled common fixed point if one of the following conditions holds.
(a) The pair F, g is called w-compatible and there exists $\mu, \alpha, \beta, \gamma \in X$ such that $\lim_{n\to\infty} g^n s = \mu \lim_{n\to\infty} g^n x = \alpha, \lim_{n\to\infty} g^n y = \beta, \lim_{n\to\infty} g^n z = \gamma$, whenever (s,x,y,z) is a quadrupled coincidence point of F and g and g is continuous at $\mu, \alpha, \beta, \gamma$.
(b) There exist $\mu, \alpha, \beta, \gamma \in X$ such that $\lim_{n\to\infty} g^n\mu = s, \lim_{n\to\infty} g^n\alpha = x, \lim_{n\to\infty} g^n\beta = y, \lim_{n\to\infty} g^n\gamma = z$, whenever (s,x,y,z) is a quadrupled coincidence point of F and g and g is continuous at s,x,y,z.

Proof. By ***Lemma* 1.5**, there exists $E \subseteq X$ such that $g : E \to X$ is one to one and $g(E) = g(X)$.

Now, define $G : g(E) \times g(E) \times g(E) \times g(E) \to K(X)$ by $G(gs, gx, gy, gz) = F(x,y,z)$ for all $gs, gx, gy, gz \in g(E)$. Since g is one-one one E, G is well defined.

Now,

$$(2.22)\quad H_M\Big(G(gs,gx,gy,gz), G(gh,gu,gv,gw), kt\Big) = H_M\Big(F(s,x,y,z), F(h,u,v,w), kt\Big) \geq \min\Big\{M(gs,gh,t), M(gx,gu,t), M(gy,gv,t), M(gz,gw,t)\Big\}.$$

Hence G satisfies (2.6) and all the conditions of ***Theorem* 2.1**.

By ***Theorem* 2.1**, G has a quadruple fixed point $(h,u,v,w) \in g(E) \times g(E) \times g(E) \times g(E)$. Thus,

$$(2.23)\quad \begin{aligned} &h \in (h,u,v,w),\\ &u \in (u,v,w,h),\\ &v \in (v,w,h,u),\\ &w \in (w,h,u,v). \end{aligned}$$

Since $F(X \times X \times X \times X) \subseteq g(x)$, there exist $h,u,v,w \in X \times X \times X \times X$ such that $gh_1 = h, gu_1 = u, gv_1 = v$ and $gw_1 = w$. so from (2.23) we have

$$\begin{aligned} gh_1 &\in G(gh_1, gu_1, gv_1, gw_1) = F(h_1, u_1, v_1, w_1)\\ gu_1 &\in G(gu_1, gv_1, gw_1, gh_1) = F(u_1, v_1, w_1, h_1)\\ gv_1 &\in G(gv_1, gw_1, gh_1, gu_1) = F(v_1, w_1, h_1, u_1)\\ gw_1 &\in G(gv_1, gw_1, gh_1, gu_1) = F(w_1, h_1, u_1, v_1). \end{aligned}$$

This implies that $h_1, u_1, v_1, w_1 \in X \times X \times X \times X$ is a quadruple fixed point of F and g.

Now following as in [15] except from the inequalities satisfied by M we can show that F and g have a quadruple fixed point.

3 Conclusion

Thus, our paper establishes the results regarding the quadruple fixed point in Hausdorff fuzzy metric space.
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(Dedicated to Honor Dr. R. C. Singh Chandel on His 75^{th} Birth Anniversary Celebrations)

HYPERGEOMETRIC FORMS OF CERTAIN COMPOSITE FUNCTIONS INVOLVING ARCSINE(x) USING MACLAURIN SERIES AND THEIR APPLICATIONS

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Abstract

In this article, we obtain hypergeometric forms of some composite functions containing arcsine(x) like: $\exp(a\sin^{-1}x)$, $\frac{\exp(a\sin^{-1}x)}{\sqrt{(1-x^2)}}$, $\cos(b\sin^{-1}x)$, $\sin(b\sin^{-1}x)$, $\cosh(d\sin^{-1}x)$, $\sinh(d\sin^{-1}x)$, $\frac{\cos(b\sin^{-1}x)}{\sqrt{(1-x^2)}}$, $\frac{\sin(b\sin^{-1}x)}{\sqrt{(1-x^2)}}$ and arcsinh (x) like: $\exp(b\sinh^{-1}x)$, $\frac{\exp(b\sinh^{-1}x)}{\sqrt{(1+x^2)}}$, $\cos(g\sinh^{-1}x)$, $\sin(g\sinh^{-1}x)$, $\cosh(b\sinh^{-1}x)$, $\sinh(b\sinh^{-1}x)$, by using Leibniz theorem for successive differentiation, Maclaurin's series expansion and Taylor's series expansion, as the proof of the hypergeometric forms of the above functions is not available in the literature.
2010 Mathematics Subject Classifications: 33C05, 34A35, 41A58, 33B10.
Keywords and phrases: Hypergeometric function; Maclaurin's series; Taylor's series; Leibniz theorem.

1 Introduction and Preliminaries

In this paper, we shall use the following standard notations:
$\mathbb{N} := \{1, 2, 3, \cdots\}$; $\mathbb{N}_0 := \mathbb{N} \bigcup \{0\}$; and $\mathbb{Z}_0^- := \mathbb{Z}^- \bigcup \{0\} = \{0, -1, -2, -3, \cdots\}$.

The symbols $\mathbb{C}, \mathbb{R}, \mathbb{N}, \mathbb{Z}, \mathbb{R}^+$ and $\mathbb{R}^-$ denote the sets of complex numbers, real numbers, natural numbers, integers, positive and negative real numbers respectively.

The Pochhammer symbol $(\alpha)_p(\alpha, p \in \mathbb{C})$ is defined by ([10, p.22 Eq.(1), p.32, Q.N.(8) and Q.N.(9)],see also [12, p.23, Eq.(22) and Eq.(23)]).

A natural generalization of the Gaussian hypergeometric series ${}_2F_1[\alpha, \beta; \gamma; z]$ is accomplished by introducing any arbitrary number of numerator and denominator parameters[12, p.42, Eq.(1)].

Relation between hyperbolic and trigonometric functions:

(1.1) $\cos(i\theta) = \cosh(\theta), \quad \sin(i\theta) = i\sinh(\theta).$

(1.2) $\sin^{-1}(x) = -i\sinh^{-1}(ix), \ \ \sinh^{-1}(x) = -i\sin^{-1}(ix).$

The Taylor's series of a real or complex-valued function $y(x)$ which is infinitely differentiable at a real or complex number a, is the power series:

(1.3) $y(x) = (y)_{x=a} + (x-a)(y_1)_{x=a} + \dfrac{(x-a)^2}{2!}(y_2)_{x=a} + \dfrac{(x-a)^3}{3!}(y_3)_{x=a} + \dfrac{(x-a)^4}{4!}(y_4)_{x=a} + \cdots$

(1.4) $= \sum_{n=0}^{\infty} \frac{(x-a)^n}{n!}(y_n)_{x=a}$

(1.5) $= \sum_{n=0}^{\infty} \frac{(x-a)^{2n}}{(2n)!}(y_{2n})_{x=a} + \sum_{n=0}^{\infty} \frac{(x-a)^{2n+1}}{(2n+1)!}(y_{2n+1})_{x=a}.$

The Maclaurin's series is a particular case of Taylor's series expansion of a function, about the origin i.e, when $a = 0$ in equation (1.3), the Maclaurin series is given as:

$$y(x) = (y)_0 + x(y_1)_0 + \frac{x^2}{2!}(y_2)_0 + \frac{x^3}{3!}(y_3)_0 + \frac{x^4}{4!}(y_4)_0 + \frac{x^5}{5!}(y_5)_0 + \cdots$$

(1.6) $= \sum_{n=0}^{\infty} \frac{x^n}{n!}(y_n)_0$

(1.7) $= \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}(y_{2n})_0 + \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}(y_{2n+1})_0,$

where, $(y_m)_0 = \left(\frac{d^m y}{dx^m}\right)_{x=0}$.

The general Leibniz rule, named after Gottfried Wilhelm Leibniz, generalizes the product rule (which is also known as "Leibniz's rule"), which states that if $U(x)$ and $T(x)$ are n-times differentiable functions, then the product $U(x).T(x)$ is also n-times differentiable and its nth derivative is given by:

$$D^n[U(x)\,T(x)] = ({}^nC_0)(D^n\,U)(D^0\,T) + ({}^nC_1)(D^{n-1}\,U)(D^1\,T) + ({}^nC_2)(D^{n-2}\,U)(D^2\,T) + \cdots +$$

(1.8) $+({}^nC_{n-1})(D^1\,U)(D^{n-1}\,T) + ({}^nC_n)(D^0\,U)(D^n\,T)$

(1.9) $= \sum_{r=0}^{n} {}^nC_r(D^rT)(D^{n-r}U)$

(1.10) $= \sum_{r=0}^{n} {}^nC_r(D^{n-r}T)(D^rU)$,

where $D = \frac{d}{dx}$.

Euler's linear transformation

(1.11) $${}_2F_1\left[\begin{array}{c}\beta,\ \lambda;\\ \\ \mu;\end{array} z\right] = (1-z)^{\mu-\beta-\lambda}\ {}_2F_1\left[\begin{array}{c}\mu-\beta,\ \mu-\lambda;\\ \\ \mu;\end{array} z\right],$$

where $\mu \in \mathbb{C}\backslash\mathbb{Z}_0^-$ and $|arg(1-z)| < \pi$.

The present article is organized as follows. In section 3 we have given the proof of the hypergeometric forms of presented functions, because their proofs are not available in the literature[1, 2, 3, 4, 5, 6, 7, 8, 9] see also [11, 13]. So we are interested to give the proof of hypergeometric forms of some composite functions containing arcsine(x), using Maclaurin series. In section 4 we have obtained hypergeometric forms of some more functions by using the relations between inverse trigonometric and inverse hyperbolic functions. In section 5 we have discussed some applications of hypergeometric forms (2.1),(2.2) and (2.3). In section 6 we discussed some applications of hypergeometric forms (5.4) and (5.5).

2 Main Hypergeometric Forms of Certain Composite Functions

When the values of numerator, denominator parameters and arguments leading to the results which do not make sense are tacitly excluded, then each of the following hypergeometric form holds true:

(2.1) $$\exp(a\sin^{-1}x) = {}_2F_1\left[\begin{array}{c}\frac{ia}{2},\ -\frac{ia}{2};\\ \\ \frac{1}{2};\end{array} x^2\right] + ax\ {}_2F_1\left[\begin{array}{c}\frac{1+ia}{2},\ \frac{1-ia}{2};\\ \\ \frac{3}{2};\end{array} x^2\right],$$

where $|x| < 1$.

(2.2) $$\exp(b\sin^{-1}x) = \exp\left(\frac{\pi b}{2}\right){}_2F_1\left[\begin{array}{c}ib,\ -ib;\\ \\ \frac{1}{2};\end{array} \frac{1-x}{2}\right],$$

where $|\frac{1-x}{2}| < 1$.

(2.3) $$\frac{\exp(a\sin^{-1}x)}{\sqrt{(1-x^2)}} = {}_2F_1\left[\begin{array}{c}\frac{1+ia}{2},\ \frac{1-ia}{2};\\ \\ \frac{1}{2};\end{array} x^2\right] + ax\ {}_2F_1\left[\begin{array}{c}\frac{2+ia}{2},\ \frac{2-ia}{2};\\ \\ \frac{3}{2};\end{array} x^2\right],$$

where $|x| < 1$.

Note: In the above hypergeometric functions x, a and b can be purely real or purely imaginary or complex numbers.

3 Proof of Hypergeometric Forms

Proof of hypergeometric form (2.1)

Let

(3.1) $y = \exp(a\sin^{-1}x)$.

Put $x = 0$ in equation (3.1), we get

(3.2) $(y)_0 = 1$.

Differentiate equation (3.1) w.r.t. x and put $x = 0$, we get

(3.3) $\sqrt{1-x^2}\,y_1 = ay$,

(3.4) $(y_1)_0 = a$.

Differentiate equation (3.3) w.r.t. x and put $x = 0$, we get

(3.5) $(1 - x^2)y_2 - xy_1 - a^2 y = 0,$

(3.6) $(y_2)_0 = a^2.$

Differentiate equation (3.5) w.r.t. x and put $x = 0$, we get

(3.7) $(1 - x^2)y_3 - 3xy_2 - (1 + a^2)y_1 = 0,$

(3.8) $(y_3)_0 = (1 + a^2)a.$

Now differentiate equation (3.5) n-times w.r.t. x, and applying Leibnitz theorem we get

$$D^n\left\{(1 - x^2)y_2\right\} - D^n(xy_1) - D^n(a^2 y) = 0 \quad ; n \geq 2,$$

(3.9) $(1 - x^2)y_{n+2} - (2n + 1)xy_{n+1} - (n^2 + a^2)y_n = 0 \quad ; n \geq 2.$

Put $x = 0$ in equation(3.9) we get

(3.10) $(y_{n+2})_0 = (n^2 + a^2)(y_n)_0 \quad ; n \geq 2.$

Put $n = 2, 3, 4, 5, 6, 7, 8...$ in equations (3.10), we get

(3.11) $(y_4)_0 = (2^2 + a^2)a^2,$

(3.12) $(y_5)_0 = (3^2 + a^2)(1 + a^2)a,$

(3.13) $(y_6)_0 = (4^2 + a^2)(2^2 + a^2)a^2,$

(3.14) $(y_7)_0 = (5^2 + a^2)(3^2 + a^2)(1 + a^2)a,$

(3.15) $(y_8)_0 = (6^2 + a^2)(4^2 + a^2)(2^2 + a^2)a^2,$

(3.16) $(y_9)_0 = (7^2 + a^2)(5^2 + a^2)(3^2 + a^2)(1 + a^2)a,$

(3.17) $(y_{10})_0 = (8^2 + a^2)(6^2 + a^2)(4^2 + a^2)(2^2 + a^2)a^2,$

$\vdots$

Recurrence Relation

In case of odd

(3.18) $(y_{2n+1})_0 = a \prod_{j=1}^{n}\left\{(2j - 1)^2 + a^2\right\}.$

In case of even

(3.19) $(y_{2n})_0 = \prod_{j=1}^{n}\left\{(2j - 2)^2 + a^2\right\}.$

We know by Maclaurin series expansion

(3.20) $y = (y)_0 + x(y_1)_0 + \dfrac{x^2}{2!}(y_2)_0 + \dfrac{x^3}{3!}(y_3)_0 + \dfrac{x^4}{4!}(y_4)_0 + \dfrac{x^5}{5!}(y_5)_0 + ...$

Substitute the values of $(y)_0, (y_1)_0, (y_2)_0, (y_3)_0, (y_4)_0, (y_5)_0, ...$ in equation (3.20), we get

$$y = \sum_{n=0}^{\infty} \tfrac{x^{2n}}{2n!} \prod_{j=1}^{n}\left\{(2j - 2)^2 + a^2\right\} + a\sum_{n=0}^{\infty} \tfrac{x^{2n+1}}{(2n+1)!} \prod_{j=1}^{n}\left\{(2j - 1)^2 + a^2\right\},$$

$$y = \sum_{n=0}^{\infty} \tfrac{x^{2n}}{2n!}\left\{\prod_{j=1}^{n}\left[(2j - 2) + ia\right]\prod_{j=1}^{n}\left[(2j - 2) - ia\right]\right\} +$$

$$+ ax\sum_{n=0}^{\infty} \tfrac{x^{2n}}{(2n+1)!}\left\{\prod_{j=1}^{n}\left[(2j - 1) + ia\right]\prod_{j=1}^{n}\left[(2j - 1) - ia\right]\right\},$$

$$y = \sum_{n=0}^{\infty} \tfrac{x^{2n}2^{2n}}{2n!}\left\{\prod_{j=1}^{n}\left[(j - 1) + \tfrac{ia}{2}\right]\prod_{j=1}^{n}\left[(j - 1) - \tfrac{ia}{2}\right]\right\} +$$

$$+ ax\sum_{n=0}^{\infty} \tfrac{x^{2n}2^{2n}}{(2n+1)!}\left\{\prod_{j=1}^{n}\left[(j - 1) + \tfrac{1}{2} + \tfrac{ia}{2}\right]\prod_{j=1}^{n}\left[(j - 1) + \tfrac{1}{2} - \tfrac{ia}{2}\right]\right\},$$

$$y = \sum_{n=0}^{\infty} \frac{x^{2n}\left(\frac{ia}{2}\right)_n\left(-\frac{ia}{2}\right)_n}{\left(\frac{1}{2}\right)_n n!} + ax\sum_{n=0}^{\infty} \frac{x^{2n}\left(\frac{1+ia}{2}\right)_n\left(\frac{1-ia}{2}\right)_n}{\left(\frac{3}{2}\right)_n n!}.$$

Using definition of generalized hypergeometric function of one variable, we get the required result (2.1).

Proof of hypergeometric forms (2.2) and (2.3)

The proof of hypergeometric form (2.2) can be given by following same approach and making use of Taylors series expansion. Similarly the proof of hypergeometric form (2.3) can be given by following same approach and making use of Maclaurin's series expansion. So we omit the details here.

4 Some Inverse Hyperbolic Sine Functions as Special Cases

Replacing x by ix in equation (2.1) and putting $a = -ib$, we get

$$(4.1)\quad \exp(b\sinh^{-1}x) = {}_2F_1\left[\begin{matrix} -\frac{b}{2}, \ \frac{b}{2}; \\ \frac{1}{2}; \end{matrix} \ -x^2\right] + bx\,{}_2F_1\left[\begin{matrix} \frac{1+b}{2}, \ \frac{1-b}{2}; \\ \frac{3}{2}; \end{matrix} \ -x^2\right].$$

Replacing x by ix in equation (2.2), and putting $b = -ia$, we get

$$(4.2)\quad \exp(a\sinh^{-1}x) = \exp\left(-\frac{i\pi a}{2}\right){}_2F_1\left[\begin{matrix} a, \ -a; \\ \frac{1}{2}; \end{matrix} \ \frac{1-ix}{2}\right].$$

Replacing x by ix in equation (2.3) and putting $a = -ib$, we get

$$(4.3)\quad \frac{\exp(b\sinh^{-1}x)}{\sqrt{(1+x^2)}} = {}_2F_1\left[\begin{matrix} \frac{1+b}{2}, \ \frac{1-b}{2}; \\ \frac{1}{2}; \end{matrix} \ -x^2\right] + bx\,{}_2F_1\left[\begin{matrix} \frac{2+b}{2}, \ \frac{2-b}{2}; \\ \frac{3}{2}; \end{matrix} \ -x^2\right].$$

5 Some Applications

5.1 Special Cases of Hypergeometric form (2.1)

Suppose $x \in \mathbb{R}$ and a is purely imaginary in equation (2.1), then put $a = ib$, where b is purely real, we get

$$(5.1)\quad \exp(ib\sin^{-1}x) = {}_2F_1\left[\begin{matrix} -\frac{b}{2}, \ \frac{b}{2}; \\ \frac{1}{2}; \end{matrix} \ x^2\right] + ibx\,{}_2F_1\left[\begin{matrix} \frac{1+b}{2}, \ \frac{1-b}{2}; \\ \frac{3}{2}; \end{matrix} \ x^2\right].$$

Applying Euler's formula on left hand side of equation (5.1), then on equating real and imaginary parts, we get

$$(5.2)\quad \cos(b\sin^{-1}x) = {}_2F_1\left[\begin{matrix} -\frac{b}{2}, \ \frac{b}{2}; \\ \frac{1}{2}; \end{matrix} \ x^2\right],$$

$$(5.3)\quad \sin(b\sin^{-1}x) = bx\,{}_2F_1\left[\begin{matrix} \frac{1+b}{2}, \ \frac{1-b}{2}; \\ \frac{3}{2}; \end{matrix} \ x^2\right].$$

Put $x = \sin(\theta)$ in equation (5.2) and (5.3), we get

$$(5.4)\quad \cos(b\theta) = {}_2F_1\left[\begin{matrix} -\frac{b}{2}, \ \frac{b}{2}; \\ \frac{1}{2}; \end{matrix} \ \sin^2\theta\right],$$

$$(5.5)\quad \sin(b\theta) = b\sin(\theta){}_2F_1\left[\begin{matrix} \frac{1+b}{2}, \ \frac{1-b}{2}; \\ \frac{3}{2}; \end{matrix} \ \sin^2\theta\right].$$

Using Euler's linear transformation (1.11) in the right hand side of equations (5.4) and (5.5), we get

$$(5.6)\quad \cos(b\theta) = \cos\theta{}_2F_1\left[\begin{matrix} \frac{1+b}{2}, \ \frac{1-b}{2}; \\ \frac{1}{2}; \end{matrix} \ \sin^2\theta\right],$$

$$(5.7)\quad \sin(b\theta) = b(\sin\theta)(\cos\theta){}_2F_1\left[\begin{matrix} \frac{2-b}{2}, \ \frac{2+b}{2}; \\ \frac{3}{2}; \end{matrix} \ \sin^2\theta\right].$$

Put $b = id$ in equation (5.2) and (5.3), where d is purely imaginary, we get

$$(5.8)\quad \cosh(d\sin^{-1}x) = {}_2F_1\left[\begin{matrix} -\frac{id}{2}, \ \frac{id}{2}; \\ \frac{1}{2}; \end{matrix} \ x^2\right],$$

$$(5.9)\quad \sinh(d\sin^{-1}x) = dx\,{}_2F_1\left[\begin{matrix}\frac{1+id}{2},\ \frac{1-id}{2}; \\ \frac{3}{2};\end{matrix}\ x^2\right].$$

Putting $x = iy$ in equation (5.2) and (5.3), where y is purely imaginary, we get

$$(5.10)\quad \cosh(b\sinh^{-1}y) = {}_2F_1\left[\begin{matrix}-\frac{b}{2},\ \frac{b}{2}; \\ \frac{1}{2};\end{matrix}\ -y^2\right],$$

$$(5.11)\quad \sinh(b\sinh^{-1}y) = by\,{}_2F_1\left[\begin{matrix}\frac{1-b}{2},\ \frac{1+b}{2}; \\ \frac{3}{2};\end{matrix}\ -y^2\right].$$

Putting $x = iy$ and $b = ig$ in equation (5.2) and (5.3), where y and g are purely imaginary, we get

$$(5.12)\quad \cos(g\sinh^{-1}y) = {}_2F_1\left[\begin{matrix}-\frac{ig}{2},\ \frac{ig}{2}; \\ \frac{1}{2};\end{matrix}\ -y^2\right],$$

$$(5.13)\quad \sin(g\sinh^{-1}y) = gy\,{}_2F_1\left[\begin{matrix}\frac{1+ig}{2},\ \frac{1-ig}{2}; \\ \frac{3}{2};\end{matrix}\ -y^2\right].$$

5.2 Special Cases of Hypergeometric form (2.2)

Suppose $x \in \mathbb{R}$ and b is purely imaginary in equation (2.2), then put $b = ia$, where a is purely real, we get

$$(5.14)\quad \exp(ia\sin^{-1}x) = \exp\left(\frac{i\pi a}{2}\right)\,{}_2F_1\left[\begin{matrix}-a,\ a; \\ \frac{1}{2};\end{matrix}\ \frac{1-x}{2}\right].$$

Applying Euler's formula on left hand side of equation (5.14), then on equating real and imaginary parts, we get

$$(5.15)\quad \cos(a\sin^{-1}x) = \cos\left(\frac{\pi a}{2}\right)\,{}_2F_1\left[\begin{matrix}-a,\ a; \\ \frac{1}{2};\end{matrix}\ \frac{1-x}{2}\right],$$

$$(5.16)\quad \sin(a\sin^{-1}x) = \sin\left(\frac{\pi a}{2}\right)\,{}_2F_1\left[\begin{matrix}-a,\ a; \\ \frac{1}{2};\end{matrix}\ \frac{1-x}{2}\right].$$

Put $a = id$ in equation (5.15) and (5.16), where d is purely imaginary, we get

$$(5.17)\quad \cosh(d\sin^{-1}x) = \cosh\left(\frac{\pi d}{2}\right)\,{}_2F_1\left[\begin{matrix}-id,\ id; \\ \frac{1}{2};\end{matrix}\ \frac{1-x}{2}\right],$$

$$(5.18)\quad \sinh(d\sin^{-1}x) = \sinh\left(\frac{\pi d}{2}\right)\,{}_2F_1\left[\begin{matrix}-id,\ id; \\ \frac{1}{2};\end{matrix}\ \frac{1-x}{2}\right].$$

Putting $x = iy$ in equation (5.15) and (5.16), where y is purely imaginary, we get

$$(5.19)\quad \cosh(a\sinh^{-1}y) = \cos\left(\frac{\pi a}{2}\right)\,{}_2F_1\left[\begin{matrix}-a,\ a; \\ \frac{1}{2};\end{matrix}\ \frac{1-iy}{2}\right],$$

$$(5.20)\quad \sinh(a\sinh^{-1}y) = -i\sin\left(\frac{\pi a}{2}\right)\,{}_2F_1\left[\begin{matrix}-a,\ a; \\ \frac{1}{2};\end{matrix}\ \frac{1-iy}{2}\right].$$

Putting $x = iy$ and $a = ig$, in equation (5.15) and (5.16), where y and g are purely imaginary, we get

$$(5.21)\quad \cos(g\sinh^{-1}y) = \cosh\left(\frac{\pi g}{2}\right)\,{}_2F_1\left[\begin{matrix}-ig,\ ig; \\ \frac{1}{2};\end{matrix}\ \frac{1-iy}{2}\right],$$

$$(5.22)\quad \sin(g\sinh^{-1}y) = -i\sinh\left(\frac{\pi g}{2}\right)\,{}_2F_1\left[\begin{matrix}-ig,\ ig; \\ \frac{1}{2};\end{matrix}\ \frac{1-iy}{2}\right].$$

5.3 Special Cases of Hypergeometric form(2.3)

Suppose $x \in \mathbb{R}$ and a is purely imaginary in equation (2.3), then put $a = ib$, where b is purely real, we get

$$(5.23)\quad \frac{\exp(ib\sin^{-1}x)}{\sqrt{(1-x^2)}} = {}_2F_1\left[\begin{matrix}\frac{1-b}{2}, & \frac{1+b}{2}; \\ & \frac{1}{2};\end{matrix}\; x^2\right] + ibx\,{}_2F_1\left[\begin{matrix}\frac{2-b}{2}, & \frac{2+b}{2}; \\ & \frac{3}{2};\end{matrix}\; x^2\right].$$

Applying Euler's formula on left hand side of equation (5.23), then on equating real and imaginary parts, we get

$$(5.24)\quad \frac{\cos(b\sin^{-1}x)}{\sqrt{(1-x^2)}} = {}_2F_1\left[\begin{matrix}\frac{1-b}{2}, & \frac{1+b}{2}; \\ & \frac{1}{2};\end{matrix}\; x^2\right],$$

$$(5.25)\quad \frac{\sin(b\sin^{-1}x)}{\sqrt{(1-x^2)}} = bx\,{}_2F_1\left[\begin{matrix}\frac{2-b}{2}, & \frac{2+b}{2}; \\ & \frac{3}{2};\end{matrix}\; x^2\right].$$

Put $b = id$, in equation (5.24) and (5.25), where d is purely imaginary, we get

$$(5.26)\quad \frac{\cosh(d\sin^{-1}x)}{\sqrt{(1-x^2)}} = {}_2F_1\left[\begin{matrix}\frac{1-id}{2}, & \frac{1+id}{2}; \\ & \frac{1}{2};\end{matrix}\; x^2\right],$$

$$(5.27)\quad \frac{\sinh(d\sin^{-1}x)}{\sqrt{(1-x^2)}} = dx\,{}_2F_1\left[\begin{matrix}\frac{2-id}{2}, & \frac{2+id}{2}; \\ & \frac{3}{2};\end{matrix}\; x^2\right].$$

Putting $x = iy$ in equation (5.24) and (5.25), where y is purely imaginary, we get

$$(5.28)\quad \frac{\cosh(b\sinh^{-1}y)}{\sqrt{(1+y^2)}} = {}_2F_1\left[\begin{matrix}\frac{1-b}{2}, & \frac{1+b}{2}; \\ & \frac{1}{2};\end{matrix}\; -y^2\right],$$

$$(5.29)\quad \frac{\sinh(b\sinh^{-1}y)}{\sqrt{(1+y^2)}} = by\,{}_2F_1\left[\begin{matrix}\frac{2-b}{2}, & \frac{2+b}{2}; \\ & \frac{3}{2};\end{matrix}\; -y^2\right].$$

Putting $x = iy$ and $b = ig$ in equation (5.24) and (5.25) where y and g are purely imaginary, we get

$$(5.30)\quad \frac{\cos(g\sinh^{-1}y)}{\sqrt{(1+y^2)}} = {}_2F_1\left[\begin{matrix}\frac{1-ig}{2}, & \frac{1+ig}{2}; \\ & \frac{1}{2};\end{matrix}\; -y^2\right],$$

$$(5.31)\quad \frac{\sin(g\sinh^{-1}y)}{\sqrt{(1+y^2)}} = gy\,{}_2F_1\left[\begin{matrix}\frac{2-ig}{2}, & \frac{2+ig}{2}; \\ & \frac{3}{2};\end{matrix}\; -y^2\right].$$

6 Some Applications of Hypergeometric Forms (5.4) and (5.5)

Replacing θ by $\left(\frac{\pi}{2}-\theta\right)$ in equations (5.4) and (5.5), we get

$$(6.1)\quad \cos(b\theta) = \cos\left(\frac{b\pi}{2}\right){}_2F_1\left[\begin{matrix}-\frac{b}{2}, & \frac{b}{2}; \\ & \frac{1}{2};\end{matrix}\; \cos^2\theta\right] + b\cos(\theta)\sin\left(\frac{b\pi}{2}\right){}_2F_1\left[\begin{matrix}\frac{1+b}{2}, & \frac{1-b}{2}; \\ & \frac{3}{2};\end{matrix}\; \cos^2\theta\right].$$

$$(6.2)\quad \sin(b\theta) = \sin\left(\frac{b\pi}{2}\right){}_2F_1\left[\begin{matrix}-\frac{b}{2}, & \frac{b}{2}; \\ & \frac{1}{2};\end{matrix}\; \cos^2\theta\right] - b\cos(\theta)\cos\left(\frac{b\pi}{2}\right){}_2F_1\left[\begin{matrix}\frac{1+b}{2}, & \frac{1-b}{2}; \\ & \frac{3}{2};\end{matrix}\; \cos^2\theta\right].$$

7 Conclusion

In this paper, we have obtained hypergeometric forms of some composite functions involving $\arcsin(x)$ and $\text{arcsinh}(x)$, by using Maclaurin's series expansion and Taylor's series expansion.We conclude our present investigation by observing that hypergeometric forms of some other functions can be derived in an analogous manner. More over the results derived are significant. These are expected to find some potential applications in the fields of Mathematics and Engineering Sciences.

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ESTIMATED SOLUTIONS OF GENERALIZED AND MULTIDIMENSIONAL CHURCHILL'S DIFFUSION PROBLEMS

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Abstract

In this paper to define a generalized Churchill's diffusion problem, we first extend the Churchill's diffusion problem. Then, we derive some of estimated and computational formulae of its solution. Further, we present a multidimensional Churchill's diffusion problem consisting of multidimensional Euler space derivatives and Caputo time fractional derivative. Then, on imposing certain boundary values, we obtain its solution and derive its many estimated formulae.
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1 Introduction

Very recently, Pathan and Kumar [17] proved the multivariable Cauchy residue theorem with the help of the Euler derivatives. On the other hand, Apostol [1] analyzed and discussed the theory of homogeneous functions in respect of Euler derivatives as if $(x_1, \ldots, x_n) \in \mathbb{R}_+^n$, and $f(x_1, \ldots, x_n)$ is a homogeneous function of degree k, then

(1.1) $\sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} f(x_1, \ldots, x_n) = k f(x_1, \ldots, x_n)$,

while in 1928, for the economists, the favorite homogeneous function in the weighted geometric mean with domain $\mathbb{R}^+$, ($\mathbb{R}^+$ being the set of positive real numbers) has presented by the Cobb-Douglas function [6] as

(1.2) $f(x_1, ..., x_n) = x_1^{\alpha_1}.x_2^{\alpha_2}...x_n^{\alpha_n}$, where each $\alpha_i > 0, i = 1, \ldots, n$.

It is homogeneous of degree $\alpha = \alpha_1 + ... + \alpha_n$.

In the theory of fractional calculus (see Diethelm [7, p. 148, Eqn. (7.12)]) following fractional differential equation, consisting of a system of equations, has been studied in the form

(1.3) ${}_t^C D_{0^+}^{\alpha} y(x) = \Lambda y(x) + q(x)$,

where, $0 < \alpha < 1$, an $N \times N$ matrix Λ, a given function $q : [0, h] \to \mathbb{C}^N, h > 0$, and the unknown solution $y : [0, h] \to \mathbb{C}^N$. $\mathbb{C}^N = \mathbb{C} \times \ldots \times \mathbb{C}$ (N times).

For any suitable vector $u \in \mathbb{C}^N$,the solution of (1.3) is found as

(1.4) $y(x) = u E_\alpha(\lambda x^\alpha)$,

where, $\lambda \in \mathbb{C}$, an eigenvalue of the matrix Λ, and the $E_\alpha(z)$, a Mittag - Leffler function, is defined by the series [15, p.80, Eqn. (2.1.1)] as

$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+1)}, \Re(\alpha) > 0, z \in \mathbb{C}$.

In 2012, Kumar [9] considered following Churchill's diffusion problem [5]

(1.5) $x\dfrac{\partial U}{\partial x} + \dfrac{\partial U}{\partial t} + U = xF(t), x, t \in \mathbb{R}^+$,

with the boundary conditions $U(x, 0) = 0 = U(0, t)$ and they analyzed its solution

(1.6) $U(x, t) = x \int_0^t e^{-2v} F(t - v) dv, x, t \in \mathbb{R}^+$.

Again by [9], the solution (1.6) is converted into a general hypergeometric series solution to obtain various results for known special functions.

On the other hand, the theory and application of fractional differential equations in the diverse field, for example, in the dynamics of sphere immersed in an incompressible viscous fluid, oscillatory process with fractional damping, a study of the tension - deformation relationship of viscoelastic materials, and anomalous diffusion problems have been described in the literature by various researchers ([7], [8], [11], [14], [15], [16], [18] and others) consisting of the general fractional differential equations along with some operators or functions in the form of matrices. By the interacting multispecies, a system of equations in the matrix form also studied by Chandel and Kumar [4] in the ecosystem. Motivated by this work, in this paper we generalize the Churchill's diffusion equation in one and multidimensional space and then obtain their estimated solutions and computational results.

2 A Generalized Churchill type problem and its computational results

In this section on extension of the problem (1.5), we discuss a generalized Churchill type problem and then by its solution, we evaluate various results to compute the problem as

(2.1) $x\dfrac{\partial U(x,t)}{\partial x} + {}_t^C D_{0^+}^{\alpha} U(x,t) + U(x,t) = x\log_e F(t) = x\ln(F(t)), t>0, x>0, F(0)\geq 1,$

$0<\alpha<1$, with following boundary conditions

(2.2) $U(x,0^+) = 0, \lim\limits_{x\to 0} U(x,t) = 0.$

In (1.3) and (2.1), the *Caputo derivative,* ${}_t^C D_{0^+}^{\alpha} f(t)$, of the function $f(t)$, whenever, $f^{(m)} \in L_1[a,b]$, $m-1<\alpha\leq m$, $\forall m\in\mathbb{N}$, is defined by [7, p. 49]

(2.3) $({}_t^C D_{0^+}^{\alpha} f)(t) = (I_a^{m-\alpha} f^{(m)})(t),$

where, $f^{(m)}(t) = D_t^m f(t), \{D_t^m \equiv \frac{d^m}{dt^m} = \frac{d}{dt}(\frac{d^{m-1}}{dt^{m-1}})\}$.

In (2.3), the $(I_a^{m-q} f)(t), f\in L_1[a,b]$ is the Riemann - Liouville fractional integral [18], given by

(2.4) $(I_a^{m-q} f)(t) = \begin{cases} \frac{1}{\Gamma(m-q)}\int_a^t (t-\tau)^{m-q-1} f(\tau)d\tau, \ \forall t\in[a,b], m-1<q<m, m\in\mathbb{N}, \\ f(t), q=m. \end{cases}$

Now, for the Laplace transform of a sufficiently well behaved function $v(t)$, denoted as $L\{V(t); s, s>0\} = v(s)$, and again defined by $v(s) = \int_0^\infty e^{-st}V(t)dt$, then, the Laplace transformation of the Caputo operator (2.3), is presented by [7, p.134]

(2.5) $L\{({}_t^C D_{0^+}^{\alpha} V)(t); s\} = s^{\alpha}v(s) - \sum_{k=0}^{m-1} s^{\alpha-1-k}V^{(k)}(0^+) \ \forall m-1<\alpha\leq m.$

Now, we solve the problem {(2.1) - (2.2)} in following manner:

***Theorem* 2.1** *Under the conditions* (2.2) *given as*

$U(x,0^+) = 0, \lim\limits_{x\to 0} U(x,t) = 0,$

then, the solution of the generalized Churchill's diffusion equation (2.1) *exists and find in the form*

(2.6) $U(x,t) = x\int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-2(t-\tau)^{\alpha})\ln(F(\tau))d\tau, 0<\alpha<1, t>0, x>0.$

Proof. By (2.5) take Laplace transformation of the both sides of (2.1) as considering

$L\{U(x,t); s, s>0\} = u(x,s), L\{\log(F(t)); s, s>0\} = \bar{F}(s),$ for $0<\alpha<1$,

then, under the given condition $U(x,0^+) = 0$, we find a linear differential equation as

(2.7) $\dfrac{du(x,s)}{dx} + \left\{\dfrac{s^{\alpha}+1}{x}\right\}u(x,s) = \bar{F}(s),$

with the initial condition $\lim_{x\to 0} u(x,s) = 0$.

The solution of Eqn. (2.7) is obtained as

(2.8) $u(x,s) = x^{-s^{\alpha}-1}\int x^{s^{\alpha}+1}\bar{F}(s)dx = x\left\{\frac{1}{s^{\alpha}+2}\right\}\bar{F}(s), x>0.$

Taking inverse Laplace transform of both the sides of the Eqn. (2.8), and employing following formula due to Kilbas, Srivastava and Trujillo [8, p. 50, Eqn. (1.10.9)]

(2.9) $t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^{\alpha}) = L^{-1}\left\{\dfrac{1}{s^{\alpha}-\lambda}\right\}, \Re(s)>0, \lambda\in\mathbb{C}, |\lambda s^{-\alpha}|<1, t>0,$

and thus applying the convolution theorem of Laplace transformation in (2.8) and making an appeal to (2.9) and (2.10), we derive the solution (2.6).

In the Eqn. (2.9), the $E_{\alpha,\beta}(z)$, a generalized Mittag - Leffler function, is defined by the series [15, p. 80, Eqn. (2.1.2)] as

(2.10) $E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+\beta)}, \Re(\alpha)>0, \Re(\beta)>0, z,\alpha,\beta\in\mathbb{C}.$

***Example* 2.1** *Consider* $F(t) = exp\Big[\sum_{n=0}^{\infty} a_n \frac{t^n}{n!}\Big]$, $\forall t > 0$, *in the problem* {(2.1) - (2.2)}, *then, make an appeal to the* ***Theorem* 2.1**, *following result exists*

(2.11) $U(x,t) = xt^{\alpha} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} a_n \frac{(-2t^{\alpha})^k}{\Gamma(\alpha k+\alpha+n+1)} t^n, 0 < \alpha < 1, t > 0, x > 0.$

Again then, the result (2.11) *gives various known results.*

Solution. In the solution (2.6) of the ***Theorem* 2.1**, choose $F(t) = \exp\Big[\sum_{n=0}^{\infty} a_n \frac{t^n}{n!}\Big]$, we get the result (2.11).

Now in the result (2.11), choosing $a_n = \frac{\Gamma(\alpha k+\alpha+n+k+1)}{\Gamma(k+1)}$ $\forall k = 0, 1, 2, \ldots$, we obtain a known generating function

(2.12) $U(x,t) = xt^{\alpha} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \binom{\alpha k+k+\alpha+n}{k} t^n, 0 < \alpha < 1, t > 0, x > 0.$

By an appeal to the formula $\sum_{k=0}^{\infty} \binom{\alpha k+k+\beta}{k} = \frac{(1+\zeta)^{\beta+1}}{1-\alpha\zeta}, \zeta(z) = z(1+\zeta(z))^{\alpha+1}, \zeta(0) = 0$, due to Srivastava and Manocha [19, p. 234, Eqn. (47, for $x = 0$)], (see also, Carlitz [2]) in the result (2.12), we derive

(2.13) $U(x,t) = xt^{\alpha}\Big\{\dfrac{(1+\zeta(t^*))^{\alpha+1}}{1-\alpha\zeta(t^*)}\Big\}\Big[\dfrac{1}{1-\{(1+\zeta(t^*))t\}}\Big]$, provided that for $|(1+\zeta(t^*))t| < 1$.

Here, $0 < \alpha < 1, t > 0, x > 0, \zeta(t^*) = t^*(1+\zeta(t^*))^{\alpha+1}, t^* = (-2t^{\alpha})$.

Further by (2.6), we establish

$$U(x,t) = x\sum_{n=0}^{\infty} \frac{a_n}{\Gamma(\alpha+n+1)} + x\sum_{n=0}^{\infty} \frac{a_n}{\Gamma(n+1)} \sum_{k=1}^{\infty} \frac{(-2)^k}{\Gamma(\alpha k+\alpha)} \int_0^t (t-\tau)^{\alpha+\alpha k-1}\tau^n d\tau.$$

Hence by (2.4), we get

(2.14) $U(x,t) = x\sum_{n=0}^{\infty} \frac{a_n}{\Gamma(\alpha+n+1)} + x\sum_{n=0}^{\infty} \frac{a_n}{\Gamma(n+1)} H_n(\alpha;t).$

Here, $H_n(\alpha;t) = \sum_{k=1}^{\infty} (-2)^k I_0^{\alpha+\alpha k}(\tau^n)(t), 0 < \alpha < 1, t > 0, x > 0.$

It is noted that some other related generating functions are obtained on application of the results concerning Laguerre polynomials found in the literature by Chandel [3].

***Corollary* 2.1** *Set* $\alpha = 1, F(t) = e^{G(t)}$ $\forall t \geq 0$, *in Eqn.* (2.1) *along with the conditions* (2.2), *then, by the* ***Theorem* 2.1,** *the solution* (2.6) *becomes the solution of the Churchill's diffusion problem* (1.5) *and is found in the form*

(2.15) $U(x,t) = xe^{-2t}\int_0^t e^{2\tau}G(\tau)d\tau, t > 0, x > 0.$

***Theorem* 2.2** *If the solution* $U(x,t)$ *of the problem* {(2.1) - (2.2)}, *given in* (2.6), *has separated in variables as*

(2.16) $U(x,t) = \varphi(x)\psi(t), x > 0, t > 0,$

then, there exists two curves in positive axis's

(2.17) $\varphi(x) = \eta x, \psi(t) = \dfrac{1}{\eta}\int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-2(t-\tau)^{\alpha})\ln(F(\tau))d\tau,$

whenever, $x > 0, t > 0$; *and* $0 < \alpha < 1$, η *is constant and* $\eta \neq 0$.

Proof. In Eqn. (2.6), consider $U(x,t) = \varphi(x)\psi(t) \neq 0$, for $0 < \alpha < 1, x > 0, \eta \neq 0$, then, it may also be written by

(2.18) $\dfrac{\varphi(x)}{x} = \dfrac{1}{\psi(t)}\int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-2(t-\tau)^{\alpha})\ln(F(\tau))d\tau = \eta.$

By the Eqn. (2.14), the parametric equations of the solution $U(x,t)$ *when*, $0 < \alpha < 1, t > 0, x > 0$,for $\eta \neq 0$, are found as

(2.19) $\varphi(x) = \eta x, \psi(t) = \dfrac{1}{\eta}\int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-2(t-\tau)^{\alpha})\ln(F(\tau))d\tau.$

Again, $\varphi(x)$ and $\psi(t)$are non-zero, when $\eta \neq 0$, and $\forall\ t, x > 0$.

Hence, the ***Theorem* 2.2** is followed.

Now, we determine various estimation formulae of the solution of the generalized Churchill type diffusion problem for computational work:

***Theorem* 2.3** *If* $0 < \alpha < 1, \eta \neq 0$, *and* $|\ln(F(t))| \leq M, M > 0$,*then by the* ***Theorem* 2.2**, *for* $t > 0$ *there exists an estimation formula*

(2.20) $|U(x,t)| \leq Mxt^{\alpha}|E_{\alpha,\alpha+1}(-2t^{\alpha})|.$

Proof. Take $G(t) = \ln(F(t))$,then by ***Theorem 2.2***, for $t > 0$ to get that

$\psi(t) = \frac{1}{\eta}\int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-2(t-\tau)^\alpha)G(\tau)d\tau$, and write it in the form

$$\psi(t) = \int_0^t \left\{\frac{1}{|\eta|}(t-\tau)^{\alpha-1}\right\}^{\frac{1}{2}}\left\{E_{\alpha,\alpha}(-2(t-\tau)^\alpha)\right\}^{\frac{1}{2}}\left\{\frac{1}{|\eta|}(t-\tau)^{\alpha-1}\right\}^{\frac{1}{2}}\left\{E_{\alpha,\alpha}(-2(t-\tau)^\alpha)\right\}^{\frac{1}{2}}G(\tau)d\tau.$$

Now, use Schwarz inequality for the integrals to find that

$$(2.21)\quad |\psi(t)| \le \frac{1}{|\eta|}\Big[t^\alpha E_{\alpha,\alpha+1}(-2t^\alpha)\int_0^t \frac{1}{\eta}(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-2(t-\tau)^\alpha)|G(\tau)|^2 d\tau\Big]^{\frac{1}{2}}.$$

An appeal to the ***Theorem 2.3,*** we have $|G(\tau)| \le M$, then, in the integral apply the definition (2.10), again, use the ***Theorem 2.2,*** with $|U(x,t)| = |\varphi(x)||\psi(t)|$, finally, we obtain the estimation formula (2.20).

Theorem 2.4 *If Laplace transformation of the function* $\{G(t)\}^2$ *is equal to* $P(s), s > 0$, *that is* $L\{\{G(t)\}^2 : s\} = P(s), s > 0$, *then by* ***Theorem 2.3****, there exists another inequality*

$$(2.22)\quad |U(x,t)| \le x\Big[L^{-1}\Big\{\frac{P(s)}{s^\alpha+2} : t\Big\}\Big]^{\frac{1}{2}}\Big[L^{-1}\Big\{\frac{1}{s}\left(\frac{1}{s^\alpha+2}\right) : t\Big\}\Big]^{\frac{1}{2}}.$$

Proof. Since $|G(t)|^2 = \{G(t)\}^2$, so that by Eqn. (20) of ***Theorem 2.1***, we write

$$(2.23)\quad |\psi(t)| \le \frac{1}{|\eta|}\Big[\int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-2(t-\tau)^\alpha)d\tau\Big]^{\frac{1}{2}}\Big[\int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-2(t-\tau)^\alpha)\{G(\tau)\}^2 d\tau\Big]^{\frac{1}{2}}.$$

Now, in Eqn. (2.19), use the formula due to Mathai and Haubold [15, p. 86, Eqn. (2.2.21)], with $L\{\{G(t)\}^2 : s\} = P(s)$, and $L\{1 : s\} = \frac{1}{s}, s > 0$, we get result (20).

Finally, the Eqn. (2.19), by an appeal to the ***Theorem 2.2***, gives the result (2.18).

Theorem 2.5 *If* $\log F(t) \in H_\mu[a,b]$, *where the Hölder space* $H_\mu[a,b] := \{\ln F(t) : [a,b] \in \mathbb{R}; \exists K > 0\ \forall (t,\tau) \in [a,b]; |\ln F(t) - \ln F(\tau)| \le K|t-\tau|^\mu$ *for some* $\mu \in [0,1]\}$, *and* $0 < \alpha < 1$. *Then, by* ***Theorem 2.2****, there exists an inequality*

$$(2.24)\quad U(x,t) \le \ln F(0)\{xt^\alpha E_{\alpha,\alpha+1}(-2t^\alpha)\} + Kx\Gamma(\mu+1)O(t^{\alpha+\mu}E_{\alpha,\alpha+\mu+1}(-2t^\alpha)).$$

Proof. Consider the Eqn. (2.1) and suppose that $F(0) \ge 1$, then by our assumption and the ***Theorem 2.2***, we write

$$(2.25)\quad \psi(t) = \frac{1}{\eta}\ln F(0)\int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-2(t-\tau)^\alpha)d\tau + \frac{1}{\eta}\int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-2(t-\tau)^\alpha)\{lnF(\tau) - \ln F(0)\}d\tau.$$

Since in ***Theorem 2.5***, $\ln F(x) \in H_\mu[a,b]$, then by an appeal to Diethelm [7, p. 15], we get $\{lnF(\tau) - \ln F(0)\} = \ln\{\frac{F(t)}{F(0)}\} \le Kt^\mu, \mu \in [0,1], K$ is a constant.

Now, making an appeal to the Eqns. (26) and (27), and the conditions of Hölder space, we obtain the inequality

$$(2.26)\quad \psi(t) \le \frac{1}{\eta}logF(0)\int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-2(t-\tau)^\alpha)d\tau + \frac{K}{\eta}\int_0^t (t-\tau)^{\alpha-1}\tau^\mu E_{\alpha,\alpha}(-2(t-\tau)^\alpha)d\tau.$$

Finally, an appeal to the Eqn. (2.25) and the ***Theorem 2.2***, gives the result (2.22).

3 Churchill's Multidimensional diffusion problem and its solution

Here, in our investigation, we introduce a general equation consisting of multidimensional Euler space derivatives, Caputo time fractional derivative and the functions in trace of a given matrix as

$$(3.1)\quad x_1\frac{\partial U(x_1,\ldots,x_r,t)}{\partial x_1} + \ldots + x_r\frac{\partial U(x_1,\ldots,x_r,t)}{\partial x_r} + {}_t^C D_{0^+}^\alpha U(x_1,\ldots,x_r,t) + \sum_{i=1}^r \beta_{ii}(U) = \prod_{i\neq j,i=1}^r V(x_i,t)\ln\prod_{j=1}^r (F_j(t))^{x_j}.$$

The Eqn. (3.1) can also be written as

$$(3.2)\quad \sum_{i=1}^r x_i\frac{\partial U(x_1,\ldots,x_r,t)}{\partial x_i} + {}_t^C D_{0^+}^\alpha U(x_1,\ldots,x_r,t) + \sum_{i=1}^r \beta_{ii}(U) = \sum_{j=1}^r x_j\ln F_j(t)\prod_{i\neq j,i=1}^r V(x_i,t).$$

In the Eqns. (3.1) and (3.2), it is provided that $0 < \alpha < 1, V(x_i,t) \neq 0, x_i > a_i, t > 0$, the function $\log F_i : \mathbb{R}^+ \to \mathbb{R}$, such that $F_i(t) \ge 1; \beta_{ii}(U) = U(x_1,\ldots,x_r,t)\ \forall i = 1,2,3,\ldots,r$ and $\sum_{i=1}^r \beta_{ii}(U)$ is the Trace of square matrix $\begin{bmatrix} U & & \\ & \ddots & \\ & & U\end{bmatrix}_{r\times r}$. An empty product, when it occurs, is taken as one.

Recently, multidimensional time diffusion problems are solved as separating in space and time variables by the theory of various authors ([10], [12], [13], [14]). To solve the equation (3.1) or (3.2) by separation in variables, we impose following initial and boundary conditions:

***Theorem* 3.1** *If consider that* $U(x_1,\ldots,x_r,t)=\prod_{i=1}^{r}V(x_i,t)$ *,* $(x_i,t)\neq 0\ \forall i=1,2,\ldots,r$ *, in the equation* (3.1) *or* (3.2)*, then, for the conditions* $U(0,x_2,\ldots,x_r,t)=\ldots=U(x_1,\ldots,x_{r-1},0,t)=0$*;* $U(x_1,\ldots,x_r,0^+)=0$*, there exists a set of problems*

$$(3.3)\quad \frac{\partial V(x_j,t)}{\partial x_j}+{}_t^C D_{0^+}^{\alpha}V(x_j,t)+V(x_j,t)=x_j\ln(F_j(t)),\ \forall j=1,2,3,\ldots.,r;$$

with the conditions given as $0<\alpha<1$, $V(0,t)=0=V(x_j,0^+)$, $\forall j=1,2,\ldots,r$.

Proof. Set $U(x_1,\ldots,x_r,t)=\prod_{i=1}^{r}V(x_i,t)$ in Eqn. (3.1) or (3.2), for $0<\alpha<1$, and then, on operating by the Caputo fractional time derivative, we find

$$(3.4)\quad \sum_{j=1}^{r}\prod_{i\neq j,i=1}^{r}V(x_i,t)x_j\frac{\partial V(x_j,t)}{\partial x_j}+\sum_{j=1}^{r}\prod_{i\neq j,i=1}^{r}V(x_i,t){}_t^C D_{0^+}^{\alpha}V(x_j,t)$$
$$+\sum_{j=1}^{r}\prod_{i\neq j,i=1}^{r}V(x_i,t)V(x_j,t)=\sum_{j=1}^{r}\prod_{i\neq j,i=1}^{r}V(x_i,t)x_j\ln(F_j(t)).$$

Equating both the sides of Eqn. (3.4), we obtain the set of equations (3.3).

Again, in the relation $U(x_1,\ldots,x_r,t)=\prod_{i=1}^{r}V(x_i,t)$, where, $V(x_i,t)\neq 0\ \forall i=1,2,\ldots,r$; set $U(0,x_2,\ldots,x_r,t)=\ldots=U(x_1,\ldots,x_{r-1},0,t)=0$; we get $V(x_i,t)\Big|_{x_i=0}=0\ \forall x_i=0, i=1,2,\ldots,r$.

By Lagrange's interpolation formula, we write

$$(3.5)\quad U(x_1,\ldots,x_r,t)\big|_{t=\tau_i,\tau_i\to 0^+}=\sum_{i=1}^{r}\left\{\frac{\prod_{j=1,j\neq i}^{r}(t-\tau_j)V(x_j,t)}{\prod_{j=1,j\neq i}^{r}(\tau_i-\tau_j)}\right\}V(x_i,t)\Big|_{t=\tau_i,\tau_i\to 0^+},$$
$$\forall i=1,2,3,\ldots,r.$$

Since, in our assumption $U(x_1,\ldots,x_r,0^+)=0$, and $V(x_i,t)\neq 0\ \forall i=1,2,\ldots,r$, thus by (3.5), we find

$$V(x_i,0^+)=0\ \forall i=1,2,\ldots,r.$$

Hence, the ***Theorem* 3.1** is proved.

***Theorem* 3.2** *The equation* (3.1) *or* (3.2) *under the given conditions* $U(0,x_2,\ldots,x_r,t)=\ldots=U(x_1,\ldots,x_{r-1},0,t)=0$; $U(x_1,\ldots,x_r,0^+)=0$, *has the solution*

$$(3.6)\quad U(x_1,\ldots,x_r,t)=\prod_{i=1}^{r}V(x_i,t)=\prod_{i=1}^{r}x_i\int_0^t(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-2(t-\tau)^{\alpha})\ln(F_i(\tau))d\tau,$$
$$\forall 0<\alpha<1, t>0, x_i>0, i=1,\ldots,r.$$

Proof. An appeal to the ***Theorem* 3.1** in the equation (3.1) or (3.2), we find r-equations

$$\frac{\partial V(x_j,t)}{\partial x_j}+{}_t^C D_{0^+}^{\alpha}V(x_j,t)+V(x_j,t)=x_j\ln(F_j(t)),\ \forall j=1,2,3,\ldots.,r;$$

with the conditions given by $0<\alpha<1$, $V(0,t)=0=V(x_j,0^+)$, $\forall j=1,2,\ldots,r$. Again, apply the ***Theorem* 2.1**, we find

$$V(x_j,t)=x_j\int_0^t(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-2(t-\tau)^{\alpha})\ln(F_j(\tau))d\tau,\ \forall j=1,2,\ldots,r.$$

Finally, use the concept and the theory of the ***Theorem* 3.1** we evaluate the solution (3.6).

***Theorem* 3.3** *For all* $0<\alpha<1, t>0, x_i>0, i=1,\ldots,r$, *if all conditions of the* ***Theorems* 3.1** *and* **3.2** *are satisfied, then, for any* $\eta'\neq 0$, *there exists* $U(x_1,\ldots,x_r,t)=\Phi(x_1,\ldots,x_r)\Psi(t)$ *such that*

$$(3.7)\quad \Phi(x_1,\ldots,x_r)=\eta' x_1\ldots x_r \text{ and } \Psi(t)=\frac{1}{\eta'}\prod_{i=1}^{r}\int_0^t(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-2(t-\tau)^{\alpha})\ln(F_i(\tau))d\tau.$$

Proof. An appeal to the ***Theorems* 3.1** and **3.2**, we find that

$$(3.8)\quad U(x_1,\ldots,x_r,t)=\Phi(x_1,\ldots,x_r)\Psi(t)=x_1\ldots x_r\prod_{i=1}^{r}\int_0^t(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-2(t-\tau)^{\alpha})\ln(F_i(\tau))d\tau.$$

By the relation (3.8), we bifurcate in separate variables as

$$(3.9)\quad \frac{\Phi(x_1,\ldots,x_r)}{x_1\ldots x_r}=\frac{\prod_{i=1}^{r}\int_0^t(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-2(t-\tau)^{\alpha})\ln(F_i(\tau))d\tau}{\Psi(t)}=\eta',\eta'\neq 0.$$

The relations (3.9) easily give the formulae (3.7).

***Theorem* 3.4** *For all* $0 < \alpha < 1, t > 0, x_i > 0, i = 1, \ldots, r$, *if all conditions of the* ***Theorems* 3.1** *and* **3.2** *are satisfied, then, for any* $\eta' \neq 0$, *and for* $|\ln(F_i(t))| \leq M_i', i = 1, 2, \ldots, r$, *then, by the* ***Theorems* 3.3,** *there exists an estimation formula via multidimensional Churchill's diffusion problem*

(3.10) $|U(x_1, \ldots, x_r, t)| \leq M_1' x_1 \ldots M_r' x_r \{t^\alpha |E_{\alpha,\alpha+1}(-2t^\alpha)|\}^r.$

Proof. Make an appeal to the ***Theorems* 3.1**, **3.2** and **3.3**, we find

(3.11) $|U(x_1, \ldots, x_r, t)| = |x_1 \ldots x_r| \prod_{i=1}^r |\int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-2(t-\tau)^\alpha)| \ln(F_i(\tau))| d\tau|.$

Now, in (3.11) use the techniques of the ***Theorem* 2.3** , and for $|\ln(F_i(t))| \leq M_i', i = 1, 2, \ldots, r$, we obtain the estimation formula

(3.12) $|U(x_1, \ldots, x_r, t)| \leq |x_1 \ldots x_r| \prod_{i=1}^r M_i' \{t^\alpha |E_{\alpha,\alpha+1}(-2t^\alpha)|\}.$

The inequality (3.12) gives us the estimation formula (3.10).

***Example* 3.1** *Let in the* ***Theorem* 3.2** $F_i(t) = exp[y_i t^{\rho_i}], y_i \in \mathbb{R}^+, \rho_i \in \mathbb{N}_0 \ \forall i = 1, 2, \ldots, r;$
$\mathbb{N}_0 = \{0, 1, 2, \ldots\}$; *then, there exists the solution of the multidimensional equation* (3.1) *or* (3.2) *with the conditions given in the* ***Theorem* 3.1***, as*

(3.13) $U(x_1, \ldots, x_r, t) = t^\alpha \prod_{i=1}^r x_i y_i \, t^{\rho_i} \, \Gamma(\rho_i + 1) E_{\alpha,\alpha+\rho_i+1}(-2t^\alpha).$

***Solution*.** In the ***Theorem* 3.2,** introduce $F_i(t) = \exp[y_i t^{\rho_i}], y_i \in \mathbb{R}^+, \rho_i \in \mathbb{N}_0 \ \forall i = 1, 2, \ldots, r$; where, $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$; we get

(3.14) $$U(x_1, \ldots, x_r, t) = \prod_{i=1}^r x_i y_i \sum_{k=0}^\infty \frac{(-2)^k}{\Gamma(\alpha k+\alpha)} \int_0^t (t-\tau)^{\alpha+\alpha k-1} \tau^{\rho_i} d\tau$$
$$= \prod_{i=1}^r x_i y_i \sum_{k=0}^\infty \frac{(-2)^k}{\Gamma(\alpha k+\alpha)} \int_0^t (t-\tau)^{\alpha+\alpha k-1} \tau^{\rho_i} d\tau.$$

By Eqn. (3.14), we easily derive the solution (3.13).

Conclusion

In 1972, Churchill studied diffusion problem and again in 2012, another form of the solution of this problem is obtained by Kumar [9] and then, converted into known and unknown hypergeometric functions. In this paper, in the ***Section* 2**, we generalize the Churchill's diffusion problem on introducing Caputo time fractional derivative and then obtain various estimation formulae and with known and unknown functions and generating relations. Again, we introduce a multidimensional time fractional diffusion problem to derive its solution on separating it in various diffusions problems. By the ***Theorem* 3.4**, at $t = x_1 \ldots x_r$, we find Cobb-Douglas [6] type functions given in (1.1).

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(Dedicated to Honor Dr. R. C. Singh Chandel on His 75^{th} Birth Anniversary Celebrations)

HYPERGEOMETRIC FORMS OF SOME MATHEMATICAL FUNCTIONS VIA DIFFERENTIAL EQUATION APPROACH

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Abstract

In this paper, by changing the independent and dependent variables in the suitable ordinary differential equations of first and second order, and comparing the resulting ordinary differential equations with standard ordinary differential equations of Leibnitz and Gauss, we obtain the hypergeometric forms of following functions:

$$-\frac{4}{x}\,\ell n\left(\frac{1+\sqrt{1-x}}{2}\right),\ \tan^{-1}(x),\ \ \ell n(1+x),\ \ \sin(b\sin^{-1}x),\ \ \cos(b\sin^{-1}x).$$

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1 Introduction and Preliminaries

In our investigations, we shall use the following standard notations:

$\mathbb{N} :=\ \{1,2,3,\cdots\}\,;\mathbb{N}_0 := \mathbb{N}\cup\{0\}\,;\mathbb{Z}_0^- := \mathbb{Z}^-\cup\{0\} = \{0,-1,-2,-3,\cdots\}\,.$

The symbols $\mathbb{C},\mathbb{R},\mathbb{N},\mathbb{Z},\mathbb{R}^+$ and $\mathbb{R}^-$ denote the sets of complex numbers, real numbers, natural numbers, integers, positive and negative real numbers respectively.

Pochhammer symbol

The Pochhammer symbol (or the *shifted* factorial) $(\lambda)_\nu$ $(\lambda,\nu\in\mathbb{C})$[16, p.22 Eqn.(1), p.32 Q.N.(8) and Q.N.(9)], see also [18, p.23, Eqn.(22) and Eqn.(23)], is defined by

$$(\lambda)_\nu := \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} = \begin{cases} 1 & (\nu=0;\ \lambda\in\mathbb{C}\backslash\{0\}),\\ \prod_{j=0}^{n-1}(\lambda+j) & (\nu=n\in\mathbb{N};\ \lambda\in\mathbb{C}),\\ \frac{(-1)^k n!}{(n-k)!} & (\lambda=-n;\ \nu=k;\ n,k\in\mathbb{N}_0;\ 0\leqq k\leqq n),\\ 0 & (\lambda=-n;\ \nu=k;\ n,k\in\mathbb{N}_0;\ k>n),\\ \frac{(-1)^k}{(1-\lambda)_k} & (\nu=-k;\ k\in\mathbb{N};\ \lambda\in\mathbb{C}\backslash\mathbb{Z}),\end{cases}$$

it being assumed tacitly that the Gamma quotient exists.

Generalized hypergeometric function of one variable

A natural generalization of the Gaussian hypergeometric series ${}_2F_1[\alpha,\beta;\gamma;z]$, is accomplished by introducing any arbitrary number of numerator and denominator parameters. Thus, the resulting series

$$(1.1)\quad {}_pF_q\left[\begin{array}{l}(\alpha_p);\\(\beta_q);\end{array} z\right] = {}_pF_q\left[\begin{array}{l}\alpha_1,\alpha_2,\ldots,\alpha_p;\\ \beta_1,\beta_2,\ldots,\beta_q;\end{array} z\right] = \sum_{n=0}^{\infty}\frac{(\alpha_1)_n(\alpha_2)_n\ldots(\alpha_p)_n}{(\beta_1)_n(\beta_2)_n\ldots(\beta_q)_n}\frac{z^n}{n!}\,,$$

is known as the generalized hypergeometric series, or simply, the generalized hypergeometric function. Here p and q are positive integers or zero and we assume that the variable z, the numerator parameters $\alpha_1,\alpha_2,\ldots,\alpha_p$ and the denominator parameters $\beta_1,\beta_2,\ldots,\beta_q$ take on complex values, provided that

$\beta_j\ \neq\ 0,-1,-2,\ldots\ ;\ \ j=1,2,\ldots,q.$

Here, none of the denominator parameters is zero or a negative integer, we note that the ${}_pF_q$ series defined by Eqn.(1.1):

(i) converges for $|z|<\infty$, if $p\leq q$,
(ii) converges for $|z|<1$, if $p=q+1$,
(iii) diverges for all z, $z\neq 0$, if $p>q+1$,
(iv) converges absolutely for $|z|=1$, if $p=q+1$ and $\Re(\omega)>0$,

(v) converges conditionally for $|z| = 1 (z \neq 1)$, if $p = q + 1$ and $-1 < \Re(\omega) \leqq 0$,

(vi) diverges for $|z| = 1$, if $p = q + 1$ and $\Re(\omega) \leqq -1$,

where by convention, a product over an empty set is interpreted as 1 and

(1.2) $\omega := \sum_{j=1}^{q} \beta_j - \sum_{j=1}^{p} \alpha_j$,

$\Re(\omega)$ being the real part of complex number ω.

(I) When $x^2 = -t$ or $x = i\sqrt{t}$ where $i = \sqrt{-1}$, then

(1.3) $\dfrac{dx}{dt} = \dfrac{i}{2\sqrt{t}}$ or $\dfrac{dt}{dx} = -2i\sqrt{t}$,

(1.4) $\dfrac{dy}{dx} = \dfrac{dy}{dt} \times \dfrac{dt}{dx} = -2i\sqrt{t}\,\dfrac{dy}{dt}$,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(-2i\sqrt{t}\,\frac{dy}{dt}\right)\frac{dt}{dx}$$

after simplification, we get

(1.5) $\dfrac{d^2y}{dx^2} = -4t\dfrac{d^2y}{dt^2} - 2\dfrac{dy}{dt}$.

(II) When $y = i\sqrt{t}z$, where z is the function of t then

(1.6) $\dfrac{dy}{dt} = \dfrac{iz}{2\sqrt{t}} + i\sqrt{t}\dfrac{dz}{dt}$,

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{d}{dt}\left(\frac{iz}{2\sqrt{t}} + i\sqrt{t}\frac{dz}{dt}\right)$$

after simplification, we get

(1.7) $\dfrac{d^2y}{dt^2} = i\left[\sqrt{t}\,\dfrac{d^2z}{dt^2} + \dfrac{1}{\sqrt{t}}\dfrac{dz}{dt} - \dfrac{z}{4t^{\frac{3}{2}}}\right]$.

(III) When $x = -t$, then

(1.8) $\dfrac{dt}{dx} = -1$,

(1.9) $\dfrac{dy}{dx} = \dfrac{dy}{dt} \times \dfrac{dt}{dx} = -\dfrac{dy}{dt}$,

(1.10) $\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = \dfrac{d}{dt}\left(-\dfrac{dy}{dt}\right)\dfrac{dt}{dx} = \dfrac{d^2y}{dt^2}$.

(IV) When $y = -tz$, where z is the function of t then

(1.11) $\dfrac{dy}{dt} = -z - t\dfrac{dz}{dt}$,

(1.12) $\dfrac{d^2y}{dt^2} = -2\dfrac{dz}{dt} - t\dfrac{d^2z}{dt^2}$.

(V) When $x^2 = t$ or $x = \sqrt{t}$, then

(1.13) $\dfrac{dx}{dt} = \dfrac{1}{2\sqrt{t}}$ or $\dfrac{dt}{dx} = 2\sqrt{t}$,

(1.14) $\dfrac{dy}{dx} = 2\sqrt{t}\,\dfrac{dy}{dt}$,

(1.15) $\dfrac{d^2y}{dx^2} = 2\dfrac{dy}{dt} + 4t\dfrac{d^2y}{dt^2}$.

(VI) Gauss' ordinary differential equation [15, Ch.(6), pp.144-148 and pp.157-158]

When $\gamma \neq 0, \pm1, \pm2, \pm3, \cdots$ and $|t| < 1$, then two linearly independent power series solutions of following Gauss'ordinary homogeneous linear differential equation of second order with variable coefficients

(1.16) $t(1-t)\dfrac{d^2z}{dt^2} + [\gamma - (1+\alpha+\beta)t]\dfrac{dz}{dt} - \alpha\beta z = 0$, are given by

(1.17) $z_1 = {}_2F_1\left[\begin{array}{c} \alpha,\ \beta; \\ \gamma; \end{array} t\right],$

and

(1.18) $z_2 = t^{1-\gamma}\ {}_2F_1\left[\begin{array}{c} \alpha+1-\gamma,\ \beta+1-\gamma; \\ 2-\gamma; \end{array} t\right],$

when γ is an integer then one solution may or may not, depending on the values of α and β, become logarithmic.

If any one solution of given differential equation is $y(x)$ then $Ay(x)$ will be the solution of same differential equation, where A is any suitable constant.

The present article is organized as follows:

In *Sections* 2, we have derived the hypergeometric forms of some mathematical functions by using differential equation approach. For hypergeometric forms of other mathematical functions and functions of mathematical physics, we refer the literature [1],[2],[3],[4],[5],[6],[7],[8],[9],[10],
[11],[12],[13],[14], [15], [17] and [19], where the proof of hypergeometric forms of related functions are not given. So we are interested to give the proof of hypergeometric forms of the functions mentioned in ***Section* 2**.

2 Hypergeometric forms of mathematical functions

Using the theory of ordinary differential equation and changing of independent and dependent variables in suitable differential equation, we can derive the following hypergeometric forms.

***Theorem* 2.1** *If $|x| < 1$, then following hypergeometric forms hold true:*

(2.1) $-\frac{4}{x}\,\ell n\left(\frac{1+\sqrt{1-x}}{2}\right) = {}_3F_2\left[\begin{array}{c} 1,\ 1,\ \frac{3}{2}; \\ 2,\ 2; \end{array} x\right],$

(2.2) $\tan^{-1}(x) = x\ {}_2F_1\left[\begin{array}{c} 1,\ \frac{1}{2}; \\ \frac{3}{2}; \end{array} -x^2\right],$

(2.3) $\ell n(1+x) = x\ {}_2F_1\left[\begin{array}{c} 1,\ 1; \\ 2; \end{array} -x\right],$

(2.4) $\sin(b\sin^{-1}x) = bx\ {}_2F_1\left[\begin{array}{c} \frac{1+b}{2},\ \frac{1-b}{2}; \\ \frac{3}{2}; \end{array} x^2\right],$

(2.5) $\cos(b\sin^{-1}x) = {}_2F_1\left[\begin{array}{c} \frac{b}{2},\ -\frac{b}{2}; \\ \frac{1}{2}; \end{array} x^2\right].$

Proof of hypergeometric form of (2.1)

Consider,

(2.1,I) $y \equiv y(x) = -\frac{4}{x}\,\ell n\left(\frac{1+\sqrt{1-x}}{2}\right).$

For $(0/0)$ indeterminate form, applying L'Hospital's rule in right hand side of the (2.1,I), the value of y at $x = 0$, will be 1. That is

(2.1,II) $y(0) = 1,$

(2.1,III) $xy = -4\,\ell n\left(\frac{1+\sqrt{1-x}}{2}\right).$

On differentiating result (2.1,III) w.r.t. x and applying product rule, after simplification we get

$$\frac{dy}{dx} + \frac{y}{x} = \frac{2}{x\{1+\sqrt{1-x}\}\sqrt{1-x}}$$
$$= \frac{2\{1-\sqrt{1-x}\}}{x\{1+\sqrt{1-x}\}\{1-\sqrt{1-x}\}\sqrt{1-x}}.$$

Therefore

(2.1,IV) $\frac{dy}{dx} + \frac{y}{x} = \frac{2}{x^2}\{(1-x)^{-\frac{1}{2}} - 1\}.$

The differential equation (2.1,IV) is written in the standard form of Leibnitz linear differential equation ($\frac{dy}{dx} + Py = Q$), therefore integrating factor for differential equation (2.1,IV) will be

(2.1,V) $I.F. = exp\left\{\int \frac{1}{x}dx\right\} = \exp\{\ell n(x)\} = x$.

The general solution of differential equation (2.1,IV) will be

(2.1,VI) $y = \dfrac{1}{x}\int x\left[\frac{2}{x^2}\left\{(1-x)^{-\frac{1}{2}} - 1\right\}\right]dx + C,$

where C is the constant of integration.

Therefore

$$y = \frac{1}{x}\int \frac{2}{x}\left\{{}_1F_0\left[\begin{array}{c}\frac{1}{2};\\ -;\end{array} x\right] - 1\right\} dx + C = \sum_{r=0}^{\infty} \frac{\left(\frac{3}{2}\right)_r (1)_r (1)_r x^r}{(2)_r (2)_r r!} + C$$

(2.1,VII) $y(x) = 1 + \sum_{r=1}^{\infty} \dfrac{\left(\frac{3}{2}\right)_r (1)_r (1)_r x^r}{(2)_r (2)_r r!} + C,$

(2.1,VIII) $y(x) = {}_3F_2\left[\begin{array}{c}1,\ 1,\ \frac{3}{2};\\ 2,\ 2;\end{array} x\right] + C.$

When $x = 0$ in the equation(2.1,VII), we get $C = 0$.

Therefore particular solution of the differential equation (2.1,IV) will be

(2.1,IX) $y = {}_3F_2\left[\begin{array}{c}1,\ 1,\ \frac{3}{2};\\ 2,\ 2;\end{array} x\right],$

therefore

(2.1,X) $-\dfrac{4}{x}\ \ell n\left(\dfrac{1+\sqrt{1-x}}{2}\right) = {}_3F_2\left[\begin{array}{c}1,\ 1,\ \frac{3}{2};\\ 2,\ 2;\end{array} x\right],$

which is satisfied by $x = 0$.

Proof of Hypergeometric Form of (2.2)

Consider,

(2.2,I) $y \equiv y(x) = \tan^{-1}(x),$

(2.2,II) $y(0) = 0.$

Differentiate the equation (2.2,I) w.r.t. x, we get

(2.2,III) $(1+x^2)\dfrac{dy}{dx} = 1.$

Again differentiate the equation (2.2,III) w.r.t. x and use product rule, after simplification we have

(2.2,IV) $(1+x^2)\dfrac{d^2y}{dx^2} + 2x\dfrac{dy}{dx} = 0$.

Put $x^2 = -t$ or $x = i\sqrt{t}$, then use values of equations (1.4) and (1.5) in above differential equation (2.2,IV), after simplification we get

(2.2,V) $t(1-t)\dfrac{d^2y}{dt^2} + \left\{\dfrac{1}{2} - \dfrac{3t}{2}\right\}\dfrac{dy}{dt} = 0$.

Now substitute $y = i\sqrt{t}z$, where z is the function of t and put the values of equations (1.6) and (1.7) in above differential equation (2.2,V), after simplification we obtain

(2.2,VI) $t(1-t)\dfrac{d^2z}{dt^2} + \left\{\dfrac{3}{2} - \dfrac{5t}{2}\right\}\dfrac{dz}{dt} - \dfrac{1}{2}z = 0$.

Now compare the coefficients of above differential equation (2.2,VI) with Gauss' standard differential equation (1.16), we get

$$\gamma = \frac{3}{2},\ \alpha+\beta+1 = \frac{5}{2},\ \alpha\beta = \frac{1}{2}\ .$$

Now solve the above algebraic equations simultaneously, we get

$$\alpha = 1,\ \beta = \frac{1}{2}\ .$$

Therefore one of the series solution of above differential equation (2.2,VI) will be

$$z = {}_2F_1\left[\begin{array}{c}1,\ \frac{1}{2};\\ \frac{3}{2};\end{array} t\right],$$

$$y = i\sqrt{t}\ {}_2F_1\left[\begin{array}{c}1,\ \frac{1}{2};\\ \frac{3}{2};\end{array} t\right],$$

$$\tan^{-1}(x) = x\ {}_2F_1\left[\begin{array}{c}1,\ \frac{1}{2};\\ \frac{3}{2};\end{array} -x^2\right],$$

which is satisfied by $x = 0$.

Proof of Hypergeometric Form of (2.3)

Consider,

(2.3,I) $y \equiv y(x) = \ell n(1 + x)$,

(2.3,II) $y(0) = 0$.

Differentiate the equation (2.3,I) w.r.t. x, we get

(2.3,III) $(1 + x)\dfrac{dy}{dx} = 1$.

Again differentiate the equation (2.3,III) w.r.t. x and use product rule, after simplification we have

(2.3,IV) $(1 + x)\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} = 0$.

Put $x = -t$, then use values of equations (1.9) and (1.10) in above differential equation (2.3,IV), after simplification we get

(2.3,V) $(1 - t)\dfrac{d^2y}{dt^2} - \dfrac{dy}{dt} = 0$.

Now substitute $y = -tz$, where z is the function of t and put the values of equations (1.11) and (1.12) in above differential equation (2.3,V), after simplification we obtain

(2.3,VI) $t(1 - t)\dfrac{d^2z}{dt^2} + \{2 - 3t\}\dfrac{dz}{dt} - z = 0$.

Now compare the coefficients of above differential equation (2.3,VI) with Gauss' standard differential equation (1.16), we get

$$\gamma = 2,\ \alpha + \beta + 1 = 3,\ \alpha\beta = 1.$$

Now solve the above algebraic equations simultaneously, we get

$$\alpha = 1,\ \beta = 1\ .$$

Therefore one of the series solution of above differential equation (2.3,VI) will be

$$z = {}_2F_1\left[\begin{matrix} 1,\ 1; \\ 2; \end{matrix}\ t\right],$$

$$y = -t\ {}_2F_1\left[\begin{matrix} 1,\ 1; \\ 2; \end{matrix}\ t\right],$$

$$\ell n(1 + x) = x\ {}_2F_1\left[\begin{matrix} 1,\ 1; \\ 2; \end{matrix}\ -x\right],$$

which is satisfied by $x = 0$.

Proof of Hypergeometric Form of (2.4)

Consider,

(2.4,I) $y \equiv y(x) = \sin(b \sin^{-1} x)$,

(2.4,II) $y(0) = 0$.

Differentiate the equation (2.4,I) w.r.t. x, we get

(2.4,III) $\sqrt{1 - x^2}\,\dfrac{dy}{dx} = b\cos(b \sin^{-1} x)$.

Again differentiate the equation (2.4,III) w.r.t. x and use product rule, after simplification we have

(2.4,IV) $(1 - x^2)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + b^2y = 0$.

Put $x^2 = t$ or $x = \sqrt{t}$, then use values of equations (1.14) and (1.15) in above differential equation (2.4,IV), after simplification we get

(2.4,V) $t(1 - t)\dfrac{d^2y}{dt^2} + \left\{\dfrac{1}{2} - t\right\}\dfrac{dy}{dt} + \dfrac{b^2}{4}y = 0$.

Now compare the coefficients of above differential equation (2.4,V) with Gauss' standard differential equation (1.16), we get

$$\gamma = \frac{1}{2},\ \alpha+\beta+1 = 1,\ \alpha\beta = -\frac{b^2}{4}.$$

Now solve the above algebraic equations simultaneously, we get

$$\alpha = \frac{b}{2},\ \beta = -\frac{b}{2}\ .$$

Therefore one of the suitable power series solution of above differential equation (2.4,V) will be

$$y = t^{\frac{1}{2}}\ {}_2F_1\left[\begin{array}{c} \frac{1+b}{2},\ \frac{1-b}{2}; \\ \frac{3}{2}; \end{array}\ t\right],$$

$$\sin(b\sin^{-1}x) = x\ {}_2F_1\left[\begin{array}{c} \frac{1+b}{2},\ \frac{1-b}{2}; \\ \frac{3}{2}; \end{array}\ x^2\right].$$

More general solution will be

$$\sin(b\sin^{-1}x) = Ax\ {}_2F_1\left[\begin{array}{c} \frac{1+b}{2},\ \frac{1-b}{2}; \\ \frac{3}{2}; \end{array}\ x^2\right],$$

$$\text{or}\quad \frac{\sin(b\sin^{-1}x)}{Ax} = {}_2F_1\left[\begin{array}{c} \frac{1+b}{2},\ \frac{1-b}{2}; \\ \frac{3}{2}; \end{array}\ x^2\right],$$

(2.4,VI) $\frac{b}{A}\left(\frac{\sin^{-1}x}{x}\right) - \frac{b^3}{3!A}\left(\frac{\sin^{-1}x}{x}\right)(\sin^{-1}x)^2 + \frac{b^5}{5!A}\left(\frac{\sin^{-1}x}{x}\right)(\sin^{-1}x)^4 - \cdots = {}_2F_1\left[\begin{array}{c} \frac{1+b}{2},\ \frac{1-b}{2}; \\ \frac{3}{2}; \end{array}\ x^2\right].$

Now taking $\lim_{x\to 0}$ in the equation (2.4,VI), we get $A = b$.

Therefore more general solution will be

$$\sin(b\sin^{-1}x) = bx\ {}_2F_1\left[\begin{array}{c} \frac{1+b}{2},\ \frac{1-b}{2}; \\ \frac{3}{2}; \end{array}\ x^2\right],$$

which is satisfied by $x = 0$ or $b = 0$ or both $b, x = 0$.

Proof of Hypergeometric Form of (2.5)

Consider,

(2.5,I) $y \equiv y(x) = \cos(b\sin^{-1}x)$,

(2.5,II) $y(0) = 1$.

Differentiate the equation (2.5,I) w.r.t. x, we get

(2.5,III) $\sqrt{1-x^2}\,\frac{dy}{dx} = -b\sin(b\sin^{-1}x)$.

Again differentiate the equation (2.5,III) w.r.t. x and use product rule, after simplification we have

(2.5,IV) $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + b^2y = 0$.

Put $x^2 = t$ or $x = \sqrt{t}$, then use values of equations (1.14) and (1.15) in above differential equation (2.5,IV), after simplification we obtain

(2.5,V) $t(1-t)\frac{d^2y}{dt^2} + \left\{\frac{1}{2} - t\right\}\frac{dy}{dt} + \frac{b^2}{4}y = 0$.

Now compare the coefficients of above differential equation (2.5,V) with Gauss' standard differential equation (1.16), we get

$$\gamma = \frac{1}{2},\ \alpha+\beta+1 = 1,\ \alpha\beta = -\frac{b^2}{4}.$$

Now solve the above algebraic equations simultaneously, we get

$$\alpha = \frac{b}{2},\ \beta = -\frac{b}{2}\ .$$

Therefore one of the series solution of above differential equation (2.5,V) will be

$$y = {}_2F_1\left[\begin{array}{c} \frac{b}{2},\ -\frac{b}{2}; \\ \frac{1}{2}; \end{array}\ t\right],$$

$$\cos(b\sin^{-1}x) = {}_2F_1\left[\begin{array}{c} \frac{b}{2},\ -\frac{b}{2}; \\ \frac{1}{2}; \end{array}\ x^2\right],$$

which is satisfied by $x = 0$ or $b = 0$ or both $b, x = 0$.

3 Conclusion

In our present investigation, we derived the hypergeometric forms of some functions by using differential equation approach. Moreover, the results derived in this paper are expected to have useful applications in wide range of problems of Mathematics, Statistics and Physical sciences. Similarly, we can derive the hypergeometric forms of other functions in an analogous manner.

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(Dedicated to Honor Dr. R. C. Singh Chandel on His 75th Birth Anniversary Celebrations)

UNIQUENESS OF ALGEBROID FUNCTIONS IN CONNECTION TO NEVANLINNA'S FIVE-VALUE THEOREM

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Abstract

In this paper, we prove a uniqueness theorem for derivatives of algebroid functions which improve and generalize the Navenlinna's five-value theorem for algebroid functions.

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1 Introduction

The value distribution theory of meromorphic functions was extended to the corresponding theory of algebroid functions by Ullarich [27] and Valiron [28] around 1930, and important results on uniqueness for algebroid functions have been obtained. It is well known that Valiron obtained a famous $4\nu+1$-valued theorem. The uniqueness theory of algebroid functions is an interesting problem in the value distribution theory. Many researchers like Valiron [28], Baganas [1], He et al.[11, 12] and others have done a lot of work in this area (see [1], [4]-[6], [9]-[26], [30], [31]). In this paper, we discuss a result of Indrajit Lahiri and Rupa Pal [13] on the Nevanlinna's value distribution theory of meromorphic functions for Nevanlinna's five values theorem to algebroid functions

Let $A_\nu(z), A_{\nu-1}(z), ..., A_0(z)$ be analytic functions with no common zeros in the complex plane, then the following equation

(1.1) $A_\nu(z)W^\nu + A_{\nu-1}(z)W^{\nu-1} + ... + A_1(z)W + A_0(z) = 0.$

Then equation (1.1) defines a ν-valued algebroid function $W(z)$ [29].

It is well known from [12] that on the complex plane with a cutting the projection of the critical points of the function W, the Nevanlinna characteristic $T(r,W)$ is defined as

$$T(r,W) = m(r,W) + N(r,W),$$

where

$$m(r,W) = \frac{1}{2\pi\nu}\sum_{j=1}^{\nu}\int_0^{2\pi}\log^+|w_j(re^{i\theta})|d\theta,$$

$$N(r,W) = \frac{1}{\nu}\int_0^r \frac{n(t,W)-n(0,W)}{t}dt + \frac{n(0,W)}{\nu}\log r,$$

where $w_j(z)(j=1,2,3,...,\nu)$ is one valued branch $W(z)$ and $n(t,W)$ is the counting function of poles of the function of $W(z)$ in the whole of the complex plane. Let $w_i(z)$ and $m_j(z)$ be one valued branches of two algebroid (μ-valued and ν-valued)functions. It follows from Prokopovich [16] that we consider their quotient in the domain of the complex plane with cutting through the projection of the critical points of both functions. The one-valued branches of the function W/M $(W.M)$ will be defined by w_i/m_j $(w_i.m_j)$, where $1 \le i \le m,\ 1 \le j \le n$. The Nevanlinna characteristic $T(r,W/M)$ or $T(r,W.M)$ is defined as follows

$$m(r,W.M) = \frac{1}{\mu\nu}\sum_{1\le i\le\mu;1\le j\le\nu} m(r,w_i(z).m_j(z))$$
$$= \frac{1}{\mu\nu}\sum_{1\le i\le\mu;1\le j\le\nu}\frac{1}{2\pi}\log^+|w_i(z).m_j(z)|d\theta$$
$$= \frac{1}{\mu\nu}\left(\nu\sum_{i=1}^{\mu}\frac{1}{2\pi}\log^+|w_i(z)|d\theta + \mu\sum_{j=1}^{\nu}\frac{1}{2\pi}\log^+|m_j(z)|d\theta\right)$$

$$= \frac{1}{\mu} \sum_{i=1}^{\mu} \frac{1}{2\pi} \log^+ |w_i(z)| d\theta + \frac{1}{\nu} \sum_{j=1}^{\nu} \frac{1}{2\pi} \log^+ |m_j(z)| d\theta$$
$$= m(r, W) + m(r, M),$$

and

$$N(r, W.M) = \frac{1}{\mu\nu} \int_0^r \frac{n(t,W.M)}{t} dt$$
$$\leq \frac{1}{\mu\nu} \left(\nu \int_0^r \frac{n(t,W.M)}{t} dt + \mu \int_0^r \frac{n(t,W.M)}{t} dt\right)$$
$$= \frac{1}{\mu} \int_0^r \frac{n(t,W.M)}{t} dt + \frac{1}{\nu} \int_0^r \frac{n(t,W.M)}{t} dt$$
$$= N(r, W) + N(r, M).$$

Therefore $T(r, W.M) \leq T(r, W) + T(r, M)$.
Similarly $T(r, W/M) \leq T(r, W) + T(r, M)$.

Let $W(z)$ be a ν-valued algebroid function and $a \in \overline{\mathbb{C}}$ be any complex number. $\overline{E}_{k)}(W = a)$ denotes the set of zeros of $W(z) - a$, whose multiplicities are not greater than k. $\overline{n}_{k)}(W = a)$ denotes the number of distinct zeros of $W(z) - a$ in $|z| \leq r$, whose multiplicities are not greater than k and are counted only once. Similarly, we define the functions $\overline{n}_{(k+1}(r, W = a)$, $\overline{N}_{k)}(r, W = a)$ and $\overline{N}_{(k+1}(r, W = a)$.

***Lemma* 1.1** [9] *Let $W(z)$ be a ν-valued algebroid function and $\{a_j\}_{j=1}^{q} \subset \overline{\mathbb{C}}$ be q distinct complex numbers and let $\{k_j\}_{j=1}^{q} \subset \mathbb{N}$ be q positive integers. Then*

$$(q - 2\nu)T(r, W) \leq \sum_{k=1}^{q} \frac{k_j}{k_j+1} \overline{N}_{k_j)} \left(r, W = a_j\right) + \sum_{k=1}^{q} \frac{1}{k_j+1} N(r, W = a_j) + S(r, W),$$
$$\left(q - 2\nu - \sum_{k=1}^{q} \frac{1}{k_j+1}\right) T(r, W) \leq \sum_{k=1}^{q} \frac{k_j}{k_j+1} \overline{N}_{k_j)} \left(r, W = a_j\right) + N(r, W = a_j) + S(r, W).$$

In 2006 *Zu-Xing Xuan and Zong-Sheng Gao* [29] *improved the Nevanlinna Five Value Theorem for algebroid functions in the following manner.*

***Theorem* 1.1** *Let $W(z)$ and $M(z)$ be two ν-valued, non-constant algebroid functions, let a_j $(j = 1, 2, ..., 4\nu + 1)$ be $4\nu + 1$ distinct complex numbers in $\overline{\mathbb{C}}$. If*

$$\overline{E}_{2\nu+1)}(a_j, W) = \overline{E}_{2\nu+1)}(a_j, M) \quad (j = 1, 2, ..., 2\nu + 1)$$

and

$$\overline{E}_{2\nu)}(a_j, W) = \overline{E}_{2\nu)}(a_j, M) \quad (j = 1, 2, ..., 4\nu + 1),$$

then $W(z) = M(z)$.

***Definition* 1.1** *For $B \subset \mathbb{A}$ and $a \in \overline{C}$, we denote by $\overline{N}_B(r, \frac{1}{f-a})$ the reduced counting function of those zeros of $f - a$ on $\mathbb{A}$, which belong to the set B.*

In 2018 *Rathod* [20] *proved the following theorem for algebroid functions*

***Theorem* 1.2** *Let $W_1(z)$ and $W_2(z)$ be two ν-valued, non-constant algebroid functions, let a_j $(j = 1, 2, ..., q)$ be $q \geq 4\nu+1$ distinct complex numbers or ∞. Suppose that $k_1 \geq k_2 \geq ... \geq k_q, m$ are positive integers or ∞; $1 \leq m \leq q$ and $\delta_j(\geq 0)(j = 1, 2, ..., q)$ are such that*

$$\left(1 + \frac{1}{k_m}\right) \sum_{j=m}^{q} \frac{1}{1+k_j} + 3\nu + \sum_{j=1}^{q} \delta_j < (q - m - 1)\left(1 + \frac{1}{k_m}\right) + m.$$

Let $B_j = \overline{E}_{k_j}(a_j, f) \backslash \overline{E}_{k_j}(a_j, g)$ for $j = 1, 2, ..., q$. If

$$\overline{N}_{B_j}(r, \frac{1}{W_1 - a_j}) \leq \delta_j T(r, W_1)$$

and

$$\liminf_{r\to\infty} \frac{\sum_{j=1}^{q} \overline{N}_{k_j}(r, \frac{1}{W_1 - a_j})}{\sum_{j=1}^{q} \overline{N}_{k_j}(r, \frac{1}{W_2 - a_j})} > \frac{\nu k_m}{(1 + k_m) \sum_{j=1}^{q} \frac{k_j}{k_j+1} - 2\nu(1 + k_m) + (m - 2\nu - \sum_{j=1}^{q} \delta_j)k_m}$$

then $W_1(z) \equiv W_2(z)$.

2 Main Results

In the paper we wish to further investigate the problem on the Nevanlinna's five value theorem for algebroid functions. To state our main Theorem, we wish to introduce the following ***Lemma* 2.1**.

***Lemma* 2.1** *Let $W(z)$ be a ν-valued algebroid function and $a_1, a_2, ..., a_q$ be $q(\geq 2\nu+1)$ distinct complex numbers. If for a non-negative integer n, $E(0;W) \subseteq E(0, W^{(n)})$, then*

$$(q-2\nu+o(1))T(r,W) \leq \sum_{j=1}^{q} \overline{N}\left(r, \frac{1}{W^{(n)}-a_j}\right).$$

Proof. By Nevanlinna's first fundamental theorem for algebroid functions, we have

$$\begin{aligned}(2.1)\quad T(r,W) =& T\left(r,\frac{1}{W}\right)+O(1)\\ \leq& N\left(r,\frac{1}{W}\right)+m\left(r,\frac{W^{(n)}}{W}\right)+m\left(r,\frac{1}{W^{(n)}}\right)+O(1)\\ \leq& N\left(r,\frac{1}{W}\right)+T(r,W^{(n)})-N\left(r,\frac{1}{W^{(n)}}\right)+S(r,W).\end{aligned}$$

By the Nevanlinna's second fundamental theorem for algebroid functions, we get

$$(q-1)\,T(r,W^{(n)}) \leq \overline{N}(r,W^{(n)})+\sum_{j=1}^{q-1}\overline{N}\left(r,\frac{1}{W^{(n)}-a_j}\right)+\overline{N}\left(r,\frac{1}{W^{(n)}}\right)+S(r,W).$$

Without loss of generality, we may assume that $a_q = 0$. Otherwise a suitable linear transformation is done. Then the above inequality reduces to

$$(2.2)\quad (q-1)\,T(r,W^{(n)}) \leq \overline{N}(r,W^{(n)})+\sum_{j=1}^{q}\overline{N}\left(r,\frac{1}{W^{(n)}-a_j}\right)+S(r,W).$$

Using (2.2) in (2.1), we obtain

$$\begin{aligned}(q-1)T(r,W) \leq& (q-1)\,T\left(r,\frac{1}{W}\right)+\overline{N}(r,W^{(n)})+\sum_{j=1}^{q}\overline{N}\left(r,\frac{1}{W^{(n)}-a_j}\right)\\ &-(q-1)N\left(r,\frac{1}{W^{(n)}}\right)+S(r,W).\end{aligned}$$

Thus

$$\begin{aligned}(2.3)\quad (q-1)T(r,W) \leq& (q-1)\,T\left(r,\frac{1}{W}\right)+\overline{N}(r,W)+\sum_{j=1}^{q}\overline{N}\left(r,\frac{1}{W^{(n)}-a_j}\right)\\ &-(q-1)N\left(r,\frac{1}{W^{(n)}}\right)+S(r,W).\end{aligned}$$

Since $E(0,W) \subseteq E(0,W^{(n)})$, we have from (2.3)

$$(q-1)T(r,W) \leq \overline{N}(r,W)+\sum_{j=1}^{q}\overline{N}\left(r,\frac{1}{W^{(n)}-a_j}\right)+S(r,W).$$

Hence

$$(q-2\nu+o(1))T(r,W) \leq \sum_{j=1}^{q}\overline{N}\left(r,\frac{1}{W^{(n)}-a_j}\right).$$

This completes the proof of the ***Lemma* 2.1**.

In this paper we wish to obtain a generalization of ***Theorem* 1.2**. Now we state and prove our main result in the following way.

***Theorem* 2.1** *Let $W_1(z)$ and $W_2(z)$ be two ν-valued, non-constant algebroid functions, let $a_j(j=1,2,...,q)$ be $q \geq 4\nu+1$ distinct complex numbers or ∞. Suppose that $k_1 \geq k_2 \geq ... \geq k_q$ are positive integers or ∞ and $\delta_j(\geq 0)(j=1,2,...,q)$ are such that*

$$\frac{1}{k_1}+\left(1+\frac{1}{k_m}\right)\sum_{j=2\nu}^{q}\frac{1}{1+k_j}+1+\delta < \frac{q-2\nu}{n+1}\left(1+\frac{1}{k_1}\right).$$

for a non-negative integer n. Let $B_j = \overline{E}_{k_j}(a_j,W_1)\backslash\overline{E}_{k_j}(a_j,W_2)$ for $j=1,2\nu,...,q$ and $E(0,W_i) \subseteq E(0,W_i^{(n)})$ for $i=1,2$. If

$$\overline{N}_{B_j}(r,\frac{1}{W_1^{(n)}-a_j}) \leq \delta_j T(r,W_1^{(n)})$$

and

$$\liminf_{r\to\infty}\frac{\sum_{j=1}^{q}\overline{N}_{k_{j)}}(r,\frac{1}{W_1^{(n)}-a_j})}{\sum_{j=1}^{q}\overline{N}_{k_{j)}}(r,\frac{1}{W_2^{(n)}-a_j})} > \frac{(n+1)k_1}{(p-2\nu)(1+k_1)-(n+1)(1+k_1)\sum_{j=2\nu}^{q}\frac{1}{1+k_j}-(n+1)\{(1+\delta)k_1+1\}},$$

then $W_1^{(n)}(z) \equiv W_2^{(n)}(z)$.

Proof. By ***Lemma* 2.1**, we have

(2.4) $(q-2\nu+o(1))T(r,W_1)\le\sum_{j=1}^{q}\overline{N}\left(r,\frac{1}{W_1^{(n)}-a_j}\right)$

and

(2.5) $(q-2\nu+o(1))T(r,W_2)\le\sum_{j=1}^{q}\overline{N}\left(r,\frac{1}{W_2^{(n)}-a_j}\right)$.

From (2.4), we have

$$\begin{aligned}(q-2\nu+o(1))T(r,W_1)&\le\sum_{j=1}^{q}\left\{\overline{N}_{k_j)}\left(r,\frac{1}{W_1^{(n)}-a_j}\right)+\overline{N}_{(k_j+1}\left(r,\frac{1}{W_1^{(n)}-a_j}\right)\right\}\\&\le\sum_{j=1}^{q}\left\{\overline{N}_{k_j)}\left(r,\frac{1}{W_1^{(n)}-a_j}\right)+\frac{1}{1+k_j}N_{(k_q+1}\left(r,\frac{1}{W_1^{(n)}-a_j}\right)\right\}\\&\le\sum_{j=1}^{q}\left\{\frac{k_j}{1+k_j}\overline{N}_{k_j)}\left(r,\frac{1}{W_1^{(n)}-a_j}\right)+\frac{1}{1+k_j}N\left(r,\frac{1}{W_1^{(n)}-a_j}\right)\right\}\\&\le\sum_{j=1}^{q}\frac{k_j}{1+k_j}\overline{N}_{k_j)}\left(r,\frac{1}{W_1^{(n)}-a_j}\right)+\sum_{j=1}^{q}\frac{1}{1+k_j}T\left(r,W_1^{(n)}\right)\\&\le\sum_{j=1}^{q}\frac{k_j}{1+k_j}\overline{N}_{k_j)}\left(r,\frac{1}{W_1^{(n)}-a_j}\right)+(n+1)\sum_{j=1}^{q}\frac{1}{1+k_j}T\left(r,W_1^{(n)}\right).\end{aligned}$$

Therefore

$$(q-2\nu-(n+1)\sum_{j=1}^{q}\frac{1}{1+k_j}+o(1))T(r,W_1)\le\sum_{j=1}^{q}\frac{k_j}{1+k_j}\overline{N}_{k_j)}\left(r,\frac{1}{W_1^{(n)}-a_j}\right).$$

Similarly from (2.5), we get

$$(q-2\nu-(n+1)\sum_{j=1}^{q}\frac{1}{1+k_j}+o(1))T(r,W_2)\le\sum_{j=1}^{j}\frac{k_j}{1+k_j}\overline{N}_{k_j)}\left(r,\frac{1}{W_2^{(n)}-a_j}\right).$$

Let $B_j=\overline{E}_{k_j}(a_j,W_1^{(n)})\backslash A_j$ for $j=1,2\nu,...,q$.
Now

$$\begin{aligned}\sum_{j=1}^{q}\overline{N}_{k_j)}\left(r,\frac{1}{W_1^{(n)}-a_j}\right)&=\sum_{j=1}^{q}\overline{N}_{A_j}\left(r,\frac{1}{W_1^{(n)}-a_j}\right)+\sum_{j=1}^{q}\overline{N}_{B_j}\left(r,\frac{1}{W_1^{(n)}-a_j}\right)\\&\le\delta T(r,W_1^{(n)})+N\left(r,\frac{1}{W_1^{(n)}-W_2^{(n)}}\right)\\&\le(1+\delta)(n+1)T(r,W_1)+(n+1)T(r,W_2).\end{aligned}$$

Hence

$$\begin{aligned}&\left(q-2\nu-(n+1)\sum_{j=1}^{q}\frac{1}{1+k_j}+o(1)\right)\sum_{j=1}^{q}\overline{N}_{k_j)}\left(r,\frac{1}{W_1^{(n)}-a_j}\right)\\&\le(1+\delta)(n+1)\sum_{j=1}^{q}\frac{k_j}{1+k_j}\overline{N}_{k_j)}\left(r,\frac{1}{W_1^{(n)}-a_j}\right)+(n+1)\sum_{j=1}^{q}\frac{k_j}{1+k_j}\overline{N}_{k_j)}\left(r,\frac{1}{W_2^{(n)}-a_j}\right).\end{aligned}$$

Since $1\ge\dfrac{k_1}{k_1+1}\ge\dfrac{k_2}{k_2+1}\ge...\ge\dfrac{k_q}{k_q+1}\ge\dfrac{1}{2}$, we get from the above inequality

$$\begin{aligned}&\left(q-2\nu-(n+1)\sum_{j=1}^{q}\frac{1}{1+k_j}+o(1)\right)\sum_{j=1}^{q}\overline{N}_{k_j)}\left(r,\frac{1}{W_1^{(n)}-a_j}\right)\\&\le(1+\delta)(n+1)\frac{k_1}{1+k_1}\sum_{j=1}^{q}\overline{N}_{k_j)}\left(r,\frac{1}{W_1^{(n)}-a_j}\right)+(n+1)\frac{k_1}{1+k_1}\sum_{j=1}^{q}\overline{N}_{k_j)}\left(r,\frac{1}{W_2^{(n)}-a_j}\right).\end{aligned}$$

Since that implies

$$\begin{aligned}&\left(q-2\nu-(n+1)\sum_{j=1}^{q}\frac{1}{1+k_j}-(1+\delta)(n+1)\frac{k_1}{1+k_1}+o(1)\right)\sum_{j=1}^{q}\overline{N}_{k_j)}\left(r,\frac{1}{W_1^{(n)}-a_j}\right)\\&\le(n+1)\frac{k_1}{1+k_1}\sum_{j=1}^{q}\overline{N}_{k_j)}\left(r,\frac{1}{W_2^{(n)}-a_j}\right).\end{aligned}$$

Therefore

$$\begin{aligned}&\liminf_{r\to\infty}\frac{\sum_{j=1}^{q}\overline{N}_{k_j}(r,\frac{1}{W_1-a_j})}{\sum_{j=1}^{q}\overline{N}_{k_j}(r,\frac{1}{W_2-a_j})}\\&\le\frac{(n+1)k_1}{(q-2\nu)(1+k_1)-(n+1)(1+k_1)\sum_{j=1}^{q}\frac{1}{1+k_j}-(n+1)\{(1+\delta)k_1}\end{aligned}$$

$$\leq \frac{(n+1)k_1}{(q-2\nu)(1+k_1)-(n+1)(1+k_1)\sum_{j=2\nu}^{q}\frac{1}{1+k_j}-(n+1)\{(1+\delta)k_1+1}.$$

which is a contradiction.

Thus, we have $W_1^{(n)}(z) \not\equiv W_2^{(n)}(z)$.

Therefore we complete the proof of ***Theorem* 2.1**.

From ***Theorem* 2.1**, we can get the following consequences.

***Corollary* 2.1** *Let $k_j = \infty$ for $j = 1, 2\nu, ..., q$ and*

$$\gamma = \liminf_{r\to\infty} \frac{\overline{N}_{k_j)}\left(r, \frac{1}{W_1^{(n)}-a_j}\right)}{\overline{N}_{k_j)}\left(r, \frac{1}{W_2^{(n)}-a_j}\right)} > \frac{n+1}{q-(n+2\nu+1)}.$$

If $\overline{N}_{A_j}(r, \frac{1}{W_1^{(n)}-a_j}) \leq \delta_j T(r, W_1^{(n)})$ where $\delta(\geq 0)$ satisfy $0 \leq \delta_j < \frac{q-(n+2\nu+1)}{n+1} - \frac{1}{\gamma}$.

If we assume $E_{\infty)}(a_j, W_1^{(n)}) \subseteq E_{\infty)}(a_j, W_2^{(n)})$, then $A_j = \phi$ for $j = 1, 2\nu, ..., q$ and so we can choose $\delta = 0$.

*Therefore **Theorem 2.1** is an improvement of following theorem.*

***Theorem* 2.2** *Let $W_1(z)$ and $W_2(z)$ be two ν-valued, non-constant algebroid functions, let a_j $(j = 1, 2, ..., q)$ be $q \geq 4\nu+1$ distinct complex numbers or ∞. and for a non-negative integer n, $E_{\infty)}(a_j, W_1^{(n)}) \subseteq E_{\infty)}(a_j, W_2^{(n)})$ for $1 \leq j \leq q$, $E_{\infty)}(0, W_1) \subseteq E_{\infty)}(0, W_1^{(n)})$, $E_{\infty)}(0, W_2) \subseteq E_{\infty)}(0, W_2^{(n)})$ and*

$$\liminf_{r\to\infty} \frac{\sum_{j=1}^{q} \overline{N}_{k_j)}(r, \frac{1}{W_1^{(n)}-a_j})}{\sum_{j=1}^{q} \overline{N}_{k_j)}(r, \frac{1}{W_2^{(n)}-a_j})} > \frac{(n+1)}{q-(n+2\nu+1)},$$

then $W_1^{(n)}(z) \equiv W_2^{(n)}(z)$.

***Corollary* 2.2** *Let $n = 0$, $k_j = \infty$ for $j = 1, 2\nu, ..., q$ and*

$$\gamma = \liminf_{r\to\infty} \frac{\overline{N}_{k_j)}\left(r, \frac{1}{W_1-a_j}\right)}{\overline{N}_{k_j)}\left(r, \frac{1}{W_2-a_j}\right)} > \frac{1}{q-2\nu+1}.$$

*If $\overline{N}_{B_j}(r, \frac{1}{W_1-a_j}) \leq \delta_j T(r, W_1)$ where $\delta(\geq 0)$ satisfy $0 \leq \sum_{j=1}^{q} \delta_j < k-(2\nu+1)-\frac{1}{\gamma}$, then $W_1(z) \equiv W_2(z)$. If we take $q = 4\nu+1$ and $\overline{E}(a_j, f) \subseteq \overline{E}(a_j, f)$, then $A_j = \phi$ for $j = 1, 2, ..., 4\nu+1$. Therefore, if we choose $\delta_j = 0$ for $j = 1, 2, ..., 4\nu+1$ and take any constant γ, such that $0 \leq 2\nu - \frac{1}{\gamma}$ in **Corollary 2.2**; we can get that $W_1(z) \equiv W_2(z)$. Especially, if $q = 4\nu+1$ and $\overline{E}(a_j, W_1) = \overline{E}(a_j, W_2)$, then $\gamma = 1$ and $\delta_j = 0$ for $j = 1, 2, ..., 4\nu+1$. We can obtain $W_1(z) \equiv W_2(z)$. Then **Corollary 2.2** is an improvement of **Theorem 1.1**.*

***Corollary* 2.3** *Let $W_1(z)$ and $W_2(z)$ be two ν-valued, non-constant algebroid functions, let a_j $(j = 1, 2, ..., q)$ be $q \geq 5$ distinct complex numbers or ∞. Suppose that $k_1, k_2, ..., k_q$ are positive integers or ∞; with $k_1 \geq k_2 \geq ... \geq k_q$ if $\overline{E}_{k_j)}(a_j, W_1) \subseteq \overline{E}_{k_j)}(a_j, W_2)$ and :*

$$\sum_{j=2\nu}^{q} \frac{k_j}{k_j+1} - \frac{k_1}{\gamma(k_1+1)} - 2\nu > 0,$$

where γ is stated as in Corollary 2.2; then $W_1(z) \equiv W_2(z)$.

***Corollary* 2.4** *Under the assumptions of **Corollary 2.2**, $\overline{E}_{k_j)}(a_j, W_1) = \overline{E}_{k_j)}(a_j, W_2)$ and :*

$$\sum_{j=2\nu}^{q} \frac{k_j}{k_j+1} - \frac{k_1}{(k_1+1)} - 2\nu > 0,$$

***Corollary* 2.5** *Let $W_1(z)$ and $W_2(z)$ be two ν-valued, non-constant algebroid functions, let a_j $(j = 1, 2, ..., q)$ be $q \geq 5$ distinct complex numbers or ∞. Suppose that $k_1, k_2, ..., k_q$ are positive integers or ∞; with $k_1 \geq k_2 \geq ... \geq k_q$ if $\overline{E}_{k_j)}(a_j, f) \subseteq \overline{E}_{k_j)}(a_j, g)$ and :*

$$\sum_{j=2\nu}^{q} \frac{k_j}{k_j+1} - 2\nu + \frac{(m-2\nu-\frac{1}{\gamma})k_m}{\gamma(k_m+1)} - 2\nu > 0,$$

where γ is stated as in Corollary 2.2; then $W_1(z) \equiv W_2(z)$.

In Corollary 2.1 if $n = 0$ and $q = 4\nu+1$ then we get the following theorem.

***Theorem* 2.3** *Let $W_1(z)$ and $W_2(z)$ be two ν-valued, non-constant algebroid functions such that $E_{\infty)}(a_j, W_1) \subseteq E_{\infty)}(a_j, W_2)$ for $a_1, a_2, ..., a_5$ of $\mathbb{C} \cup \infty$. If*

$$\liminf_{r\to\infty} \frac{\sum_{j=1}^{4\nu+1} \overline{N}_{k_j)}(r, \frac{1}{W_1-a_j})}{\sum_{j=1}^{4\nu+1} \overline{N}_{k_j)}(r, \frac{1}{W_2-a_j})} > \frac{1}{2},$$

then $W_1(z) \equiv W_2(z)$.

3 Conclusion

In this paper, we discussed on the Nevanlinna's value distribution theory of meromorphic functions to Nevanlinna's five values theorem for algebroid functions and we further investigated the problems on the Nevanlinna's five value theorem for algebroid functions.

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(Dedicated to Honor Dr. R. C. Singh Chandel on His 75^{th} Birth Anniversary Celebrations)

NONLINEAR ABSTRACT MEASURE HYBRID DIFFERENTIAL EQUATIONS WITH A LINEAR PERTURBATION OF SECOND TYPE

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Abstract

In this paper, an existence result for perturbed abstract measure differential equations is proved via hybrid fixed point theorems of Dhage [4] under the mixed generalized Lipschitz and Carathéodory conditions. The existence of the extremal solutions is also proved under certain monotonicity conditions and using a hybrid fixed point theorem of Dhage [4] on ordered Banach spaces. Our existence results include the existence results of Sharma [23], Joshi [19] and Shendge and Joshi [25] as special cases under weaker continuity condition.
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1 Introduction

Sharma [23, 24] introduced the abstract measure differential equations as the generalizations of the ordinary differential equations in which ordinary derivative is replaced with the Radon-Nykodym derivative of vector measures in abstract spaces. The basic results concerning the existence and uniqueness of solutions for such equations in the above papers via fixed point techniques from nonlinear functional analysis. Later, such abstract measure differential equations are studied by various authors for different aspects of the solutions (see Joshi [19], Shendge and Joshi [25], Dhage [1, 2, 3], Dhage et al. [13], Dhage and Graef [14], Dhage and Reddy [16] and the references therein).

It is quite familiar that if a nonlinear differential equation is not solvable, but when we perturb it, we obtain very interesting results along with existence of solution. The classifications of different types of perturbations appear in Dhage [5]. The perturbed differential equation of any type is called a hybrid differential equations and studied extensively in the literature for different aspects of the solutions via hybrid fixed point theory initiated by Krasnoseslskii [20] and Dhage [4, 5, 7]. In the present paper, we consider a nonlinear abstract measure differential equation with linear perturbation of second type and deal with a variant of Krasnoselskii [20] fixed point theorem due to Dhage [4]. The results of this paper complement and generalize the results of Sharma [23, 24], Joshi [19], Shendge and Joshi [25], Dhage [1, 2, 3] on abstract measure differential equations under suitable conditions.

The rest of the paper is organized as follows. Section 2 deals with the statement of the problem of abstract measure differential equation and Section 3 deals with the auxiliary results needed in the subsequent sections of the paper. The main existence result is presented in Section 4 and an existence result concerning the extremal solutions is given in Section 5. Finally, a comparison of our AMDE with the ordinary differential equations along with a couple of examples are presented in Section 6.

2 Statement of the Problem

Let X be a real Banach space with a convenient norm $\|\cdot\|$ and let $x, y \in X$ be any two elements. Then the line segment $\overline{xy}$ in X is defined by

(2.1) $\overline{xy} = \{z \in X \mid z = x + r(y - x), 0 \leq r \leq 1\}$.

Let $x_0 \in X$ be a fixed point and $z \in X$. Then for any $x \in \overline{x_0 z}$, we define the sets S_x and $\overline{S}_x$ in X by

(2.2) $S_x = \{rx \mid -\infty < r < 1\}$,

and

(2.3) $\overline{S}_x = \{rx \mid -\infty < r \leq 1\}$.

Let $x_1, x_2 \in \overline{xy}$ be arbitrary. We say $x_1 < x_2$ if $S_{x_1} \subset S_{x_2}$, or equivalently, $\overline{x_0 x_1} \subset \overline{x_0 x_2}$. In this case we also write $x_2 > x_1$.

Let M denote the σ-algebra of all subsets of X such that (X, M) is a measurable space. Let $ca(X, M)$ be the space of all vector measures (real signed measures) and define a norm $|\cdot|$ on $ca(X, M)$ by

(2.4) $\|p\| = |p|(X)$,

where $|p|$ is a total variation measure of p and is given by

(2.5) $|p|(X) = \sup_{\sigma} \sum_{i=1}^{\infty} |p(E_i)|, \; E_i \subset X$,

where the supremum is taken over all possible partitions $\sigma = \{E_i : i \in \mathbb{N}\}$ of measurable subsets of X. It is known that $ca(X, M)$ is a Banach space with respect to the norm $\|\cdot\|$ given by (2.4).

Let μ be a σ-finite positive measure on X, and let $p \in ca(X, M)$. We say p is absolutely continuous with respect to the measure μ if $\mu(E) = 0$ implies $p(E) = 0$ for $E \in M$. In this case we also write $p << \mu$.

Let $x_0 \in X$ be fixed and let M_0 denote the σ- algebra on S_{x_0}. Let $z \in X$ be such that $z > x_0$ and let M_z denote the σ-algebra of all sets containing M_0 and the sets of the form S_x, $x \in \overline{x_0 z}$.

Given a $p \in ca(X, M)$ with $p << \mu$, consider the abstract measure differential equation (AMDE) with a linear perturbation of second type of the form

(2.6) $\dfrac{d}{d\mu}\Big[p(\overline{S}_x) - f(x, p(\overline{S}_x))\Big] = g(x, p(\overline{S}_x)) \quad$ a.e. $[\mu]$ on $\overline{x_0 z}$,

and

(2.7) $p(E) = q(E), \quad E \in M_0$,

where q is a given known vector measure, $\dfrac{dp}{d\mu}$ is a Radon-Nikodym derivative of p with respect to μ, $f : S_z \times \mathbb{R} \to \mathbb{R}$ is a continuous function with the map $E \mapsto [p(E) - f(x, p(E))] = \lambda(E)$ is an absolutely continuous measure w.r.t. the measure μ for each $x \in S_z$ and the function $g : S_z \times \mathbb{R} \to \mathbb{R}$ is such that the map $x \mapsto g(x, p(\overline{S}_x))$ is μ-integrable for each $p \in ca(S_z, M_z)$.

***Remark* 2.1** *Let $\lambda(E) = [p(E) - f(x, p(E))]$ for $x \in S_z$ and $E \in M_z$. If $p << \mu$ and $\lambda << \mu$, then $f(x, 0) = 0$ for each $x \in S_z$.*

***Definition* 2.1** Given an initial real measure q on M_0, a vector $p \in ca(S_z, M_z)$ $(z > x_0)$ is said to be a *solution* of AMDE (2.6)-(2.7) if

(i) $p(E) = q(E)$, $E \in M_0$,
(ii) $p << \mu$ on $\overline{x_0 z}$, and
(iii) p satisfies (2.6) a.e. $[\mu]$ on $\overline{x_0 z}$.

The following result from measure theory is often times used for transforming the abstract measure differential equation into an equivalent abstract measure integral equation.

***Theorem* 2.1** (Radon-Nikkodym theorem) *Let λ and μ be two σ-finite measures defined on a measurable space (X, M) such that $\lambda << \mu$. Then there exists a M-measurable function $f : X \to [0, \infty)$ such that*

(2.8) $\lambda(E) = \int_E f \, d\mu$

for any $E \in M$. The function f is unique upto the set of measure zero.

Note that the function f in the expression (2.8) is called the Radon-Nikodym derivative of the measure λ with respect to the measure μ and in this case we write

(2.9) $\dfrac{d\lambda}{d\mu} = f \quad$ a. e. $[\mu]$ on X.

The details of Radon-Nikodym derivative and its integral representation appear in Ruddin [22], Sharma [23, 24], Dhage [1] and the references therein.

***Remark* 2.2** *By an application of Radon-Nikodym theorem given in **Theorem 2.1**, the AMDE (2.6)-(2.7) is equivalent to the abstract measure integral equation (in short AMIE)*

(2.10) $p(E) = f(x, p(E)) + \int_E g(x, p(\overline{S}_x)) \, d\mu$,

if $E \in M_z$, $E \subset \overline{x_0 z}$. and

(2.11) $p(E) = q(E) \quad$ *if* $E \in M_0$.

A solution p of the AMDE (2.6)-(2.7) on $\overline{x_0 z}$ will be denoted by $p(\overline{S}_{x_0}, q)$.

Note that our AMDE (2.6)-(2.7) includes the abstract measure differential equation considered in the previous papers as special case. To see this, define $f(x, y) = 0$ for all $x \in \overline{x_0 z}$ and $y \in \mathbb{R}$, then AMDE (2.6)-(2.7) reduces to

(2.12) $\dfrac{dp}{d\mu} = g(x, p(\overline{S}_x))$ a.e. $[\mu]$ on $\overline{x_0 z}$.

and

(2.13) $p(E) = q(E)$ if $E \in M_0$.

The AMDE (2.11)-(2.12) has been studied in Joshi [19] and Dhage et. al [13] which further includes the abstract measure differential equations studied by Sharma [23, 24] and Leela [21] as special cases. Thus our AMDE (2.6)-(2.7) is more general and we claim that it is a new to the literature on measure differential equations. As a result the results of the present study are new and original contribution to the theory of nonlinear measure differential equations. In the following section we shall prove the existence and monotonicity theorems for AMDE (2.6)-(2.7).

3 Auxiliary Results

***Definition* 3.1** (Dhage [4, 5, 6, 8]) *An upper semi-continuous and nondecreasing function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is called a $\mathcal{D}$-function if $\psi(0) = 0$. The class of all $\mathcal{D}$-functions on $\mathbb{R}$+ is denoted by $\mathfrak{D}$.*

***Remark* 3.1** It is clear that if ϕ, ψ are $\mathcal{D}$-functions, then (i) $\phi + \psi$, (ii) $\lambda\phi$, $\lambda > 0$, and (iii) $\phi \circ \psi$ are also $\mathcal{D}$-functions, where " $\circ$ " is the composite operation of two functions on $\mathbb{R}_+$.

***Definition* 3.2** (Dhage [4, 5, 6, 8, 9]) *Let $\mathfrak{X}$ be a Banach space. An operator $\mathcal{T} : \mathfrak{X} \to \mathfrak{X}$ is called $\mathcal{D}$-Lipschitz if there exists a $\mathcal{D}$-function $\psi_{\mathcal{T}} \in \mathfrak{D}$ such that*

(3.1) $\|\mathcal{T}x - \mathcal{T}y\| \le \psi_{\mathcal{T}}(\|x - y\|)$

for all elements $x, y \in \mathfrak{X}$. If $\psi_{\mathcal{T}}(r) = k\,r$, $k > 0$, $\mathcal{T}$ is called a Lipschitz operator on $\mathfrak{X}$ with the Lipschitz constant k. Again, if $0 \le k < 1$, then $\mathcal{T}$ is called a contraction on $\mathfrak{X}$ with contraction constant k. Furthermore, if $\psi_{\mathcal{T}}(r) < r$ for $r > 0$, then $\mathcal{T}$ is called a nonlinear $\mathcal{D}$-contraction on $\mathfrak{X}$. The class of all $\mathcal{D}$-functions satisfying the condition of nonlinear $\mathcal{D}$-contraction is denoted by $\mathfrak{D}\mathfrak{N}$.

An operator $\mathcal{T} : \mathfrak{X} \to \mathfrak{X}$ is called compact if $\overline{\mathcal{T}(\mathfrak{X})}$ is a compact subset of $\mathfrak{X}$. $\mathcal{T}$ is called totally bounded if for any bounded subset S of $\mathfrak{X}$, $\mathcal{T}(S)$ is a totally bounded subset of $\mathfrak{X}$. T is called completely continuous if $\mathcal{T}$ is continuous and totally bounded on $\mathfrak{X}$. Every compact operator is totally bounded, but the converse may not be true, however, two notions are equivalent on bounded subsets of $\mathfrak{X}$. The details of different types of nonlinear contraction, compact and completely continuous operators appear in Granas and Dugundji [17].

To prove the main existence result of this section, we need the following variant of Krasnoselskii fixed point theorem proved in Dhage [4, 9, 11, 12] for the sum of two operators in a Banach space $\mathfrak{X}$. Also see Dhage [10] for related results and applications.

***Theorem* 3.1** *Let S be a closed convex and bounded subset of a Banach space $\mathfrak{X}$ and let $\mathcal{A} : \mathfrak{X} \to \mathfrak{X}$ and $\mathcal{B} : S \to \mathfrak{X}$ be two operators satisfying the following conditions.*

(a) $\mathcal{A}$ is nonlinear $\mathcal{D}$-contraction,

(b) $\mathcal{B}$ is completely continuous, and

(c) $\mathcal{A}x + \mathcal{B}y = x \implies x \in S$ for all $y \in S$.

Then the operator equation

(3.2) $\mathcal{A}x + \mathcal{B}x = x$

has a solution in S.

An interesting corollary to Theorem 3.1 in the applicable form is

***Corollary* 3.1** *Let S be a closed convex and bounded subset of a Banach space $\mathfrak{X}$ and let $\mathcal{A} : \mathfrak{X} \to \mathfrak{X}$ and $\mathcal{B} : S \to \mathfrak{X}$ be two operators satisfying the following conditions.*

(a) $\mathcal{A}$ is linear contraction,

(b) $\mathcal{B}$ is compact and continuous, and

(c) $\mathcal{A}x + \mathcal{B}y = x \implies x \in S$ for all $y \in S$.

Then the operator equation (3.2) *has a solution in S.*

In the following section we state our perturbed abstract measure differential equations to be discussed qualitatively in the subsequent part of this paper.

4 Existence Theorem

We need the following definition in the sequel.

***Definition* 4.1** *A function $\beta : S_z \times \mathbb{R} \to \mathbb{R}$ is called Carathéodory if*

(i) *$x \to \beta(x, y)$ is μ-measurable for each $y \in \mathbb{R}$, and*

(ii) *$y \to \beta(x, y)$ is continuous almost everywhere $[\mu]$ on $\overline{x_0 z}$.*

Further a Carathéodory function $\beta(x, y)$ is called $L^{\mu}_{\mathbb{R}}$-Carathéodory if

(iii) *there exists a μ-integrable function $h : S_z \to \mathbb{R}$ such that*

$$|\beta(x, y)| \le h(x) \quad a.\ e.\ [\mu], \quad x \in \overline{x_0 z}$$

for all $y \in \mathbb{R}$.

We consider the following set of assumptions.

(A_0) For any $z > x_0$, the σ-algebra M_z is compact with respect to the topology generated by the pseudo-metric d defined on M_z by

$$d(E_1, E_2) = |\mu|(E_1 \Delta E_2)$$

for all $E_1, E_2 \in M_z$.

(A_1) $\mu(\{x_0\}) = 0$.

(A_2) There exist real numbers $L > 0$ and $M > 0$ such that

$$|f(x, y_1) - f(x, y_2)| \le \frac{L\,|y_1 - y_2|}{M + |y_1 - y_2|} \quad \text{a.e. } [\mu], \quad x \in \overline{x_0 z},$$

for all $y_1, y_2 \in \mathbb{R}$. Moreover, $L \le M$.

(B_0) q is continuous on M_z with respect to the pseudo-metric d defined in (A_0).

(B_1) The function $g(x, y)$ is $L^{\mu}_{\mathbb{R}}$-Carathéodory.

***Theorem* 4.1** *Suppose that the assumptions $(A_0) - (A_2)$ and $(B_0) - (B_1)$ hold. Then the AMDE (2.6) - (2.7) has a solution on $\overline{x_0 z}$.*

Proof. By expressions (2.2) and (2.3), we have a decreasing sequence $\{r_n\}$ of positive real numbers such that $r_n \to 1$ as $n \to \infty$ and $S_{r_n x_0} \supset S_{x_0}$. Then, from hypothesis (H_1), it follows that

$$\bigcap_{r \to 1} \left(\overline{S}_{rx_0} - S_{x_0}\right) = \{x_0\}$$

and so,

$$\mu\left(\overline{S}_{rx_0} - S_{x_0}\right) = \mu(\{x_0\}) = 0 \quad \text{as} \quad r \to 1.$$

Therefore, we can choose a real number $r^* > 1$ such that $S_{r^* x_0} \supset S_{x_0}$ and

$$\mu(\overline{S}_{r^* x_0} - S_{x_0}) < 1 \quad \text{and} \quad \int\limits_{\overline{S}_{r^* x_0} - S_{x_0}} h(x)\, d\mu < 1.$$

Let $z^* = r^* x_0$ and consider the measure p_0 on M_{z^*} which is a continuous extension of the measure q on M_0 defined by

$$p_0(E) = \begin{cases} q(E) & \text{if} \quad E \in M_0, \\ 0 & \text{if} \quad E \notin M_0. \end{cases}$$

Now define a subset $S(\rho)$ of $ca(S_{z^*}, M_{z^*})$ by

(4.1) $S(\rho) = \{p \in ca(S_{z^*}, M_{z^*}) \mid \|p - p_0\| \le \rho\}$

where $\rho = F_0 + L + 1$. Clearly, $S(\rho)$ is a closed convex ball in $ca(S_{z^*}, M_{z^*})$ centred at p_0 of radius ρ and $q \in S(\rho)$.

Define two operators $\mathcal{A} : ca(S_{z^*}, M_{z^*}) \to ca(S_{z^*}, M_{z^*})$ and $\mathcal{B} : S(\rho) \to ca(S_{z^*}, M_{z^*})$ by

$$(4.2) \quad \mathcal{A}p(E) = \begin{cases} f(x, p(\overline{S}_x)), & \text{if } E \in M_z, E \subset \overline{x_0 z^*}, \\ 0, & \text{if } E \in M_0, \end{cases}$$

and

$$(4.3) \quad \mathcal{B}p(E) = \begin{cases} \int_E g(x, p(\overline{S}_x))\, d\mu & \text{if} \in M_{z^*}, \text{if } E \subset \overline{x_0 z^*}, \\ q(E) & \text{if } E \in M_0. \end{cases}$$

We shall show that the operators $\mathcal{A}$ and $\mathcal{B}$ satisfy all the conditions of ***Corollary* 2.1** on S.

Step I: First we show that $\mathcal{A}$ is a contraction on $ca(S_{z^*}, M_{z^*})$. Let $p_1, p_2 \in ca(S_{z^*}, M_{z^*})$ be arbitrary. Then by assumption (A_2),

$$\begin{aligned}|\mathcal{A}p_1(E) - \mathcal{A}p_2(E)| &= |f(x, p_1(E)) - f(x, p_2(E))| \\ &\leq \frac{L\,|p_1(E) - p_2(E)|}{M + |p_1(E) - p_2(E)|} \\ &\leq \frac{L\,|p_1 - p_2|(E)}{M + |p_1 - p_2|(E)}\end{aligned}$$

for all $E \in M_{z^*}$. Hence by definition of the norm in $ca(S_{z^*}, M_{z^*})$ one has

$$\|\mathcal{A}p_1 - \mathcal{A}p_2\| \leq \frac{L\,\|p_1 - p_2\|}{M + \|p_1 - p_2\|}$$

for all $p_1, p_2 \in ca(S_{z^*}, M_{z^*})$. As a result $\mathcal{A}$ is a nonlinear $\mathcal{D}$-contraction on $ca(S_{z^*}, M_{z^*})$ with the $\mathcal{D}$-function ψ given by $\psi(r) = \dfrac{Lr}{M + r}$.

Step II : We show that $\mathcal{B}$ is continuous on S. Let $\{p_n\}$ be a sequence of vector measures in S converging to a vector measure p. Then by dominated convergence theorem,

$$\begin{aligned}\lim_{n\to\infty} \mathcal{B}p_n(E) &= \lim_{n\to\infty} \int_E g(x, p_n(\overline{S}_x))\,d\mu \\ &= \int_E \left[\lim\nolimits_{n\to\infty}\ g(x, p_n(\overline{S}_x))\right]\,d\mu \\ &= \int_E g(x, p(\overline{S}_x))\,d\mu \\ &= \mathcal{B}p(E)\end{aligned}$$

for all $E \in M_{z^*}$, $E \subset \overline{x_0 z}$. Similarly, if $E \in M_{z^*}$, then

$$\lim_{n\to\infty} \mathcal{B}p_n(E) = q(E) = \mathcal{B}p(E),$$

and so $\mathcal{B}$ is a pointwise continuous operator on S.

Next we show that $\{\mathcal{B}p_n : n \in \mathbb{N}\}$ is a equi-continuous sequence in $ca(S_{z^*}, M_{z^*})$. Let $E_1, E_2 \in M_{z^*}$. Then there exist subsets $F_1, F_2 \in M_0$ and $G_1, G_2 \in M_z$, $G_1 \subset \overline{x_0 z^*}$, $G_2 \subset \overline{x_0 z^*}$ such that

$$E_1 = F_1 \cup G_1 \text{ with } F_1 \cap G_1 = \emptyset$$

and

$$E_2 = F_2 \cup G_2 \text{ with } F_2 \cap G_2 = \emptyset.$$

We know the identities

(4.4) $G_1 = (G_1 - G_2) \cup (G_2 \cap G_1)$,

and

(4.5) $G_2 = (G_2 - G_1) \cup (G_1 \cap G_2)$.

Therefore, we have

$$\mathcal{B}p_n(E_1) - \mathcal{B}p_n(E_2) \leq q(F_1) - q(F_2) + \int\limits_{G_1 - G_2} g(x, p_n(\overline{S}_n))d\mu + \int\limits_{G_2 - G_1} g(x, p_n(\overline{S}_x))d\mu.$$

Since $g(x, y)$ is L^l_μ- Carathéodory, we have that

$$\begin{aligned}|\mathcal{B}p_n(E_1) - \mathcal{B}p_n(E_2)| &\leq |q(F_1) - q(F_2)| + \int\limits_{G_1 \Delta G_2} |g(x, p_n(\overline{S}_x))| d\mu \\ &\leq |q(F_1) - q(F_2)| + \int\limits_{G_1 \Delta G_2} h(x)\,d\mu.\end{aligned}$$

Assume that

$$d(E_1, E_2) = |\mu|(E_1 \Delta E_2) \to 0.$$

Then we have that $E_1 \to E_2$. As a result $F_1 \to F_2$ and $|\mu|(G_1 \Delta G_2) \to 0$. As q is continuous on compact M_z , it is uniformly continuous and so

$$|\mathcal{B}p_n(E_1) - \mathcal{B}p_n(E_2)| \leq |q(F_1) - q(F_2)| + \int\limits_{G_1 \Delta G_2} h_r(x) d\mu$$

$\rightarrow 0$ as $E_1 \rightarrow E_2$

uniformly for all $n \in \mathbb{N}$. This shows that $\{\mathcal{B}p_n : n \in \mathbb{N}\}$ is a equi-continuous set in $ca(S_{z^*}, M_{z^*})$. As a result, $\{\mathcal{B}p_n\}$ converges to $\mathcal{B}p$ uniformly on M_{z^*} and a fortipri $\mathcal{B}$ is a continuous operator on $S(\rho)$.

Step III: Next we show that $\mathcal{B}(S)$ is a totally bounded set in $ca(S_{z^*}, M_{z^*})$, where $S = S(\rho)$. We shall show that the set is uniformly bounded and equi-continuous set in $ca(S_{z^*}, M_{z^*})$. Firstly, we show that $\mathcal{B}(S)$ is a uniformly bounded set in $ca(S_{z^*}, M_{z^*})$.

Let $\lambda \in \mathcal{B}(S)$ be an arbitrary element. Then, there is a member $p \in S$ such that $\lambda(E) = \mathcal{B}p(E)$ for all $E \in M_{z^*}$. Let $E \in M_{z^*}$. Then there exist two subsets $F \in M_0$ and $G \in M_{z^*}$, $G \subset \overline{x_0 z^*}$ such that

$E = F \cup G$ and $F \cap G = \phi$.

Hence by definition of $\mathcal{B}$,

$$
\begin{aligned}
(4.6)\quad |\lambda(E)| &= |\mathcal{B}p(E)| \\
&\leq |q(F)| + \int_G |g(x, p(\overline{S}_x))| d\mu \\
&\leq \|q\| + \int_G h(x) d\mu \\
&\leq \|q\| + \int_E h(x) d\mu \\
&= \|q\| + \|h\|_{L^l_\mu}
\end{aligned}
$$

for all $E \in M_{z^*}$.

From (4.6) it follows that

$$
\begin{aligned}
\|\lambda\| &= \|\mathcal{B}p\| \\
&= |\mathcal{B}p|(E) \\
&= \sup_{\sigma} \textstyle\sum_{i=1}^{\infty} |Bp(E_i)| \\
&= \|q\| + \|h\|_{L^l_\mu}
\end{aligned}
$$

for all $\lambda \in \mathcal{B}(S)$.

Hence the sequence $\mathcal{B}(S)$ is uniformly bounded set in $ca(S_{z^*}, M_{z^*})$.

Next we show that $\mathcal{B}(S)$ is a equi-continuous set of measures in $ca(S_{z^*}, M_{z^*})$. Let $E_1, E_2 \in M_{z^*}$. Then there exist subsets $F_1, F_2 \in M_0$ and $G_1, G_2 \in M_{z^*}$, $G_1 \subset \overline{x_0 z^*}$, $G_2 \subset \overline{x_0 z^*}$ such that

$E_1 = F_1 \cup G_1$ with $F_1 \cap G_1 = \emptyset$

and

$E_2 = F_2 \cup G_2$ with $F_2 \cap G_2 = \emptyset$.

We know the identities

(4.7) $G_1 = (G_1 - G_2) \cup (G_2 \cap G_1)$,

and

(4.8) $G_2 = (G_2 - G_1) \cup (G_1 \cap G_2)$.

Therefore, we have

$$|\lambda(E_1) - \lambda(E_2)| = |Bp(E_1) - Bp(E_2)| \leq |q(F_1) - q(F_2)| + \int\limits_{G_1 - G_2} |g(x, p(\overline{S}_n))| \, d\mu + \int\limits_{G_2 - G_1} |g(x, p(\overline{S}_x))| \, d\mu.$$

Since $g(x, y)$ is L^l_μ- Carathéodory, we have that

$$
\begin{aligned}
|\lambda(E_1) - \lambda(E_2)| &\leq |q(F_1) - q(F_2)| + \int\limits_{G_1 \Delta G_2} |g(x, p(\overline{S}_x))| \, d\mu \\
&\leq |q(F_1) - q(F_2)| + \int\limits_{G_1 \Delta G_2} h(x) \, d\mu.
\end{aligned}
$$

Assume that

$d(E_1, E_2) = |\mu|(E_1 \Delta E_2) \rightarrow 0$.

Then we have that $E_1 \rightarrow E_2$. As a result $F_1 \rightarrow F_2$ and $|\mu|(G_1 \Delta G_2) \rightarrow 0$. As q is continuous on compact M_{z^*}, it is uniformly continuous and so

$$|\lambda(E_1) - \lambda(E_2)| \leq |q(F_1) - q(F_2)| + \int\limits_{G_1 \Delta G_2} h_r(x) d\mu$$

$$\to 0 \text{ as } E_1 \to E_2$$

uniformly for all $\lambda \in \mathcal{B}(S)$. This shows that $\mathcal{B}(S)$ is a equi-continuous set in the banach space $ca(S_{z^*}, M_{z^*})$. Now an application of the Arzela-Ascolli theorem yields that $\mathcal{B}$ is a totally bounded operator on S. Now $\mathcal{B}$ is continuous and totally bounded, it is completely continuous operator on S.

Step IV: Finally, we show that the hypothesis (c) of ***Theorem* 3.1** is satisfied. Let $p \in S$ be arbitrary and let there is an element $u \in ca(S_{z^*}, M_{z^*})$ such that $\mathcal{A}u + \mathcal{B}p = u$. We show that $u \in S$. Now, by definitions of the operators $\mathcal{A}$ and $\mathcal{B}$,

$$u(E) = \begin{cases} f(x, u(E)) + \int_E g(x, p(\overline{S}_x))\, d\mu, & \text{if } E \in M_z,\ E \subset \overline{x_0 z^*} \\ q(E), & \text{if } E \in M_0. \end{cases}$$

for all $E \in M_{z^*}$.

If $E \in M_{z^*}$, then there exist sets $F \in M_0$ and $G \in M_{z^*}$, $G \subset \overline{x_0 z^*}$ such that $E = F \cup G$ and $F \cap G = \emptyset$. Then we have

$$u(E) = q(F) + f(x, u(G)) + \int_G g(x, p(\overline{S}_x))\, d\mu.$$

Hence,

$$\begin{aligned} |u(E) - p_0(E)| \leq& \left| f(x, u(G)) - f(x, 0) \right| + \int_G |g(x, p(\overline{S}_x))|\, d\mu \\ \leq& \frac{L|u(G)|}{M + |u(G)|} + \int_G h(x)\, d\mu \\ \leq& L + \int_{\overline{x_0 z^*}} h(x)\, d\mu \\ \leq& L + 1 \\ =& \rho \end{aligned}$$

which further implies that

$$\|u - p_0\| \leq L + 1 = \rho.$$

As a result, we have $u \in S(\rho)$ and so hypothesis (c) of ***Theorem* 3.1** is satisfied. In consequence, the operator equation $\mathcal{A}p(E) + \mathcal{B}p(E) = p(E)$ has a solution $p(S_{x_0}, q)$ in $ca(S_{z^*}, M_{z^*})$. This further implies that the AMDE (2.6)-(2.7) has a solution on $\overline{x_0 z}$. This completes the proof.

5 Existence of Extremal Solutions

In this section we prove the existence of the extremal solutions for the AMDE (2.6)-(2.7) on $\overline{x_0 z}$ under certain monotonicity conditions. We define an order relation $\preceq$ in $ca(S_z, M_z)$ with the help of the cone K in $ca(S_z, M_z)$ given by

(5.1) $K = \{p \in ca(S_z, M_z) \mid p(E) \geq 0 \text{ for all } E \in M_z\}.$

Thus for any $p_1, p_2 \in ca(S_z, M_z)$, one has

(5.2) $p_1 \preceq p_2 \iff p_2 - p_1 \in K$

or, equivalently,

$$p_1 \preceq p_2 \iff p_1(E) \leq p_2(E)$$

for all $E \in M_z$.

A cone K in $ca(S_z, M_z)$ is called normal if the norm is semi-monotone on K. The details of different properties of cones in Banach spaces appear in Heikkilä and Lakshmikantham [18].

The following lemma follows immediately from the definition of the cone K in $ca(S_z, M_z)$.

***Lemma* 5.1** *The cone K is normal in the Banach space $ca(S_z, M_z)$.*

Proof. To finish, it is enough to prove that the norm $\|\cdot\|$ is semi-monotone on K. Let $p_1, p_2 \in K$ be such that $p_1 \preceq p_2$ on M_z . Then we have

$$0 \leq p_1(E) \leq p_2(E)$$

for all $E \in M_z$. Now, for a countable partition $\sigma = \{E_n : n \in \mathbb{N}\}$ of measurable subsets of S_z, by definition of the norm in $ca(S_z, M_z)$, one has

$$\|p_1\| = |p_1|(S_z) = \sup_\sigma \sum_{i=1}^{\infty} |p_1(E_i)| = \sup_\sigma \sum_{i=1}^{\infty} p_1(E_i)$$

$$\leq \sup_{\sigma} \sum_{i=1}^{\infty} p_2(E_i) = \sup_{\sigma} \sum_{i=1}^{\infty} |p_2(E_i)| = |p_2|(S_z) = \|p_2\|.$$

This shows that $\|\cdot\|$ is semi-monotone on K and consequently the order cone K is normal in $ca(S_z, M_z)$. The proof of the lemma is complete.

We need the following fixed point theorem of Dhage [5] involving the sum of two operators in a ordered Banach space.

***Theorem* 5.1** *Let K be a cone in a real Banach space $\mathfrak{X}$ and let $\mathcal{A}, \mathcal{B} : \mathfrak{X} \to \mathfrak{X}$ be two nondecreasing operators such that*

(a) *$\mathcal{A}$ is linear contraction,*

(b) *$\mathcal{B}$ is completely continuous, and*

(c) *there exist elements $u, v \in \mathfrak{X}$ such that $u \leq v$ satisfying $u \leq \mathcal{A}u + \mathcal{B}u$ and $\mathcal{A}v + \mathcal{B}v \leq v$.*

Further if the cone K is normal, then the operator equation $\mathcal{A}x + \mathcal{B}x = x$ has a minimal and a maximal solution in $[u, v]$.

We need the following definitions in the sequel.

***Definition* 5.1** *A vector measure $u \in ca(S_z, M_z)$ is called a lower solution of AMDE (2.6)-(2.7) if*

$$\frac{d}{d\mu}\Big[u(\overline{S}_x) - f(x, u(\overline{S}_x))\Big] \leq g(x, u(\overline{S}_x)) \text{ a.e. } [\mu] \text{ on } \overline{x_0 z},$$

and

$$u(E) \leq q(E), \quad E \in M_0.$$

Similarly, a vector measure $v \in ca(S_z, M_z)$ is called an upper solution to AMDE (2.6)-(2.7) if

$$\frac{d}{d\mu}\Big[v(\overline{S}_x) - f(x, v(\overline{S}_x))\Big] \geq g(x, v(\overline{S}_x)) \text{ a.e. } [\mu] \text{ on } \overline{x_0 z},$$

and

$$v(E) \geq q(E), \quad E \in M_0.$$

A vector measure $p \in ca(S_z, M_z)$ is a solution to AMDE (2.6)-(2.7) if it is upper as well as lower solution to AMDE (2.6)-(2.7) on $\overline{x_0 z}$.

***Definition* 5.2** *A solution p_M is called a maximal solution for the AMDE (2.6)-(2.7) if for any other solution $p(\overline{S}_{x_0}, q)$ of the AMDE (2.6)-(2.7) we have that*

$$p(E) \leq p_M(E) \quad \forall E \in M_z.$$

Similarly, a minimal solution $p_m(\overline{S}_{x0}, q)$ for the AMDE (2.6)-(2.7) is defined on $\overline{x_0 z}$.

We consider the following assumptions:

(C_1) The functions $f(x, y)$ and $g(x, y)$ are nondecreasing in y a.e. $[\mu]$ for $x \in \overline{x_0 z}$.

(C_3) AMDE (2.6)-(2.7) has a lower solution u and an upper solution v such that $u \leq v$ on M_z.

***Theorem* 5.2** *Suppose that the assumptions (A_0) - (A_2), (B_1)-(B_2) and (C_1)-(C_2) hold. Then the AMDE (2.6)-(2.7) has a minimal and a maximal solution defined on $\overline{x_0 z}$.*

Proof. Now, AMDE (2.6)-(2.7) is equivalent to the abstract measure integral equation (in short AMIE)

$$(5.3)\quad p(E) = f(x, p(E))\, d\mu + \int_E g(x, p(\overline{S}_x))\, d\mu, \quad E \in M_z, E \subset \overline{x_0 z},$$

and

$$(5.4)\quad p(E) = q(E), \quad E \in M_0.$$

Define two operators $\mathcal{A}, \mathcal{B} : [u, v] \to ca(S_z, M_z)$ by

$$(5.5)\quad \mathcal{A}p(E) = \begin{cases} f(x, p(E)) & \text{if } E \in M_z,\ E \subset \overline{x_0 z}, \\ 0 & \text{if } E \in M_0. \end{cases}$$

and

$$(5.6)\quad \mathcal{B}p(E) = \begin{cases} \int_E g(x, p(\overline{S}_x))d\mu, & \text{if } E \in M_z,\ E \subset \overline{x_0 z}, \\ q(E) & \text{if } E \in M_0. \end{cases}$$

Then the AMIE (2.6)-(2.7) is equivalent to the operator equation

(5.7) $\mathcal{A}p(E) + \mathcal{B}p(E) = p(E), \quad E \in M_z.$

We shall show that the operators $\mathcal{A}$ and $\mathcal{B}$ satisfy all the conditions of ***Theorem* 5.1** on $ca(S_z, M_z)$. Since μ is a positive measure , from assumption (C_1) it follows that $\mathcal{A}$ and $\mathcal{B}$ are nondecreasing operators on $ca(S_z, M_z)$. To show this , let $p_1, p_2 \in ca(S_z, M_z)$ be such that $p_1 \leq p_2$ on M_z. From hypothesis (C_2), it follows that

$$\begin{aligned}\mathcal{A}p_1(E) &= f(x, p_1(E)\, d\mu \\ &\leq f(x, p_2(E))\, d\mu \\ &= \mathcal{A}p_2(E)\end{aligned}$$

for all $E \in M_z$, $E \subset \overline{x_0 z}$ and

$$\mathcal{A}p_1(E) = 0 = \mathcal{A}p_2(E)$$

for $E \in M_0$. Hence $\mathcal{A}$ is nondecreasing on $ca(S_z, M_z)$.

Similarly, we have

$$\begin{aligned}\mathcal{B}p_1(E) &= \int_E g(x, p_1(\overline{S}_x))\, d\mu \\ &\leq \int_E g(x, p_2(\overline{S}_x))\, d\mu \\ &= \mathcal{B}p_2(E)\end{aligned}$$

for all $E \in M_z$, $E \subset \overline{x_0 z}$. Again if $E \in M_0$, then

$$\mathcal{B}p_1(E) = q(E) = \mathcal{B}p_2(E).$$

Therefore, the operator $\mathcal{B}$ is also nondecreasing on $ca(S_z, M_z)$. Now it can be shown as in the proof of ***Theorem* 4.1** that the operators $\mathcal{A}$ is a nonlinear $\mathcal{D}$-contraction on $ca(S_z, M_z)$ with the $\mathcal{D}$-function ψ given by $\psi(r) = \frac{Lr}{M+r}$ and the operator $\mathcal{B}$ is completely continuous on S. Since u is a lower solution of AMDE (2.6)-(2.7), we have

(5.8) $u(E) \leq f(x, u(E)) + \int_E g(x, u(\overline{S}_x))d\mu, \quad E \in M_z,\ E \subset \overline{x_0 z},$

and

(5.9) $u(E) \leq q(E), \quad E \in M_0.$

From (6.8) and (6.9) it follows that

$$u(E) \leq \mathcal{A}u(E) + \mathcal{B}u(E) \quad \text{if } E \in M_z$$

and so, $u \leq \mathcal{A}u + \mathcal{B}u$. Similarly since $v \in ca(S_z, M_z)$ is an upper solution of AMDE (2.6)-(2.7), it can be proved that $\mathcal{A}v(E) + \mathcal{B}v(E) \leq v(E)$ for all $E \in M_z$ and consequently $\mathcal{A}v + \mathcal{B}v \leq v$ on M_z. Thus hypotheses (a)-(c) of ***Theorem* 5.1** are satisfied.

Thus the operators $\mathcal{A}$ and $\mathcal{B}$ satisfy all the conditions of ***Theorem* 5.1** and so an application of it yields that the operator equation $\mathcal{A}p + \mathcal{B}p = p$ has a maximal and a minimal solution in $[u, v]$. This further implies that AMDE (2.6)-(2.7) has a maximal and a minimal solution on $\overline{x_0 z}$. This completes the proof.

6 Special Case

In this section it is shown that, in a certain situation, the AMDE (2.6)-(2.7) reduces to an ordinary perturbed differential equation, viz.,

(6.1) $$\frac{d}{dx}\Big[y(x) - f(x, y(x))\Big] = g(x, y(x)), \quad x \geq x_0,$$
$$y(x_0) = y_0,$$

where f is a continuous real-valued function on $[x_0, x_0 + T] \times \mathbb{R}$, and the function g satisfies Carathéodory conditions on $[x_0, x_0 + T] \times \mathbb{R}$. Note that the hybrid ordinary differential equation (6.1) is discussed first time in Dhage and Jadhav [15] for some basic results related to its solution.

Let $X = \mathbb{R}$, $\mu = m$, the Lebesgue measure on $\mathbb{R}$, $\overline{S}_x = (-\infty, x]$, $x \in \mathbb{R}$, and q a given real Borel measure on M_0. Then equations (2.6)-(2.7) take the form

(6.2) $$\frac{d}{dm}\Big[p((-\infty, x]) - f(x, p((-\infty, x]))\Big] = g(x, p((-\infty, x])),$$
$$p(E) = q(E), \quad E \in M_0.$$

It will now be shown that, the equations (6.1) and (6.2) are equivalent in the sense of the following theorem.

***Theorem* 6.1** *Let $q : M_0 \to \mathbb{R}$ be a given initial measure such that $q(E) = 0$ for all $E \in M_0$ and $q(\{x_0\}) = 0$. Then*

(a) to each solution $p = p(\overline{S}_{x_0}, q)$ of (6.2) existing on $[x_0, x_1)$, there corresponds a solution y of (6.1) satisfying $y(x_0) = y_0$.
(b) Conversely, to every solution $y(x)$ of (6.1), there corresponds a solution $p(\overline{S}_{x_0}, q)$, of (6.2) existing on $[x_0, x_1)$ with a suitable initial measure q provided f satisfies the relation $f(x_0, 0) = 0$.

Proof. (a) Let $p = p(\overline{S}_{x_0}, q)$ be a solution of (6.2), existing on $[x_0, x_1)$. Define a real Borel measure p_1 on $\mathbb{R}$ as follows.

$$(6.3)\quad p_1((-\infty, x)) = \begin{cases} 0, & \text{if } x \leq x_0, \\ p((-\infty, x]) - p((-\infty, x_0]), & \text{if } x_0 < x < x_1 \\ p((-\infty, x_1)), & \text{if } x \geq x_1, \end{cases}$$

and

$$p_1(-\infty, x_0]) = p(-\infty, x_0]).$$

Define the functions $y_1(x)$ and $y(x)$ by

$$(6.4)\quad y_1(x) = p_1((-\infty, x)),\ x \in \mathbb{R}$$
$$y(x) = y_1(x) + p((-\infty, x_0]), x \in [x_0, x_1).$$

The condition $q(\{x_0\}) = 0$, the definition of the solution p, and the definitions of $y(x)$ imply that

$$p_1(\{x_0\}) = p(\{x_0\}) = 0.$$

Now for each $x \in [x_0, x_1)$ we obtain from (6.2) and the definition of $y(x)$

$$\begin{aligned}(6.5)\quad y(x) &= y_1(x) + p((-\infty, x_0]) \\ &= p_1((-\infty, x)) + p((-\infty, x_0]) \\ &= p(\overline{S}_x).\end{aligned}$$

Since p is a solution of (6.2) we have $p << m$ on $[x_0, x_1)$. Hence $y(x)$ is absolutely continuous on $[x_0, x_1)$. The details concerning these arguments appear in Rudin [22, pages 163-165]. This shows that $y'(x)$ exists a.e. on $[x_0, x_1)$. Now for each $x \in [x_0, x_1)$, we have, by virtue of (6.3) and (6.4)

$$p([x_0, x]) - f(x, p([x_0, x]) = \int_{[x_0,x]} \tfrac{d}{dm}[p((-\infty, t]) - f(t, p((-\infty, t]))]\, dm.$$

Therefore,

$$[p((-\infty, x]) - p((-\infty, x_0]) - f(x, p((-\infty, x]) - p((-\infty, x_0]))$$
$$= \int_{[x_0,x]} \tfrac{d}{dm}[p((-\infty, t]) - f(t, p((-\infty, t]))]\, dm.$$

This further implies that

$$p(\overline{S}_x) - f(x, p(\overline{S}_x)) = p(\overline{S}_{x_0}) + \int_{x_0}^{x} g(t, p(\overline{S}_t))\, dm.$$

That is,

$$y(x) - f(x, y(x)) = y(x_0) + \int_{x_0}^{x} g(t, y(t))dt.$$

Hence,

$$\frac{d}{dx}[y(x) - f(x, y(x))] = g(x, y(x)) \quad \text{a.e on } [x_0, x_1).$$

This proves that $y(x)$ is a solution of (6.1) on $[x_0, x_1)$ satisfying

$$y(x_0) = y_0.$$

(b) Conversely, suppose that $y(x)$ be a solution of (6.1) existing on $[x_0, x_1]$. Then, y is absolutely continuous on $[x_0, x_1]$. Now, corresponding to the absolutely continuous function $y(x)$ which is a solution of (6.1) on $[x_0, x_1)$, we can construct a absolutely continuous real Borel measure p on M_{x_1} such that,

$$(6.6)\quad p(E) = 0, \quad \text{for all } E \in M_0,$$
$$p(\overline{S}_x) = y(x), \text{ if } x \in [x_0, x_1).$$

The details concerning these arguments appear in Rudin [22, pages 163-165]. Since $y(x)$ is a solution of (6.1) we have for $x \in [x_0, x_1)$,

$$y(x) - f(x, y(x)) = y(x_0) + f(x_0, y(x_0)) + + \int_{x_0}^{x} f(t, y(t))\, dt.$$

Now, $y(x_0) = p(S_{x_0}) = 0$ and so, $f(x_0, y_0) = 0$. Hence by (6.6) it follows that

$$[p(\overline{S}_x) - p(\overline{S}_{x_0})] - f(x, p(\overline{S}_x) - p(\overline{S}_{x_0})) = \int_{[x_0,x]} g(t, p(\overline{S}_t))\, dm.$$

That is,

$$p([x_0, x]) = f(x, p([x_0, x])) + \int_{[x_0,x]} f(t, p(\overline{S}_t))\, dm.$$

In general, if $E \in M_{x_1}$, $E \subset \overline{x_0 x_1}$, then

$$p(E) = f(x, p(E)) + \int_E g(x, p((-\infty, x]))\, dm.$$

By definition of Radon-Nykodym derivative, we obtain

$$\frac{d}{dm}\Big[p((-\infty, x]) - f(x, p((-\infty, x]))\Big] = g(x, p((-\infty, x])) \ \ a.e.\ [\mu] \ \text{ on } \overline{x_0 z}$$

$$p(E) = 0 \ \text{ for } \ E \in M_0.$$

This shows that p is a solution of (6.2) on $[x_0, x_1)$ and the proof of (b) is complete.

***Remark* 6.1** ***Theorem* 6.1** *shows that our results for the equation* (6.2) *are general in the sense that they include the corresponding results for the equation* (6.1).

***Example* 6.1** *Given a* $p \in \mathrm{ca}(X, M)$ *with* $p << \mu$, *consider the abstract measure differential equation (AMDE) with a linear perturbation of second type of the form*

$$(6.7)\quad \frac{d}{d\mu}\left[p(\overline{S}_x) - \frac{|p(\overline{S}_x)|}{1 + |p(\overline{S}_x)|}\right] = \frac{\ln(1 + |p(\overline{S}_x)|)}{1 + |p(\overline{S}_x)|} \quad a.e.\ [\mu] \text{ on } \overline{x_0 z}.$$

and

$$(6.8)\quad p(\overline{S}_{x_0})) = 1,$$

where $\dfrac{dp}{d\mu}$ *is a Radon-Nikodym derivative of p with respect to* μ.

Here, $f(x, y) = \dfrac{|y|}{1 + |y|}$ *and* $g(x, y) = \dfrac{\ln(1 + |y|)}{1 + |y|}$ *for all* $x \in \overline{x_0 z}$ *and* $y \in \mathbb{R}$. *Clearly, the function* $(x, y) \mapsto \dfrac{|y|}{1 + |y|} = f(x, y)$ *is continuous and bounded on* $S_z \times \mathbb{R}$ *with ound* $M_f = 1$. *Also, after simple computation it can be shown that f satisfies the assumption* (A_3) *with* $L = 1$ *and* $M = 1$. *Next, the function g is continuous and bounded on* $S_z \times \mathbb{R}$ *with bound* $M_g = 1$. *Therefore, if the assumptions* (A_0)-(A_1) *hold, then the AMDE* (6.7) - (6.8) *has a solution defined on* $\overline{x_0 z}$.

***Example* 6.2** *Given a* $p \in \mathrm{ca}(X, M)$ *with* $p << \mu$, *consider the abstract measure differential equation (AMDE) with a linear perturbation of second type of the form*

$$(6.9)\quad \frac{d}{d\mu}\Big[p(\overline{S}_x) - \gamma \sin p(\overline{S}_x)\Big] = \frac{1 + |p(\overline{S}_x)|}{2 + p^2(\overline{S}_x)} \quad a.e.\ [\mu] \text{ on } \overline{x_0 z}.$$

and

$$(6.10)\quad p(\overline{S}_{x_0})) = 1,$$

where $\dfrac{dp}{d\mu}$ *is a Radon-Nikodym derivative of p with respect to* μ *and* $0 \le \gamma < 1$.

Here, $f(x, y) = \gamma \sin y$ and $g(x, y) = \dfrac{1 + |y|}{2 + y^2}$ for all $x \in \overline{x_0 z}$ and $y \in \mathbb{R}$. Clearly, the function $(x, y) \mapsto \gamma \sin y = f(x, y)$ is continuous and bounded on $S_z \times \mathbb{R}$ with bound $M_f = 1$. Also, after simple computation it can be shown that f satisfies the assumption (A_3) with $L = \gamma M \le M$. Next, the function g is continuous and bounded on $S_z \times \mathbb{R}$ with bound $M_g = 1$. Therefore, if the assumptions (A_0)-(A_1) hold, then the AMDE (6.9) - (6.10) has a solution defined on $\overline{x_0 z}$.

***Remark* 6.2** If we define the initial vector measure q on M_0 by

$$q(\overline{S}_{x_0}) = \alpha, \ \text{ and } \ q(E) = 0 \ \text{ if } \ E \ne \overline{S}_{x_0},$$

where α is a real number and $f \equiv 0$ on $S_z \times \mathbb{R}$, then the equations (2.6)-(2.7) is reduced to the form

$$(6.11)\quad \frac{dp}{d\mu} = g(x, p(\overline{S}_x)), \quad p(\overline{S}_{x_0}) = \alpha$$

which is the AMDE studied in Sharma [23, 24]. Thus our existence results of this paper include as particular cases, the results in Sharma [23, 24] under weaker Carathoédory conditions.

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(Dedicated to Honor Dr. R. C. Singh Chandel on His 75^th^ Birth Anniversary Celebrations)

GROWTH ANALYSIS OF COMPOSITE *P*-ADIC ENTIRE FUNCTIONS FROM THE VIEW POINT OF RELATIVE (p, q)-TH ORDER AND RELATIVE (p, q)-TH TYPE

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Abstract

Suppose $\mathbb{K}$ be a complete ultrametric algebraically closed field and suppose $\mathcal{A}(\mathbb{K})$ be the $\mathbb{K}$-algebra of entire functions on $\mathbb{K}$. In this paper we study some growth properties of composite p-adic entire functions on the basis of their relative (p, q)-th order, relative (p, q)-th type and relative (p, q)-th weak type.

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1 Introduction and Definitions

Let us consider $\mathbb{K}$ be an algebraically closed field of characteristic 0, complete with respect to a p-adic absolute value $|\cdot|$ (example $\mathbb{C}_p$). For any $\alpha \in \mathbb{K}$ and $R \in]0, +\infty[$, the closed disk $\{x \in \mathbb{K} : |x-\alpha| \le R\}$ and the open disk $\{x \in \mathbb{K} : |x-\alpha| < R\}$ are denoted by $d(\alpha, R)$ and $d(\alpha, R^-)$ respectively. Also $C(\alpha, r)$ denotes the circle $\{x \in \mathbb{K} : |x - a| = r\}$. Moreover $\mathcal{A}(\mathbb{K})$ represent the $\mathbb{K}$-algebra of analytic functions in $\mathbb{K}$ i.e. the set of power series with an infinite radius of convergence. For the most comprehensive study of analytic functions inside a disk or in the whole field $\mathbb{K}$, we refer the reader to the books [11, 12, 15, 17]. During the last several years the ideas of p-adic analysis have been studied from different aspects and many important results were gained (see [2] to [10], [13, 14]).

Let $f \in \mathcal{A}(\mathbb{K})$ and $r > 0$, then we denote by $|f|(r)$ the number $\sup\{|f(x)| : |x| = r\}$ where $|\cdot|(r)$ is a multiplicative norm on $\mathcal{A}(\mathbb{K})$. Moreover, if f is not a constant, the $|f|(r)$ is strictly increasing function of r and tends to $+\infty$ with r therefore there exists its inverse function $\widehat{|f|} : (|f(0)|, \infty) \to (0, \infty)$ with $\lim\limits_{s\to\infty} \widehat{|f|}(s) = \infty$.

For $x \in [0, \infty)$ and $k \in \mathbb{N}$, we define $\log^{[k]} x = \log(\log^{[k-1]} x)$ and $\exp^{[k]} x = \exp(\exp^{[k-1]} x)$ where $\mathbb{N}$ be the set of all positive integers. We also denote $\log^{[0]} x = x$ and $\exp^{[0]} x = x$. Throughout the paper, log denotes the Neperian logarithm. Further we assume that throughout the present paper p, q, m, n and l always denote positive integers. Now taking this into account the (p, q)-th order and (p, q)-th lower order of an entire function $f \in \mathcal{A}(\mathbb{K})$ are defined as follows:

***Definition* 1.1** [5] *Let $f \in \mathcal{A}(\mathbb{K})$ and $p, q \in \mathbb{N}$. Then the (p, q)-th order and (p, q)-th lower order of f are respectively defined as:*

$$\rho^{(p,q)}(f) = \limsup_{r\to\infty} \frac{\log^{[p]} |f|(r)}{\log^{[q]} r} \text{ and } \lambda^{(p,q)}(f) = \liminf_{r\to\infty} \frac{\log^{[p]} |f|(r)}{\log^{[q]} r}.$$

Definition 1.1 avoids the restriction $p \ge q$ of the original definition of (p, q)-th order (respectively (p, q)-th lower order) of entire functions introduced by Juneja et al. [16] *in complex context.*

When $q = 1$, we get the definitions of generalized order and generalized lower order of an entire function $f \in \mathcal{A}(\mathbb{K})$ which symbolize as $\rho^{(p)}(f)$ and $\lambda^{(p)}(f)$ respectively. If $p = 2$ and $q = 1$ then we write $\rho^{(2,1)}(f) = \rho(f)$ and $\lambda^{(2,1)}(f) = \lambda(f)$ where $\rho(f)$ and $\lambda(f)$ are respectively known as order and lower order of $f \in \mathcal{A}(\mathbb{K})$ introduced by Boussaf et al. [2].

In this connection we just introduce the following definition:

***Definition* 1.2** *An entire function $f \in \mathcal{A}(\mathbb{K})$ is said to have index-pair (p,q) if $b < \rho^{(p,q)}(f) < \infty$ and $\rho^{(p-1,q-1)}(f)$ is not a nonzero finite number, where $b = 1$ if $p = q$ and $b = 0$ for otherwise. Moreover if $0 < \rho^{(p,q)}(f) < \infty$, then*

$$\begin{cases} \rho^{(p-n,q)}(f) = \infty & \text{for } n < p, \\ \rho^{(p,q-n)}(f) = 0 & \text{for } n < q, \\ \rho^{(p+n,q+n)}(f) = 1 & \text{for } n = 1, 2, \cdots \end{cases}.$$

Similarly for $0 < \lambda^{(p,q)}(f) < \infty$, one can easily verify that

$$\begin{cases} \lambda^{(p-n,q)}(f) = \infty & \text{for } n < p, \\ \lambda^{(p,q-n)}(f) = 0 & \text{for } n < q, \\ \lambda^{(p+n,q+n)}(f) = 1 & \text{for } n = 1, 2, \cdots \end{cases}.$$

An entire function $f \in \mathcal{A}(\mathbb{K})$ of index-pair (p,q) is said to be of regular (p,q)-th growth if its (p,q)-th order coincides with its (p,q)-th lower order, otherwise f is said to be of irregular (p,q)-th growth.

Next, to compare the growth of entire functions on $\mathbb{K}$ having the same (p,q)-th order, we give the definitions of (p,q)-th type and (p,q)-th lower type in the following manner :

***Definition* 1.3** [5] *Let $f \in \mathcal{A}(\mathbb{K})$. The (p,q)-th type $\sigma^{(p,q)}(f)$ and the (p,q)-th lower type $\overline{\sigma}^{(p,q)}(f)$ of f having finite positive (p,q)-th order $\rho^{(p,q)}(f)$ $(0 < \rho^{(p,q)}(f) < \infty)$ are defined as:*

$$\sigma^{(p,q)}(f) = \limsup_{r\to\infty} \frac{\log^{[p-1]} |f|(r)}{(\log^{[q-1]} r)^{\rho^{(p,q)}(f)}} \text{ and } \overline{\sigma}^{(p,q)}(f) = \liminf_{r\to\infty} \frac{\log^{[p-1]} |f|(r)}{(\log^{[q-1]} r)^{\rho^{(p,q)}(f)}}.$$

***Remark* 1.1** *If $p = 2$ and $q = 1$ then we write $\sigma^{(p,q)}(f) = \sigma(f)$ where $\sigma(f)$ is known as type of $f \in \mathcal{A}(\mathbb{K})$ introduced by Boussaf et al.* [2].

Likewise, to compare the growth of entire functions on $\mathbb{K}$ having the same (p,q)-th lower order, one can also introduce the concepts of (p,q)-th weak type in the following manner :

***Definition* 1.4** [5] *Let $f \in \mathcal{A}(\mathbb{K})$. The (p,q)-th weak type $\tau^{(p,q)}(f)$ of f having finite positive (p,q)-th lower order $\lambda^{(p,q)}(f)$ $(0 < \lambda^{(p,q)}(f) < \infty)$ is defined as :*

$$\tau^{(p,q)}(f) = \liminf_{r\to\infty} \frac{\log^{[p-1]} |f|(r)}{(\log^{[q-1]} r)^{\lambda^{(p,q)}(f)}}.$$

Similarly one may define the growth indicator $\overline{\tau}^{(p,q)}(f)$ of an entire function $f \in \mathcal{A}(\mathbb{K})$ in the following way :

$$\overline{\tau}^{(p,q)}(f) = \limsup_{r\to\infty} \frac{\log^{[p-1]} |f|(r)}{(\log^{[q-1]} r)^{\lambda^{(p,q)}(f)}},\ 0 < \lambda^{(p,q)}(f) < \infty.$$

The notion of relative order was first introduced by Bernal [1]. *In order to make some progress in the study of p-adic analysis, recently Biswas* [4] *introduce the definition of relative order $\rho_g(f)$ and relative lower order $\lambda_g(f)$ of entire function $f \in \mathcal{A}(\mathbb{K})$ with respect to another entire function $g \in \mathcal{A}(\mathbb{K})$ in the following way:*

$$\rho_g(f) = \limsup_{r\to\infty} \frac{\log \widehat{|g|}(|f|(r))}{\log r} \text{ and } \lambda_g(f) = \liminf_{r\to\infty} \frac{\log \widehat{|g|}(|f|(r))}{\log r}.$$

Further the function $f \in \mathcal{A}(\mathbb{K})$, for which relative order and relative lower order with respect to another function $g \in \mathcal{A}(\mathbb{K})$ are the same is called a function of regular relative growth with respect to g otherwise, f is said to be irregular relative growth.with respect to g.

In the case of relative order, it therefore seems reasonable to define suitably the relative (p,q)-th order of entire function belonging to $\mathcal{A}(\mathbb{K})$. With this in view one may introduce the definition of relative (p,q)-th order $\rho_g^{(p,q)}(f)$ and relative (p,q)-th lower order $\lambda_g^{(p,q)}(f)$ of an entire function $f \in \mathcal{A}(\mathbb{K})$ with respect to another entire function $g \in \mathcal{A}(\mathbb{K})$, in the light of index-pair which are as follows:

***Definition* 1.5** [5] *Let f, $g \in \mathcal{A}(\mathbb{K})$. Also let the index-pairs of f and g are (m,q) and (m,p), respectively. Then the relative (p,q)-th order $\rho_g^{(p,q)}(f)$ and relative (p,q)-th lower order $\lambda_g^{(p,q)}(f)$ of f with respect to g are defined as*

$$\rho_g^{(p,q)}(f) = \limsup_{r\to\infty} \frac{\log^{[p]} \widehat{|g|}(|f|(r))}{\log^{[q]} r} \text{ and } \lambda_g^{(p,q)}(f) = \liminf_{r\to\infty} \frac{\log^{[p]} \widehat{|g|}(|f|(r))}{\log^{[q]} r}.$$

In order to refine the above growth scale, now we introduce the definitions of an another growth indicator, called relative (p, q)-th type and relative (p, q)-th lower type respectively of entire function belonging to $\mathcal{A}(\mathbb{K})$ with respect to another entire function belonging to $\mathcal{A}(\mathbb{K})$ in the light of their index-pair which are as follows:

***Definition* 1.6** [5] *Let $f, g \in \mathcal{A}(\mathbb{K})$. Also let the index-pairs of f and g are (m, q) and (m, p), respectively. The relative (p, q)-th type and relative (p, q)-th lower type of f with respect to g having finite positive relative (p, q)-th order $\rho_g^{(p,q)}(f)$ $(0 < \rho_g^{(p,q)}(f) < \infty)$ are defined as:*

$$\sigma_g^{(p,q)}(f) = \limsup_{r\to\infty} \frac{\log^{[p-1]} \widehat{|g|}(|f|(r))}{(\log^{[q-1]} r)^{\rho_g^{(p,q)}(f)}} \text{ and } \overline{\sigma}_g^{(p,q)}(f) = \liminf_{r\to\infty} \frac{\log^{[p-1]} \widehat{|g|}(|f|(r))}{(\log^{[q-1]} r)^{\rho_g^{(p,q)}(f)}}.$$

Analogously, to determine the relative growth of two entire functions belonging to $\mathcal{A}(\mathbb{K})$ and having same non zero finite relative (p, q)-th lower order with respect to another entire function belonging to $\mathcal{A}(\mathbb{K})$, one can introduce the definition of relative (p, q)-th weak type of an entire function $f \in \mathcal{A}(\mathbb{K})$ with respect to another entire function $g \in \mathcal{A}(\mathbb{K})$ of finite positive relative (p, q)-th lower order $\lambda_g^{(p,q)}(f)$ in the following way:

***Definition* 1.7** [5] *Let $f, g \in \mathcal{A}(\mathbb{K})$. Also let the index-pair of f and g are (m, q) and (m, p), respectively. The relative (p, q)-th weak type $\tau_g^{(p,q)}(f)$ and the growth indicator $\overline{\tau}_g^{(p,q)}(f)$ of f with respect to g having finite positive relative (p, q)-th lower order $\lambda_g^{(p,q)}(f)$ $(0 < \lambda_g^{(p,q)}(f) < \infty)$ are defined as:*

$$\overline{\tau}_g^{(p,q)}(f) = \limsup_{r\to\infty} \frac{\log^{[p-1]} \widehat{|g|}(|f|(r))}{(\log^{[q-1]} r)^{\lambda_g^{(p,q)}(f)}} \text{ and } \tau_g^{(p,q)}(f) = \liminf_{r\to\infty} \frac{\log^{[p-1]} \widehat{|g|}(|f|(r))}{(\log^{[q-1]} r)^{\lambda_g^{(p,q)}(f)}}.$$

The main purpose of this paper is to ascertain some results associated to the growth properties of composite p-adic entire functions on the basis of relative (p, q)-th order, relative (p, q)-th type and relative (p, q)-th weak type.

2 Lemma

In this section we present the following lemma which can be found in [2] or [3] and will be needed in the sequel.

Lemma 2.1. Let $f, g \in \mathcal{A}(\mathbb{K})$. Then for all sufficiently large positive numbers of r the following equality holds

$$|f \circ g|(r) = |f|(|g|(r)).$$

3 Main Results

In this section we present the main results of the paper.

***Theorem* 3.1** *Let f, g, $h \in \mathcal{A}(\mathbb{K})$. Also let $0 < \lambda_h^{(p,q)}(f) \le \rho_h^{(p,q)}(f) < \infty$, $\lambda^{(m,n)}(g) > 0$ and γ be a positive continuous function defined on $[0, +\infty)$ increasing to ∞ as $r \to \infty$. Then for any number $\alpha \ge 0$,*

$$\lim_{r\to\infty} \frac{\log^{[p]} \widehat{|h|}(|f \circ g|(\exp^{[n-1]} r))}{\{\log^{[p]} \widehat{|h|}(|f|(\exp^{[q]} \gamma(r)))\}^{1+\alpha}} = \infty \text{ when } q < m \text{ and } \lim_{r\to+\infty} \frac{\log \gamma(r)}{\log r} = 0,$$

and

$$\lim_{r\to\infty} \frac{\log^{[p]} \widehat{|h|}(|f \circ g|(\exp^{[n-1]} r))}{\{\log^{[p]} \widehat{|h|}(|f|(\exp^{[q]} \gamma(r)))\}^{1+\alpha}} = \infty \text{ when } q \geqslant m \text{ and } \lim_{r\to+\infty} \frac{\log \gamma(r)}{\log^{[q-m+1]} r} = 0.$$

Proof. From the definition of $\rho_h^{(p,q)}(f)$, it follows for all sufficiently large positive numbers of r that

(3.1) $\log^{[p]} \widehat{|h|}(|f|(\exp^{[q]} \gamma(r))) \le (\rho_h^{(p,q)}(f) + \varepsilon)\gamma(r).$

Since $\widehat{|h|}(r)$ is an increasing function of r, it follows from ***Lemma* 2.1** and for all sufficiently large positive numbers of r that

(3.2) $\log^{[p]} \widehat{|h|}(|f \circ g|(\exp^{[n-1]} r)) \ge (\lambda_h^{(p,q)}(f) - \varepsilon) \log^{[q]} |g|(\exp^{[n-1]} r).$

***Case* I.** Let $q < m$. Then from (3.2) it follows for all sufficiently large positive numbers of r that

(3.3) $\log^{[p]} \widehat{|h|}(|f \circ g|(\exp^{[n-1]} r)) \ge (\lambda_h^{(p,q)}(f) - \varepsilon) \exp^{[m-q-1]} \log^{[m-1]} |g|(\exp^{[n-1]} r)$

i.e.,

(3.4) $\log^{[p]} \widehat{|h|}(|f \circ g|(\exp^{[n-1]} r)) \ge (\lambda_h^{(p,q)}(f) - \varepsilon) \exp^{[m-q-1]} r^{(\lambda^{(m,n)}(g)-\varepsilon)}.$

***Case* II.** Let $q \geqslant m$. Then from (3.2) we get for all sufficiently large positive numbers of r that

(3.5) $\log^{[p]} \widehat{|h|}(|f \circ g|(\exp^{[n-1]} r)) \ge (\lambda_h^{(p,q)}(f) - \varepsilon) \log^{[q-m]} \log^{[m]} |g|(\exp^{[n-1]} r)$

i.e.,

$$\log^{[p]} \widehat{|h|}(|f \circ g|(\exp^{[n-1]} r)) \geq (\lambda_h^{(p,q)}(f) - \varepsilon) \log^{[q-m+1]} r + O(1)$$

i.e.,

$$\log^{[p-1]} \widehat{|h|}(|f \circ g|(\exp^{[n-1]} r)) \geqslant (\log^{[q-m]} r)^{(\lambda_h^{(p,q)}(f)-\varepsilon)} + O(1). \tag{3.6}$$

Now combining (3.1) and (3.4) of ***Case* I** it follows for all sufficiently large positive numbers of r that

$$\frac{\log^{[p]} \widehat{|h|}(|f \circ g|(\exp^{[n-1]} r))}{\{\log^{[p]} \widehat{|h|}(|f|(\exp^{[q]} \gamma(r)))\}^{1+\alpha}} \geq \frac{(\lambda_h^{(p,q)}(f) - \varepsilon) \exp^{[m-q-1]} r^{(\lambda^{(m,n)}(g)-\varepsilon)}}{(\rho_h^{(p,q)}(f) + \varepsilon)^{1+\alpha}\{\gamma(r)\}^{1+\alpha}}.$$

Since $\lim\limits_{r\to+\infty} \frac{\log \gamma(r)}{\log r} = 0$, therefore $\frac{\exp^{[m-q-1]} r^{(\lambda^{(m,n)}(g)-\varepsilon)}}{\{\gamma(r)\}^{1+\alpha}} \to +\infty$ as $r \to +\infty$, then from above it follows that

$$\lim_{r\to\infty} \frac{\log^{[p]} \widehat{|h|}(|f \circ g|(\exp^{[n-1]} r))}{\{\log^{[p]} \widehat{|h|}(|f|(\exp^{[q]} \gamma(r)))\}^{1+\alpha}} = \infty,$$

from which the first part of ***Theorem* 3.1** follows.

Again combining (3.1) and (3.6) of ***Case* II** it follows for all sufficiently large positive numbers of r that

$$\frac{\log^{[p-1]} \widehat{|h|}(|f \circ g|(\exp^{[n-1]} r))}{\{\log^{[p]} \widehat{|h|}(|f|(\exp^{[q]} \gamma(r)))\}^{1+\alpha}} \geq \frac{(\log^{[q-m]} r)^{(\lambda_h^{(p,q)}(f)-\varepsilon)} + O(1)}{(\rho_h^{(p,q)}(f) + \varepsilon)^{1+\alpha}\{\gamma(r)\}^{1+\alpha}}.$$

As $\lim\limits_{r\to+\infty} \frac{\log \gamma(r)}{\log^{[q-m+1]} r} = 0$, so $\frac{(\log^{[q-m]} r)^{(\lambda_h^{(p,q)}(f)-\varepsilon)}}{\{\gamma(r)\}^{1+\alpha}} \to +\infty$ as $r \to +\infty$. Thus it follows from above that

$$\lim_{r\to\infty} \frac{\log^{[p-1]} \widehat{|h|}(|f \circ g|(\exp^{[n-1]} r))}{\{\log^{[p]} \widehat{|h|}(|f|(\exp^{[q]} \gamma(r)))\}^{1+\alpha}} = \infty.$$

This proves the second part of ***Theorem* 3.1**.

Thus ***Theorem* 3.1** follows.

***Remark* 3.1** ***Theorem* 3.1** *is still valid with "superior limit" instead of "limit" if we replace the condition " $0 < \lambda_h^{(p,q)}(f) \leq \rho_h^{(p,q)}(f) < \infty$" by " $0 < \lambda_h^{(p,q)}(f) < \infty$".*

In the line of ***Theorem* 3.1** one may state the following theorem without proof:

***Theorem* 3.2** *Let f, g, h, $k \in \mathcal{A}(\mathbb{K})$. Also let g is of finite (m,n)-th lower order, $\lambda_h^{(p,q)}(f) > 0$, $\rho_k^{(l,n)}(g) < \infty$ and γ be a positive continuous function defined on $[0, +\infty)$ increasing to ∞ as $r \to \infty$. Then for any number $\alpha \geq 0$,*

$$\lim_{r\to\infty} \frac{\log^{[p]} \widehat{|h|}(|f \circ g|(\exp^{[n-1]} r))}{\{\log^{[l]} \widehat{|k|}(|g|(\exp^{[n]} \gamma(r)))\}^{1+\alpha}} = \infty \text{ when } q < m \text{ and } \lim_{r\to+\infty} \frac{\log \gamma(r)}{\log r} = 0,$$

and

$$\lim_{r\to\infty} \frac{\log^{[p-1]} \widehat{|h|}(|f \circ g|(\exp^{[n-1]} r))}{\{\log^{[l]} \widehat{|k|}(|g|(\exp^{[n]} \gamma(r)))\}^{1+\alpha}} = \infty \text{ when } q \geqslant m \text{ and } \lim_{r\to+\infty} \frac{\log \gamma(r)}{\log^{[q-m+1]} r} = 0.$$

***Remark* 3.2** *In* ***Theorem* 3.2** *if we take the condition $\lambda_k^{(l,n)}(g) < \infty$ instead of $\rho_k^{(l,n)}(g) < \infty$, then also* ***Theorem* 3.2** *remains true with "superior limit" in place of " limit ".*

***Theorem* 3.3** *Let f, g, $h \in \mathcal{A}(\mathbb{K})$. Also let $0 < \lambda_h^{(p,q)}(f) \leq \rho_h^{(p,q)}(f) < \infty$, $\rho^{(m,n)}(g) < \infty$ and γ be a positive continuous function defined on $[0, +\infty)$ increasing to ∞ as $r \to \infty$. Then for any number $\alpha \geq 0$,*

$$\lim_{r\to\infty} \frac{\{\log^{[p]} \widehat{|h|}(|f \circ g|(r))\}^{1+\alpha}}{\log^{[p]} \widehat{|h|}(|f|(\exp^{[q]}\{\gamma(r)\}))} = 0 \text{ if } q \geqslant m$$

and

$$\lim_{r\to+\infty} \frac{\{\log^{[p+m-q-1]} \widehat{|h|}(|f \circ g|(r))\}^{1+\alpha}}{\log^{[p]} \widehat{|h|}(|f|(\exp^{[q]} \gamma(r)))} = 0 \text{ if } q < m,$$

where

$$\lim_{r\to\infty} \frac{\log \gamma(r)}{\log r} = \infty.$$

Proof. Since $\widehat{|h|}(r)$ is an increasing function of r, it follows from ***Lemma* 2.1** and for all sufficiently large positive numbers of r that

(3.7) $\log^{[p]} \widehat{|h|}(|f \circ g|(r)) \leqslant (\rho_h^{(p,q)}(f) + \varepsilon) \log^{[q]} |g|(r).$

Now the following cases may arise :

***Case* I.** Let $q \geqslant m$. Then we get from (3.7) for all sufficiently large positive numbers of r that

(3.8) $\log^{[p]} \widehat{|h|}(|f \circ g|(r)) \leqslant (\rho_h^{(p,q)}(f) + \varepsilon) \log^{[m-1]} |g|(r).$

Now from the definition of (m,n)-th order of g, for arbitrary positive ε and for all sufficiently large positive numbers of r, we have

$$\log^{[m]} |g|(r) \leqslant (\rho^{(m,n)}(g) + \varepsilon) \log^{[n]} r$$

i.e.,

(3.9) $\log^{[m]} |g|(r) \leq (\rho^{(m,n)}(g) + \varepsilon) \log r$

and

(3.10) $\log^{[m-1]} |g|(r) \leq r^{(\rho^{(m,n)}(g)+\varepsilon)}.$

So from (3.8) and (3.10) it follows for all sufficiently large positive numbers of r that

(3.11) $\log^{[p]} \widehat{|h|}(|f \circ g|(r)) \leqslant (\rho_h^{(p,q)}(f) + \varepsilon) r^{(\rho^{(m,n)}(g)+\varepsilon)}.$

***Case* II**. Let $q < m$. Then we obtain from (3.7) for all sufficiently large positive numbers of r that

(3.12) $\log^{[p]} \widehat{|h|}(|f \circ g|(r)) \leqslant (\rho_h^{(p,q)}(f) + \varepsilon) \exp^{[m-q]} \log^{[m]} |g|(r).$

Also we get from (3.9) for all sufficiently large positive numbers of r that

$$\exp^{[m-q]} \log^{[m]} |g|(r) \leqslant \exp^{[m-q]} \log r^{(\rho^{(m,n)}(g)+\varepsilon)}$$

i.e.,

(3.13) $\exp^{[m-q]} \log^{[m]} |g|(r) \leqslant \exp^{[m-q-1]} r^{(\rho^{(m,n)}(g)+\varepsilon)}.$

Now from (3.12) and (3.13) we have for all sufficiently large positive numbers of r that

$$\log^{[p]} \widehat{|h|}(|f \circ g|(r)) \leqslant (\rho_h^{(p,q)}(f) + \varepsilon) \exp^{[m-q-1]} r^{(\rho^{(m,n)}(g)+\varepsilon)}$$

i.e.,

(3.14) $\log^{[p+m-q-1]} \widehat{|h|}(|f \circ g|(r)) \leqslant r^{(\rho^{(m,n)}(g)+\varepsilon)} + O(1).$

Again for all sufficiently large positive numbers of r that

(3.15) $\log^{[p]} \widehat{|h|}(|f|(\exp^{[q]} \gamma(r))) \geq (\lambda_h^{(p,q)}(f) - \varepsilon)\gamma(r).$

Now if $q \geqslant m$, we obtain from (3.11) and (3.15) for all sufficiently large positive numbers of r that

$$\frac{\{\log^{[p]} \widehat{|h|}(|f \circ g|(r))\}^{1+\alpha}}{\log^{[p]} \widehat{|h|}(|f|(\exp^{[q]} \gamma(r)))} \leqslant \frac{(\rho_h^{(p,q)}(f) + \varepsilon)^{1+\alpha} r^{(\rho^{(m,n)}(g)+\varepsilon)(1+\alpha)}}{(\lambda_h^{(p,q)}(f) - \varepsilon)\gamma(r)}.$$

Since $\lim\limits_{r\to\infty} \frac{\log \gamma(r)}{\log r} = \infty$, therefore $\frac{r^{(\rho^{(m,n)}(g)+\varepsilon)(1+\alpha)}}{\gamma(r)} \to 0$ as $r \to \infty$, then the first part of ***Theorem* 3.3** follows from above.

Further when $q < m$, we get from (3.14) and (3.15) for all sufficiently large positive numbers of r that

$$\frac{\{\log^{[p+m-q-1]} \widehat{|h|}(|f \circ g|(r))\}^{1+\alpha}}{\log^{[p]} \widehat{|h|}(|f|(\exp^{[q]} \gamma(r)))} \leqslant \frac{r^{(\rho^{(m,n)}(g)+\varepsilon)(1+\alpha)}(1 + \frac{O(1)}{r^{(\rho^{(m,n)}(g)+\varepsilon)}})^{1+\alpha}}{(\lambda_h^{(p,q)}(f) - \varepsilon)\gamma(r)}$$

i.e.,

$$\lim_{r\to\infty} \frac{\{\log^{[p+m-q-1]} \widehat{|h|}(|f \circ g|(r))\}^{1+\alpha}}{\log^{[p]} \widehat{|h|}(|f|(\exp^{[q]} \gamma(r)))} = 0,$$

This proves the second part of ***Theorem* 3.3**.

***Remark* 3.3** *In* ***Theorem* 3.3** *if we take the condition* $\rho_h^{(p,q)}(f) > 0$ *instead of* $\lambda_h^{(p,q)}(f) > 0$*, the theorem remains true with "inferior limit" in place of "limit ".*

***Theorem* 3.4** *Let $f,\ g,\ h,\ k\in\mathcal{A}(\mathbb{K})$. Also let g is of finite (m,n)-th order, $\rho_h^{(p,q)}(f)<+\infty$, $\lambda_k^{(l,n)}(g)>0$ and γ be a positive continuous function defined on $[0,+\infty)$ increasing to ∞ as $r\to\infty$. Then for any number $\alpha\geq 0$,*

$$\lim_{r\to\infty}\frac{\{\log^{[p]}\widehat{|h|}(|f\circ g|(r))\}^{1+\alpha}}{\log^{[l]}\widehat{|k|}(|g|(\exp^{[n]}\gamma(r)))}=0 \text{ if } q\geqslant m$$

and

$$\lim_{r\to\infty}\frac{\{\log^{[p+m-q-1]}\widehat{|h|}(|f\circ g|(r))\}^{1+\alpha}}{\log^{[l]}\widehat{|k|}(|g|(\exp^{[n]}\gamma(r)))}=0 \text{ if } q<m,$$

where

$$\lim_{r\to+\infty}\frac{\log\gamma(r)}{\log r}=\infty.$$

*The proof of **Theorem** 3.4 would run parallel to that of **Theorem** 3.3. We omit the details.*

***Remark* 3.4** *In **Theorem** 3.4, if we take the condition $\rho_k^{(l,n)}(g)>0$ instead of $\lambda_k^{(l,n)}(g)>0$, **Theorem 3.4** remains true with "limit replaced by limit inferior".*

***Theorem* 3.5** *Let $f,\ g,\ h\in\mathcal{A}(\mathbb{K})$. Also let $\rho^{(m,n)}(g)<\lambda_h^{(p,q)}(f)\leq\rho_h^{(p,q)}(f)<\infty$. Then*

$$(i)\ \lim_{r\to\infty}\frac{\log^{[p]}\widehat{|h|}(|f\circ g|(r))}{\log^{[p-1]}\widehat{|h|}(|f|(\exp^{[q-1]}r^A))}=0 \text{ if } q\geqslant m$$

and

$$(ii)\ \lim_{r\to\infty}\frac{\log^{[p+m-q-1]}\widehat{|h|}(|f\circ g|(r))}{\log^{[p-1]}\widehat{|h|}(|f|(\exp^{[q-1]}r^A))}=0 \text{ if } q<m$$

where $A>0$.

Proof. From the definition of relative (p,q)-th lower order, we get for all sufficiently large positive numbers of r that

(3.16) $\log^{[p-1]}\widehat{|h|}(|f|(\exp^{[q-1]}r^A))\geqslant r^{A(\lambda_h^{(p,q)}(f)-\varepsilon)}$.

As $\rho^{(m,n)}(g)<\lambda_h^{(p,q)}(f)$, we can choose $\varepsilon(>0)$ in such a way that

(3.17) $\rho^{(m,n)}(g)+\varepsilon<A(\lambda_h^{(p,q)}(f)-\varepsilon)$.

Now if $q\geqslant m$, combining (3.11), (3.16) and in view of (3.17) we obtain for all sufficiently large positive numbers of r that

$$\frac{\log^{[p]}\widehat{|h|}(|f\circ g|(r))}{\log^{[p-1]}\widehat{|h|}(|f|(\exp^{[q-1]}r^A))}\leqslant\frac{(\rho_h^{(p,q)}(f)+\varepsilon)r^{(\rho^{(m,n)}(g)+\varepsilon)}}{r^{A(\lambda_h^{(p,q)}(f)-\varepsilon)}}$$

i.e.,

$$\lim_{r\to\infty}\frac{\log^{[p]}\widehat{|h|}(|f\circ g|(r))}{\log^{[p-1]}\widehat{|h|}(|f|(\exp^{[q-1]}r^A))}=0.$$

This proves the first part of ***Theorem* 3.5**.

When $q<m$, combining (3.14) and (3.16) it follows for all sufficiently large positive numbers of r that

$$\frac{\log^{[p+m-q-1]}\widehat{|h|}(|f\circ g|(r))}{\log^{[p-1]}\widehat{|h|}(|f|(\exp^{[q-1]}r^A))}\leqslant\frac{r^{(\rho^{(m,n)}(g)+\varepsilon)}(1+\frac{O(1)}{r^{(\rho^{(m,n)}(g)+\varepsilon)}})}{r^{A(\lambda_h^{(p,q)}(f)-\varepsilon)}}.$$

Since $\rho^{(m,n)}(g)<\lambda_h^{(p,q)}(f)$ and $\varepsilon(>0)$ is arbitrary, we get from above

$$\lim_{r\to\infty}\frac{\log^{[p+m-q-1]}\widehat{|h|}(|f\circ g|(r))}{\log^{[p-1]}\widehat{|h|}(|f|(\exp^{[q-1]}r^A))}=0,$$

which is the second part of ***Theorem* 3.5**.

***Remark* 3.5** *In **Theorem** 3.5, if we take the condition $\lambda^{(m,n)}(g)<\lambda_h^{(p,q)}(f)\leq\rho_h^{(p,q)}(f)<\infty$ instead of $\rho^{(m,n)}(g)<\lambda_h^{(p,q)}(f)\leq\rho_h^{(p,q)}(f)<\infty$, **Theorem 3.5** remains true with "inferior limit" in place of "limit ".*

***Theorem* 3.6** *Let f, g, $h \in \mathcal{A}(\mathbb{K})$. Also let $0 < \lambda_h^{(p,q)}(f) \leq \rho_h^{(p,q)}(f) < \infty$ and $\rho^{(m,q)}(g) < \infty$. Then*

$$\limsup_{r\to\infty} \frac{\log^{[p+m-q]} \widehat{|h|}(|f\circ g|(r))}{\log^{[p]} \widehat{|h|}(|f|(r))} \leq \frac{\rho^{(m,q)}(g)}{\lambda_h^{(p,q)}(f)},$$

where $m > q$.

Proof. Since $q < m$, we have from (3.7) for all sufficiently large positive numbers of r that

$$\log^{[p+m-q]} \widehat{|h|}(|f\circ g|(r)) \leqslant \log^{[m]} |g|(r) + O(1)$$

i.e.,

$$\frac{\log^{[p+m-q]} \widehat{|h|}(|f\circ g|(r))}{\log^{[p]} \widehat{|h|}(|f|(r))} \leq \frac{\log^{[m]} |g|(r) + O(1)}{\log^{[q]} r} \cdot \frac{\log^{[q]} r}{\log^{[p]} \widehat{|h|}(|f|(r))}$$

i.e.,

$$\limsup_{r\to\infty} \frac{\log^{[p+m-q]} \widehat{|h|}(|f\circ g|(r))}{\log^{[p]} \widehat{|h|}(|f|(r))} \leq \frac{\rho^{(m,q)}(g)}{\lambda_h^{(p,q)}(f)}.$$

This proves ***Theorem* 3.6**.

In the line of ***Theorem* 3.6** we may state the following theorem without proof.

***Theorem* 3.7** *Let f, g, $h, k \in \mathcal{A}(\mathbb{K})$. Also let $\rho_h^{(p,q)}(f) < \infty$, $\lambda_k^{(l,n)}(g) > 0$ and $\rho^{(m,n)}(g) < \infty$. Then*

$$\limsup_{r\to\infty} \frac{\log^{[p+m-q]} \widehat{|h|}(|f\circ g|(r))}{\log^{[l]} \widehat{|k|}(|g|(r))} \leq \frac{\rho^{(m,n)}(g)}{\lambda_k^{(l,n)}(g)},$$

where $m > n$.

***Theorem* 3.8** *Let f, g, $h \in \mathcal{A}(\mathbb{K})$. Also let $0 < \lambda_h^{(p,q)}(f) \leq \rho_h^{(p,q)}(f) < \infty$ and $\sigma^{(m,n)}(g) < \infty$ where $q = m - 1$. Then*

$$\limsup_{r\to\infty} \frac{\log^{[p]} \widehat{|h|}(|f\circ g|(r))}{\log^{[p]} \widehat{|h|}(|f|(\exp^{[q]}(\log^{[n-1]} r)^{\rho^{(m,n)}(g)}))} \leq \frac{\sigma^{(m,n)}(g)\cdot \rho_h^{(p,q)}(f)}{\lambda_h^{(p,q)}(f)}.$$

Proof. Since $q = m - 1$, we obtain from (3.7) for all sufficiently large positive numbers of r that

$$\log^{[p]} \widehat{|h|}(|f\circ g|(r)) \leqslant (\rho_h^{(p,q)}(f) + \varepsilon) \log^{[m-1]} |g|(r)$$

i.e.,

$$\text{(3.18)}\quad \log^{[p]} \widehat{|h|}(|f\circ g|(r)) \leq (\rho_h^{(p,q)}(f) + \varepsilon)(\sigma^{(m,n)}(g) + \varepsilon)(\log^{[n-1]}(r))^{\rho^{(m,n)}(g)}.$$

Now from the definition of $\lambda_h^{(p,q)}(f)$, we get for all sufficiently large positive numbers of r that

$$\text{(3.19)}\quad \log^{[p]} \widehat{|h|}(|f|(\exp^{[q]}(\log^{[n-1]} r)^{\rho^{(m,n)}(g)})) \geq (\lambda_h^{(p,q)}(f) - \varepsilon)(\log^{[n-1]} r)^{\rho^{(m,n)}(g)}.$$

Therefore from (3.18) and (3.19), it follows for all sufficiently large positive numbers of r that

$$\frac{\log^{[p]} \widehat{|h|}(|f\circ g|(r))}{\log^{[p]} \widehat{|h|}(|f|(\exp^{[q]}(\log^{[n-1]} r)^{\rho^{(m,n)}(g)}))} \leq \frac{(\rho_h^{(p,q)}(f) + \varepsilon)(\sigma^{(m,n)}(g) + \varepsilon)(\log^{[n-1]}(r))^{\rho^{(m,n)}(g)}}{(\lambda_h^{(p,q)}(f) - \varepsilon)(\log^{[n-1]} r)^{\rho^{(m,n)}(g)}}.$$

Since $\varepsilon(> 0)$ is arbitrary, we get from above that

$$\limsup_{r\to\infty} \frac{\log^{[p]} \widehat{|h|}(|f\circ g|(r))}{\log^{[p]} \widehat{|h|}(|f|(\exp^{[q]}(\log^{[n-1]} r)^{\rho^{(m,n)}(g)}))} \leq \frac{\sigma^{(m,n)}(g)\cdot \rho_h^{(p,q)}(f)}{\lambda_h^{(p,q)}(f)}.$$

Thus ***Theorem* 3.8** is established.

***Remark* 3.6** *In **Theorem 3.8**, if we will replace "$\sigma^{(m,n)}(g)$" by "$\overline{\sigma}^{(m,n)}(g)$", then **Theorem 3.8** remains valid with "inferior limit" replacing "superior limit".*

*Now we state the following theorem without its proof as it can easily be carried out in the line of **Theorem 3.8**.*

***Theorem* 3.9** *Let $f, g, h, k \in \mathcal{A}(\mathbb{K})$. Also let $\lambda_k^{(l,n)}(g) > 0$, $\rho_h^{(p,q)}(f) < \infty$ and $\sigma^{(m,n)}(g) < \infty$ where $q = m - 1$. Then*

$$\limsup_{r\to\infty} \frac{\log^{[p]} \widehat{|h|}(|f\circ g|(r))}{\log^{[l]} \widehat{|k|}(|g|(\exp^{[n]}(\log^{[n-1]} r)^{\rho^{(m,n)}(g)}))} \leq \frac{\sigma^{(m,n)}(g)\cdot \rho_h^{(p,q)}(f)}{\lambda_k^{(l,n)}(g)}.$$

***Remark* 3.7** *In **Theorem** 3.9, if we will replace "$\sigma^{(m,n)}(g)$" by "$\overline{\sigma}^{(m,n)}(g)$", then **Theorem** 3.9 remains valid with "inferior limit" in place of "superior limit".*

***Remark* 3.8** *We remark that in **Theorem** 3.9, if we will replace the condition "$\rho_h^{(p,q)}(f) < \infty$" by "$\lambda_h^{(p,q)}(f) < \infty$", then*

$$(3.20)\quad \liminf_{r\to\infty} \frac{\log^{[p]} |\widehat{h}|(|f\circ g|(r))}{\log^{[l]} |\widehat{k}|(|g|(\exp^{[n]}(\log^{[n-1]} r)^{\rho^{(m,n)}(g)}))} \le \frac{\sigma^{(m,n)}(g)\cdot \lambda_h^{(p,q)}(f)}{\lambda_k^{(l,n)}(g)}.$$

***Remark* 3.9** *In **Remark** 3.8, if we will replace the conditions "$\lambda_k^{(l,n)}(g) > 0$ and $\lambda_h^{(p,q)}(f) < \infty$" by "$\rho_k^{(l,n)}(g) > 0$ and $\rho_h^{(p,q)}(f) < \infty$" respectively, then is need to go the same replacement in right part of* (3.20).

*Using the concept of the growth indicator $\overline{\tau}^{(m,n)}(g)$ of a p-adic entire function g, we may state the subsequent two theorems without their proofs since those can be carried out in the line of **Theorem** 3.8 and **Theorem** 3.9 respectively.*

***Theorem* 3.10** *Let f, g, $h \in \mathcal{A}(\mathbb{K})$. Also let $0 < \lambda_h^{(p,q)}(f) \le \rho_h^{(p,q)}(f) < \infty$ and $\overline{\tau}^{(m,n)}(g) < \infty$ where $q = m-1$. Then*

$$\limsup_{r\to\infty} \frac{\log^{[p]} |\widehat{h}|(|f\circ g|(r))}{\log^{[p]} |\widehat{h}|(|f|(\exp^{[q]}(\log^{[n-1]} r)^{\lambda^{(m,n)}(g)}))} \le \frac{\overline{\tau}^{(m,n)}(g)\cdot \rho_h^{(p,q)}(f)}{\lambda_h^{(p,q)}(f)}.$$

***Remark* 3.10** *We remark that in **Theorem** 3.10, if we will replace the condition "$0 < \lambda_h^{(p,q)}(f) \le \rho_h^{(p,q)}(f) < \infty$ and $\overline{\tau}^{(m,n)}(g) < \infty$" by "$0 < \lambda_h^{(p,q)}(f) < \infty$ or $0 < \rho_h^{(p,q)}(f) < \infty$ and $\sigma^{(m,n)}(g) < \infty$", then*

$$\liminf_{r\to\infty} \frac{\log^{[p]} |\widehat{h}|(|f\circ g|(r))}{\log^{[p]} |\widehat{h}|(|f|(\exp^{[q]}(\log^{[n-1]} r)^{\rho^{(m,n)}(g)}))} \le \sigma^{(m,n)}(g).$$

***Theorem* 3.11** *Let f, g, $h, k \in \mathcal{A}(\mathbb{K})$. Also let $\lambda_k^{(l,n)}(g) > 0$, $\rho_h^{(p,q)}(f) < \infty$ and $\overline{\tau}^{(m,n)}(g) < \infty$ where $q = m-1$. Then*

$$\limsup_{r\to\infty} \frac{\log^{[p]} |\widehat{h}|(|f\circ g|(r))}{\log^{[l]} |\widehat{k}|(|g|(\exp^{[n]}(\log^{[n-1]} r)^{\lambda^{(m,n)}(g)}))} \le \frac{\overline{\tau}^{(m,n)}(g)\cdot \rho_h^{(p,q)}(f)}{\lambda_k^{(l,n)}(g)}.$$

*Further using the notion of (p,q)-th weak type we may also state the following two theorems without proof because it can be carried out in the line of **Theorem** 3.10 and **Theorem** 3.11 respectively.*

***Theorem* 3.12** *Let f, g, $h \in \mathcal{A}(\mathbb{K})$. Also let $0 < \lambda_h^{(p,q)}(f) \le \rho_h^{(p,q)}(f) < \infty$ and $\tau^{(m,n)}(g) < \infty$ where $q = m-1$. Then*

$$\liminf_{r\to\infty} \frac{\log^{[p]} |\widehat{h}|(|f\circ g|(r))}{\log^{[p]} |\widehat{h}|(|f|(\exp^{[q]}(\log^{[n-1]} r)^{\lambda^{(m,n)}(g)}))} \le \frac{\tau^{(m,n)}(g)\cdot \rho_h^{(p,q)}(f)}{\lambda_h^{(p,q)}(f)}.$$

***Remark* 3.11** *We remark that in **Theorem** 3.12, if we will replace the condition "$0 < \lambda_h^{(p,q)}(f) \le \rho_h^{(p,q)}(f) < \infty$ and $\tau^{(m,n)}(g) < \infty$" by "$0 < \lambda_h^{(p,q)}(f) < \infty$ or $0 < \rho_h^{(p,q)}(f) < \infty$ and $\overline{\tau}^{(m,n)}(g) < \infty$", then*

$$\liminf_{r\to\infty} \frac{\log^{[p]} |\widehat{h}|(|f\circ g|(r))}{\log^{[p]} |\widehat{h}|(|f|(\exp^{[q]}(\log^{[n-1]} r)^{\lambda^{(m,n)}(g)}))} \le \overline{\tau}^{(m,n)}(g).$$

***Theorem* 3.13** *Let f, g, $h, k \in \mathcal{A}(\mathbb{K})$. Also let $\lambda_k^{(l,n)}(g) > 0$, $\rho_h^{(p,q)}(f) < \infty$ and $\tau^{(m,n)}(g) < \infty$ where $q = m-1$. Then*

$$\liminf_{r\to\infty} \frac{\log^{[p]} |\widehat{h}|(|f\circ g|(r))}{\log^{[l]} |\widehat{k}|(|g|(\exp^{[n]}(\log^{[n-1]} r)^{\lambda^{(m,n)}(g)}))} \le \frac{\tau^{(m,n)}(g)\cdot \rho_h^{(p,q)}(f)}{\lambda_k^{(l,n)}(g)}.$$

***Remark* 3.12** *We remark that in **Theorem** 3.13, if we will replace the condition "$\rho_h^{(p,q)}(f) < \infty$ and $\tau^{(m,n)}(g) < \infty$" by "$\lambda_h^{(p,q)}(f) < \infty$ and $\overline{\tau}^{(m,n)}(g) < \infty$", then*

$$(3.21)\quad \liminf_{r\to\infty} \frac{\log^{[p]} |\widehat{h}|(|f\circ g|(r))}{\log^{[l]} |\widehat{k}|(|g|(\exp^{[n]}(\log^{[n-1]} r)^{\lambda^{(m,n)}(g)}))} \le \frac{\overline{\tau}^{(m,n)}(g)\cdot \lambda_h^{(p,q)}(f)}{\lambda_k^{(l,n)}(g)}.$$

***Remark* 3.13** *In **Remark** 3.12, if we will replace the conditions "$\lambda_k^{(l,n)}(g) > 0$ and $\lambda_h^{(p,q)}(f) < \infty$" by "$\rho_k^{(l,n)}(g) > 0$ and $\rho_h^{(p,q)}(f) < \infty$" respectively, then is need to go the same replacement in right part of* (3.21).

***Theorem* 3.14** *Let $f, g, h \in \mathcal{A}(\mathbb{K})$. Also let $0 < \rho_h^{(p,q)}(f) < \infty$, $\rho_h^{(p,q)}(f) = \rho^{(m,n)}(g)$, $\sigma^{(m,n)}(g) < \infty$ and $0 < \sigma_h^{(p,q)}(f) < \infty$ where $q = n = m - 1$. Then*

$$\liminf_{r\to\infty} \frac{\log^{[p]} \widehat{|h|}(|f \circ g|(r))}{\log^{[p-1]} \widehat{|h|}(|f|(r))} \le \frac{\rho_h^{(p,q)}(f) \cdot \sigma^{(m,n)}(g)}{\sigma_h^{(p,q)}(f)}. \tag{3.22}$$

Proof. Since $\rho_h^{(p,q)}(f) = \rho^{(m,n)}(g)$ and $q = m-1$, we have from (3.7) for all sufficiently large positive numbers of r that

$$\log^{[p]} \widehat{|h|}(|f \circ g|(r)) \le (\rho_h^{(p,q)}(f) + \varepsilon)(\sigma^{(m,n)}(g) + \varepsilon)(\log^{[n-1]}(r))^{\rho_h^{(p,q)}(f)}. \tag{3.23}$$

As $q = n$, then we obtain in view of the definition of $\sigma_h^{(p,q)}(f)$ for a sequence of positive numbers of r tending to infinity that

$$\log^{[p-1]} \widehat{|h|}(|f|(r)) \ge (\sigma_h^{(p,q)}(f) - \varepsilon)(\log^{[n-1]}(r))^{\rho_h^{(p,q)}(f)}. \tag{3.24}$$

Now from (3.23) and (3.24), it follows for a sequence of positive numbers of r tending to infinity that

$$\frac{\log^{[p]} \widehat{|h|}(|f \circ g|(r))}{\log^{[p-1]} \widehat{|h|}(|f|(r))} \le \frac{(\rho_h^{(p,q)}(f) + \varepsilon)(\sigma^{(m,n)}(g) + \varepsilon)(\log^{[n-1]}(r))^{\rho_h^{(p,q)}(f)}}{(\sigma_h^{(p,q)}(f) - \varepsilon)(\log^{[n-1]}(r))^{\rho_h^{(p,q)}(f)}}.$$

Since $\varepsilon(> 0)$ is arbitrary, it follows from above that

$$\liminf_{r\to\infty} \frac{\log^{[p]} \widehat{|h|}(|f \circ g|(r))}{\log^{[p-1]} \widehat{|h|}(|f|(r))} \le \frac{\rho_h^{(p,q)}(f) \cdot \sigma^{(m,n)}(g)}{\sigma_h^{(p,q)}(f)}.$$

***Remark* 3.14** *In **Theorem** **3.14**, if we will replace the conditions " $\sigma^{(m,n)}(g) < \infty$" and "$0 < \sigma_h^{(p,q)}(f) < \infty$" by "$\overline{\sigma}^{(m,n)}(g) < \infty$" and "$0 < \overline{\sigma}_h^{(p,q)}(f) < \infty$", then is need to go the same replacement in right part of* (3.22). *Also if we replace the conditions $0 < \rho_h^{(p,q)}(f) < \infty$ and $0 < \sigma_h^{(p,q)}(f) < \infty$ of **Theorem** **3.14** by $0 < \lambda_h^{(p,q)}(f) \le \rho_h^{(p,q)}(f) < \infty$ and $0 < \overline{\sigma}_h^{(p,q)}(f) < \infty$ respectively, then*

$$\liminf_{r\to\infty} \frac{\log^{[p]} \widehat{|h|}(|f \circ g|(r))}{\log^{[p-1]} \widehat{|h|}(|f|(r))} \le \frac{\lambda_h^{(p,q)}(f) \cdot \sigma^{(m,n)}(g)}{\overline{\sigma}_h^{(p,q)}(f)}.$$

*Further if, in **Theorem** **3.14**, we replace $\sigma_h^{(p,q)}(f)$ by $\overline{\sigma}_h^{(p,q)}(f)$, then **Theorem** **3.14** remains valid with "superior limit" in place of "inferior limit".*

*Now we state the following three theorems without their proofs as those can easily be carried out in the line of **Theorem** **3.14**.*

***Theorem* 3.15** *Let $f, g, h \in \mathcal{A}(\mathbb{K})$. Also let $0 < \lambda_h^{(p,q)}(f) \le \rho_h^{(p,q)}(f) < \infty$, $\lambda_h^{(p,q)}(f) = \lambda^{(m,n)}(g)$, $\overline{\tau}^{(m,n)}(g) < \infty$ and $0 < \overline{\tau}_h^{(p,q)}(f) < \infty$ where $q = n = m - 1$. Then*

$$\liminf_{r\to\infty} \frac{\log^{[p]} \widehat{|h|}(|f \circ g|(r))}{\log^{[p-1]} \widehat{|h|}(|f|(r))} \le \frac{\rho_h^{(p,q)}(f) \cdot \overline{\tau}^{(m,n)}(g)}{\overline{\tau}_h^{(p,q)}(f)}. \tag{3.25}$$

***Remark* 3.15** *In **Theorem** **3.15**, if we will replace the conditions " $\overline{\tau}^{(m,n)}(g) < \infty$" and "$0 < \overline{\tau}_h^{(p,q)}(f) < \infty$" by "$\tau^{(m,n)}(g) < \infty$" and "$0 < \tau_h^{(p,q)}(f) < \infty$", then is need to go the same replacement in right part of* (3.25). *Also if we replace the conditions $0 < \lambda_h^{(p,q)}(f) \le \rho_h^{(p,q)}(f) < \infty$ and $0 < \overline{\tau}_h^{(p,q)}(f) < \infty$ of **Theorem** **3.15** by $0 < \lambda_h^{(p,q)}(f) < \infty$ and $0 < \tau_h^{(p,q)}(f) < \infty$ respectively, then*

$$\liminf_{r\to\infty} \frac{\log^{[p]} \widehat{|h|}(|f \circ g|(r))}{\log^{[p-1]} \widehat{|h|}(|f|(r))} \le \frac{\lambda_h^{(p,q)}(f) \cdot \overline{\tau}^{(m,n)}(g)}{\tau_h^{(p,q)}(f)}.$$

*Further, in **Theorem** **3.15**, if we replace $\overline{\tau}_h^{(p,q)}(f)$ by $\tau_h^{(p,q)}(f)$, then **Theorem** **3.15** remains valid with "superior limit" instead of "inferior limit".*

***Theorem* 3.16** *Let $f, g, h \in \mathcal{A}(\mathbb{K})$. Also let $0 < \lambda_h^{(p,q)}(f) \le \rho_h^{(p,q)}(f) < \infty$, $\lambda_h^{(p,q)}(f) = \rho^{(m,n)}(g)$, $\sigma^{(m,n)}(g) < \infty$ and $0 < \overline{\tau}_h^{(p,q)}(f) < \infty$ where $q = n = m - 1$. Then*

$$\liminf_{r\to\infty} \frac{\log^{[p]} \widehat{|h|}(|f \circ g|(r))}{\log^{[p-1]} \widehat{|h|}(|f|(r))} \le \frac{\rho_h^{(p,q)}(f) \cdot \sigma^{(m,n)}(g)}{\overline{\tau}_h^{(p,q)}(f)}. \tag{3.26}$$

***Remark* 3.16** *In **Theorem** 3.16, if we will replace the conditions " $\sigma^{(m,n)}(g) < \infty$" and "$0 < \overline{\tau}_h^{(p,q)}(f) < \infty$" by "$\overline{\sigma}^{(m,n)}(g) < \infty$" and "$0 < \tau_h^{(p,q)}(f) < \infty$", then is need to go the same replacement in right part of* (3.26). *Also if we replace the conditions* $0 < \lambda_h^{(p,q)}(f) \le \rho_h^{(p,q)}(f) < \infty$ *and* $0 < \overline{\tau}_h^{(p,q)}(f) < \infty$ *of **Theorem** 3.16 by* $0 < \lambda_h^{(p,q)}(f) < \infty$ *and* $0 < \tau_h^{(p,q)}(f) < \infty$ *respectively, then*

$$\liminf_{r\to\infty} \frac{\log^{[p]} \widehat{|h|}(|f \circ g|(r))}{\log^{[p-1]} \widehat{|h|}(|f|(r))} \le \frac{\lambda_h^{(p,q)}(f) \cdot \sigma^{(m,n)}(g)}{\tau_h^{(p,q)}(f)}.$$

*Further if, in **Theorem** 3.16, we replace* $\overline{\tau}_h^{(p,q)}(f)$ *by* $\tau_h^{(p,q)}(f)$*, then **Theorem** 3.16 remains valid with "superior limit" replacing "inferior limit".*

***Theorem* 3.17 *Theorem* 3.17.** *Let* $f, g, h \in \mathcal{A}(\mathbb{K})$. *Also let* $0 < \rho_h^{(p,q)}(f) < \infty$, $\rho_h^{(p,q)}(f) = \lambda^{(m,n)}(g)$, $\overline{\tau}^{(m,n)}(g) < \infty$ *and* $0 < \sigma_h^{(p,q)}(f) < \infty$ *where* $q = n = m - 1$. *Then*

$$\liminf_{r\to\infty} \frac{\log^{[p]} \widehat{|h|}(|f \circ g|(r))}{\log^{[p-1]} \widehat{|h|}(|f|(r))} \le \frac{\rho_h^{(p,q)}(f) \cdot \overline{\tau}^{(m,n)}(g)}{\sigma_h^{(p,q)}(f)}. \tag{3.27}$$

***Remark* 3.17** *In **Theorem** 3.17, if we will replace the conditions " $\overline{\tau}^{(m,n)}(g) < \infty$" and "$0 < \sigma_h^{(p,q)}(f) < \infty$" by "$\tau^{(m,n)}(g) < \infty$" and "$0 < \overline{\sigma}_h^{(p,q)}(f) < \infty$", then is need to go the same replacement in right part of* (3.27). *Also if we replace the conditions* $0 < \rho_h^{(p,q)}(f) < \infty$ *and* $0 < \sigma_h^{(p,q)}(f) < \infty$ *of **Theorem** 3.17 by* $0 < \lambda_h^{(p,q)}(f) \le \rho_h^{(p,q)}(f) < \infty$ *and* $0 < \overline{\sigma}_h^{(p,q)}(f) < \infty$ *respectively, then*

$$\liminf_{r\to\infty} \frac{\log^{[p]} \widehat{|h|}(|f \circ g|(r))}{\log^{[p-1]} \widehat{|h|}(|f|(r))} \le \frac{\lambda_h^{(p,q)}(f) \cdot \overline{\tau}^{(m,n)}(g)}{\overline{\sigma}_h^{(p,q)}(f)}.$$

*Further if, in **Theorem** 3.17, we replace* $\sigma_h^{(p,q)}(f)$ *by* $\overline{\sigma}_h^{(p,q)}(f)$*, then **Theorem** 3.17 remains valid with "superior limit" in place of "inferior limit".*

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ON BURR (4P) DISTRIBUTION: APPLICATION OF BREAKING STRESS DATA

By

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Abstract

In many problems of quality and reliability engineering processes and designs, fitting of a probability distribution to the tensile strength or breaking stress data may be helpful in predicting the probability or forecasting the frequency of occurrence of the breaking stress, and planning beforehand. In this paper, Burr (4P) distribution functions was fitted to such breaking stress data, and compared with Burr (3P), Dagum (3P) and Dagum (4P) distribution functions. Goodness of fit has been tested using the Kolmogorov-Smirnov, Anderson-Darling and Chi-Squared distribution tests. It is observed that the Burr (4P) distribution fits the best among these four distributions.

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1 Introduction

In order to deal with the random phenomena and data occurring in many applied problems in the fields of actuarial science, biological sciences, engineering, finance, hydrology, medical sciences, reliability, transportation, etc., probability distributions can be applied to make predictions and informed decisions under uncertainty. For example, according to Birnbaum and Saunders [3], "for the amount of fatigue data which can usually be obtained almost any two-dimensional parametric family of distributions can be made to fit reasonably well. In fact, in the region of central tendency the lognormal, the Weibull, the Gamma etc., can all be fitted by parametric estimation and because of the relatively small sample sizes hardly any can be rejected by, say a Chi-square Goodness of Fit test. However, when it becomes a question of predicting the "safe life", say, the one thousandth percentile, there is a wide discrepancy between these models. For this reason, a family of distributions which is obtained from considerations of the basic characteristics of the fatigue process should be more persuasive in its implications than any ad hoc family chosen for extraneous." As pointed out by Lio and Park [15] "the two-parameter Birnbaum–Saunders distributions have been shown to provide a better fit to the strength or breaking stress data such as carbon fiber or composite tensile strengths than the more commonly used Weibull distributions by Durham and Padgett [9], in addition to cycles to failure data, as was investigated by Birnbaum and Saunders [3, 4]."

The observed frequency distributions of the brittle materials, such as 6061-T6 aluminum, are the results of many complex parameters such as their tensile strength, among others, and it may not be possible to predict them exactly. Therefore, the statistical treatment of such data is an important aspect of their analysis and interpretation. As stated above, different probability models such as Birnbaum–Saunders and Weibull distributions have been applied to characterize the strength or breaking stress data such as carbon fiber or composite tensile strengths in the past, see Birnbaum and Saunders [2, 3], and Durham and Padgett [9], among others. However, it appears from the literature that no such studies have performed for the breaking stress of 6061-T6 aluminum, except the bootstrap control chart for Birnbaum-Saunders percentiles by Lio and Park [15]. As most of the distributions of the tensile strength or breaking stress of 6061-T6 aluminum data are continuous and skewed in nature, some of the various continuous skewed probability distributions, developed recently, seem to be better choices for such studies. Thus, a better selection of the best fitting probability distribution to the tensile strength or breaking stress of 6061-T6 aluminum data may help us in extrapolating the observed values to those which are more significant from the point of view of quality and reliability engineering standards. Motivated by the importance of such studies described above, we have considered the fitting of Burr (4P) distribution functions was fitted to to the breaking stress of 6061-T6 aluminum data set, which has been taken from Lio and Park [15] consisting of 200 observations on the breaking stress of 6061-T6 aluminum. Then, Burr (4P) was compared with Burr (3P), Dagum (3P) and Dagum (4P) distribution functions.

The organization of this paper is as follows: In ***Section* 2**, we briefly provide a review of the Burr and Dagum type distributions and some of their basic distributional properties. ***Section* 3** contains the data description, parameters estimation and fitting the distributions to breaking stress data. Some concluding remarks are given in ***Section* 4**.

2 Review of the models and distributional properties

In what follows, we shall briefly provide a review of the Burr (4P) and Burr (3P) distribution models and some of their essential basic distributional properties. For details on these distributions, see, for example, Blischke and Murthy [1], Burr [5], Dey et al. [8], Johnson et al. [12], Kleiber and Kotz [13], Kleiber [14], and Tadikamalla [18], among others.

2.1 Burr (4P) Distribution:

A continuous non-negative random variable, X, is said to have a Burr (4P) distribution if its probability density function (pdf), cumulative distribution function (cdf) and hazard function (hf) are respectively given by

(2.1) $$f(x) = \frac{\alpha k \left(\frac{x-\gamma}{\beta}\right)^{\alpha-1}}{\beta \left(1 + \left(\frac{x-\gamma}{\beta}\right)^{\alpha}\right)^{k+1}},$$

(2.2) $$F(x) = 1 - \left(1 + \left(\frac{x-\gamma}{\beta}\right)^{\alpha}\right)^{-k},$$

and

(2.3) $$h(x) = \frac{\alpha k \left(\frac{x-\gamma}{\beta}\right)^{\alpha-1}}{\beta \left(1 + \left(\frac{x-\gamma}{\beta}\right)^{\alpha}\right)},$$

where k (>0): shape parameter ; α (>0): shape parameter ; β (>0): scale parameter ; $-\infty < \gamma < +\infty$: location parameter ; and domain: $\gamma \le x < \infty$. The possible shapes of the pdf (1) and cdf (2) are given for some selected values of the parameters in ***Figures* 2.1(a)** and **2.1(b)** respectively.

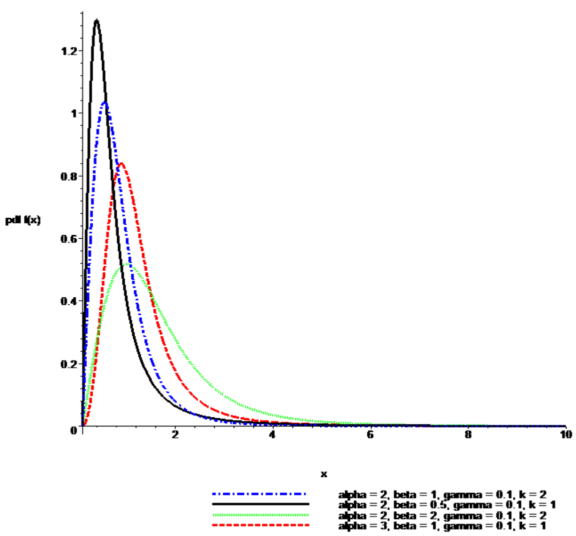

Figure **2.1(a)**: Plots of the Burr (4P) pdf 1

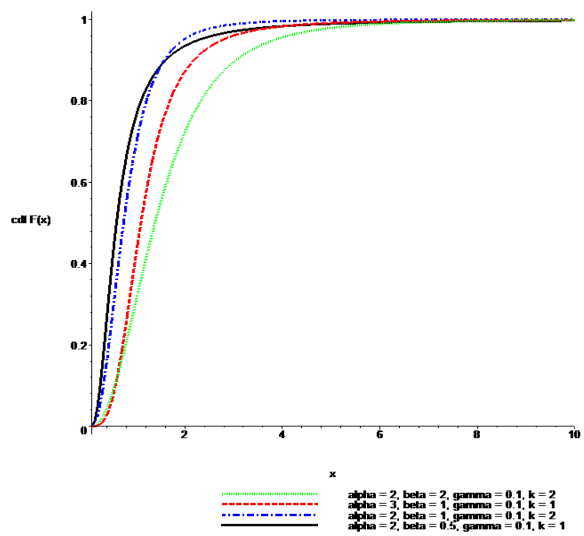

Figure 2.1(b): Plots of the Burr (4P) cdf (2)

The effects of the parameters can easily be seen from these graphs. For example, it is clear from these plots that the Burr (4P) distribution is positively right skewed with longer and heavier right tails for the selected values of the parameters.

2.2 Moments:

It is interesting to note that, after thorough search of literature, we did not find any expression for the j^{th} moment of the Burr (4P) distribution. Therefore, in what follows, we shall first derive the j^{th} moment of the Burr (4P) distribution independently. Then, various moments of the Burr (4P) distribution will be derived.

j^{th} Moment of the Burr (4P) Distribution: For a positive integer j, the j^{th} moment of the random variable X of the Burr (4P) distribution is given by

(2.4) $$E\left(X^j\right) = \int_{\gamma}^{\infty} x^j \frac{\alpha k \left(\frac{x-\gamma}{\beta}\right)^{\alpha-1}}{\beta \left(1+\left(\frac{x-\gamma}{\beta}\right)^{\alpha}\right)^{k+1}} \, dx.$$

Letting $\frac{x-\gamma}{\beta} = u$ in equation (2.4), we have

(2.5) $E\left(X^j\right) = \left(\alpha k \beta^j\right) \int_0^\infty u^{\alpha-1} \left(u + \frac{\gamma}{\beta}\right)^j (1 + u^\alpha)^{-(k+1)}\, du.$

Now, using the binomial expansion for $\left(u + \frac{\gamma}{\beta}\right)^j$ in equation (2.5) and simplifying, we obtain

(2.6) $E\left(X^j\right) = \left(\alpha k \beta^j\right) \sum_{m=0}^{j} \binom{j}{m} \left(\frac{\gamma}{\beta}\right)^{j-m} \int_0^\infty u^{\alpha+m-1} (1 + u^\alpha)^{-(k+1)}\, du.$

Thus, using Gradshteyn and Ryzhik [10, page 295] Eq. 3.251.11, of the j^{th} moment of the Burr (4P) distribution is easily given by

(2.7) $E\left(X^j\right) = \left(k \beta^j\right) \sum_{m=0}^{j} \binom{j}{m} \left(\frac{\gamma}{\beta}\right)^{j-m} B\left(1 + \frac{m}{\alpha},\, k - \frac{m}{\alpha}\right),$

where $0 \le m \le j < k\alpha$, $j > 0$ (a positive integer), $k > 0, \alpha > 0, \beta > 0, -\infty < \gamma < +\infty$, and $B()$ denotes the complete beta function.

First Moment (or Mean) of the Burr (4P) Distribution: Taking $j = 1$ in (2.7) and simplifying, the first moment (or the mean) of the Burr (4P) distribution is easily given by

(2.8) $E(X) = \gamma + (k\beta)\, B\left(1 + \frac{1}{\alpha},\, k - \frac{1}{\alpha}\right),\ 0 < \frac{1}{\alpha} < k,$

where $k > 0, \alpha > 0, \beta > 0, -\infty < \gamma < +\infty$.

***j^{th}* (Central) Moment**: The j^{th} (central) moment of the Burr (4P) distribution can easily be derived as follows:

(2.9) $E\,[X - E\,(X)]^j = \int_0^\infty [x - E\,(X)]^j f_X(x)\, dx$

$$= \sum_{m=0}^{j} (-1)^m \binom{j}{m} (E\,(X))^m\, E\left(X^{j-m}\right),$$

where $E\left(X^{j-m}\right)$ and $(E\,(X))^m$ can be obtained from the equations (2.7) and (2.8) respectively. From the equation (2.9), one can easily obtain the second, third, and higher central moments.

Variance: Taking $j = 2$ in equation (2.9), the variance (or the second central moment) is given by

(2.10) Variance $= E\,[X - E\,(X)]^2 = \int_0^\infty [x - E\,(X)]^2 f_X(x)\, dx = E\left[X^2\right] - (E\,[X])^2.$

Coefficients of Skewness and Kurtosis: By taking $j = 3$ and $j = 4$ in the equation (2.9), the third and fourth central moments are respectively given by

(2.11) $E\,[X - E\,(X)]^3 = \sum_{m=0}^{3} (-1)^m \binom{3}{m} (E\,(X))^m\, E\left(X^{3-m}\right),$

and

(2.12) $E\,[X - E\,(X)]^4 = \sum_{m=0}^{4} (-1)^m \binom{4}{m} (E\,(X))^m\, E\left(X^{4-m}\right).$

Thus, using equations (2.11) and (2.12), the measure of skewness and kurtosis are respectively given by

(2.13) $\text{Skewness} = \dfrac{\sum_{m=0}^{3} (-1)^m \binom{3}{m} (E\,(X))^m\, E\left(X^{3-m}\right)}{\left(E\,[X^2] - E\,[X]^2\right)^{\frac{3}{2}}},$

and

(2.14) $\text{Kurtosis} = \dfrac{\sum_{m=0}^{4} (-1)^m \binom{4}{m} (E\,(X))^m\, E\left(X^{4-m}\right)}{\left(E\,[X^2] - E\,[X]^2\right)^{2}},$

where $E\left(X^{j-m}\right)$ and $(E\,(X))^m$ can be obtained from the equations (2.7) and (2.8) respectively.

2.3 Burr (3P) Distribution:

It is special case of Burr (4P) distribution, and can be obtained by taking the location parameter $\gamma = 0$ in Burr (4P) distribution, with its probability density function (pdf) and cumulative distribution function (cdf), respectively, given by

$$f\,(x) = \frac{\alpha k \left(\frac{x}{\beta}\right)^{\alpha-1}}{\beta\left(1 + \left(\frac{x}{\beta}\right)^\alpha\right)^{k+1}}, \text{ and } F\,(x) = 1 - \left(1 + \left(\frac{x}{\beta}\right)^\alpha\right)^{-k},$$

where k (> 0): shape parameter ; α (> 0): shape parameter ; β (> 0): scale parameter ; and domain: $0 \le x < \infty$.

***Remark* 2.1** *Proceeding in the same manner as in **sub-sections** 2.1 and 2.2 above, we can draw the graphs of the pdf and cdf of Burr (3P) distribution for some selected values of the parameters, and also we can derive various moments of the Burr (3P) distribution.*

3 Data Analysis and Fitting Distributions

To illustrate the performance of the Burr (3P), Burr (4P), Dagum (3P) and Dagum (4P) distributions, we have considered the breaking stress of 6061-T6 aluminum data, consisting of 200 observations, as reported in Lio and Park [15], and determine their best fit.

3.1 Data Description:

In what follows, the breaking stress of 6061-T6 aluminum data is provided in ***Table* 3.1** below. The descriptive statistics of the data are computed in ***Table* 3.2.** Using the software statdisk (https://www.triolastats.com/statdisk), the histogram, boxplot and the probability plot of the data are drawn in ***Figure* 3.1,** followed by the testing of the normality of the data by Ryan-Joiner Test (Similar to Shapiro-Wilk Test), which is given in ***Table* 3.3**, along with a description of the methods of parameter estimates and goodness of fit tests, which are given subsequently.

***Table* 3.1:** Breaking Stress of 6061-T6 Aluminum

0.2187, 0.2802, 0.3026, 0.3693, 0.4136, 0.4155, 0.4348, 0.4357, 0.4436, 0.5193, 0.5239, 0.5319, 0.5322, 0.5355, 0.5365, 0.5459, 0.5653, 0.5654, 0.5680, 0.5950, 0.6235, 0.6270, 0.7027, 0.7174, 0.7382, 0.7527, 0.7666, 0.7666, 0.7707, 0.7716, 0.7735, 0.7842, 0.8068, 0.8069, 0.8156, 0.8168, 0.8243, 0.8541, 0.8861, 0.8899, 0.9028, 0.9263, 0.9522, 0.9569, 0.9583, 0.9660, 0.9880, 0.9917, 1.0150, 1.0380, 1.0440, 1.0470, 1.0760, 1.0800, 1.0830, 1.1040, 1.1140, 1.1150, 1.1170, 1.1210, 1.1260, 1.1400, 1.1420, 1.1450, 1.1450, 1.1520, 1.1550, 1.1570, 1.1590, 1.1670, 1.1730, 1.1860, 1.1880, 1.1900, 1.1980, 1.2000, 1.2090, 1.2160, 1.2210, 1.2210, 1.2260, 1.2470, 1.2470, 1.2650, 1.2900, 1.2970, 1.3000, 1.3080, 1.3080, 1.3160, 1.3200, 1.3360, 1.3370, 1.3380, 1.3410, 1.3430, 1.3440, 1.3520, 1.3580, 1.3690, 1.3840, 1.3870, 1.3870, 1.3870, 1.3910, 1.3990, 1.4090, 1.4170, 1.4300, 1.4320, 1.4360, 1.4360, 1.4630, 1.4730, 1.4890, 1.4900, 1.4940, 1.5040, 1.5060, 1.5110, 1.5130, 1.5140, 1.5180, 1.5260, 1.5310, 1.5510, 1.5560, 1.5750, 1.5860, 1.5870, 1.5920, 1.5920, 1.5950, 1.6030, 1.6030, 1.6100, 1.6150, 1.6310, 1.6390, 1.6460, 1.6490, 1.6500, 1.6510, 1.6690, 1.6800, 1.6830, 1.6880, 1.6890, 1.6890, 1.7080, 1.7260, 1.7360, 1.7540, 1.7670, 1.7790, 1.8260, 1.8450, 1.8570, 1.8760, 1.8850, 1.9000, 1.9070, 1.9460, 1.9810, 2.0220, 2.0280, 2.0400, 2.0510, 2.0660, 2.1080, 2.1130, 2.1900, 2.2010, 2.3440, 2.3460, 2.3460, 2.4700, 2.5640, 2.6140, 2.6250, 2.9400, 3.0470, 3.0870, 3.1030, 3.1470, 3.5950, 3.6320, 3.8360, 3.9910, 4.2170, 4.4480, 4.4740, 4.5500, 4.5620, 5.0100, 5.2110, 5.8540, 5.9990, 7.2170, 8.5320

***Table* 3.2:** Descriptive Statistics of the Breaking Stress data

Statistic	**Value**	**Statistic**	**Value**	**Percentile**	**Value**
Sample Size	200	Kurtosis	13.051	Min	0.2187
Range	8.3133	Mode	1.3870	5%	0.5216
Mean	1.6234	Midrange	4.37535	10%	0.60925
Variance	1.3699			25% (Q1)	1.041
Std. Deviation	1.1704			50% (Median)	1.3765
Coef. of Variation	0.72097			75% (Q3)	1.717
Std. Error	0.08276			90%	2.7825
Skewness	2.7615			95%	4.3325
Excess Kurtosis	10.005			Max	8.532

Table **3.3:** Ryan-Joiner Test of Normality Assessment

Ryan-Joiner Test
Test statistic, Rp: 0.849
Critical value for 0.05 significance level: 0.993
Critical value for 0.01 significance level: 0.99
Reject normality with a 0.05 significance level.
Reject normality with a 0.01 significance level.
Number of data values below Q1 by more than 1.5 IQR: 0
Number of data values above Q3 by more than 1.5 IQR: 20

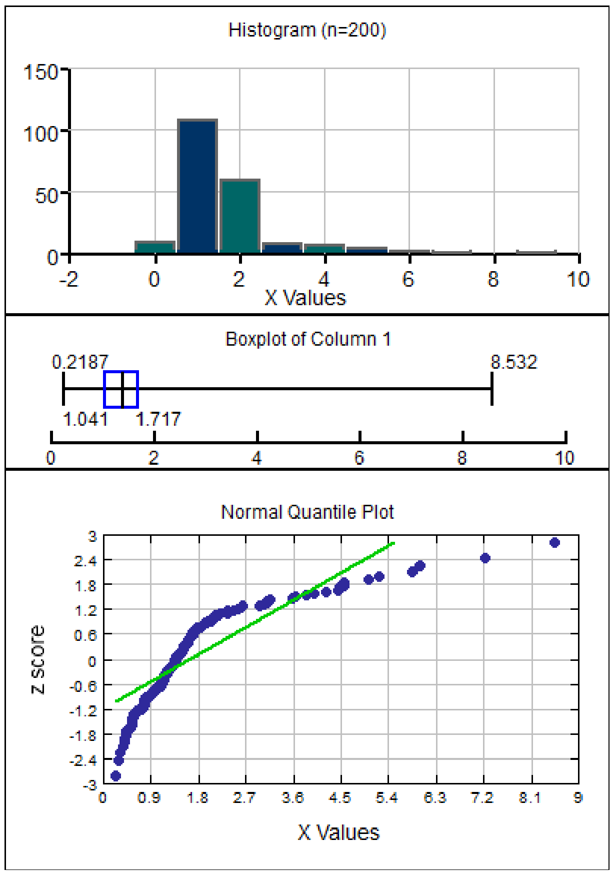

***Figure* 3.1:** Histogram, Boxplot and the Probability Plot of the Breaking Stress Data

From ***Table* 3.3** of Ryan-Joiner Test of Normality Assessment and ***Figure* 3.1** (for the histogram, boxplot and the probability plot), it is obvious that the shape of the breaking stress of 6061-T6 aluminum data is not normally distributed, instead of skewed to the right. This is also confirmed from the skewness (2.7615) and kurtosis (13.051) of the breaking stress of 6061-T6 aluminum data as computed in ***Table* 3.2.**

3.2 Estimation of parameters:

In what follows, we provide the estimation of the parameters of the Burr (3P), Burr (4P), Dagum (3P) and Dagum (4P) distributions by the method of moment and the method of maximum likelihood.

3.2.1 The Method of Moments:

If $\{X_i\}_{i=1}^{n}$ be an *iid* sample from a distribution with a m-dimensional parameter vector φ, then, according to the method of moment (MOM), the estimator $\tilde{\varphi}$ is the solution of the following system of equations:

$$(3.1)\quad E_{\tilde{\varphi}}\left(X^j\right) = \frac{\sum_{i=1}^{n} X_i^j}{n}, \quad j = 1, 2, 3, \ldots, m.$$

Thus, using the above-mentioned definition of MOM, we can obtain the respective moments from the equation (3.1) of the jth moment, $E\left(X^j\right)$, of the Burr (3P), Burr (4P), Dagum (3P) and Dagum (4P) distributions by taking the respective values of j, and evaluating the respective expressions of the respective moments numerically. Then, the moment estimations of the respective parameters of the Burr (3P), Burr (4P), Dagum (3P) and Dagum (4P) distributions can be determined by solving the system of respective equations thus obtained by Newton-Raphson's iteration method, and using some computer packages such as Maple, or Mathematica, or R, or MathCAD, or other software.

3.2.2 The Method of Maximum Likelihood:

Given a sample $\{x_i\}$, $i = 1, 2, 3, \ldots, n$, the likelihood functions of the respective pdf's of the Burr (3P), Burr (4P), Dagum (3P) and Dagum (4P) distributions are given by $L = \prod_{i=1}^{n} f(x_i)$. The objective of the likelihood function approach is to determine those values of the parameters that maximize the function L. Suppose $R = \ln(L) = \sum_{i=1}^{n} \ln[f(x_i)]$..Then, upon differentiation, the maximum likelihood estimates *(MLE)* of the respective parameters of the Burr (3P), Burr (4P), Dagum (3P) and Dagum (4P) distributions can be obtained by solving the respective maximum likelihood system of equations, applying the Newton-Raphson's iteration method and using some computer packages such as Maple, or Mathematica, or R, or MathCAD14, or other software.

3.3 Goodness-of-fit tests:

Since fitting of a probability distribution to of the breaking stress data may be helpful in predicting the probability or forecasting the frequency of occurrence of the breaking stress, this suggests that the breaking stress could possibly be modeled by some skewed distributions. As such we have tested the fitting of the Burr (3P), Burr (4P), Dagum (3P) and Dagum (4P) distributions based on their goodness of fit to the breaking stress of 6061-T6 aluminum data (***Table*** **3.1**). For this, we have used the Easyfit software for estimating the parameters of these distributions (http://www.mathwave.com/easyfit-distribution-fitting.html), which are provided in the ***Table*** **3.4.** The goodness of fit (GOF) tests, namely,

- Chi-Squared test
- Kolmogorov-Smirnov
- Anderson-Darling

are provided in the ***Tables*** **3.5, 3.6** and **3.7** respectively. For GOF tests, see, for example, Hogg and Tanis (2006), among others

Table **3.4:** Estimation of the Parameters

#	Distributions	Parameters
1	Burr (4P), with pdf as in Eq. (1).	k = 0.7002, α = 4.3802, β = 1.4014, γ = - 0.22955
2	Burr (3P), with pdf as given in sub-section 2.2.	k = 0.88229, α = 3.3412, β = 1.2771
3	Dagum (3P), with pdf as $f(x) = \dfrac{\alpha k \left(\frac{x}{\beta}\right)^{\alpha k - 1}}{\beta \left(1 + \left(\frac{x}{\beta}\right)^{\alpha}\right)^{k+1}}$, see Dagum [6, 7].	k = 0.94777, α = 3.2545, β = 1.3825
4	Dagum (4P), with pdf as $f(x) = \dfrac{\alpha k \left(\frac{x-\gamma}{\beta}\right)^{\alpha k - 1}}{\beta \left(1 + \left(\frac{x-\gamma}{\beta}\right)^{\alpha}\right)^{k+1}}$, see Dagum [6, 7].	k = 0.77762, α = 3.1754, β = 1.3882, γ = 0.11121

Table **3.5:** Comparison Criteria / Ranking of Fitted Distributions (P-Values and Test Statistics Analysis) (Based on the Chi-Square Test for Goodness-of-Fit at the Level of Significance = 0.05)

	Burr (4P) (Rank 1)	**Dagum (3P) (Rank 2)**	**Burr (3P) (Rank 3)**	**Dagum (4P) (Rank 4)**
Test Statistic	7.8209	9.609	10.277	10.926
Critical Value	14.067	14.067	14.067	14.067
P-Value	0.34865	0.21184	0.17341	0.14188

Table **3.6:** Comparison Criteria / Ranking of Fitted Distributions (P-Values and Test Statistics Analysis) (Based on the Kolmogorov Smirnov Test at the Level of Significance = 0.05)

		Kolmogorov Smirnov			
#	**Distribution**	**Test Statistic**	**Critical Value**	**P-Value**	**Rank**
1	Burr (4P)	0.07091	0.09603	0.25475	1
2	Burr (3P)	0.07244	0.09603	0.23316	2
3	Dagum (3P)	0.07307	0.09603	0.22475	3
4	Dagum (4P)	0.0735	0.09603	0.21913	4

Table **3.7:** Comparison Criteria / Ranking of Fitted Distributions (Test Statistics Analysis) (Based on the Anderson-Darling Goodness-of-Fit Test at the Level of Significance = 0.05)

		Anderson-Darling		
#	**Distribution**	**Test Statistic**	**Critical Value**	**Rank**
1	Burr (4P)	1.2487	2.5018	1
2	Dagum (3P)	1.3759	2.5018	2
3	Burr (3P)	1.3852	2.5018	3
4	Dagum (4P)	1.4157	2.5018	4

For the parameters estimated in ***Table*** **3.4**, the probability density functions (pdf's) of the Burr (3P), Burr (4P), Dagum (3P) and Dagum (4P) distributions respectively have been superimposed on the histogram of the breaking stress of 6061-T6 aluminum data, which is provided in ***Figure*** **3.2** below. For these distributions, we have also plotted the cumulative distribution function (cdf's), and probability difference, in ***Figures*** **3.3** and **3.4** respectively.

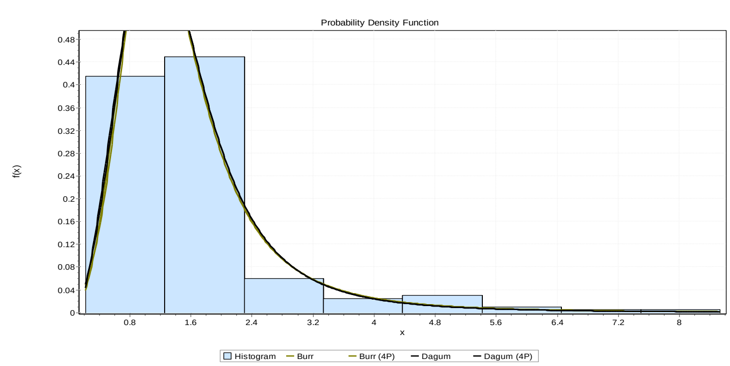

Figure **3.2:** Plots of the pdf's of the Fitted Burr (3P), Burr (4P), Dagum (3P) and Dagum (4P) Distributions to the Breaking Stress of 6061-T6 Aluminum Data

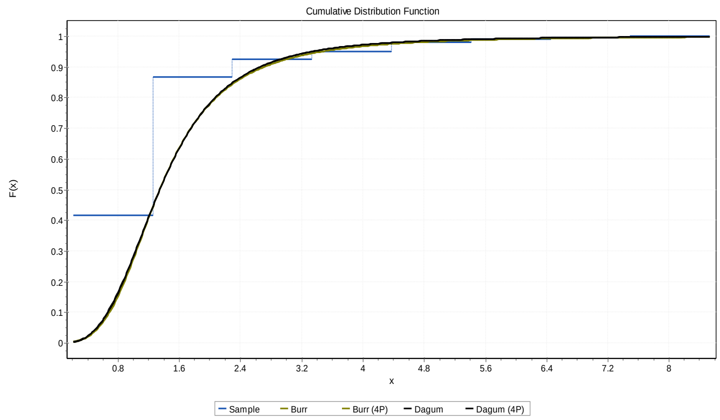

***Figure* 3.3:** Plots of the cdf's of the Fitted Burr (3P), Burr (4P), Dagum (3P) and Dagum (4P) Distributions to the Breaking Stress of 6061-T6 Aluminum Data

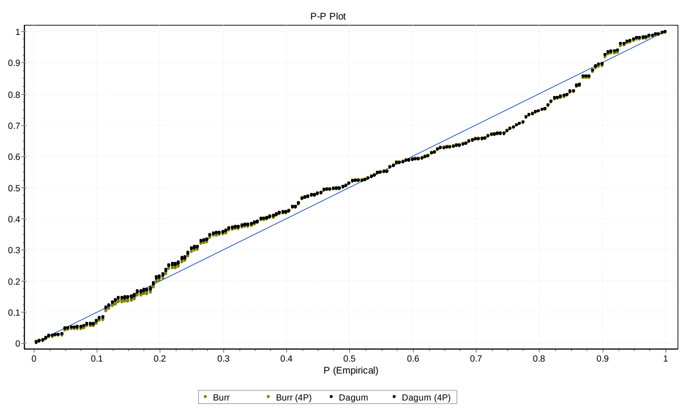

***Figure* 3.4:** P-P Plots of the Fitted Burr (3P), Burr (4P), Dagum (3P) and Dagum (4P) Distributions to the Breaking Stress of 6061-T6 Aluminum Data

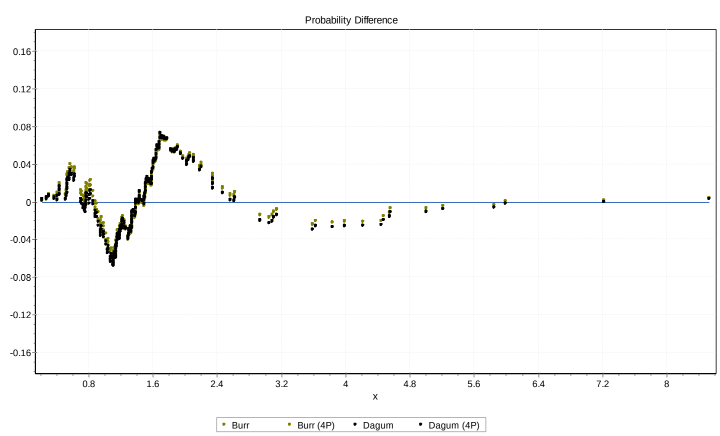

***Figure* 3.5:** Probability Difference of the Fitted Burr (3P), Burr (4P), Dagum (3P) and Dagum (4P) Distributions to the Breaking Stress of 6061-T6 Aluminum Data

3.4 Results Discussions

- From ***Table*** **3.3** of Ryan-Joiner Test of Normality Assessment and ***Figure*** **3.1** (for the histogram, boxplot and the probability plot of the data respectively), it is obvious that the shape of the breaking stress of 6061-T6 aluminum data is skewed to the right. This is also confirmed from the skewness (2.7615) and kurtosis (13.051) as computed in ***Table*** **3.2.**
- Based on the Chi-Squared test for goodness-of-fit, using the P-values and test statistics analysis, as provided in ***Table*** **3.5**, Burr (4P) distribution was found to be the best fit (Rank 1) for the breaking stress data, followed by the Dagum (3P) (Rank 2), Burr (4P) (Rank 3) and Dagum (4P) (Rank 4) distributions.
- From the Kolmogorov-Smirnov and Anderson-Darling GOF tests as provided in ***Tables*** **3.6** and **3.7** respectively, we observed that the Burr (4P) is the best fit amongst the four continuous probability distributions to the breaking stress data, since it has the lowest test statistic.
- The effects of the parameters can also be easily seen from the plots of the pdf's of the fitted Burr (3P), Burr (4P), Dagum (3P) and Dagum (4P) distributions to the breaking stress data in ***Figure*** **3.2**. For example, it is clear from these plots that the above-said distributions are positively right skewed with longer and heavier right tails for the estimated values of the parameters.
- The ***Figure*** **3.4** displays the P-P plot of the empirical cdf values plotted against the theoretical (fitted) cdf values. It is observed that the graph points fall approximately along on the diagonal line implying that the Burr (3P), Burr (4P), Dagum (3P) and Dagum (4P) distributions fit reasonably well to the observed data.
- The ***Figure*** **3.5** displays the probability difference graph, which is defined as a plot of the difference between the empirical cumulative distribution function and the fitted cdf. It is well-known that if the value of the maximum absolute difference is less than 0.05 (or 5%), we may consider the fit to be good. If the maximum absolute difference value is less than 0.01 (or 1%), then the fitting of the distributions are considered to be very good. These fact also confirmed from the probability difference plots of the fitted Burr (3P), Burr (4P), Dagum (3P) and Dagum (4P) distributions to the breaking stress data in ***Figure*** **3.5.**

4 Some Concluding Remarks

As we pointed out above, the strength or breaking stress data such as 6061-T6 aluminum or carbon fiber or composite tensile strengths are fundamental issues in many problems of quality and reliability engineering processes and designs. The statistical treatment of such data is an important aspect of their analysis and interpretation, and is therefore very crucial, and can play an important role in many studies quality and reliability engineering processes and designs. Fitting of a probability distribution to the breaking stress of 6061-T6 aluminum data may be helpful in predicting the probability or forecasting the frequency of occurrence of the breaking stress of 6061-T6 aluminum, and planning beforehand.

Motivated by the importance of such studies, in this paper, we have investigated the goodness of fit of the Burr (3P), Burr (4P), Dagum (3P) and Dagum (4P) distributions to a random sample of 200 observations of the breaking stress of 6061-T6 aluminum data set to determine their applicability and best fit to these data based on the Kolmogorov-Smirnov, Anderson-Darling, and Chi-Squared Goodness-of-Fit Tests. Based on the Chi-Squared test for goodness-of-fit, using the P-values and test statistics analysis, Burr (4P) distribution was found to be the best fit (Rank 1) to the breaking stress of 6061-T6 aluminum data. Moreover, Burr (4P) distribution is the best fit amongst the four continuous probability distributions to the breaking stress of 6061-T6 aluminum data based on the Kolmogorov-Smirnov and Anderson-Darling Goodness-of-Fit Tests, since it has the lowest test statistic.

It is hoped that this study will be helpful in many problems of quality and reliability engineering processes and designs. One can also consider of developing bootstrap control charts for the percentiles of the above-said distributions, which is an important area of studies in quality and reliability engineering.

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(Dedicated to Honor Dr. R. C. Singh Chandel on His 75^{th} Birth Anniversary Celebrations)

A NEW APPROACH TO GENERATE FORMULAE FOR PYTHAGOREANS TRIPLES, QUADRUPLES AND THEIR GENERALISATION TO *N*-TUPLES

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Abstract

In this paper, innovative methods have been devised to generate formulae for Pythagorean's Triples, Quadruples and these are finally generalised to generate Pythagorean's n-tuples. First method utilises formula for solution of a quadratic equation and generate two sets of Pythagorean's Triples. Second method determines universal identities that satisfy Pythagorean's Triples, Quadruples so on up to n-tuples. These methods are unprecedented, easy to derive at and hence are comprehensible to students and scholars alike.
2010 Mathematics Subject Classifications: Number Theory 11D09.
Keywords and phrases: Pythagorean Triples, Quadruples, Quintuples, Sextuples, Integers, n-tuples, Rational Numbers, Universal Identity.

1 Introduction

Integers X, Y and Z are said to be Pythagorean's Triple if these satisfy the relation

(1.1) $X^2 + Y^2 = Z^2$.

When $X = Y$, in that event, Equation (1.1) yields Z equal to $\sqrt{2}.X$ which is an irrational quantity and can not be Pythagorean's triple. It is amply explained in basic books of number theory [3], [5]. On the other hand, when X and Y are unequal say $X = x, Y = x + a, Z = x + b$, a and b are real and rational quantities then algebraic Equation (1.1) takes the form

$$(x)^2 + (x + a)^2 = (x + b)^2.$$

Such algebraic equations have also been used [6] while deriving identities for Pythagorean's quadruples. On expansion

(1.2) $x^2 - 2x(b - a) - (b^2 - a^2) = 0.$

Equation (1.2) being a quadratic has two roots given by

(1.3) $x = (b - a) \pm \sqrt{(b - a)^2 + (b^2 - a^2)}.$

2 Theory and Concept

***Lemma* 2.1** *Integers $x, (x+a)$ and $(x+b)$ will be unnormalised Pythagorean's Triples if $x = (b-a) \pm \sqrt{(b - a) + (b^2 - a^2)}$ and a and b are rational quantities such that $\pm \sqrt{(b - a) + (b^2 - a^2)}$ is real, perfect square and hence rational. If $x, (x + a)$ and $(x + b)$ so determined are fractions (unnormalised) , then multiplying them with their lowest common multiplier LCM will give normalised Pythagorean's Triples in integer form.*

***Lemma* 2.2** *If real and rational quantities $x, (x + a)$ and $(x + b)$ have $x = (b - a) \pm \sqrt{(b - a) + (b^2 - a^2)} = p/q, a = p_1/q_1$, $b = p_2/q_2$ and p, q, p_1, q_1, p_2, q_2 are integers then $(p.q_1.q_2), (p.q_1.q_2 + p_1.q.q_2)and(p.q_1.q_2 + p_2.q.q_1)$ will be Pythagorean's Triple.*

2.1 Values of x, a and b Satisfying Equation (1.3)

We will assume different values of a and b so that Equation (1.3) has real and rational solutions. That is term $\pm \sqrt{(b - a) + (b^2 - a^2)}$ should be a perfect square. After assumption of different values of a and b, values of x are calculated by Equation (1.3). Two cases then arise, one when x is found integer, two when x found is a fraction as p/q. If x is found to be an integer, no further operation is required,

2.1a. Normalisation: When x Found is a Fraction p/q.

If x is found to be a fraction of form p/q where p, q, a and b are integers then by Equation (1.1)

$$(p/q)^2 + (p/q + a)^2 = (p/q + b)^2.$$

On expansion and simplification

(2.1) $(p)^2 + (p + q.a)^2 = (p + q.b)^2$.

Since p, q, a and b are all integers then $p, (p + q.a)$and $(p + q.b)$ being sum and product of integers are always integers. Such explanation can be found in basic books of number theory [3], [5]. Equation (2.1) is obtained on multiplying$(p/q)^2 + (p/q + a)^2 = (p/q + b)^2$ with lowest common multiplier *LCM*. On putting different integer values of a and b, values of x are calculated from Equation (1.3) and if values of x are found to be fractions, these are normalised as discussed above. Explanation of *LCM* and conversion of fraction to integers are given in books on number theory [3], [5]. If x found is of form p/q , a of form p_1/q_1 and b of form p_2/q_2 then

$$(p/q)^2 + (p/q + p_1/q_1)^2 = (p/q + p_2/q_2)^2$$

or

(2.2) $(p.q_1.q_2)^2 + (p.q_1.q_2 + p_1.q.q_2)^2 = (p.q_1.q_2 + p_2.q.q_1)^2$.

Values of x calculated with different values of a and b are given in ***Table*** **1.1** after normalisation. It is also worth mentioning if value of x is found to be negative and as its square will always be positive, therefore, negative value of x will not have any effect on Pythagorean's triple. In the ***Table*** **1.1** , at one place, x is calculated negative 11/2, after normalisation it is taken as positive 11 on account of the fact already discussed.

Table **2.1:** Pythagorean's Triples

x	a	b	Normalised x	Normalised $y = x + a$	Normalised $z = x + b$
3	1	2	3	4	5
5/4	7/4	2	5	12	13
8/3	7/3	3	8	15	17
7/6	17/6	3	7	24	25
12/5	23/5	5	12	35	37
9/16	31/16	2	9	40	41
28/5	17/5	5	28	45	53
−11/12	71/12	6	11	60	61
33/8	23/8	4	33	56	65
48/5	7/5	5	48	55	73
13/12	71/12	6	13	84	85
39/10	41/10	5	39	80	89
65/8	7/8	4	65	72	97
88/7	17/7	7	88	105	137
85/12	47/12	6	85	132	157
119/10	1/10	5	119	120	169
133/12	23/12	6	133	156	205
207/14	17/1	7	207	224	305

2.2 Pythagorean's Quadruples

Pythagorean's quadruples x, y, w and z are given by equation given below. Studies for derivation of identifies for Pythagorean's quadruples from the point of view of area of one face of tetrahedron that equals areas of three opposite faces has been done and also Wikipedia provides an analytic account of it [1], [9].

(2.3) $x^2 + y^2 + w^2 = z^2$.

Again by putting $y = x + a$, $z = x + b$ and $w = x + c$ where$x, a, b$and c are all rational quantities, In this regards, theory of representation of rational quantities in algebraic is given in basic books on number theory [3], [5]. Equation (2.3) transforms to $(x)^2 + (x + a)^2 + (x + b)^2 = (x + c)^2$ or

(2.4) $x^2 - x(c - b - a) - \frac{1}{2}(c^2 - b^2 - a^2) = 0$.

Equation (2.4) being quadratic has roots given by

$$(2.5)\quad x = \frac{1}{2}(c-b-a) \pm \frac{1}{2}\left\{(c-b-a)^2 + 2.(c^2-b^2-a^2)\right\}^{\frac{1}{2}}.$$

***Lemma* 2.3** *Integers x, y, z and w are Pythagorean's quadruples if*

$$x = \frac{1}{2}(c-b-a) \pm \frac{1}{2}\left\{(c-b-a)^2 + 2.(c^2-b^2-a^2)\right\}^{\frac{1}{2}}, y = (x+a), w = (x+b)$$

and $z = (x+c)$ where x, a, b and c are rational quantities such that quantity $\pm\frac{1}{2}\left\{(c-b-a)^2 + 2(c^2-b^2-a^2)\right\}^{\frac{1}{2}}$ is rational, in other words quantity $\pm\frac{1}{2}\left\{(c-b-a)^2 + 2(c^2-b^2-a^2)\right\}^{\frac{1}{2}}$ is real and a perfect square. If x, y, z and w so determined are fractions, then multiplying x, y, z and w with their lowest common multiplier LCM will give normalised Pythagorean's quadruples in integer form.

***Lemma* 2.4** *If x, y, z and w are rational quantities and x is is given by*

$$x = \frac{1}{2}(c-b-a) \pm \frac{1}{2}\left\{(c-b-a)^2 + 2(c^2-b^2-a^2)\right\}^{\frac{1}{2}}, y = (x+a), w = (x+b)$$

and $z = (x+c)$ where a, b and c are rational quantities such that quantity $\pm\frac{1}{2}\left\{(c-b-a)^2 + 2.(c^2-b^2-a^2)\right\}^{\frac{1}{2}}$ is rational, in other words quantity $\pm\frac{1}{2}\left\{(c-b-a)^2 + 2(c^2-b^2-a^2)\right\}^{\frac{1}{2}}$ is real and perfect square and if x is of form p/q, a of form p_1/q_1, b of form p_2/q_2 and c of form p_3/q_3 then $(p.q_1.q_2.q_3), (p.q_1.q_2.q_3 + p_1.q.q_2.q_3), (p.q_1.q_2.q_3 + p_2.q.q_1.q_3)and(p.q_1.q_2.q_3 + p_3.q.q_1.q_2)$ will be Pythagorean's Quadruples..

2.2a. Values of x, a and b Satisfying Equation (2.5)

Different values of$a, b$and c are assumed so that Equation (2.5) has rational solutions i.e. $\left\{(c-b-a)^2+2(c^2-b^2-a^2)\right\}^{\frac{1}{2}}$ is rational and is a perfect square. After assumption of values of a, b and c, values of x are calculated by equations (2.5). On calculations two cases arise, one when x is found integer, no further operation is required, two if x is found to be a fraction of the form p/q, it requires normalisation.

2.2b. Normalisation: When x on Calculation is a Fraction of the Form p/q

When x is a fraction say of kind p/q where p, q, a and b are integers then by Equation (2.3),

$$(p/q)^2 + (p/q+a)^2 + (p/q+b)^2 = (p/q+c)^2.$$

After normalisation by multiplying with LCM,

$$(2.6)\quad p^2 + (p+q.a)^2 + (p+q.b)^2 = (p+q.c)^2.$$

Sincep, q, a and b are all integers, therefore,$p, (p+q.a), (p+q.b)$ and $(p+q.c)$ being sum and product of integers are always integers. Such explanation is given in books [3], [5]. On putting different integer values of a, b and c, values of x are calculated from equation (2.5) and if values of x are found to be fractions, these are normalised as discussed above. If x calculated is of form p/q, a of form p_1/q_1, b of form $p_2/q_2$and c of the form p_3/q_3 then

$$(p/q)^2 + (p/q+p_1/q_1)^2 + (p/q+p_2/q_2)^2 = (p/q+p_3/q_3)^2.$$

After normalisation on multiplying with LCM,

$$(2.7)\quad (p.q_1.q_2.q_3)^2 + (p.q_1.q_2.q_3 + p_1.q.q_2.q_3)^2 + (p.q_1.q_2.q_3 + p_2.q.q_1.q_3)^2(p.q_1.q_2.q_3 + p_3.q.q_1.q_2)^2.$$

Values of x are calculated by putting different values of a and b in equations (2.5). These values of $x, a, b$and c are given in ***Table* 2.2**. It is is also worth mentioning that if value of x is found to be negative and as its square will always be positive, therefore, assuming this negative value of x or $(x+a)$ or$(x+b)$ or $(x+c)$ as positive will not have any effect on Pythagorean's quadruples. In the ***Table* 2.2***,* at some places, x is calculated negative, but after normalisation, it is taken as positive on account of the facts as have already been discussed.

***Table* 2.2:** Pythagorean's Quadruples given by equation $x^2 + y^2 + z^2 = w^2$

x	a	b	c	Norm. x	Norm. $y = x + a$	Norm. $z = x + b$	Norm. $w = x + c$
1	1	1	2	1	2	2	3
1/4	7/4	3/4	2	1	8	4	9
1/8	11/8	11/8	2	1	12	12	17
2/3	7/3	4/3	3	2	9	6	11
4	0	−2	2	4	4	2	6
−7/8	11/8	11/8	2	7	4	4	9
−7/6	13/6	−13/6	3	7	6	6	11
7/2	23/2	23/2	3	7	30	30	43
9/4	−17/4	−21/4	2	9	8	12	17
9/8	−21/8	−29/8	2	9	12	20	25
−17/12	23/12	23/12	3	17	6	6	19
21/4	−5/4	−9/4	2	21	16	12	29
29/8	−15/8	−15/8	2	29	14	14	45
31/12	11/12	11/12	3	31	42	42	67
37/4	-17/4	−21/4	2	37	20	16	45
57/8	-21/8	−29/8	2	57	36	28	73

2.2c. Pythagorean Quadruples of the Form $x^2 + y^2 = z^2 + w^2$

Let $y = x + a, z = x + b$ and $w = x + c$ then $x^2 + (x + a)^2 = (x + b)^2 + (x + c)^2$.

Therefore,

$$x = \frac{b^2 + c^2 - a^2}{2(a - b - c)}, \tag{2.8}$$

where$x, a, b$and c are rationals. Amongst others, Wikipedia describes the method of generation of Pythagorean's triples [8]. Let x is of form p/q , a of form p_1/q_1 , b of form p_2/q_2 , c of form p_3/q_3 where $p, q, p_1q_1, p_2, q_2, p_3andq_3$ are all integers then after normalisation,

$$(p.q_1.q_2.q_3)^2 + (p.q_1.q_2.q_3 + p_1.q.q_2.q_3)^2 = (p.q_1.q_2.q_3 + p_2.q.q_1.q_3)^2 + (p.q_1.q_2.q_3 + p_3.q.q_1.q_2)^2. \tag{2.9}$$

Based on the above said formula, some Pythagorean's quadruples are given in ***Table* 2.3**.

***Table* 2.3:** Pythagorean Quadruples of the Form $x^2 + y^2 = z^2 + w^2$

x	a	b	c	Normalised x	Normalised $y = x + a$	Normalised $z = x + b$	Normalised $w = x + c$
-7/6	1	2	2	7	1	5	5
-17/10	1	3	3	17	7	13	13
-31/4	1	4	4	31	17	25	25
-19/2	1	-2	4	19	17	23	11
-19/14	-1	2	4	19	33	9	7
19/6	1	2	-4	19	25	31	5
1/6	2	1	-2	1	13	7	11
-1/10	-2	1	2	1	21	9	19
7/2	2	3	-3	7	11	13	1
7/8	2	-3	-3	7	23	17	17
-1/3	3	-2	-1	1	8	7	4
-1/10	3	-3	1	1	31	29	11
1/14	3	-3	-1	1	43	41	13
-9/2	1	-1	3	9	7	11	13

2.2d. Pythagorean Quadruples of the Form $x^2+y^2+w^2 = v^2$ where $x^2+y^2 = z^2$ and $z^2+w^2 = v^2$

Pythagorean's triples of the form$x^2+y^2 = z^2$have already been formulated in paragraphs **1** to **2.1a** and are not repeated here. After finding value of z, value of c and d are to be found so that these may satisfy the equation $z^2 + w^2 = v^2$. This equation is same as $x^2 + y^2 = z^2$and values of z can be found by Equation (1.3). Values of x, y, z and w are given in ***Table* 2.4.**

***Table* 2.4:** Pythagorean Quadruples of the Form $x^2 + y^2 = z^2 + w^2$

x	y	$z = \left(x^2 + y^2\right)^{\frac{1}{2}}$	w	$v = \left(z^2 + w^2\right)^{\frac{1}{2}}$
3	4	5	12	13
5	12	13	84	85
9	12	15	8	17
13	84	85	132	157
15	36	39	80	89
28	96	100	105	145
33	56	65	420	425
33	56	65	72	97

2.2e. Pythagorean Quintuples of the Form $x^2 + y^2 + z^2 + w^2 = v^2$

Let $y = (x + a), z = (x + b), w = (x + c)$ and $v = (2x + d)$ [8] then $(x)^2 + (x + a)^2 + (x + b)^2 + (x + c)^2 = (2.x + d)^2$.

Therefore,

$$x = \frac{d^2 - a^2 - b^2 - c^2}{2(a + b + c - 2.d)}. \tag{2.10}$$

On putting different integer values of a, b, c and d, different values of x are obtained. From those values of x, values ofy, z, w and v are calculated and normalised. Wikipedia describes the method of generation of Pythagorean's quintuples [8]. ***Table* 2.5** gives Pythagorean's Quintuples based on the above said formula.

***Table* 2.5:** Pythagorean Pentagonal Numbers of the Form $x^2 + y^2 + z^2 + w^2 = v^2$

a	b	c	d	x	Norm.x	Norm.$y = x + a$	Norm.$z = x + b$	Norm.$w = x + c$	Norm.$v = 2x+d = (x^2 + y^2 + z^2 + w^2)^{\frac{1}{2}}$
1	2	3	4	-1/2	1	1	3	5	6
1	3	4	5	1/4	1	5	13	17	22
1	2	3	-\4	1/14	1	15	29	43	54
1	2	-\3	4	-1/8	1	7	15	25	30
-1	2	3	4	-1/4	1	5	7	11	14
-1	-2	3	4	-1/8	1	9	17	23	30
-1	2	-3	4	-1/10	1	11	19	31	38
-1	2	3	-\4	1/12	1	11	25	37	46
1	-3	4	5	1/16	1	17	47	65	82
1	3	-\4	5	1/20	1	21	61	79	102

2.2f. Pythagorean *N*-Tuples of the Form $x_1^2 + x_2^2 + x_3^2 + \ldots + x_n^2 = y^2$

Let $x_2 = (x_1 + a_2)$, $x_3 = (x_1 + a_3)$, $x_4 = (x_1 + a_4), \ldots, x_n = (x_1 + a_n)$ and $y = (x_1 + a_1)$then putting these values in equation $x_1^2 + x_2^2 + x_3^2 + \ldots x_n^2 = y^2$ it will take the form

$$x_1^2 + (x_1 + a_2)^2 + (x_1 + a_3)^2 + (x_1 + a_4)^2 + \ldots + (x_1 + a_n)^2 = (x_1 + a_1)^2.$$

Methods of generating such generalised Pythagorean's n-tuples has been described by many mathematicians [4], [10], [1], [2]. On expansion and rearrangement,

$$(n - 1)x_1^2 + 2x_1(a_2 + a_3 + a_4 + \ldots + a_n - a_1) + a_2^2 + a_3^2 + a_4^2 + \ldots a_n^2 - a_1^2 = 0$$

This is a quadratic in x_1 and its roots are

$$x_1 = \frac{-Q \pm \sqrt{Q^2 - 4P.R}}{2P},$$

where $P = (n-1), Q = 2(a_2 + a_3 + a_4 + \ldots + a_n - a_1)$ and $R = a_2^2 + a_3^2 + a_4^2 + \ldots a_n^2 - a_1^2$. Values of $a_1, a_2, a_3, \ldots a_n$ can be assumed arbitrarily such that part under square root is rational and that determines x_1. Once x_1 is determined, $a_1, a_2, a_3, \ldots, a_n$ are known, $x_2, x_3, \ldots, x_n$ and y can be calculated. By assuming different set of $a_1, a_2, a_3, \ldots, a_n$, different set of $x_1, x_2, x_3, \ldots, x_n$ can be found out, hence n-tuples are generated.

2.3 Second Method: To Generate Pythagorean's Numbers

2.3a. Pythagorean's Triples

Pythagorean's Triples are given by the equation $a^2 + b^2 = c^2$. This can also be written as $a^2 = c^2 - b^2$ or $a^2 = (c-b)(c+b)$or $(\frac{c}{a} - \frac{b}{a})(\frac{c}{a} + \frac{b}{a}) = 1$. Let $x = (\frac{c}{a} + \frac{b}{a})$then from above equation, $\frac{1}{x} = (\frac{c}{a} - \frac{b}{a})$. On adding and subtracting,

(2.11) $x + \dfrac{1}{x} = 2\dfrac{c}{a}$

(2.12) $x - \dfrac{1}{x} = 2\dfrac{b}{a}.$

Since $c^2/a^2 - b^2/a^2 = 1$, therefore using Equations (2.11) and (2.12),

(2.13) $\dfrac{1}{4}(x + \dfrac{1}{x})^2 - \dfrac{1}{4}(x - \dfrac{1}{x})^2 = 1,$

(2.14) $(x - \dfrac{1}{x})^2 + 2^2 = (x + \dfrac{1}{x})^2,$

identity (2.14) generates Pythagorean's Triples $(x - 1/x)$, 2 and $(x + 1/x)$ satisfying $a^2 + b^2 = c^2$.[8].Let $x = \frac{3}{2}$then $(\frac{3}{2} - \frac{2}{3})$, 2 and $(\frac{3}{2} + \frac{2}{3})$ on normalisation are Pythagorean's triple as 5, 12 and 13. Procedure analogous to this has also been adopted by some mathematicians and it also makes mention in Wikipedia.[7], [8]. Some Pythagorean's triples using Equation (2.14) are given in ***Table* 2.6**.

***Table* 2.6:** Pythagorean's Triples

x	$(x-1/x)$	2	$(x+1/x)$	Normalised a	Normalised b	Normalised c
2	3/2	2	5/2	3	4	5
3/2	5/6	2	13/6	5	12	13
4/3	7/12	2	25/12	7	24	25
5/4	9/20	2	41/20	9	40	41
4	15/4	2	17/4	15	8	17
6	35/6	2	37/6	35	12	37
8/3	55/24	2	73/24	55	48	73
8/5	65/40	2	89/40	65	80	89
8/7	15/56	2	113/56	15	112	113
8	63/8	2	65	63	16	65

2.3b. Pythagorean's Quadruples of the Form $a^2+b^2+c^2 = d^2$ where $a^2+b^2 = e^2$and $e^2+c^2 = d^2$

Let $a^2 + b^2 = e^2$ then as proved in paragraph **2.3a**, $x - \frac{1}{x} = 2\frac{b}{a}$ and $x + \frac{1}{x} = 2\frac{e}{a}$. That makes $a^2 + b^2 = e^2$ as $\frac{1}{4}(x - \frac{1}{x})^2 + 1 = \frac{1}{4}(x + \frac{1}{x})^2$. On dividing left hand side and right hand side by $\frac{1}{4}(x + \frac{1}{x})^2$

(2.15) $(x - \dfrac{1}{x})^2(x + \dfrac{1}{x})^{-2} + 4(x + \dfrac{1}{x})^{-2} = 1.$

Since $a^2+b^2+c^2 = d^2$ and it is assumed $a^2+b^2 = e^2$, therefore,$e^2+c^2 = d^2$. That makes $y - \frac{1}{y} = 2\frac{c}{e}$ and $y + \frac{1}{y} = 2\frac{d}{e}$. Therefore,

(2.16) $1 + \dfrac{1}{4}(y - \dfrac{1}{y})^2 = \dfrac{1}{4}(y + \dfrac{1}{y})^2.$

On putting value of 1 from equation (2.15) in Equation (2.16),

(2.17) $(x - \dfrac{1}{x})^2(x + \dfrac{1}{x})^{-2} + 4(x + \dfrac{1}{x})^{-2} + \dfrac{1}{4}(y - \dfrac{1}{y})^2 = \dfrac{1}{4}(y + \dfrac{1}{y})^2.$

Equation (2.17) is an equation of Pythagorean's Quadruples for all real and rational values of x and y except $x = y = 1$. An example where $x = 2, y = 3$ is taken up below,

$(x - \frac{1}{x})^2(x + \frac{1}{x})^{-2} = (\frac{9}{4})(\frac{4}{25}) = \frac{9}{25}$,

$4(x + \frac{1}{x})^{-2} = 4(\frac{4}{25}) = \frac{16}{25}$,

$4(x + \frac{1}{x})^{-2} = 4(\frac{4}{25}) = \frac{16}{25}$,

$\frac{1}{4}(y - \frac{1}{y})^2 = \frac{1}{4}(\frac{64}{9}) = \frac{16}{9}$,

$\frac{1}{4}(y + \frac{1}{y})^2 = \frac{1}{4}(\frac{100}{9}) = \frac{25}{9}$.

That makes $\frac{9}{25} + \frac{16}{25} + \frac{16}{9} = \frac{25}{9}$ or $(\frac{3}{5})^2 + (\frac{4}{5})^2 + (\frac{4}{3})^2 = (\frac{5}{3})^2$ and after normalisation,

$9^2 + 12^2 + 20^2 = 25^2$.

In this way, using equation (2.17), Pythagorean's Quadruples are generated and are given in ***Table* 2.7**.

***Table* 2.7:** Pythagorean's Quadruples of the Form $a^2 + b^2 + c^2 = d^2$ where $a^2 + b^2 = e^2$ and $e^2 + c^2 = d^2$

x	y	$\frac{(x-\frac{1}{x})}{(x+\frac{1}{x})}$	$2(x + \frac{1}{x})$	$\frac{1}{2}(y - \frac{1}{y})$	$\frac{1}{2}(y + \frac{1}{y})$	a	b	e	c	d
2	3	3/5	4/5	4/3	5/3	9	12	15	20	25
3	4	4/5	3/5	15/8	17/8	32	24	40	75	80
2	5	3/5	4/5	12/5	13/5	3	4	5	12	13
2	7	3/5	4/5	24/7	25/7	21	28	35	120	125
3	2	4/5	3/5	3/4	5/4	16	12	20	15	25
3	8	4/5	3/5	63/16	65/16	64	48	80	315	325
2	9	3/5	4/5	40/9	41/9	27	36	45	200	205
3	11	4/5	3/5	60/11	61/11	44	33	55	300	305
4	5	15/17	8/17	12/5	13/5	75	40	85	204	221
3	3/2	4/5	3/5	5/12	13/12	48	36	60	25	65

2.3c. Pythagorean's Quintuples of the Form $a^2+b^2+c^2+d^2 = g^2$ where $a^2+b^2 = e^2$, $e^2+c^2 = f^2$ and $f^2 + d^2 = g^2$.

Pythagorean's Quadruples, generating equation has already been derived at (2.17), we shall proceed further from this equation which can also be written as

$$4(x - \frac{1}{x})^2(x + \frac{1}{x})^{-2}(y + \frac{1}{y})^{-2} + 4^2(x + \frac{1}{x})^{-2}(y + \frac{1}{y})^{-2} + (y - \frac{1}{y})^2(y + \frac{1}{y})^{-2} = 1$$

or

$$\left\{2(x - \frac{1}{x})(x + \frac{1}{x})^{-1}(y + \frac{1}{y})^{-1}\right\}^2 + \left\{4(x + \frac{1}{x})^{-1}(y + \frac{1}{y})^{-1}\right\}^2 + \left\{(y - \frac{1}{y})(y + \frac{1}{y})^{-1}\right\}^2 = 1, \tag{2.18}$$

since $a^2 + b^2 = e^2$, $e^2 + c^2 = f^2$ and $f^2 + d^2 = g^2$.

From discussion made in the paragraph related to generation of Quadruples, f, d and g can be given by relations, $\frac{f}{d} = \frac{1}{2}(z - \frac{1}{z})$ and $\frac{g}{d} = \frac{1}{2}(z + \frac{1}{z})$ where z is real rational number. Equation $f^2 + d^2 = g^2$ can be written as $\frac{1}{4}(z - \frac{1}{z})^2 + 1 = \frac{1}{4}(z + \frac{1}{z})^2$ and substituting 1 as given by equation (2.18), this equation takes the form

$$\left\{2(x - \frac{1}{x})(x + \frac{1}{x})^{-1}(y + \frac{1}{y})^{-1}\right\}^2 + \left\{4(x + \frac{1}{x})^{-1}(y + \frac{1}{y})^{-1}\right\}^2 + \left\{(y - \frac{1}{y})(y + \frac{1}{y})^{-1}\right\}^2 + \left\{\frac{1}{2}(z - \frac{1}{z})\right\}^2 = \left\{\frac{1}{2}(z + \frac{1}{z})\right\}^2. \tag{2.19}$$

Equation (2.19) after normalisation generates Pythagorean's Quintuples for all real rational values of x, y and z. Let $A = \left\{2(x - \frac{1}{x})(x + \frac{1}{x})^{-1}(y + \frac{1}{y})^{-1}\right\}$, $B = \left\{4(x + \frac{1}{x})^{-1}(y + \frac{1}{y})^{-1}\right\}$, $C = \left\{(y - \frac{1}{y}).(y + \frac{1}{y})^{-1}\right\}$, $D = \left\{\frac{1}{2}(z - \frac{1}{z})\right\}$ and $G = \left\{\frac{1}{2}.(z + \frac{1}{z})\right\}$. Un-normaised Pythagorean's Quintuples are generated by equation $A^2 + B^2 + C^2 + D^2 = G^2$ and after normalisation, we get $a^2 + b^2 + c^2 + d^2 = g^2$.

Let us take an example where $x = 2, y = 3$ and $z = 4$. Then $A^2 = \left\{2(\frac{3}{2})(\frac{2}{5})(\frac{3}{10})\right\}^2 = (\frac{9}{25})^2$, $B^2 = \left\{4(\frac{2}{5})(\frac{3}{10})\right\}^2 = (\frac{12}{25})^2$, $C^2 = \left\{(\frac{8}{3})(\frac{3}{10})\right\}^2 = (\frac{4}{5})^2$, $D^2 = \left\{\frac{1}{2}(\frac{15}{4})\right\}^2 = (\frac{15}{8})^2$ and $G^2 = \left\{\frac{1}{2}(\frac{17}{4})\right\}^2 = (\frac{17}{8})^2$. Since $A^2 + B^2 + C^2 + D^2 = G^2$, therefore, $(\frac{9}{25})^2 + (\frac{12}{25})^2 + (\frac{4}{5})^2 + (\frac{15}{8})^2 = (\frac{17}{8})^2$. After normalisation, $a^2+b^2+c^2+d^2 = g^2$ or $72^2+96^2+160^2+375^2 = 425^2$. In this way, using equation (2.19), Pythagorean's Quintuples are generated and are given in the ***Table* 2.8.**

Table **2.8:** Pythagorean's Quintuples Of The Form $a^2 + b^2 + c^2 + d^2 = g^2$

x	y	z	A	B	C	D	G	a	b	e	c	f	d	g
2	3	4	9/25	12/25	4/5	15/8	17/8	72	96	120	160	200	375	425
2	3	5	9/25	12/25	4/5	12/5	13/5	9	12	15	20	25	60	65
2	3	2	9/25	12/25	4/5	3/4	5/4	36	48	60	80	100	75	125
3	2	4	16/25	12/25	3/5	15/8	17/8	128	96	160	120	200	375	425
2	2	2	12/25	16/25	3/5	3/4	5/4	48	64	80	60	100	75	125
2	2	3	12/25	16/25	3/5	4/3	5/3	36	48	60	45	75	100	125
2	2	3/2	12/25	16/25	3/5	5/12	13/12	144	192	240	180	300	125	325
2	3	3	9/25	12/25	4/5	4/3	5/3	27	36	45	60	75	100	125
3	3	3/2	12/25	9/25	4/5	5/12	13/12	144	108	180	240	300	125	325
2	2	5	12/25	16/25	3/5	12/5	13/5	12	16	20	15	25	60	65
3	3	5/2	12/25	9/25	4/5	21/20	29/20	48	36	60	80	100	105	145

2.3d. Pythagorean's Sextuples of the Form $a^2 + b^2 + c^2 + d^2 + g^2 = h^2$ where $a^2 + b^2 = e^2$, $e^2 + c^2 = f^2$, $f^2 + d^2 = k^2$ and $k^2 + g^2 = h^2$

Equation (2.19) can also be written as

$$(2.20)\ \left\{2^3(x+\frac{1}{x})^{-1}(y+\frac{1}{y})^{-1}(z+\frac{1}{z})^{-1}\right\}^2 + \left\{2^2(x-\frac{1}{x})(x+\frac{1}{x})^{-1}(y+\frac{1}{y})^{-1}(z+\frac{1}{z})^{-1}\right\}^2$$
$$+\left\{2(y-\frac{1}{y})(y+\frac{1}{y})^{-1}(z+\frac{1}{z})^{-1}\right\}^2 + \left\{2^0(z-\frac{1}{z})(z+\frac{1}{z})^{-1}\right\}^2 = 1.$$

Also $k^2 + g^2 = h^2$ can be written as $\left\{\frac{1}{2}(w-\frac{1}{w})\right\}^2 + 1 = \left\{\frac{1}{2}(w+\frac{1}{w})\right\}^2$ where $w+\frac{1}{w} = 2\frac{h}{k}$ and $w-\frac{1}{w} = 2\frac{g}{k}$ from equations (2.11) and (2.12). On substituting 1 as given in equation (2.20) in the above equation, we get

$$(2.21)\ \left\{2^3(x+\frac{1}{x})^{-1}(y+\frac{1}{y})^{-1}(z+\frac{1}{z})^{-1}\right\}^2 + \left\{2^2(x-\frac{1}{x})(x+\frac{1}{x})^{-1}(y+\frac{1}{y})^{-1}(z+\frac{1}{z})^{-1}\right\}^2$$
$$+\left\{2(y-\frac{1}{y})(y+\frac{1}{y})^{-1}(z+\frac{1}{z})^{-1}\right\}^2 + \left\{2^0(z-\frac{1}{z})(z+\frac{1}{z})^{-1}\right\}^2 + \left\{2^{-1}(w-\frac{1}{w})\right\}^2 = \left\{2^{-1}(w+\frac{1}{w})\right\}^2.$$

Equation (2.21) after normalisation generates Pythagorean's Sextuples for all real rational values of x, y, z and w. Let

$A^2 = \left\{2^3(x+\frac{1}{x})^{-1}(y+\frac{1}{y})^{-1}(z+\frac{1}{z})^{-1}\right\}^2$

$B^2 = \left\{2^2(x-\frac{1}{x})(x+\frac{1}{x})^{-1}(y+\frac{1}{y})^{-1}(z-\frac{1}{z})^{-1}\right\}^2$

$C^2 = \left\{2(y-\frac{1}{y})(y+\frac{1}{y})^{-1}(z+\frac{1}{z})^{-1}\right\}^2$

$D^2 = \left\{(z-\frac{1}{z})(z+\frac{1}{z})^{-1}\right\}^2$

$G^2 = \left\{\frac{1}{2}(w-\frac{1}{w})\right\}^2$

$H^2 = \left\{\frac{1}{2}(w+\frac{1}{w})\right\}^2,$

then unnormalised Pythagorean's Sextuples are generated byA, B, C, D, G and H by the equation (2.21) where

$A^2 + B^2 + C^2 + D^2 + G^2 = H^2$.

After normalisation, we get

$a^2 + b^2 + c^2 + d^2 + g^2 = h^2$.

Let us take an example where$x = 2, y = 2, z = 2$ and $w = 3$ then

$A^2 = \left\{2^3(\frac{2}{5})(\frac{2}{5})(\frac{2}{5})\right\}^2 = (\frac{64}{125})^2,$

$B^2 = \left\{2^2(\frac{3}{2})(\frac{2}{5})(\frac{2}{5})(\frac{2}{5})\right\}^2 = (\frac{48}{125})^2,$

$$C^2 = \left\{2(\tfrac{3}{2})(\tfrac{2}{5})(\tfrac{2}{5})\right\}^2 = (\tfrac{12}{25})^2,$$
$$D^2 = \left\{(\tfrac{3}{2})(\tfrac{2}{5})\right\}^2 = (\tfrac{3}{5})^2,$$
$$G^2 = \left\{(\tfrac{1}{2})(\tfrac{8}{3})\right\}^2 = (\tfrac{4}{3})^2,$$
$$H^2 = \left\{(\tfrac{1}{2})(\tfrac{10}{3})\right\}^2 = (\tfrac{5}{3})^2.$$

Therefore,
$(\frac{64}{125})^2 + (\frac{48}{125})^2 + (\frac{12}{25})^2 + (\frac{3}{5})^2 + (\frac{4}{3})^2 = (\frac{5}{3})^2.$
After normalisation
$192^2 + 144^2 + 180^2 + 225^2 + 500^2 = 625^2.$

In this way, using Equation (2.21), Pythagorean's Sextuples are generated and unnormalised are given in the ***Table* 2.9 A** and normalised in ***Table* 2.9 B.**

***Table* 2.9 A:** Unnormalised Pythagorean's Sextuples

x	y	z	w	A	B	C	D	G	H
2	2	2	3	64/125	48/125	12/25	3/5	4/3	5/3
2	2	2	2	64/125	48/125	12/25	3/5	3/4	5/4
2	2	2	4	64/125	48/125	12/25	3/5	15/8	17/8
2	2	2	3/2	64/125	48/125	12/25	3/5	5/12	13/12
2	2	2	5	64/125	48/125	12/25	3/5	12/5	13/5
2	2	2	10	64/125	48/125	12/25	3/5	99/20	101/20
2	2	2	25	64/125	48/125	12/25	3/5	312/25	313/25
2	2	2	5/3	64/125	48/125	12/25	3/5	8/15	17/15

***Table* 2.9 B:** Pythagorean's Sextuples of the Form $a^2 + b^2 + c^2 + d^2 + g^2 = h^2$

x	y	z	w	a	b	c	d	g	h
2	2	2	3	192	144	180	225	500	625
2	2	2	2	256	192	240	300	375	625
2	2	2	4	512	384	480	600	1875	2125
2	2	2	3/2	768	576	720	900	625	1625
2	2	2	5	64	48	60	75	300	325
2	2	2	10	256	192	240	300	2475	2525
2	2	2	25	64	48	60	75	1560	1565
2	2	2	5/3	192	144	180	225	200	425

2.3e. Pythagorean's *N*-Tuples of the Form $x_1^2 + x_2^2 + x_3^2 + \ldots + x_n^2 = y^2$

Equations 2.17), (2.19) and (2.20) on changing $xtox_1, ytox_2, ztox_3$ so on can be written in the following way for number of Pythagorean's terms $n = 4$,

$$\left\{2(x_1 + \frac{1}{x_1})^{-1}\right\}^2 + \left\{2^0(x_1 - \frac{1}{x_1})(x_1 + \frac{1}{x_1})^{-1}\right\}^2 + \left\{2^{-1}(x_2 - \frac{1}{x_2})\right\}^2 = \left\{2^{-1}(x_2 + \frac{1}{x_2})\right\}^2.$$

For number of Pythagorean's terms $n = 5$,

$$\left\{2^2(x_1 + \frac{1}{x_1})^{-1}(x_2 + \frac{1}{x_2})^{-1}\right\}^2 + \left\{2^1(x_1 - \frac{1}{x_1})(x_1 + \frac{1}{x_1})^{-1}(x_2 + \frac{1}{x_2})^{-1}\right\}^2$$
$$+\left\{2^0(x_2 - \frac{1}{x_2})(x_2 + \frac{1}{x_2})^{-1}\right\}^2 + \left\{2^{-1}(x_3 - \frac{1}{x_3})\right\}^2 = \left\{2^{-1}(x_3 + \frac{1}{x_3})\right\}^2.$$

For number of Pythagorean's terms $n = 6$,

$$\left\{2^3(x_1 + \frac{1}{x_1})^{-1}(x_2 + \frac{1}{x_2})^{-1}(x_3 + \frac{1}{x_3})^{-1}\right\}^2 + \left\{2^2(x_1 - \frac{1}{x_1})(x_1 + \frac{1}{x_1})^{-1}(x_2 + \frac{1}{x_2})^{-1}(x_3 + \frac{1}{x_3})^{-1}\right\}^2$$

$$+\left\{2^1(x_2-\frac{1}{x_2})(x_2+\frac{1}{x_2})^{-1}(x_3+\frac{1}{x_3})^{-1}\right\}^2+\left\{2^0(x_3-\frac{1}{x_3}).(x_3+\frac{1}{x_3})^{-1}\right\}^2+\left\{2^{-1}(x_4-\frac{1}{x_4})\right\}^2=\left\{2^{-1}(x_4+\frac{1}{x_4})\right\}^2.$$

By mathematical induction for Pythagorean's terms n,

(2.22) $$\left\{2^{-1}(x_{n-2}+\frac{1}{x_{n-2}})\right\}^2$$
$$=\left\{2^{n-3}(x_1+\frac{1}{x_1})^{-1}(x_2+\frac{1}{x_2})^{-1}.(x_3+\frac{1}{x_3})^{-1}\dots(x_{n-3}+\frac{1}{x_{n-3}})^{-1}\right\}^2$$
$$+\left\{2^{n-4}(x_1-\frac{1}{x_1})(x_1+\frac{1}{x_1})^{-1}(x_2+\frac{1}{x_2})^{-1}\dots.(x_{n-3}+\frac{1}{x_{n-3}})^{-1}\right\}^2$$
$$+\left\{2^{n-5}(x_2-\frac{1}{x_2})(x_2+\frac{1}{x_2})^{-1}(x_3+\frac{1}{x_3})^{-1}\dots(x_{n-3}+\frac{1}{x_{n-3}})^{-1}\right\}^2$$
$$+\left\{2^{n-6}(x_3-\frac{1}{x_3})(x_3+\frac{1}{x_3})^{-1}(x_4+\frac{1}{x_4})^{-1}\dots(x_{n-3}+\frac{1}{x_{n-3}})^{-1}\right\}^2$$
$$+\dots+\left\{2^0(x_{n-3}-\frac{1}{x_{n-3}}).(x_{n-3}+\frac{1}{x_{n-3}})^{-1}\right\}^2+\left\{2^{-1}(x_{n-2}-\frac{1}{x_{n-2}})\right\}^2.$$

This can also be written in mathematical notation,

$$\left\{2^{2(n-3)}\prod_{k=1}^{n-3}(x_k+\tfrac{1}{x_k})^{-2}\right\}+\left\{2^{2(n-4)}(x_1-\tfrac{1}{x_1})^2\prod_{k=1}^{n-3}(x_k+\tfrac{1}{x_k})^{-2}\right\}+\left\{2^{2(n-5)}(x_2-\tfrac{1}{x_2})^2\prod_{k=2}^{n-3}(x_k+\tfrac{1}{x_k})^{-2}\right\}$$
$$+\left\{2^{2(n-6)}(x_3-\frac{1}{x_3})^2\prod_{k=3}^{n-3}(x_k+\tfrac{1}{x_k})^{-2}\right\}+\dots+\left\{2^2(x_{n-4}-\tfrac{1}{x_{n-4}})^2\prod_{k=n-4}^{n-3}(x_k+\tfrac{1}{x_k})^{-2}\right\}$$
$$+\left\{2^0(x_{n-3}-\frac{1}{x_{n-3}})^2(x_{n-3}+\frac{1}{x_{n-3}})^{-2}\right\}+\left\{2^{-2}(x_{n-2}-\frac{1}{x_{n-2}})^2\right\}=\left\{2^{-2}(x_{n-2}+\frac{1}{x_{n-2}})^2\right\}.$$

Symbol $\prod_{k=1}^{n-3}(x_k+\frac{1}{x_k})^{-2}$ denotes product of terms $(x_k+\frac{1}{x_k})^{-2}$ when k varies from 1 to $n-3$. Let us take an example of generation of Pythagorean's numbers when its terms n equals to 9 and $x_1=2, x_2=2, x_3=2, x_4=2, x_5=3, x_6=2$ and $x_7=3$. Identity (2.22) for $n=9$, transforms to

$$2^{2(6)}\prod_{k=1}^{6}(x_k+\tfrac{1}{x_k})^{-2}+2^{2(5)}(x_1-\tfrac{1}{x_1})^2\prod_{k=1}^{6}(x_k+\tfrac{1}{x_k})^{-2}+2^{2(4)}(x_2-\tfrac{1}{x_2})^2\prod_{k=2}^{6}(x_k+\tfrac{1}{x_k})^{-2}$$
$$+2^{2(3)}(x_3-\frac{1}{x_3})^2\prod_{k=3}^{6}(x_k+\tfrac{1}{x_k})^{-2}+\dots+2^2(x_5-\tfrac{1}{x_5})^2\prod_{k=5}^{6}(x_k+\tfrac{1}{x_k})^{-2}$$
$$+2^0(x_6-\frac{1}{x_6})^2(x_6+\frac{1}{x_6})^{-2}+2^{-2}(x_7-\frac{1}{x_7})^2=2^{-2}(x_7+\frac{1}{x_7})^2.$$

Let us denote above terms as $A^2, B^2, C^2, D^2, E^2, F^2, G^2, H^2 and I^2$ respectively so that $A^2+B^2+C^2+D^2+E^2+F^2+G^2+H^2=I^2$. Values of $A^2, B^2, C^2, D^2, E^2, F^2, G^2, H^2$ and I^2 are calculated by putting values of say $x_1=2, x_2=2, x_3=2, x_4=2, x_5=3, x_6=2$ and $x_7=3$ in above identity as solved below.

$$A^2=\left\{2^6(\frac{2}{5})^6\right\}^2=(\frac{2^{12}}{5^6})^2, B^2=\left\{2^5(\frac{3}{2})(\frac{2}{5})^6\right\}^2=\left\{3(\frac{2^{10}}{5^6})\right\}^2, C^2=\left\{2^4(\frac{3}{2})(\frac{2}{5})^5\right\}^2=\left\{3(\frac{2^8}{5^5})\right\}^2,$$
$$D^2=\left\{2^3(\frac{3}{2})(\frac{2}{5})^4\right\}^2=\left\{3(\frac{2^6}{5^4})\right\}^2, E^2=\left\{2^2(\frac{3}{2})(\frac{2}{5})^3\right\}^2=\left\{3(\frac{2^4}{5^3})\right\}^2, F^2=\left\{2(\frac{3}{2})(\frac{2}{5})^2\right\}^2=\left\{3(\frac{2}{5})^2\right\}^2,$$
$$G^2=\left\{(\frac{3}{2})(\frac{2}{5})\right\}^2=(\frac{3}{5})^2, H^2=\left\{(\frac{1}{2})(\frac{8}{3})\right\}^2=(\frac{4}{3})^2, I^2=\left\{(\frac{1}{2})(\frac{10}{3})\right\}^2=(\frac{5}{3})^2.$$

Therefore, above identity can be written as

$$\left\{\frac{2^{12}}{5^6}\right\}^2+\left\{3(\frac{2^{10}}{5^6})\right\}^2+\left\{3(\frac{2^8}{5^5})\right\}^2+\left\{3(\frac{2^6}{5^4})\right\}^2+\left\{3(\frac{2^4}{5^3})\right\}^2+\left\{3(\frac{2}{5})^2\right\}^2+\left\{\frac{3}{5}\right\}^2+\left\{\frac{4}{3}\right\}^2=\left\{\frac{5}{3}\right\}^2.$$

On normalisation,

$$\left\{3(2^{12})\right\}^2+\left\{3^2(2^{10})\right\}^2+\left\{5(3^2)(2^8)\right\}^2+\left\{(5^2)(3^2)(2^6)\right\}^2+\left\{(5^3)(3^2)(2^4)\right\}^2+\left\{(5^4)(3^2)(2^2)\right\}^2+\left\{(5^5)(3^2)\right\}^2+\left\{(5^6)(2^2)\right\}^2$$
$$=\left\{5^7\right\}^2,\text{ or } 12288^2+9216^2+11520^2+14400^2+18000^2+22500^2+28125^2+62500^2=78125^2.$$

3 Results And Conclusions

Pythagorean's triples $x, y$and z satisfying equation $x^2 + y^2 = z^2$ can be given by equation $(x)^2 + (x+a)^2 = (x+b)^2$ or $x^2 - 2x(b-a) + (b^2 - a^2) = 0$ where $y = (x+a)$ and $z = (x+b)$. Above quadratic equation has two roots as $x = (b-a) \pm \left\{(b-a)^2 + (b^2 - a^2)\right\}^{\frac{1}{2}}$ wherex, a and b are real and rational quantities. Values of a and b are so assumed that roots are real and rationals. If the roots found are real but fractions or a or b or both a and b are real and rationals but fractions, then quantities $x, (x+a)$ and $(x+b)$ are normalised by multiplying with LCM to make these integers. Since different values of a and b can be chosen, therefore, a number of values of x can be generated, hence a number of Pythagorean's triples.

Pythagorean's quadruples x, y, z and w satisfying equation $x^2 + y^2 + w^2 = z^2$ can be found by equation $(x)^2 + (x+a)^2 + (x+b)^2 = (x+c)^2$ where $y = (x+a), z = (x+b)$ and $w = (x+c)$. This quadratic has roots as $x = \frac{1}{2}(c-b-a) \pm \frac{1}{2}\left\{(c-b-a)^2 + 2(c^2 - b^2 - a^2)\right\}^{\frac{1}{2}}$ where x, a, b and c are real and rational quantities. In assuming values of $a, b$and c care should be taken that roots are real and rationals. If the roots found are real but fractions or a or b or c or all are real and rationals but fractions, then quantities $x, (x+a), (x+b)$ and $(x+c)$are normalised by multiplying these with LCM to make these integers. Since different values of $a, b$and c can be chosen, therefore, a number of values of x can be generated, hence a number of Pythagorean's quadruples.

Pythagorean quadruples x, y, z and w satisfying equation $x^2 + y^2 = z^2 + w^2$ can be written as $(x)^2 + (x+a)^2 = (x+b)^2 + (x+c)^2$ where $y = (x+a), z = (x+b)$ and $w = (x+c)$. This equation on simplification, reduces to

$$x = \frac{b^2 + c^2 - a^2}{2(a-b-c)}.$$

Assigning different real and rational values of $a, b$and c will yield different values of x and hence different quadruples after normalisation.

Pythagorean Quintuples of form $x^2 + y^2 + z^2 + w^2 = v^2$can also be reduced to linear equation $(x)^2 + (x+a)^2 + (x+b)^2 + (x+c)^2 = (2.x+d)^2$ where $y = (x+a), z = (x+b), w = (x+c)$ and $v = (2.x+d)$. The above equation on simplification reduces to

$$x = \frac{d^2 - a^2 - b^2 - c^2}{2(a+b+c-2d)}.$$

Different real rational values are assigned to a, b, c and d so that real rational values of x are obtained. After normalisation, Pythagorean's Quintuples are generated.

Pythagorean n-tuples of form $x_1^2 + x_2^2 + x_3^2 + \ldots x_n^2 = y^2$can be reduced to quadratic equation $(n-1)x_1^2 + 2x_1(a_2 + a_3 + a_4 + \ldots + a_n - a_1) + a_2^2 + a_3^2 + a_4^2 + \ldots a_n^2 - a_1^2 = 0$ where $x_2 = (x_1 + a_2), x_3 = (x_1 + a_3), x_4 = (x_1 + a_4) \ldots x_n = (x_1 + a_n)$ and $y = (x_1 + a_1)$. Different real rational values of $a_1, a_2, a_3, \ldots a_n$are assumed so that real rational values of $x_1$are obtained as

$$x_1 = \frac{-Q \pm \sqrt{Q^2 - 4P.R}}{2P},$$

where $P = (n-1), Q = 2(a_2 + a_3 + a_4 + \ldots + a_n - a_1)$ and $R = a_2^2 + a_3^2 + a_4^2 + \ldots a_n^2 - a_1^2$.

Apart from this, there is a second method to generate Pythagorean's n-tuples. In this method, an identity is derived satisfying Pythagorean's n-tuples and then assigning different real rational values to variables in the identity, Pythagorean's numbers are generated.

Pythagorean's triples a, b and c given by the equation $a^2 + b^2 = c^2$ can be generated from identity

$$\left\{\frac{1}{2}(x - \frac{1}{x})\right\}^2 + 1^2 = \left\{\frac{1}{2}(x + \frac{1}{x})\right\}^2,$$

where $\frac{1}{2}(x - \frac{1}{x}) = \frac{b}{a}$, and $\frac{1}{2}(x + \frac{1}{x}) = \frac{c}{a}$ and $x$is real rational quantity. It may be integer or fraction of the form p/q where p and q are integers.

Pythagorean's Quadruples of the form $a^2 + b^2 + c^2 = d^2$ can be generated after normalisation, from the identity

$$(x - \frac{1}{x})^2(x + \frac{1}{x})^{-2} + 4(x + \frac{1}{x})^{-2} + \frac{1}{4}(y - \frac{1}{y})^2 = \frac{1}{4}(y + \frac{1}{y})^2,$$

where x and y are real rational quantities and these may be integers or fractions of kind p/q.

Pythagorean's Quintuples can be generated by the identity

$$\left\{2(x - \frac{1}{x})(x + \frac{1}{x})^{-1}(y + \frac{1}{y})^{-1}\right\}^2 + \left\{2^2(x + \frac{1}{x})^{-1}(y + \frac{1}{y})^{-1}\right\}^2 + \left\{2^0(y - \frac{1}{y})(y + \frac{1}{y})^{-1}\right\}^2 + \left\{2^{-1}(z - \frac{1}{z})\right\}^2 = \left\{2^{-1}(z + \frac{1}{z})\right\}^2,$$

after normalisation where x, y and z are real rational quantities and these may be integers or fractions of kind p/q .

Pythagorean's Sextuples can be generated by the identity

$$\left\{2^3(x+\frac{1}{x})^{-1}(y+\frac{1}{y})^{-1}(z+\frac{1}{z})^{-1}\right\}^2+\left\{2^2(x-\frac{1}{x})(x+\frac{1}{x})^{-1}(y+\frac{1}{y})^{-1}(z+\frac{1}{z})^{-1}\right\}^2$$
$$+\left\{2(y-\frac{1}{y})(y+\frac{1}{y})^{-1}(z+\frac{1}{z})^{-1}\right\}^2+\left\{2^0(z-\frac{1}{z})(z+\frac{1}{z})^{-1}\right\}^2+\left\{2^{-1}(w-\frac{1}{w})\right\}^2=\left\{2^{-1}(w+\frac{1}{w})\right\}^2,$$

after normalisation where $x, y, z$and w are real rational quantities and these may be integers or fractions of kind p/q.

Pythagorean's n-tuples can be generated by the identity

$$\left\{2^{-1}(x_{n-2}+\frac{1}{x_{n-2}})\right\}^2$$
$$=\left\{2^{n-3}(x_1+\frac{1}{x_1})^{-1}(x_2+\frac{1}{x_2})^{-1}(x_3+\frac{1}{x_3})^{-1}\dots(x_{n-3}+\frac{1}{x_{n-3}})^{-1}\right\}^2$$
$$+\left\{2^{n-4}(x_1-\frac{1}{x_1})(x_1+\frac{1}{x_1})^{-1}(x_2+\frac{1}{x_2})^{-1}\dots.(x_{n-3}+\frac{1}{x_{n-3}})^{-1}\right\}^2$$
$$+\left\{2^{n-5}(x_2-\frac{1}{x_2})(x_2+\frac{1}{x_2})^{-1}(x_3+\frac{1}{x_3})^{-1}\dots(x_{n-3}+\frac{1}{x_{n-3}})^{-1}\right\}^2$$
$$+\left\{2^{n-6}(x_3-\frac{1}{x_3})(x_3+\frac{1}{x_3})^{-1}(x_4+\frac{1}{x_4})^{-1}\dots(x_{n-3}+\frac{1}{x_{n-3}})^{-1}\right\}^2$$
$$+\dots+\left\{2^0(x_{n-3}-\frac{1}{x_{n-3}})(x_{n-3}+\frac{1}{x_{n-3}})^{-1}\right\}^2+\left\{2^{-1}(x_{n-2}-\frac{1}{x_{n-2}})\right\}^2$$

after normalisation where $x_1, x_2, x_3, \dots x_{n-2}$ are real rational quantities and these may be integers or fractions of kind p/q.

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(Dedicated to Honor Dr. R. C. Singh Chandel on His 75^{th} Birth Anniversary Celebrations)

MODELING THE EFFECT OF ECOLOGICAL CONDITIONS IN THE HABITAT ON THE SPREAD OF TUBERCULOSIS

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Abstract

In this paper, the cumulative effect of ecological conditions in the habitat on the spread of *TB* in human population is modeled and analyzed. The total human population is divided into two classes, susceptibles and infectives where the infective class is further subdivided into latent and actively infected subclasses. It is assumed that *TB* is spread by direct contact between members of the population as well as indirectly by bacteria which are emitted by infectives in the environment, survive and get accumulated due to favorable ecological conditions in the habitat. The cumulative density of ecological factors determining conditions in the habitat is assumed to follow a population density dependent logistic model. The analysis of the model shows that as parameters governing the ecological factors in the habitat increase, the spread of *TB* increases. The same result is also found with the increase in the parameter defining the accumulation of bacteria in the habitat. It is further found that due to immigration of the population *TB* becomes more endemic. A numerical study of the model is also carried out to see the role of key parameters on the spread of tuberculosis and to support the analytical results.
2010 Mathematics Subject Classifications: 37C75, 92B05.
Keywords and phrases: Mycobacterium tuberculosis; Ecological status in the habitat; latently-infected; actively-infected.

1 Introduction

Tuberculosis (*TB*) is an infectious disease which has world-wide prevalence been declining due to vaccination and other preventive strategies [5, 19, 22], but its recent reappearance in developing countries with high burden of infection in regions of Southeast Asia have sparked renewed research in *TB*. Mycobacterium tuberculosis is the bacterium that causes most cases of tuberculosis. It is an obligate aerobe mycobacterium that divides every 16-20 hrs, extremely slow as compared to other bacteria which tend to have division times measured in minutes (for example, *E*. Coli can divide roughly every 20 min.) [13]. It is small rod like bacillus which can withstand weak disinfectants and can survive in a dry state for weeks but can only grow within a host organism [13].

Recent quantitative monitoring estimates are that over 30% of the population in developing countries is infected with *TB*, which results in approximately 2-3 million deaths each year [1, 2, 6]. Every year, 8 to 10 million new cases of tuberculosis occur and this figure is growing with the advent of *HIV* infection [21]. Socio-economic status, family size, crowding, malnutrition and limited access to health care or effective treatment also play important roles in the transmission [3, 14]. The reason for the increase in such cases in developed countries is principally immigration, poverty, living conditions, food security, etc. [12]. It is reported that eight million people develop active *TB* every year, each of which can infect between 10 and 15 people in one year just by breathing [2, 4, 20]. Overall, the mortality from tuberculosis is approximately 8%, being over 30% in the elderly cases but less than 1% in the young's [20, 23].

Humans are the natural reservoir of *TB*, which spreads from person to person by direct contact via airborne droplets [18] and indirectly from environment, by inhalation of small (1-10μ_m) droplets containing only tubercle bacilli, which are expelled during coughing, sneezing, talking or singing by a *TB* infected person [10]. *TB* also spreads indirectly by the use of contaminated utensils, contaminated dust, flowers, etc.

Mathematical models for the spread of infectious diseases have played a major role in providing deeper insight into the understanding of the transmission as well as control strategies [7, 8, 9, 11,12, 16, 17], including *HIV-TB* co-infection [15]. For example, Feng et al. [7] formulated a two strain *TB* model with an arbitrary distributed delay in the latent stage of individual infected with the drug-sensitive strain and investigated the effects of variable periods of latency on the disease dynamics. Naresh and Tripathi [15] have also modeled and studied the co-infection of *HIV* and *TB* in a variable size population.

It is noted here that in recent years the spread of infectious diseases have been modeled and analyzed by considering environmental and ecological conditions in the habitat [8, 9, 16, 17]. In particular, Singh et al. [16] have studied the spread of carrier dependent infectious diseases by considering the effect of environmental factors which are conducive to the growth of carrier population. They have shown that the spread of the disease increases due to conducive environmental factors. They [17] have also studied the spread of malaria by taking into account environmental and ecological factors which are conducive to the growth of mosquito population. Ghosh et al. [8, 9] have studied the spread of bacteria infected diseases such as *TB* by considering environmental effect as well as by considering the effect of migration. As pointed out earlier, that in the case of *TB* the bacteria emitted from the infected persons get accumulated in the habitat as these settle down on fomites or remain suspended in the air. These bacteria then affect the suceptibles indirectly and the rate of infection depends upon the ecological conditions in the habitat. Our aim in this paper is to model and analyze the effect of accumulation of bacteria which survive due to conducive ecological factors in the habitat acting as a reservoir, on the spread of *TB*.

2 Mathematical Model

In the model presented here, the total human population, $N(t)$, is divided into three sub-populations: susceptibles, latently infected individuals and actively infected individuals with densities $S(t)$, $L(t)$, and $T(t)$ respectively. It is assumed that all susceptibles are infected by both the direct and indirect contacts with bacteria. The following system of nonlinear, ordinary differential equations is assumed to model the dynamics of the spread of *TB*,

$$
\begin{aligned}
(2.1)\quad \frac{dS}{dt} &= A - \beta ST - \lambda SB - dS + \alpha_1 T + \alpha_2 L,\\
\frac{dL}{dt} &= (1-p)\beta ST + (1-q)\lambda SB - (\sigma + d + \alpha_2)L,\\
\frac{dT}{dt} &= p\beta ST + q\lambda SB + \sigma L - (d + \alpha + \alpha_1)T,\\
\frac{dN}{dt} &= A - dN - \alpha T,\\
\frac{dB}{dt} &= sT - s_0 B + s_1 BE,\\
\frac{dE}{dt} &= \gamma E - \gamma_0 E^2 + \gamma_1 NE.
\end{aligned}
$$

Here A is the immigration rate of susceptible β and λ are the transmission coefficients for susceptibles due to person to person contact with infectives and by inhalation of bacteria from environment respectively; $p > 0$ and $q > 0$ are the fraction of infected individuals who develop active *TB* soon after initial infection; σ is the rate of progression of latently infected individuals to active *TB*; d is the natural death rate and is the death rate due to *TB* infection. The parameters and are the therapeutic treatment rate of actively infected and latently infected individuals respectively. The second last differential equation represents change in bacterial population $B(t)$ in the environment. Since bacteria of *TB* grows only in the host (human) body and it only survives in the environment, therefore, no growth term is taken into consideration. In the environment, growth in the density of bacterial population is all due to number of bacteria released from actively infected *TB* patients and also because of accumulation due to conducive ecological conditions in the habitat. The parameter s is the rate of release of bacteria from the actively infected individuals, s_0 is their decay coefficient due to natural factors or control measures and s_1 is the rate of accumulation of bacteria population due to conducive ecological factors in the habitat; $E(t)$ is the cumulative density of ecological factors governing the condition in the habitat which is conducive to the accumulation of bacteria population; γ is the growth rate of cumulative density of ecological factors in the habitat, $\frac{\gamma}{\gamma_0}$ is the carrying capacity of the habitat, γ_1 is the interaction coefficient with respect to total human population.

In the following lines, we analyze the model (2.1) using stability theory of differential equations. We need the bounds of dependent variables involved in the model. For this, we give the region of attraction in the form of following lemma, stated without proof.

***Lemma* 2.1** *The region of attraction for the system* (2.1) *is given by,*

$$(2.2)\quad \Omega = \{(L, T, N, B, E) : 0 \le A/d,\ 0 \le T \le N \le A/d,\ 0 \le B \le B_m,\ 0 \le E \le E_m\}$$

which attracts all solutions initiating in the positive octant,

$$(2.3)\quad \textit{where, } B_m = \left[\frac{s\gamma_0 A/d}{s_0\gamma_0 - s_1(\gamma + \gamma_1 A/d)}\right] \textit{ and } E_m = \frac{\gamma + \gamma_1 A/d}{\gamma_0}.$$

3 Equilibrium Analysis

It is sufficient to consider the reduced system of model system (2.1) (since $S + L + T = N$), as follows,

$$(3.1)\quad \frac{dL}{dt} = (1-p)\beta(N-L-T)T + (1-q)\lambda(N-L-T)B - (\sigma + d + \alpha_2)L,$$

$$\frac{dT}{dt} = p\beta(N-L-T)T + q\lambda(N-L-T)B - \sigma L - (d+\alpha+\alpha_1)T,$$

$$\frac{dN}{dt} = A - dN - \alpha T,$$

$$\frac{dB}{dt} = sT - s_0B + s_1BE,$$

$$\frac{dE}{dt} = \gamma E - \gamma_0E^2 + \gamma_1NE.$$

The equilibrium analysis of the model system (3.1) has been carried out and the results are given as follows:

There exist following four nonnegative equilibria of the model system (3.1),

(I) Disease free equilibrium $W_0\left(0, 0, \frac{A}{d}, 0, 0\right)$.

This equilibrium exists without any condition. It explains that if the bacterial population is absent, due to non-conducive ecological conditions in the habitat and the TB infected individuals are not present, the disease would not persist and population remains at its equilibrium A/d.

(II) The equilibrium $W_1\left(0, 0, \frac{A}{d}, 0, E_m\right)$.

This equilibrium also exists without any condition in the absence of disease and bacterial population. However, in that case the population remains at its equilibrium A/d and the ecological status of the habitat is maintained at the level E_m.

(III) The equilibrium $W_2(\bar{L}, \bar{T}, \bar{N}, \bar{B}, 0)$

In this case the disease would still persist due to release of bacteria from the infected individuals even if the bacteria population is not accumulated further as it does not depend on the ecological conditions in the habitat. The explicit equilibrium values of different variables are given as follows,

$$(3.2)\quad \bar{T} = \frac{\{\beta s_0[\sigma + p(d+\alpha_2)] + s\lambda[\sigma + q(d+\alpha_2)]\}A - (d+\alpha+\alpha_1)(\sigma+d+\alpha_2)s_0d}{(\alpha+d)\{\beta s_0[\sigma + p(d+\alpha_2)] + s\lambda[\sigma + q(d+\alpha_2)]\} + d(d+\alpha+\alpha_1)[\beta s_0(1-p) + \lambda s(1-q)]},$$

$$(3.3)\quad \bar{L} = \frac{[(1-p)s_0\beta + (1-q)\lambda s][A - (\alpha+d)T]T}{d[(1-p)s_0\beta\bar{T} + (1-q)\lambda s\bar{T} + s_0(\sigma+d+\alpha_2)]},$$

$$(3.4)\quad \bar{N} = \frac{A - \alpha\bar{T}}{d}, \bar{B} = \frac{s\bar{T}}{s_0}, \text{ as } \bar{T} < \frac{A}{\alpha},$$

provided that $p\beta\frac{A}{d} > (d + \alpha + \alpha_1)$.

(IV) The endemic equilibrium, $W_3(L^*, T^*, N^*, B^*, E^*,)$

The endemic equilibrium W_3 is given by the solution of following algebraic equations and a quadratic equation, obtained from (3.1),

$$(3.5)\quad N = \frac{A - \alpha T}{d},$$

$$(3.6)\quad E = \frac{\gamma d + \gamma_1(A - \alpha T)}{d\gamma_0},$$

$$(3.7)\quad B = \frac{ds\gamma_0T}{ds_0\gamma_0 - s_1[\gamma d + \gamma_1(A-\alpha T)]},$$

$$(3.8)\quad L = \frac{[(1-p)\beta T + (1-q)\lambda B][A - (\alpha+d)T]}{d[(1-p)\beta T + (1-q)\lambda B + (\sigma+d+\alpha_2)]},$$

$$(3.9)\quad aT^2 + bT - c = 0,$$

where

$$a=\beta s_1\gamma_1\alpha\{(1-p)[d^2(d+\alpha+\alpha_1)+\sigma(\alpha+d)]+(\alpha+d)(\sigma+d+\alpha_2)\},$$
$$b=\alpha_2(\alpha+d)p\beta[ds_0\gamma_0-s_1(\gamma d+\gamma_1A)]+(\alpha+d)(\alpha_2+d)\{dsq\lambda\gamma_0-\beta s_1\gamma_1\alpha A[\sigma+p(\alpha_2+d)]\}$$
$$+\sigma(\alpha+d)\{\beta[ds_0\gamma_0-s_1(\gamma d+\gamma_1A)]+ds\lambda\gamma_0\}+\alpha_1d(1-p)\beta[ds_0\gamma_0-s_1(\gamma d+\gamma_1A)]$$
$$+d(d+\alpha+\alpha_1)[(1-q)s\lambda\gamma_0+s_1\gamma_1\alpha(\sigma+d+\alpha_2)],$$
$$c=[ds_0\gamma_0-s_1(\gamma d+\gamma_1A)]\{\beta A[\sigma+p(\alpha_2+d)]-d(d+\alpha+\alpha_1)(\sigma+d+\alpha_2)\}+ds\lambda\gamma_0A[\sigma+q(\alpha_2+d)].$$

There exists unique positive root of eq.(3.9) is given as $T^*=\frac{-b+\sqrt{b^2+4ac}}{2a}$ if $p\beta\frac{A}{d}>(d+\alpha+\alpha_1)$and $s_0>s_1E_m$. Substituting the value of T^* in eqs. (3.5-3.8), we can compute the value of L^*, N^*, B^* and E^*.

4 Stability Analysis

Now, we analyze the stability of each of the equilibrium W_0, W_1, W_2 and W_3.

***Theorem* 4.1** *The equilibrium W_0, W_1 and W_2 are unstable and the endemic equilibrium W_3 is locally asymptotically stable provided the following conditions are satisfied,*

(4.1) $\alpha\gamma_1^2E^*<\frac{2}{3}d(p\beta T^*+q\lambda B^*)$,

(4.2) $q^2\lambda^2(N^*-L^*-T^*)^2<\frac{1}{5}(s_0-s_1E^*)^2\xi_1\min.\left\{\frac{\gamma_0^2E^*}{2s_1^2B^{*2}},\ \frac{\xi_1}{5s^2}\right\}$,

(4.3) $(p\beta T^*+q\lambda B^*-\sigma)^2<\xi_1\xi_2^2\min.\left\{\frac{\xi_1}{5\xi_3^2},\frac{d(p\beta T^*+q\lambda B^*)}{3\alpha[(1-p)\beta T^*+(1-q)\lambda B^*]^2},\frac{k_3(s_0-s_1E^*)}{4(1-q)^2\lambda^2(N^*-L^*-T^*)^2}\right\}$

where,

$$\xi_1=\left[p\beta T^*+q\lambda(N^*-L^*)\tfrac{B^*}{T^*}+\sigma\tfrac{L^*}{T^*}\right],\quad \xi_2=\left[(1-p)\beta(N^*-T^*)\tfrac{T^*}{L^*}+(1-q)\lambda(N^*-T^*)\tfrac{B^*}{L^*}\right],$$
$$\xi_3=[(1-p)\beta(N^*-L^*-T^*)-(1-p)\beta T^*-(1-q)\lambda B^*].$$

Proof. See ***Appendix*-I**

***Theorem* 4.2** *The endemic equilibrium W_3 is nonlinearly asymptotically stable in the region Ω provided the following inequalities are satisfied:*

(4.4) $\alpha q^2\lambda^2B_m^2<\frac{1}{3}d\,p^2\beta^2T^{*2}$,

(4.5) $s\,q\lambda(N^*-L^*)^2<\frac{1}{3}p\beta(s_0-s_1E^*)T^{*2}$,

(4.6) $\alpha\,q\lambda\,\gamma_1^2s_1^2B_m^2<\frac{4}{9}\gamma_0^2\,s\,p\beta\,d(s_0-s_1E^*)$,

(4.7) $(p\beta T^*+q\lambda B_{\max}-\sigma)^2<\frac{1}{4}p\beta(\sigma+d+\alpha_2)^2T^{*2}$,

$$\min.\left\{\frac{p\beta}{4\xi_4^2},\frac{p\beta\,d}{3\alpha\left[(1-p)\beta A/d+(1-q)\lambda B_{\max}\right]^2},\frac{q\lambda(s_0-s_1E^*)}{3s(1-q)^2\lambda^2(N^*-L^*-T^*)^2}\right\},$$

where $\xi_4=\left[(1-p)\beta\frac{A}{d}+(1-q)\lambda B_{\max}-(1-p)\beta(N^*-L^*-T^*)\right]$.

Proof. See ***Appendix*-II**.

***Remark* 4.1** *As the growth rate of cumulative density of ecological factors conducive to the accumulation of bacterial population due to human population activities tends to zero i.e., $\gamma_1\to 0$, inequalities* (4.1) *and* (4.4) *are automatically satisfied. This implies that the ecological factors conducive to the accumulation of bacterial population have a destabilizing effect on the system. If the rate of accumulation of bacteria due to conducive ecological conditions is very small i.e., $s\to 0$ then inequalities* (4.2) *and* (4.5) *are satisfied.*

The above theorems imply that under appropriate conditions, if the density of bacteria due to conducive ecological conditions increases, then the number of latently-infected and actively-infected individuals increases leading to fast spread of TB. However, the effect of immigration is to make TB more endemic.

5 Numerical Simulation

In this section, we conduct simulation analysis of the model (3.1) to study its dynamical behavior and to prove the feasibility of local and nonlinear stability conditions of the model system. The numerical simulation of the system (3.1) is done by MAPLE 7.0 using the parameters values [8, 9, 11, 15] given below:

***Table* 5.1:** Parameter values

Parameters	Symbol	Parameter value
recruitment rate of susceptible	A	500
transmission coefficient (by infectives)	β	0.0005
transmission coefficient (through bacteria)	λ	0.0003
recovery rate of latently-infected TB patient	α_1	0.012
recovery rate of actively-infected TB class	α_2	0.01
natural death rate	d	0.15
disease-induced death rate	α	0.2
rate with which latently-infected goes to actively-infected TB class	σ	0.02
rate of release of bacteria from TB patients	s	1
accumulation of bacteria due to ecology	s_1	0.0001
decay rate of bacteria in the environment	s_0	0.3
growth rate of ecological status in the habitat	γ	25
growth rate of ecological status due to human activities	γ_1	0.002
depletion rate of ecological status	γ_0	0.1
fraction of infected individuals (by infectives) who develop active TB soon after initial infection	p	0.45
fraction of infected individuals (by bacteria population) who develop active TB soon after initial infection	q	0.6

The equilibrium values for the model system (3.1) are computed as follows:
N^* =2300.799543, L^* = 1153.414779, T^* = 774.4003430, B^* =2863.923531, E^* = 296.0159909.
The eigen values of variational matrix corresponding to the endemic equilibrium for the model system (3.1) are

$$-1.353015784, -0.1854221833, -0.3408624139, -0.2455538488, -29.601599.$$

Since all the eigen values are negative which implies that the endemic equilibrium W3 is locally asymptotically stable.

The results of numerical simulation are displayed graphically in ***Figs.* 5.1-5.11.** ***Fig.* 5.1** shows that the system (3.1) is nonlinearly asymptotically stable in T-N plane. All the trajectories starting from different initial starts reaches to equilibrium point.

(i) $L(0) = 1500$, $T(0) = 600$, $N(0) = 3000$, $B(0) = 2863$, $E(0) = 296$.
(ii) $L(0) = 1000$, $T(0) = 1000$, $N(0) = 3000$, $B(0) = 2863$, $E(0) = 296$.
(iii) $L(0) = 400$, $T(0) = 400$, $N(0) = 1000$, $B(0) = 2863$, $E(0) = 296$.
(iv) $L(0) = 200$, $T(0) = 1000$, $N(0) = 1400$, $B(0) = 2863$, $E(0) = 296$.

In ***Figs.* 5.2 - 5.3**, the variation of density of bacteria population and the actively-infected *TB* population with time is shown respectively for different values of accumulation rates (s_1) of bacteria due to conducive ecological status of the habitat. It is found that as the accumulation rate of bacteria increases, bacteria population also increases which results in increasing the spread of tuberculosis. Thus ecological conditions conducive to the accumulation of bacterial population help in spreading the tuberculosis infection. In ***Figs.* 5.4 - 5.5**, we show the variation of bacterial population density and actively-infected *TB* population with time for different values of rate of release of bacteria from actively-infected population. From these figures, we infer that as the rate of emission of bacteria, (s) from actively-infected *TB* population increases, the accumulation of the bacterial population in the habitat also increases due to conducive ecological conditions. These bacteria when comes in contact with susceptibles through contaminated clothes, utensils, etc., further increases the spread of tuberculosis which ultimately results in rise in the actively infected *TB* population. ***Figs.* 5.6** and **5.7** depict the role of decay coefficients (s_0) of bacteria on the variation of bacteria population density and actively-infected *TB* population. When there is a rise in the decay coefficient due to natural factors or control measures, the density of bacteria population decreases significantly and consequently the actively infected *TB* population declines. This decline in actively-infected population does not seem to be much significant. It seems, it is due to the fact that disease spreads not only through bacteria but also through direct interaction of susceptibles with actively-infected *TB* individuals. It is, therefore, speculated that not only the accumulation of bacteria be curbed using effective control mechanism but the direct interaction of susceptibles with actively-infected *TB* population be also restricted.

Figs. **5.8 - 5.9**, show that as the growth rate of cumulative density of ecological factors in the habitat (γ) conducive to the accumulation of bacteria increases, there is a significant increase in the density of bacteria population. This, in turn, increases the number of actively-infected *TB* individuals. Thus, if the density of ecological factors is higher, the spread of tuberculosis is faster due to significant increase in bacterial population in a conducive environment. Also, as the growth of ecological status making the environment conducive to bacteria population due to human population activities (γ_1) increases, the density of bacteria population increases resulting in the spread of tuberculosis, see ***Figs.*** **5.10 - 5.11**. Thus, the human population related factors responsible for making the ecological conditions favourable for the accumulation of bacterial population further increases the load of tuberculosis.

Finally, from the above discussion, we infer that the spread of tuberculosis not only depends upon the interaction of susceptibles with actively-infected population but also depends upon the interaction of susceptibles with bacteria population. Moreover, the ecological status of the surroundings plays a vital role in the accumulation of *Mycobacterium Tuberculosis*. It may be possible to curb the spread of tuberculosis if the bacterial population is diminished by way of providing hyegenic environment in the habitat and restricting the interaction of *TB* patients with the susceptible population.

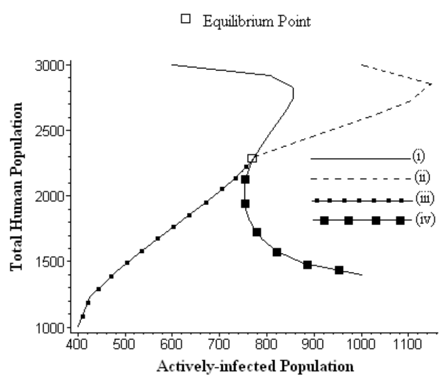

Figure **5.1:** Variation of total human population with actively-infected population

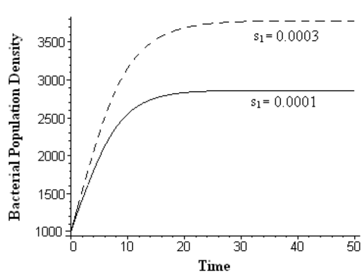

Figure **5.2:** Variation of bacterial population density with time for different values of s_1

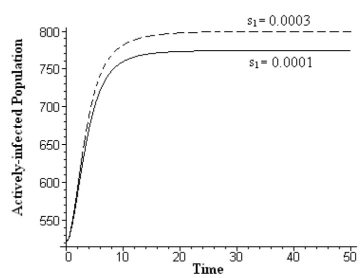

Figure **5.3:** Variation of actively-infected population with time for different values of s_1

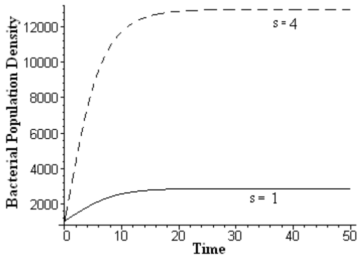

Figure **5.4:** Variation of bacterial population density with time for different values of s

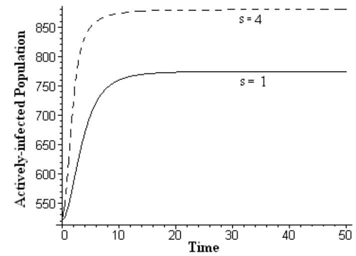

Figure **5.5:** Variation of actively-infected population with time for different values of s

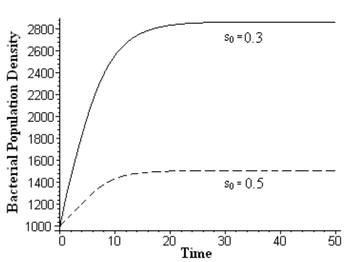

Figure **5.6:** Variation of bacterial population density with time for different values of s_0

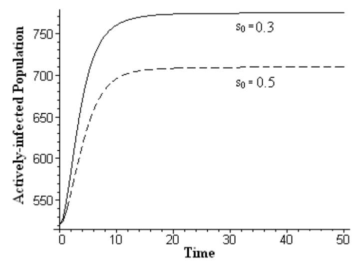

Figure **5.7:** Variation of actively-infected population with time for different values of s_0

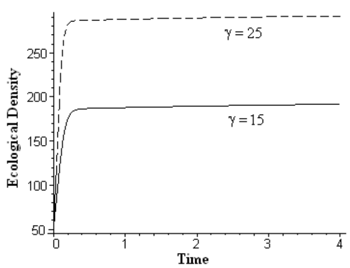

Figure **5.8:** Variation of ecological density with time for different values of γ

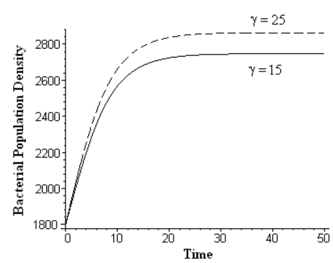

Figure **5.9:** Variation of bacterial population density with time for different values of γ

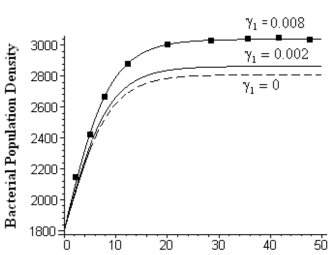

Figure **5.10:** Variation of bacterial population density with time for different values of γ_1

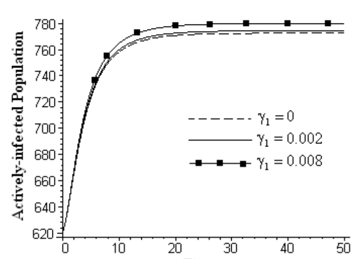

Figure **5.11:** Variation of actively-infected population with time for different values of γ_1

6 Conclusion

In this paper, a two stage SIS model for Tuberculosis, caused by *Mycobacterium Tuberculosis* is proposed and analyzed with constant migration of human population. The cumulative density of ecological factors in the habitat is assumed to be governed by a logistic model which is population density dependent. The endemic equilibrium is shown to be

locally and nonlinearly stable under certain conditions. Our analysis shows that the spread of tuberculosis not only depends upon the interaction of susceptibles with actively-infected population but also depends upon the interaction of susceptibles with bacteria population accumulated in the habitat. The ecological status of the habitat plays a vital role in the accumulation of *Mycobacterium Tuberculosis*. It is shown that the cumulative effect of ecological factors is to increase the spread of the disease. Thus, an effective control mechanism must be undertaken to curb the accumulation of bacteria in the environment and the direct interaction of susceptibles with actively-infected population be restricted.
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Appendix – I

***Proof of Theorem* 4.1.**

The variational matrix M_0 of model (3.1) corresponding to equilibrium W_0 is given by,

$$M_0=\begin{bmatrix} -(\sigma+d+\alpha_2) & \frac{(1-p)\beta A}{d} & 0 & \frac{(1-q)\lambda A}{d} & 0 \\ \sigma & \frac{p\beta A}{d}-(d+\alpha+\alpha_1) & 0 & \frac{q\lambda A}{d} & 0 \\ 0 & -\alpha & -d & 0 & 0 \\ 0 & s & 0 & -\left(s_0-\frac{s_1(\gamma+\gamma_1 A/d)}{\gamma_0}\right) & s_1 \\ 0 & 0 & 0 & 0 & \left(\gamma+\frac{\gamma_1 A}{d}\right) \end{bmatrix}.$$

The fifth eigenvalue of M_0 is positive, as all the model parameters are nonnegative. Therefore, disease free equilibrium W_0 is unstable.

The variational matrix M_1 of model (3.1) corresponding to equilibrium W_1 is given by,

$$M_1=\begin{bmatrix} -(\sigma+d+\alpha_2) & \frac{(1-p)\beta A}{d} & 0 & \frac{(1-q)\lambda A}{d} & 0 \\ \sigma & \frac{p\beta A}{d} & 0 & \frac{q\lambda A}{d} & 0 \\ 0 & -\alpha & -d & 0 & 0 \\ 0 & s & 0 & -\left(s_0-\frac{s_1(\gamma+\gamma_1 A/d)}{\gamma_0}\right) & s_1 \\ 0 & 0 & \frac{\gamma_1(\gamma+\gamma_1 A/d)}{\gamma_0} & 0 & -\left(\gamma+\frac{\gamma_1 A}{d}\right) \end{bmatrix}.$$

The characteristic polynomial corresponding to above matrix is given by,

$$(d+\psi)(\sigma+d+\alpha_2+\psi)(\gamma+\gamma_1 A/d+\psi)(\psi^2+h_1\psi+h_2)=0,$$

where $h_1=\left(s_0-\frac{s_1(\gamma+\gamma_1 A/d)}{\gamma_0}-\frac{p\beta A}{d}\right)$,

$$h_2=-spq\beta\lambda\frac{A^2}{d^2} \quad < \quad 0..$$

Using Routh-Hurwitz criteria as $h_2<0$, therefore, disease free equilibrium W_1 is unstable.

The variational matrix M_2 of model (3.1) corresponding to equilibrium W_2 is given by,

$$M_2=\begin{bmatrix} m_{11} & m_{12} & (1-p)\beta\bar{T}+(1-q)\lambda\bar{B} & (1-q)\lambda(\bar{N}-\bar{L}-\bar{T}) & 0 \\ \sigma-p\beta\bar{T}-q\lambda\bar{B} & m_{22} & p\beta\bar{T}+q\lambda\bar{B} & q\lambda(\bar{N}-\bar{L}-\bar{T}) & 0 \\ 0 & -\alpha & -d & 0 & 0 \\ 0 & s & 0 & -s_0 & s_1 \\ 0 & 0 & 0 & 0 & (\gamma+\gamma_1\bar{N}) \end{bmatrix}.$$

where, $m_{11}=-(1-p)\beta(\bar{N}-\bar{T})\frac{\bar{T}}{\bar{L}}-(1-q)\lambda(\bar{N}-\bar{T})\frac{\bar{B}}{\bar{L}}$,

$m_{12}=(1-p)\beta(\bar{N}-\bar{L}-\bar{T})-(1-p)\beta\bar{T}-(1-q)\lambda\bar{B}$ and $m_{22}=-p\beta\bar{T}-q\lambda(\bar{N}-\bar{L})\frac{\bar{B}}{\bar{T}}-\sigma\frac{\bar{L}}{\bar{T}}$.

This equilibrium is also unstable as fifth eigen value is always positive.

To establish the local stability of endemic equilibrium W_3, we consider the following positive definite function,

$U_1=\frac{1}{2}(k_0l^2+k_1t^2+k_2n^2+k_3b^2+k_4e^2)$,

where $ki(i=0,1,2,3,4)$ are positive constants to be chosen appropriately and l, t, n, b and e are small perturbations about W_3, defined as follows

$L=L^*+l$, $T=T^*+t$, $N=N^*+n$, $B=B^*+b$ and $E=E^*+e$.

Differentiating above equation, with respect to 't' and using the linearized system of model equations (3.1) corresponding to W_3, we get,

$\frac{dU_1}{dt}=-k_0\left[(1-p)\beta(N^*-T^*)\frac{T^*}{L^*}+(1-q)\lambda(N^*-T^*)\frac{B^*}{L^*}\right]l^2-k_1\left[p\beta T^*+q\lambda(N^*-L^*)\frac{B^*}{T^*}+\sigma\frac{L^*}{T^*}\right]t^2-k_2dn^2-k_3(s_0-s_1E^*)b^2-k_4\gamma_0E^*e^2+k_0[(1-p)\beta(N^*-L^*-T^*)-(1-p)\beta T^*-(1-q)\lambda B^*]\,lt$
$+k_0[(1-p)\beta T^*+(1-q)\lambda B^*]\ \ln+k_0(1-q)\lambda(N^*-L^*-T^*)\,lb+k_1(\sigma-p\beta T^*-q\lambda B^*)\,lt$
$+[k_1(p\beta T^*+q\lambda B^*)-k_2\alpha]nt+k_1q\lambda(N^*-L^*-T^*)bt+k_3s\,tb+k_3s_1B^*be+k_4\gamma E^*ne$

For $\frac{dU_1}{dt}$ to be negative definite, the following conditions must be satisfied,

(i) $k_0[(1-p)\beta(N^*-L^*-T^*)-(1-p)\beta T^*-(1-q)\lambda B^*]^2<\frac{1}{5}k_1\left[(1-p)\beta(N^*-T^*)\frac{T^*}{L^*}+(1-q)\lambda(N^*-T^*)\frac{B^*}{L^*}\right]\left[p\beta T^*+q\lambda(N^*-L^*)\frac{B^*}{T^*}+\sigma\frac{L^*}{T^*}\right]$

(ii) $k_1(\sigma-p\beta T^*q\lambda B^*)^2<\frac{1}{5}k_0\left[(1-p)\beta(N^*-T^*)\frac{T^*}{L^*}+(1-q)\lambda(N^*-T^*)\frac{B^*}{L^*}\right]\left[p\beta T^*+q\lambda(N^*-L^*)\frac{B^*}{T^*}+\sigma\frac{L^*}{T^*}\right]$

(iii) $k_0[(1-p)\beta T^*+(1-q)\lambda B^*]^2<\frac{1}{3}k_2d\left[(1-p)\beta(N^*-T^*)\frac{T^*}{L^*}+(1-q)\lambda(N^*-T^*)\frac{B^*}{L^*}\right]$

(iv) $k_0(1-q)^2\lambda^2(N^*-L^*-T^*)^2 < \frac{1}{4}k_3(s_0-s_1E^*)\left[(1-p)\beta(N^*-T^*)\frac{T^*}{L^*}+(1-q)\lambda(N^*-T^*)\frac{B^*}{L^*}\right]$
(v) $[k_1(p\beta T^*+q\lambda B^*)-k_2\alpha]^2 < \frac{4}{15}k_1k_2d\left[p\beta T^*+q\lambda(N^*-L^*)\frac{B^*}{T^*}+\sigma\frac{L^*}{T^*}\right]$
(vi) $k_1q^2\lambda^2(N^*-L^*-T^*)^2 < \frac{1}{5}k_3(s_0-s_1E^*)\left[p\beta T^*+q\lambda(N^*-L^*)\frac{B^*}{T^*}+\sigma\frac{L^*}{T^*}\right]$
(vii) $k_3s^2 < \frac{1}{5}k_1(s_0-s_1E^*)\left[p\beta T^*+q\lambda(N^*-L^*)\frac{B^*}{T^*}+\sigma\frac{L^*}{T^*}\right]$
(viii) $k_3s_1^2B^{*2} < \frac{1}{2}k_4\gamma_0E^*(s_0-s_1E^*)$
(ix) $k_4\gamma_1^2E^{*2} < \frac{2}{3}k_2d\gamma_0E^*$
After choosing $k_1 = 1$, $k_2 = \frac{p\beta T^*+q\lambda B^*}{\alpha}$ and $k_4 = \gamma_0$, we can choose k_0 and k_3 such that

$$\frac{q^2\lambda^2(N^*-L^*-T^*)^2}{(s_0-s_1E^*)\xi_1} < k_3 < \frac{1}{5}(s_0-s_1E^*)\min.\left\{\frac{\gamma_0^2E^*}{2s_1^2B^{*2}},\ \frac{\xi_1}{5s^2}\right\}$$

$$\frac{(p\beta T^*+q\lambda B^*-\sigma)^2}{\xi_1\xi_2} < k_0 < \xi_2\min.\left\{\frac{\xi_1}{5\xi_3^2},\frac{d(p\beta T^*+q\lambda B^*)}{3\alpha[(1-p)\beta T^*+(1-q)\lambda B^*]^2},\frac{k_3(s_0-s_1E^*)}{4(1-q)^2\lambda^2(N^*-L^*-T^*)^2}\right\}\alpha\gamma_1^2E^* < \frac{2}{3}d(p\beta T^*+q\lambda B^*)$$

Hence, we obtain the conditions as stated in the ***Theorem* 4.1**.
Thus, dU_1/dt is a negative definite under the conditions (4.1), (4.2) and (4.3) as stated in the ***Theorem* 4.1**, showing that W_3 is locally asymptotically stable.

Appendix – II

***Proof of Theorem* 4.2**
Consider the following positive definite function, corresponding to the model system (3.1) about W_3,

$$U_2 = \frac{k_0}{2}(L-L^*)^2+\frac{k_1}{2}\left(T-T^*-T^*\ln\frac{T}{T^*}\right)+\frac{k_2}{2}(N-N^*)^2+\frac{k_3}{2}(B-B^*)^2+\frac{k_4}{2}\left(E-E^*-E^*\ln\frac{E}{E^*}\right),$$

where the coefficients k_0, k_1, k_2, k_3 and k_4 can be chosen appropriately.
Differentiating the above equation with respect to 't' and using (3.1), we get,
$\frac{dU_2}{dt} = -k_0[(1-p)\beta T+(1-q)\lambda B](L-L^*)^2-k_1\left[\frac{q\lambda B(N-L)+\sigma L}{TT^*}\right](T-T^*)^2$
$-k_0(\sigma+d+\alpha_2)(L-L^*)^2-k_1p\beta(T-T^*)^2-k_2d(N-N^*)^2-k_3(s_0-s_1E^*)(B-B^*)^2-k_4\gamma_0(E-E^*)^2+k_0\{(1-p)\beta(N^*-L^*-T^*)-(1-p)\beta T-(1-q)\lambda B\}(L-L^*)(T-T^*)+k_1\left(\frac{\sigma}{T^*}-\frac{q\lambda B}{T^*}-p\beta\right)(L-L^*)(T-T^*)+k_0[(1-p)\beta T+(1-q)\lambda B](L-L^*)(N-N^*)+k_0(1-q)\lambda(N^*-L^*-T^*)(L-L^*)(B-B^*)$
$+\left[k_1\left(p\beta+\frac{q\lambda B}{T^*}\right)-k_2\alpha\right](T-T^*)(N-N^*)+k_1\frac{q\lambda(N^*-L^*-T^*)}{T^*}(T-T^*)(B-B^*)+k_3s(T-T^*)(B-B^*)+k_3s_1B(B-B^*)(E-E^*)+k_4\gamma_1(E-E^*)(N-N^*)$.
Assuming $k_1 = 1$, $k_2 = \frac{p\beta}{\alpha}$ and $k_3 = \frac{q\lambda}{s}$, the above equation reduces to the form,
$\frac{dU_2}{dt} = -k_0[(1-p)\beta T+(1-q)\lambda B](L-L^*)^2-\left[\frac{q\lambda B(N-L)+\sigma L}{TT^*}\right](T-T^*)^2$
$-k_0(\sigma+d+\alpha_2)(L-L^*)^2-p\beta(T-T^*)^2-\frac{p\beta d}{\alpha}(N-N^*)^2-\frac{q\lambda(s_0-s_1E^*)}{s}(B-B^*)^2-k_4\gamma_0(E-E^*)^2$
$+k_0\{(1-p)\beta(N^*-L^*-T^*)-(1-p)\beta T-(1-q)\lambda B\}(L-L^*)(T-T^*)+\left(\frac{\sigma}{T^*}-\frac{q\lambda B}{T^*}-p\beta\right)(L-L^*)(T-T^*)$ For dU_2/dt
$+k_0[(1-p)\beta T+(1-q)\lambda B](L-L^*)(N-N^*)+k_0(1-q)\lambda(N^*-L^*-T^*)(L-L^*)(B-B^*)$
$+\frac{q\lambda B}{T^*}(T-T^*)(N-N^*)+\frac{q\lambda(N^*-L^*)}{T^*}(T-T^*)(B-B^*)+\frac{qs_1\lambda B}{s}(B-B^*)(E-E^*)+k_4\gamma_1(E-E^*)(N-N^*)$.
to be negative definite, the following conditions must be satisfied,
(i) $k_0[(1-p)\beta(N^*-L^*-T^*)-(1-p)\beta T-(1-q)\lambda B]^2 < \frac{1}{4}p\beta(\sigma+d+\alpha_2)$,
(ii) $\left(\frac{\sigma-p\beta T^*-q\lambda B}{T^*}\right)^2 < \frac{1}{4}k_0p\beta(\sigma+d+\alpha_2)$,
(iii) $k_0[(1-p)\beta T+(1-q)\lambda B]^2 < \frac{1}{3}\frac{p\beta d}{\alpha}(\sigma+d+\alpha_2)$,
(iv) $k_0(1-q)^2\lambda^2(N^*-L^*-T^*)^2 < \frac{q\lambda(s_0-s_1E^*)(\sigma+d+\alpha_2)}{3s}$,
(v) $\frac{q^2\lambda^2B^2}{T^{*2}} < \frac{p^2\beta^2d}{3\alpha}$,
(vi) $\frac{q^2\lambda^2(N^*-L^*)^2}{T^{*2}} < \frac{pq\beta\lambda(s_0-s_1E^*)}{3s}$,
(vii) $\frac{q\lambda s_1^2B^2}{s} < \frac{2}{3}k_4\gamma_0(s_0-s_1E^*)$,
(ix) $k_4\gamma_1^2 < \frac{2}{3}\frac{p\beta d\gamma_0}{\alpha}$.
Now choosing k_0 and k_4 such that,

$$\frac{4(p\beta T^*+q\lambda B_{\max}-\sigma)^2}{p\beta(\sigma+d+\alpha_2)T^{*2}} < k_0 < (\sigma+d+\alpha_2,)$$

$$
\begin{aligned}
&\min.\left\{\frac{p\beta}{4\xi_4^2}, \frac{p\beta\, d}{3\alpha\left[(1-p)\beta A/d + (1-q)\lambda B_{\max}\right]^2}, \frac{q\lambda(s_0 - s_1 E^*)}{3s(1-q)^2\lambda^2(N^* - L^* - T^*)^2}\right\}\\
&\frac{3q\lambda\, s_1^2 B_m^2}{2\, s\, \gamma_0(s_0 - s_1 E^*)} < k_4 < \frac{2\gamma_0 p\beta\, d}{3\alpha\gamma_1^2},\\
&\alpha\, q^2\lambda^2 B_m^2 < \frac{1}{3} d\, p^2\beta^2 T^{*2},\\
&s\, q\lambda(N^* - L^*)^2 < \frac{1}{3} p\beta(s_0 - s_1 E^*) T^{*2}.
\end{aligned}
$$

Hence, we obtain the conditions as stated in the ***Theorem*** **4.2**. Thus, $\frac{dU_2}{dt}$ is a negative definite under the conditions (4.4 - 4.7) as given in the statement of the theorem, showing that W_3 is nonlinearly asymptotically stable inside the region Ω.

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(Dedicated to Honor Dr. R. C. Singh Chandel on His 75^{th} Birth Anniversary Celebrations)

MATHEMATICAL MODELLING OF FOOD MANAGEMENT FOR WILD LIFE POPULATION WITH MILD ENVIRONMENTAL EFFECT

By

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Abstract

In this paper we constructed a Mathematical Model of food management for animal species interacting with natural environment. The mathematical framework has been generated with generations as the time scale. Efforts have been made to mathematically formalize the environmental changes dependent food availability and consequent changes. The aim to develop a mathematical perspective to understand the environmental impact on wild life population and its food management. The carrying capacity of environment in terms of primary food has been assumed to be limited. The environmental changes lead to changes in availability of vegetation for animal species. A Mathematical model is constructed in terms of a system of nonlinear difference equations. The solutions have been worked out for some special cases. These solutions can be expressed as finite polynomials. Graphical patterns have been worked out as examples

1 Introduction

The environment is the source of the primary food production in the nature and therefore it has strong bearing upon the interplay of animal population along the food chain [6]. The environmental changes have led to drastic changes in food production and natural habitats of wildlife species which have given rise to the changes in the interactional patterns among the various trophic levels of the food chain ([4], [5]). Despite the inherent complexity of interactions between animal species and environment. Mathematical Models of single species interacting with natural environment has been a subject matter of interest for the mathematicians. The single species models have been primarily inspired by their simplicity of mathematical treatment and scope for further development [11]. We consider the growth of animal population is dependent on limited carrying capacity of the environment like water, vegetation and temperature. A mathematical model is constructed in terms of a system of nonlinear difference equations incorporating all the significant parameters. The mathematical framework has been generated with generations as the time scale ([7], [3]). It has been observed that the interactions between the species in various generation does not remain confined to those specific generations but decides the population patterns in all the successive generation in the form of a series ([12],[14], [17]).

Sustainable Development Goals (SDG) of United Nations has raised several challenges for development planners particularly in developing countries where the policy planners are not adequately capacitated to plan the strategies and monitor the outcomes. As SDGs are time bound their effective implementation will requires predictive methods ([15], [16], [18], [19]). The Mathematical Tools can prove handy to adequately plan the developmental interventions and project the outcomes ([8], [9], [10]). There has been a growing realization that Mathematical Modelling can be a viable tool for Sustainable Development Planning [11].

Earlier researchers ([1], [2], [10]) have developed and employed single species finite animal population models to understand patterns have growth of populations.

2 Mathematical Framework for Food Availability

Following factors will attend the growth of animal population surviving on the limited carrying capacity of the environment

1. The food availability per animal of single species for a specific generation,
2. the competition with other community member to access the food is avoided.
3. the migration of the species from the ecosystem,

4. the rate of supply of food availability and

5. the food competition will enhance the growth of species whereas it will tend to refer the growth of vegetative.

We assume that F_n denotes the food availability per animal for the n^{th} generation of the species. It has also been presumed that the food requirements are the same across the age groups which means that infants adults and elder species have same requirements and food accessing behaviour from the environment. The food availability for different generation will be different primarily because of the climate change, consumption of food by earlier generation and requirement of the food by environment. Hence change in food availability at n^{th} generation can be written as below:

$$\Delta F_n = \alpha_n F_{n-1} - \beta_n F_{n-1} - \Delta E_n,$$

where

α_n = Rate of growth of food availability i.e. because the food availability gets continuously regeneration in the environment. The growth has been calculated over unit area.

β_n = rate of consumption for the food per unite area.

ΔE_n = change in environmental conditions per unite area.

$$\Delta F_n = F_n - F_{n-1}.$$

Then

$$F_n = (\alpha_n F_{n-1} - \beta_n F_{n-1} - \Delta E_n) + F_{n-1},$$

$$F_n = (1 + \alpha_n - \beta_n) F_{n-1} - \Delta E_n.$$

Let

$$1 + \alpha_n - \beta_n = g_n \qquad \text{(effective growth rate).}$$

Then

$$F_n = g_n F_{n-1} - \Delta E_n. \tag{2.1}$$

The solution of the equation (2.1) can be given as

$$F_n = g_n g_{n-1} g_{n-2} \ldots g_2 g_1 F_0 - g_n g_{n-1} g_{n-2} \ldots g_4 g_3 g_2 \Delta E_1 - g_n g_{n-1} g_{n-2} \cdots g_5 g_4 g_3 \Delta E_2 - \ldots$$
$$- g_n g_{n-1} g_{n-2} \Delta E_{n-3} - g_n g_{n-1} \Delta E_{n-2} - g_n \Delta E_{n-1} - \Delta E_n$$

or

$$F_n = \textstyle\prod_{i=1}^{n} g_i F_0 - (\sum_{j=1}^{n-1} \prod_{i=j+1}^{n} g_i \Delta E_j) - \Delta E_n. \tag{2.2}$$

Assuming that the effective growth rate is uniform i.e.

If

$$g_1 = g_2 = g_3 = g_4 = \ldots g_n = g.$$

Then

$$F_n = g^n F_0 - (\Delta E_n + g \Delta E_{n-1} + g^2 \Delta E_{n-2} + g^3 \Delta E_{n-3} + \ldots + g^{n-1} \Delta E_1)$$

$$F_n = g^n F_0 - \textstyle\sum_{r=0}^{n-1} (\Delta E)_{n-r} g^r$$

$$F_n = g^n F_0 - \textstyle\sum_{r=0}^{n-1} M_{n-r} g^r, \tag{2.3}$$

where

$$M_{n-r} = (\Delta E)_{n-r}, \qquad r \neq n,$$

$$M_{n-r} = E_0, \qquad r = n.$$

The right hand side is a polynomial in g, which gives a very important conclusion that the production and consumption has cumulative impact across the generation. Any displacement in the pattern of production and consumption get magnified many times across the generation. Saxena, V. P. ([13]) has introduced classical polynomials for the study of finite animal population. We extend this concept in our problem and give some specific polynomials applications as given below

2.1 Hermite Polynomials

$H_n(x) = \sum_{m=0}^{[\frac{n}{2}]} \frac{(-1)^m n!(2x)^{n-2m}}{m!(n-2m)!}$.

Assuming that the environmental effect will follow the pattern of this polynomial.

Accordingly from the equation (2.3) we get,

(2.4) $\sum_{r=0}^{[n-1]} M_{n-r} g^r = \sum_{m=0}^{[\frac{n}{2}]} \frac{(-1)^m n!(2x)^{n-2m}}{m!(n-2m)!}$,

where

$$n = \frac{n}{2} + 1.$$

After comparing the coefficients of equation (2.4) we derive,

(2.5) $M_{n-r} = \dfrac{(-1)^m n!(2x)^{n-2m}}{m!(n-2m)!}$.

2.2 Laguerre Polynomials

$L_n(x) = \sum_{r=0}^{[n]} \frac{(-1)^r n!(x)^r}{(n-r)!(r!)^2}$.

Comparing with equation (2.3) we get,

(2.6) $\sum_{r=0}^{[n-1]} M_{n-r} g^r = \sum_{r=0}^{[n]} \frac{(-1)^r n!}{(n-r)!(r!)^2}$,

where $n = n + 1$.

After comparing the coefficients of the equation (2.6), we get

(2.7) $M_{n-r} = \dfrac{(-1)^r n!}{(n-r)!(r!)^2}$.

2.3 Jacobi Polynomials

$P_n^{(\alpha,\delta)}(1-\alpha) = \sum_{k=0}^{n} \frac{(-n)_k(1+\gamma+\delta+n)_k(1+\gamma)_n \alpha^k}{n!(1+\gamma)_k 2^k}$.

From equation (2.3) we get,

(2.8) $\sum_{r=0}^{[n-1]} M_{n-r} g^r = \sum_{k=0}^{n} \frac{(-n)_k(1+\gamma+\delta+n)_k(1+\gamma)_n \alpha^k}{n!(1+\gamma)_k 2^k}$,

where $n = n + 1$.

After comparing the coefficients of equation (2.8) we get,

(2.9) $M_{n-r} = \dfrac{(-n)_k(1+\gamma+\delta+n)_k(1+\gamma)_n}{n!(1+\gamma)_k 2^k}$.

2.4 Gauss Hypergeometric Function

(2.10) $2_{F_1}[-n, b; c; y] = \sum_{r=0}^{n} \frac{(-n)_r(b)_r y^r}{(c)_r r!}$.

Comparing with the equation (2.3) we get,

(2.11) $\sum_{r=0}^{[n-1]} M_{n-r} g^r = \sum_{r=0}^{n} \frac{(-n)_r(b)_r y^r}{(c)_r r!}$,

where $n = n + 1$.

After comparing the coefficients of equation (2.11) we get,

(2.12) $M_{n-r} = \dfrac{(-n)_r(b)_r}{(c)_r r!}$.

The evolution of species will be given as following equation

$$\Delta P_n = B_n F_n P_{n-1} - D_n P_{n-1} - M_n,$$

where

B_n = Birth rate of species which should be further proportional to food availability,

D_n = Natural Death rate of the species,

M_n = Migration of the species.

Therefore,

$$\Delta P_n = B_n(g_n F_{n-1} - \Delta E_n)P_{n-1} - D_n P_{n-1} - M_n$$

$$\Delta P_n = B_n g_n F_{n-1} P_{n-1} - B_n g_n P_{n-1}(\Delta E_n) - D_n P_{n-1} - M_n$$

$$\Delta P_n = P_n - P_{n-1}.$$

Hence

$$P_n = \left[B_n g_n F_{n-1} - B_n g_n(\Delta E_n) - D_n\right]P_{n-1} - M_n + P_{n-1},$$

$$P_n = \left[B_n g_n F_{n-1} - B_n g_n(\Delta E_n) - D_n + 1\right]P_{n-1} - M_n.$$

Let $B_n g_n F_{n-1} - B_n g_n(\Delta E_n) - D_n + 1 = Z_n$.
Then

(2.13) $P_n = Z_n P_{n-1} - M_n$

The Solution of the equation (2.13) can be given as

$$P_n = z_n z_{n-1} z_{n-2} \ldots z_3 z_2 z_1 P_0 - z_n z_{n-1} z_{n-2} \ldots z_4 z_3 z_2 M_1 - z_n z_{n-1} z_{n-2} \ldots z_5 z_4 z_3 M_2 - \ldots - z_n z_{n-1} z_{n-2} M_{n-3} - z_n z_{n-1} M_{n-2} - z_n M_{n-1} - M_n,$$

or

(2.14) $P_n = \prod_{i=1}^{n} z_i P_0 - \left(\sum_{j=1}^{n-1} \prod_{i=j+1}^{n} z_i M_j\right) - M_n.$

Let us further assume that Z_n is uniform across the generation i.e.
If $z_1 = z_2 = z_3 = z_4 = \ldots z_n = z$
Then

$$P_n = z^n P_0 - \left(M_n + zM_{n-1} + z^2 M_{n-2} + z^3 M_{n-3} + \ldots + z^{n-1} M_1\right),$$

$$P_n = z^n P_0 - \sum_{r=0}^{n-1} M_{n-r} z^r.$$

Hence

(2.15) $P_n = z^n P_0 - \sum_{r=0}^{n-1} M_{n-r} z^r.$

The right hand side is a polynomial n in z.

3 Numerical Examples

If

$$\alpha_1 = \alpha_2 = \alpha = 0.3$$
$$\beta_1 = \beta_2 = \beta = 0.001$$
$$M_1 = M_2 = M = 10$$

The calculation of population have been shown in ***Figures* 3.1, 3.2** and **3.3** excluding the hunting or natural death of the species. This factor will significantly bring down the values of population figures as depicted in the figures. The inclusion of this factor will help us to fine-tune the population figures to acceptable values.

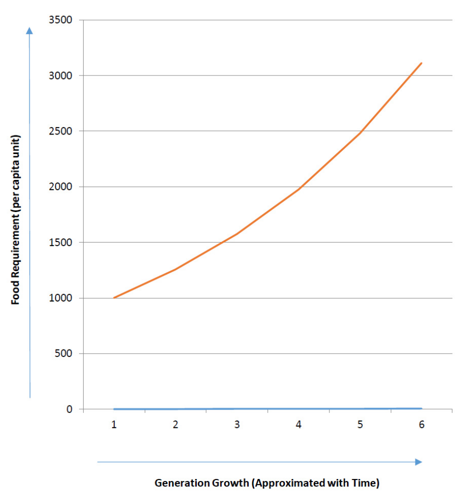

***Figure* 3.1:** Graph between food and generation for $\alpha = 0.3, \beta = 0.001$ and $M = 10$
.

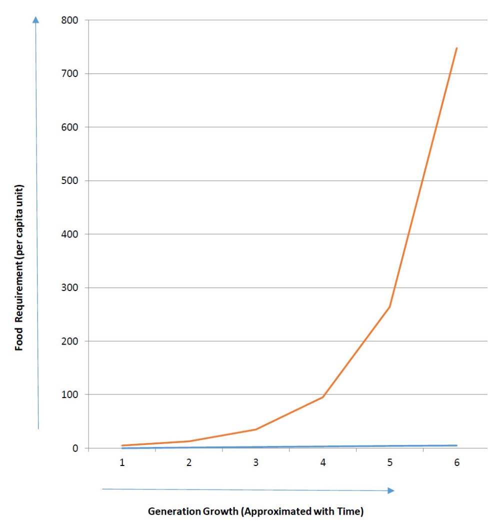

***Figure* 3.2:** Graph between food and generation for $\alpha = 0.2, \beta = 0.002$ and $M = 10$
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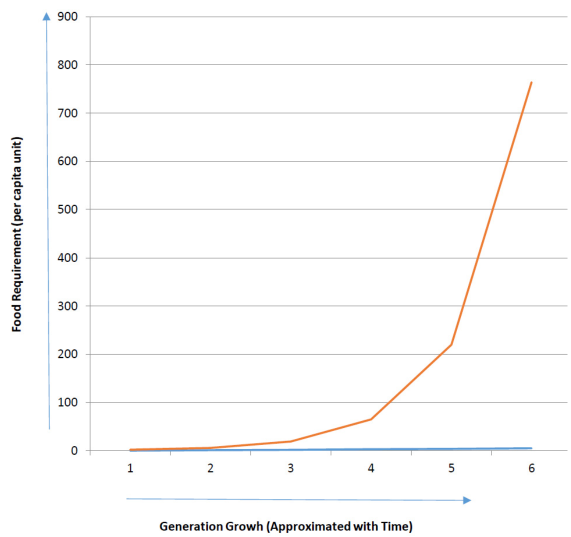

***Figure* 3.3:** Graph between food and generation for $\alpha = 0.1, \beta = 0.003$ and $M = 10$

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4 Conclusion and Way Forward

The study of animal species living in natural environment is quite useful to develop more realistic mathematical models in future. In this context this study is quite useful. The production of primary food, its consumption by animal species and likely impact of climatic change has been included in a realistic way in this model. There is however ample scope for further development of this mathematical model. Pursuits for Sustainable Development Goals require a better understanding of ecological dynamics and enhanced predictive capacity of ecological evolution. It will help us to take timely measures to mitigate the undesirable impact in ecological dynamics. This paper tries to develop a perspective to comprehend the impact of environment on evolution of wildlife species. Prey Predator Model has been revisited and the changing environmental factors have been included in the mathematical formalism.

Some of the important points which make the study useful for further studies are given below

4.1 The impact of adverse environmental impact on wild life evolution is a new trend. Our study is quite general and broad based which includes the impact of environmental impacts on population growth. Future studies can use different trends of environmental change and calculate the figures. The results will help in the food management of animal species under observation.

4.2 The study is unique in the sense that the time scale has been chosen in the terms of generation of species.

4.3 The traditional studies on ecological modelling have not used the application of polynomials. The properties of polynomials can be utilized to calculate the figures easily. It gives rise to new perspectives in population modelling. The future studies can enlarge these Mathematical treatments to make the model more realistic.

4.4 The study highlights the fact that effective growth rate of primary food in the environment will depend upon comparative rate of consumption, regeneration and environmental degradation. The population will undergo sharp decline if the rates of consumption are persistently higher than its regeneration. The rate of decline will increase over the generation.

4.5 The population of the species which survives on the primary food will ultimately depend upon a mix of the parameters consumption, regeneration, environmental degradation, natural birth/death rate of species and its consumption habits. Over the generations these parameters will have a complex dependence on these parameters which will be governed by a polynomial.

The future studies can employ different trends of environmental degradation to work out the evolution of the primary food production. These figures will help to calculate the trends of animal population. The studies can also explore the impact of more number of species in the evolution of primary food and animal population. Such efforts will help to proceed step by step towards more realistic models in ecology.

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(Dedicated to Honor Dr. R. C. Singh Chandel on His 75th Birth Anniversary Celebrations)

NUMERICAL STUDY OF MHD BOUNDARY LAYER FLOW OF WILLIAMSON FLUID WITH VARIABLE FLUID PROPERTIES

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Abstract

This numerical study investigates the *MHD* boundary layer flow and heat transfer of the Williamson fluid over a permeable nonlinearly stretching sheet. The partial differential equations corresponding to the momentum and energy are converted into ordinary differential equations with the help of similarity transformations. The numerical solution is computed by Runge-Kutta fourth order method with shooting technique. The effects of various parameters such as viscosity variation parameter, thermal conductivity variation parameter, magnetic field parameter, suction parameter, Williamson fluid parameter , radiation parameter and Eckert number on velocity and temperature profiles are discussed through graphs.
2010 Mathematics Subject Classifications: 76D05, 76D10, 76W05, 80A05.
Keywords and phrases: Williamson fluid, *MHD*, Viscous dissipation, Thermal radiation, Non-linearly permeable stretching sheet.

1 Introduction

The heat transfer of boundary layer flow over a stretching sheet has been attracted many researchers in last two decades due to its wide range of applications in many standard manufacturing and chemical engineering processes such as polymer manufacturing, crystal manufacturing, petroleum filtering operation,paper production and food preserving processes. Williamson [21] represented the model of the flow of pseudoplastic materials. Chen and Char [1] discussed the effects of suction or blowing on the heat transfer of a continuous stretching surface. Free convection on a vertical stretching surface with suction or blowing has been studied by Gorla and Sidawi [2]. Rapits [15] described the effects of radiation on the flow of a micropolar fluid past a continuously moving plate. Raptis [16] described the radiation and viscoelastic flow. Mohammadein and Gorla [10] studied the heat transfer in a micropolar fluid over a stretching sheet with viscous dissipation and internal heat generation. Mukhopadhyay et al. [11] studied of the effects of variable viscosity on *MHD* boundary layer flow over a heated stretching sheet. Vajravelu et al. [20] analyzed the peristaltic transport of a Williamson fluid in asymmetric channels with permeable walls. Jat and Chand [4] presented the *MHD* flow and heat transfer over an exponentially stretching sheet with viscous dissipation and radiation effects. Mukhopadhyay [12] investigated the slip effect on *MHD* boundary layer flow over an exponentially stretching sheet with suction/blowing and thermal radiation. Nadeem et al. [13] studied the flow of a Williamson fluid over a stretching sheet. Hayat et al. [3] investigated the effects of joule heating and thermal radiation in flow of third-grade fluid over radiative surface. Nadeem and Hussain [14] discussed the flow and heat transfer analysis of Williamson nanofluid. Rashidi et al. [17] studied the mixed convective heat transfer for *MHD* viscoelastic fluid flow over a porous wedge with thermal radiation. Yadav and Sharma [22] analyzed the effects of radiation and viscous dissipation on *MHD* boundary layer flow due to an exponentially moving stretching sheet in porous medium. Megahed [8] examined the effects of heat flux and thermal radiation on heat transfer of a non-Newtonian power-law fluid over a non-linearly stretching vertical surface.Reddy et al. [18] presented the *MHD* flow and heat transfer characteristics of Williamson nanofluid over a stretching sheet with variable thickness and variable thermal conductivity. Shafiq and Sindhu [19] discussed the statistical study of hydromagnetic boundary layer flow of Williamson fluid regarding a radiative surface. Kumar et al. [5] investigated the Slip effect on *MHD* stagnation point flow and heat transfer of cross fluid with heat generation in a porous medium. Lund et al. [6] analyzed the dual solution for *MHD* flow of Williamson fluid with slippage. Williamson fluid flow over a nonlinearly stretching sheet with viscous dissipation and thermal radiation by Megahed [9].

In this paper we use the Williamson fluid flow model which was given by Williamson [21]. The objective of present paper is to analyze the *MHD* Williamson fluid flow over a nonlinearly permeable stretching sheet with thermal radiation, viscous dissipation and variable fluid properties.

2 Formulation of the Problem

We assume moving fluid as a Williamson fluid with a time constant Γ. The fluid is flowing on nonlinearly permeable stretching sheet in the presence of radiation, viscous dissipation phenomena and magnetic field. Here x-axis is along the sheet whereas y-axis is taken perpendicular to stretching sheet. The stretching may create the velocity $U_w = cx^m$ for the fluid, where c is a constant and m is a exponent. Here it is assumed that both thermal conductivity and the fluid viscosity are varying with temperature while remaining properties are constant.

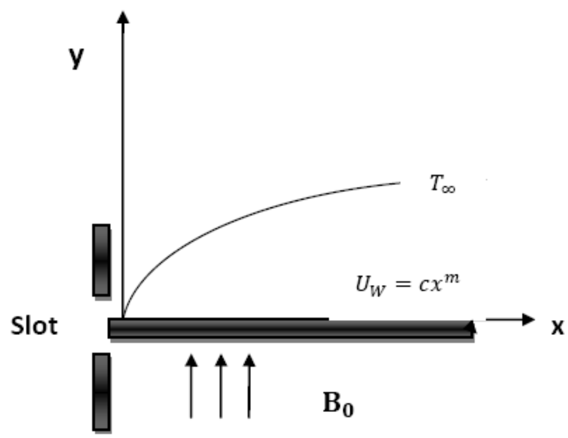

Figure **2.1:** Sketch of problem.

The governing equations of present fluid flow can be introduced in the following form [13]:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{2.1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial y} = \frac{1}{\rho_\infty}\frac{\partial}{\partial y}\left(\mu(T)\frac{\partial u}{\partial y} + \mu(T)\frac{\Gamma}{\sqrt{2}}\left(\frac{\partial u}{\partial y}\right)^2\right) - \frac{\sigma B_0^2}{\rho_\infty}u, \tag{2.2}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \frac{1}{\rho_\infty c_p}\frac{\partial}{\partial y}\left(\kappa(T)\frac{\partial T}{\partial y}\right) + \frac{\mu(T)}{\rho_\infty c_p}\left(1 + \frac{\Gamma}{\sqrt{2}}\frac{\partial u}{\partial y}\right)\left(\frac{\partial u}{\partial y}\right)^2 - \frac{1}{\rho_\infty c_p}\frac{\partial q_r}{\partial y}, \tag{2.3}$$

with boundary conditions

$$u = cx^m, \quad v = -V_w, \quad T_w(x) = T_\infty + Ax^r \quad at \quad y = 0, \tag{2.4}$$

$$u \to 0, \quad T_w(x) \to T_\infty \quad at \quad y \to 0, \tag{2.5}$$

where u and v are the components of velocity in the x and y directions, respectively. Also, T is the temperature for the Williamson fluid, ρ_∞ refers to the fluid density at the ambient, q_r is the radiation heat flux and c_p is the specific heat at constant pressure. T_∞ is the constant ambient temperature, A and r are constants.

By taking Rosseland approximation [15], q_r can be described in the following form:

$$q_r = -\frac{4\sigma^*}{3k^*}\frac{\partial T^4}{\partial y}, \tag{2.6}$$

where the constant σ^* is the Stefan-Boltzmann and k^* is the absorption coefficient.The term T^4 can be simplified by using Taylor expansion about the constant value T_∞ as $T^4 \cong 4T_\infty^3 T - 3T_\infty^4$, after neglecting all higher order terms [16].

Using the similarties transformation

$$\eta = \left(\frac{cx^{m-1}}{\nu_\infty}\right)^{1/2} y, \quad \psi(x,y) = \left(cx^{m+1}\nu_\infty\right)^{1/2} f(\eta), \quad \theta(\eta) = \frac{T - T_\infty}{T_w - T_\infty}, \tag{2.7}$$

(2.8) $u = \dfrac{\partial \psi}{\partial y}, \qquad v = -\dfrac{\partial \psi}{\partial x},$

where ν_∞ is the kinematic viscosity at the ambient and $\psi(x, y)$ is the stream function.

The important assumptions in this research is that the viscosity is altering exponentially with the temperature and the thermal conductivity is changing linearly with temperature for these equations [7]:

(2.9) $\mu = \mu_\infty e^{-\alpha\theta}, \qquad \kappa = \kappa_\infty (1 + \epsilon\theta),$

where μ_∞ is the viscosity at the ambient, κ_∞ represents the thermal conductivity at the ambient, α is the viscosity variation parameter and ϵ is the thermal conductivity variation parameter.

Equations (2.2)-(2.3) converted into

(2.10) $e^{-\alpha\theta}\left(\left(1 + \delta f^{''}\right) f^{'''} - \alpha\theta^{'} f^{''}\left(1 + \dfrac{\delta}{2}\right)\right) - Mf^{'} + \left(\dfrac{m+1}{2}\right) f f^{''} - m f^{'2} = 0,$

(2.11) $\dfrac{1}{Pr}\left(\epsilon\theta^{'2} + (1 + R + \epsilon\theta)\,\theta^{''}\right) + \left(\dfrac{m+1}{2}\right) f\theta^{'} - r f^{'}\theta + Ec\left(1 + \dfrac{\delta}{2}\right) f^{''2} e^{-\alpha\theta} = 0,$

and reduced boundary conditions are

(2.12) $f(0) = S, \quad f^{'}(0) = 1, \quad \theta(0) = 1,$

(2.13) $f^{'} \to 0, \quad \theta \to 0, \quad at \quad \eta \to \infty,$

where $\delta = \left(\frac{\sqrt{2}c^{3/2} x^{\frac{3m-1}{2}}}{\sqrt{\nu_\infty}}\right)\Gamma$ is the local Williamson fluid parameter, M is magnetic field parameter, S is the suction parameter, R is the radiation paramter, $Ec = \frac{U_w^2}{c_p(T_w - T_\infty)} = \frac{c^2 x^{2m-r}}{Ac_p}$ is the Eckert number, $Pr = \dfrac{\mu_\infty c_p}{\kappa_\infty}$ is the Prandtl number. Here we take $r = 2m = \frac{2}{3}$, so,these parameters take the form $\delta = \left(\frac{\sqrt{2}c^{3/2}}{\nu_\infty}\right)\Gamma$ and $Ec = \frac{c^2}{Ac_p}$.

The local skin - friction coefficient Cf_x and the local Nusselt number Nu_x are given as

(2.14) $Cf_x = -2Re_x^{-1/2}\left(1 + \dfrac{\delta}{2} f^{''}(0)\right) f^{''}(0) e^{-\alpha\theta(0)},$

(2.15) $Nu_x = -Re_x^{-1/2}\left(1 + R + \epsilon\theta(0)\right)\theta^{'}(0),$

where $Re_x = \frac{U_w x}{\nu_\infty}$ is the local Reynolds number.

3 Method of Solution

Numerical shooting technique with Runge-Kutta fourth order method was adopted to solve the problem. Equations (2.10)-(2.11) subject to the boundary conditions (2.12)-(2.13) are transformed into the following system of first-order differential equations:

(3.1) $f_1^{'} = f_2,$

(3.2) $f_2^{'} = f_3,$

(3.3) $f_3^{'} = \dfrac{e^{\alpha f_4}}{(1 + \delta f_3)}\left(m f_2^2 - \left(\dfrac{m+1}{2}\right) f_1 f_3 + M f_2\right) + \dfrac{\alpha f_5 f_3}{(1 + \delta f_3)}\left(1 + \dfrac{\delta}{2} f_3\right),$

(3.4) $f_4^{'} = f_5,$

(3.5) $f_5^{'} = \dfrac{1}{(1 + R + \epsilon f_4)}\left(Pr\left(r f_2 f_4 - \left(\dfrac{m+1}{2}\right) f_1 f_5 - Ec\left(1 + \dfrac{\delta}{2} f_3\right) f_3^2 e^{-\alpha f_4}\right) - \epsilon f_5^2\right),$

where $f = f_1$, $f^{'} = f_2$, $f^{''} = f_3$, $\theta = f_4$ and $\theta^{'} = f_5$
and initial conditions are $f_1(0) = S$, $f_2(0) = 1$,$f_4(0) = 1$.

This system can not be solved with the infinite conditions which appear in eq.(2.13). So, these conditions are replaced by appropriate finite guessing values $f^{''}(0)$ and $\theta^{'}(0)$.

4 Results and Discussion

The influence of viscosity variation parameter α on velocity and temperature profile is shown in Fig. **4.1** and Fig. **4.2**. From Fig. **4.1** and Fig. **4.2**, it is clear that velocity decrease with respect to increasing value of α and temperature profile increase with respect to increasing value of α.

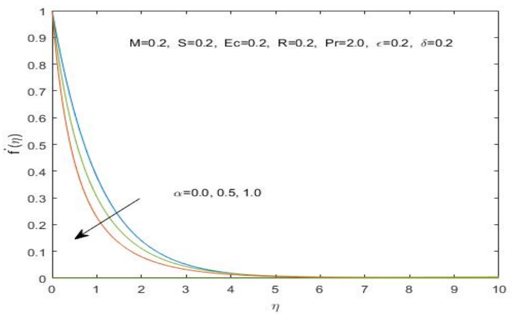

Figure **4.1:** Velocity profiles for different values of viscosity variation parameter α.

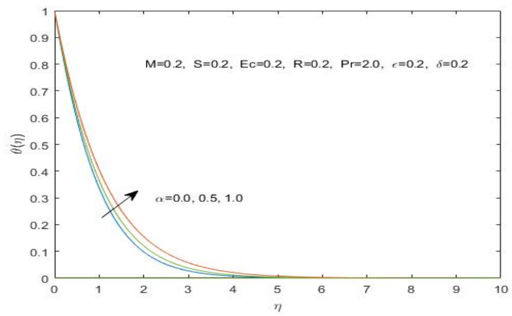

Figure **4.2:** Temperature profiles for different values of viscosity variation parameter α.

The varaition of local Williamson fluid parameter δ on velocity and temperature distributions is shown in Fig. **4.3** and Fig. **4.4**. From Fig. **4.3**, it is noticed that increase in δ leads to decrease in velocity. From Fig. **4.4**, it is observed that increase in δ tends to increase in temperature.

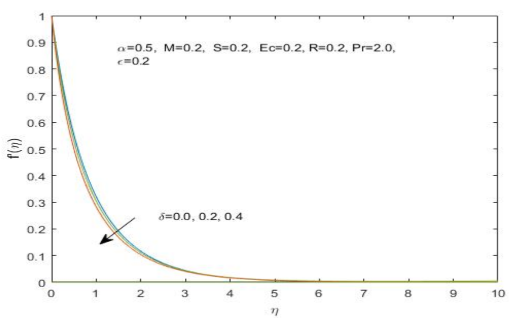

Figure **4.3:** Velocity profiles for different values of Williamson fluid parameter δ.

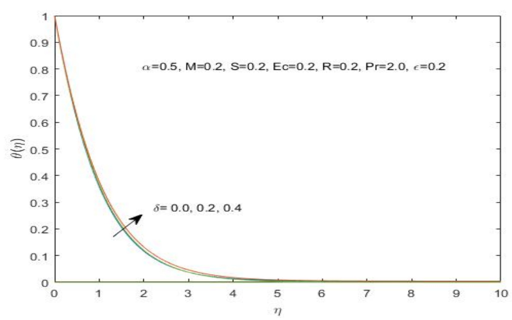

Figure **4.4:** Temperature profiles for different values of Williamson fluid parameter δ.

The influence of magnetic field parameter M on both velocity and temperature profiles are shown in Fig. **4.5** and Fig. **4.6**. From Fig. **4.5**, it is observed that velocity of fluid decreases with increase in M. The magnetic field causes a amount of resistance to its motion due to Lorentz force, which reduces the fluid velocity. From Fig. **4.6**, it is clear that increase in M leads to increase in temperature. Fig. **4.7** and Fig. **4.8** depict the variation of suction parameter S on velocity and temperature profile. From Fig. **4.7** and Fig. **4.8**, it is observed that velocity and temperature decrease with increase in S.

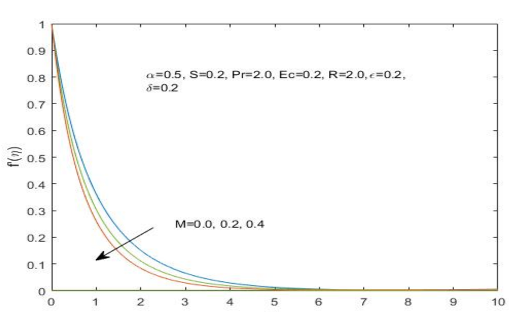

Figure **4.5:** Velocity profiles for different values of magnetic field parameter M.

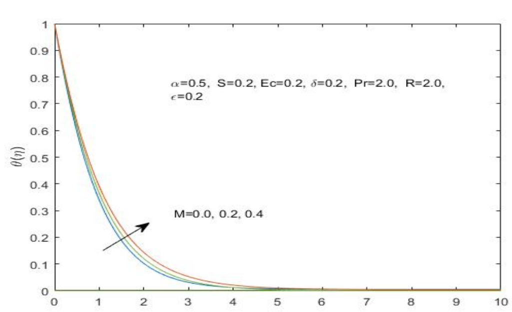

Figure **4.6:** Temperature profiles for different values of magnetic field parameter M.

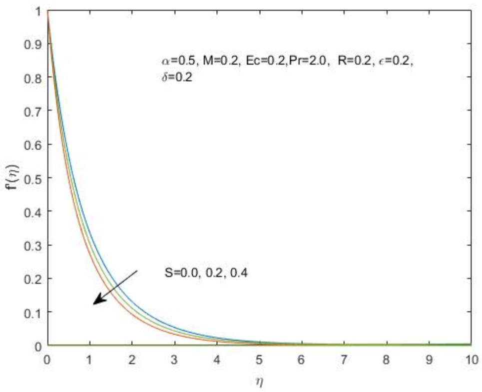

Figure **4.7:** Velocity profiles for different values of suction parameter S.

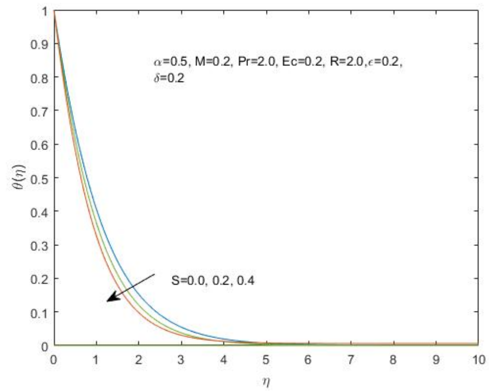

Figure **4.8:** Temperature profiles for different values of suction parameter S.

The effect of thermal conductivity parameter ϵ and radiation parameter R on temperature profile are given in Fig. **4.9** and Fig. **4.10** respectively, it is observed that increase in ϵ and R, leads to increase in temperature. Because increase in radiation parameter R provides more heat to fluid that leads increase in temperature profile.

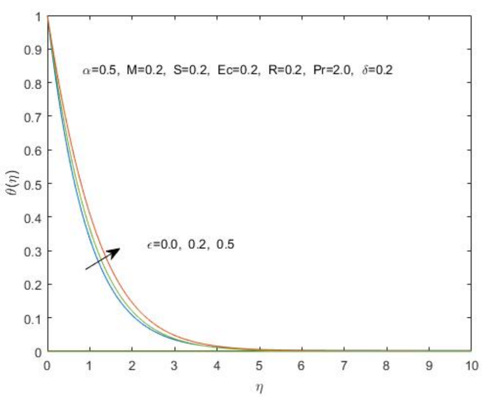

Figure **4.9:** Temperature profiles for different values of thermal conductivity parameter ϵ.

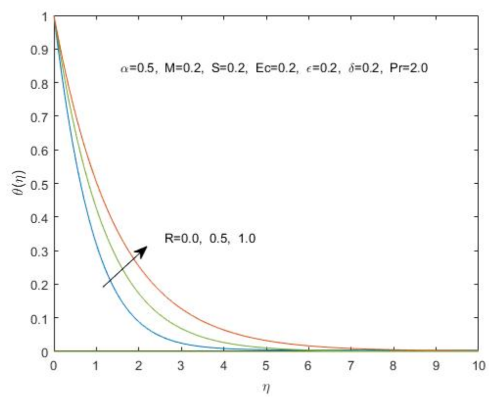

Figure **4.10:** Temperature profiles for different values of radiation parameter R.

The effect of Eckert number Ec on temperature profile is shown in Fig. **4.11**. From Fig. **4.11**, it is clear that increase in Ec leads to increase in temperature profile because viscosity of fluid converts the energy from motion into the internal energy of fluid which results in increasing of temperature.

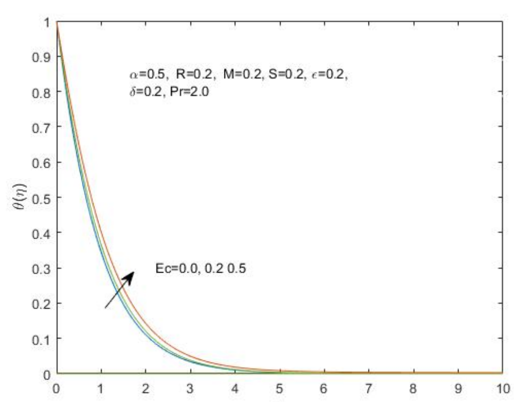

Figure **4.11:** Temperature profiles for different values of Eckert number Ec.

Table **4.1:** Comparison of Nusselt number $Re_x^{-1/2} Nu_x$ for various values of Pr when $\delta = \alpha = \epsilon = R = Ec = M = S = r = 0$ and $m = 1$

Pr	Gorla and Sidawi (1994)	Megahed (2019)	Present study
2.0	0.91142	0.911358	0.911355
7.0	1.89046	1.895453	1.894460
20.0	3.35391	3.353902	3.353904

The comparison of present problem with previous research works done by Megahed[9] and Gorla and Sidawi[2] are shown in Table **4.1**. The present problem can be transformed into the previous published work when $\delta = \alpha = \epsilon = R = Ec = M = S = r = 0$ and $m = 1$.

Table **4.2** presents the numerical values of both the local Nusselt number and the local skin friction parameter for various values of Eckert number Ec, magnetic field parameter M, suction parameter S, Williamson fluid parameterδ, radiation parameter R, viscosity variation parameter α and thermal conductivity parameter ϵ. Analysis of table shows that the Eckert number, the Williamson fluid parameter, the viscosity parameter lead to decresing behaviour for both the local Nusselt number and the local skin friction coefficent and the suction parameter leads to a increasing behaviour for both. Magnetic field parameter leads a diminishing behaviour for the local Nusselt number and increasing behaviour for the the local skin friction.

Table **4.2:** Values of $1/2 Re_x^{1/2} Cf_x$ and $Re_x^{-1/2} Nu_x$ for various values of α, δ, ϵ , M, S, R, Ec with $m = 1/3$, $r = 2/3$, $Pr = 2.0$

α	δ	ϵ	M	S	R	Ec	$\frac{1}{2} Re_x^{1/2} Cf_x$	$Re_x^{-1/2} Nu_x$
0.0	0.2	0.2	0.2	0.2	0.2	0.2	0.8541	1.3875
0.5	0.2	0.2	0.2	0.2	0.2	0.2	0.7168	1.3122
1.0	0.2	0.2	0.2	0.2	0.2	0.2	0.5862	1.2173
0.5	0.0	0.2	0.2	0.2	0.2	0.2	0.7491	1.3298
0.5	0.2	0.2	0.2	0.2	0.2	0.2	0.7168	1.3122
0.5	0.4	0.2	0.2	0.2	0.2	0.2	0.6744	1.2789
0.5	0.2	0.0	0.2	0.2	0.2	0.2	0.7201	1.2639
0.5	0.2	0.2	0.2	0.2	0.2	0.2	0.7168	1.3122
0.5	0.2	0.4	0.2	0.2	0.2	0.2	0.7128	1.3738
0.5	0.2	0.2	0.0	0.2	0.2	0.2	0.6195	1.3881
0.5	0.2	0.2	0.2	0.2	0.2	0.2	0.7168	1.3122
0.5	0.2	0.2	0.4	0.2	0.2	0.2	0.7981	1.2415
0.5	0.2	0.2	0.2	0.0	0.2	0.2	0.6443	1.1907
0.5	0.2	0.2	0.2	0.2	0.2	0.2	0.7168	1.3122
0.5	0.2	0.2	0.2	0.4	0.2	0.2	0.7952	1.4412
0.5	0.2	0.2	0.2	0.2	0.0	0.2	0.7208	1.2412
0.5	0.2	0.2	0.2	0.2	0.5	0.2	0.7123	1.3981
0.5	0.2	0.2	0.2	0.2	1.0	0.2	0.7068	1.5092
0.5	0.2	0.2	0.2	0.2	0.2	0.0	0.7189	1.4377
0.5	0.2	0.2	0.2	0.2	0.2	0.2	0.7168	1.3122
0.5	0.2	0.2	0.2	0.2	0.2	0.5	0.7156	1.1139

5 Conclusions

In this paper we have studied the boundary layer flow and heat transfer of Williamson fluid over a permeable nonlinearly stretching sheet in the presence of magnetic field. We draw the following conclusions from our study

(1) Both the thermal radiation parameter and Eckert number enhance the temperature distribution, thicken thermal region.
(2) An increase in both the viscosity parameter and the Williamson parameter results in a rise in the temperature distribution.
(3) Thermal conductivity leads to an increasing behaviour for the local Nusselt number.

(4) Magnetic field parameter leads to decreasing behaviour for the local Nusselt number and increasing behaviour for the local skin-friction coefficient.
(5) Velocity and temperature profiles are decreasing as the value of suction parameter is increasing

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(Dedicated to Honor Dr. R. C. Singh Chandel on His 75th Birth Anniversary Celebrations)

STUDY OF Alfvén WAVES USING MAGNETO HYDRODYNAMIC EQUATIONS IN SOLAR ATMOSPHERE

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Abstract

This paper presents a three dimensional analysis of the properties of magneto hydrodynamic waves. A quantitative formulation of magneto hydrodynamic Alfvén waves is studied. On considering the motion in ordinary fluid, it is observed that the propagation of low frequency waves is only in the form of pressure waves. We have focused on such modes of propagation of incompressible fluid or plasma in magnetic field for which, the speed does not surpass the speed of sound. The value of conductivity is taken to be infinity and viscosity negligible. In such a case, the sum of the pressures of the field induced and the pressure of the plasma is independent of all the coordinates. The kinetic energy density of wave motion and the energy density of the field induced by the perturbation come out to be equal in both phase and amplitude. The study of the effect of large but finite conductivity for small "Joule losses" reveal that the waves remain periodic, but it is observed that the amplitude of the waves decreases continuously and exponentially at a slow pace.
2010 Mathematics Subject Classifications: 76W05.
Keywords and phrases: *MHD*, Alfvén waves, plasma pressure, magnetic field.

1 Introduction

Most of the physical phenomena that take place in our Sun can be described in terms of waves or oscillations. The behavior of plasma and magnetic fields of Sun is described by solar magneto hydrodynamics *(MHD)*. Solar *MHD* deals with the propagation of *MHD* waves in the solar atmosphere. It is the study of the magnetic properties of electrically conducting fluids. Some of such magneto fluids include plasmas, liquid metals and salt water or electrolytes.It is a macroscopic theory that is valid when the smallest length-scale, namely, the width of the diffusion region, is larger than the mean-free path for collisions([10]). The basic concept behind the theory of *MHD* is that the magnetic fields are able to induce currents in a moving conductive fluid, which in turn polarizes the fluid and reciprocally changes the magnetic field itself. Hence, the conductive fluids can support magnetic fields. The main concern for a particular conducting fluid is the relative strength of the advecting motions in the fluid, compared to the diffusive effects caused by the electrical resistivity. Different solar activities are due to the interaction of the plasma of both solar interior and atmosphere with the magnetic field of the Sun as well as the convection and differential rotation of the Sun. These interactions are studied by taking into consideration the plasma physics and how it deals with the magnetic field. The interactions play a main role in the physical properties of the medium.

The main aim of the paper is to study and analyze three dimensionally the properties of magneto hydrodynamic waves. A quantitative formulation of magneto hydrodynamic Alfvén waves is studied. An attempt is made to study the Sun from magneto hydrodynamic point of view.

2 Basic *MHD* equations and Alfvén waves

MHD is a combination of the equations of hydrodynamics, electrodynamics and associated equations (see,[8,11,6,3]).The study of *MHD* combines the Eulers equation describing fluid dynamics with Maxwells equations of electromagnetism. These differential equations are resolved together, either in an analytical or numerical manner. From above equations we obtain a set of four equations, referred as ideal or basic *MHD* equations (see,[9]). These consist of two vector and two scalar partial differential equations. The equations are:

$$(2.1)\quad \frac{\partial \rho}{\partial t} + \nabla.(\rho \vec{V}) = 0,$$

$$(2.2)\quad \frac{\partial(\rho(\vec{V}))}{\partial t} + \nabla.(\rho \vec{V}) = -\nabla P + \vec{J} \times \vec{B},$$

(2.3) $\dfrac{\partial\rho}{\partial t} + \vec{V}.\nabla P + \gamma P\nabla.\vec{V} = 0,$

(2.4) $\dfrac{\partial\vec{B}}{\partial t} = -\nabla\times\vec{E} = \nabla\times(\vec{V}\times\vec{B}) + \eta\nabla^2\vec{B},$

where ρ is the mass density, $\vec{V}$ the fluid velocity, $\vec{(}B)$ the magnetic field or magnetic flux density, γ is the adiabatic coefficient (generally taken as 5/3), $\vec{J}$ is the current density and η is the magnetic diffusivity or the electrical resistivity. Here P is the plasma pressure which is given as,

(2.5) $P = 2(\rho/m_i)kT,$

where m_i is the ionic mass, k is the Boltzmann constant and T is the temperature. The magnetic diffusivity or the electrical resistivity is usually taken as zero; hence equation (2.4) takes the form,

(2.6) $\dfrac{\partial\vec{B}}{\partial t} - \nabla\times(\vec{V}\times\vec{B}) = 0.$

In the above equation, as $\eta = 0$, it indicates the presence of frozen field, i.e., the magnetic field remains tightly coupled to the fluid. Here, the relative strength of resistivity is calculated with the help of a dimensionless number known as the magnetic Reynolds number, R_{me} given as:

(2.7) $R_{me} = \dfrac{VL}{\eta},$

where V is the amplitude of the plasma velocity and L is the global length scale.

In solar magneto hydrodynamics, Reynolds number attains a large value as the length scales are large enough.

The magnetic pressure is given as $\frac{B^2}{2\mu_o}$. The ratio of plasma pressure P to magnetic pressure $\frac{B^2}{2\mu_o}$ is known as the plasma beta β,

(2.8) $\beta = \dfrac{P}{B^2/2\mu_o},$

where $\mu_o$is the magnetic permeability in vacuum.

In the above equation if $\beta = 1$, it implies that gas pressure is equal to the magnetic pressure. If $\beta >> 1$, the magnetic field is weak and it spins along the plasma fluid. On the contrary if $\beta << 1$, the magnetic field dominates and forces the plasma to move along with it. Both Reynolds number and plasma beta are dimensionless numbers.

Alfvén waves are the magneto hydrodynamic waves described by frequencies which are below the gyro frequency and their wavelength is much larger than the inter-particle distance1. The Lorentz force in magnetic field $\vec{B}$ is

(2.9) $\vec{J}\times\vec{B} = \left(\dfrac{1}{4\pi}\right)(B\vec{.}\nabla)\vec{B} - \nabla(B)^2/8\pi,$

where $\vec{J}$is the current density,$(B\vec{.}\nabla)$ represents tension due to curvature of field lines and $\nabla(B)^2/8\pi$ pressure transverse to the field lines. This magnetic tension is responsible for the generation of transverse waves propagating along $\vec{B}$ in the x-direction. The equation of motion is given as,

(2.10) $\left(\dfrac{\partial^2\vec{B}}{\partial t^2}\right) = (V_A)^2\left(\dfrac{\partial^2\vec{B}}{\partial x^2}\right),$

where, V_A is the Alfvén speed given by the equation,

(2.11) $(V_A) = \dfrac{B}{\sqrt{4\pi\rho}}2.2\times\left(\dfrac{10^{11}B}{\sqrt{ne}}\right),$

It represents the speed of propagation of all the magnetic disturbances and controls the growth rate of magnetic instabilities.

When a magnetic disturbance travels at a speed greater than the Alfvén speed, it leads to the production of an *MHD* shock wave. It results in an increase in magnetic field behind the waves. Shock waves are non-linear waves whose character is determined and defined by the conservation laws of mass, total energy and momentum. Such an *MHD* shock exists at the periphery of terrestrial magnetosphere. At this boundary, the solar wind travelling with a velocity of 400 km/s carries charged particles emanating from the Sun. This shock displays a stationary mode with the boundary of the magnetosphere. Alfvén waves provide an exact solution of the non-linear *MHD* equations.

3 Study of Alfvén waves using magneto hydrodynamic equations in solar atmosphere

The properties of Alfvén waves are found to vary in magnetic field. Many studies in the past have been done using magneto hydro dynamical equations (see [7, 15, 5]). Let us consider an incompressible fluid with infinite conductivity, lying in a homogeneous magnetic field (B_o). Any plasma or gas acts as an incompressible fluid till the speed of the mode does not surpass the speed of sound.Quantitatively, on considering the motion in the ordinary fluid, it is observed that the propagation of low frequency waves is only in the form of pressure waves.

Our focus lies on such modes only, and hence the viscosity is considered negligible and conductivity infinite ([1,2]). Initially, let the Alfvén wave propagate in a uniform magnetic field along the z-direction, from $z = -\infty$ to $+\infty$(independent of x and y directions), it satisfies the following differential equations

$$(3.1)\quad \frac{\partial \rho}{\partial t} + \nabla.(\rho \vec{V}) = 0,$$

$$(3.2)\quad \frac{\partial \vec{B}}{\partial t} = Curl(\vec{V} \times \vec{B}) + \nu \nabla^2 B,$$

$$(3.3)\quad \frac{\partial \vec{V}}{\partial t} + (\vec{V}.\nabla)\vec{V} = \frac{\vec{F}}{\rho} + \frac{1}{4\pi\rho} Curl\vec{B} \times \vec{B} - \frac{1}{\rho}\nabla P.$$

From equation (3.2), it follows that $\frac{\partial V_z}{\partial z}$, suggesting V_z to be a constant w.r.t to height. It indicates that the Alfvén wave propagates with constant speed along the z-direction (height). Let us consider a small variation in magnetic field ($\vec{B_o}$) as ($\vec{b}$). On resolving b and V in component form we get equations (3.4) and (3.5).

$$(3.4)\quad \frac{\partial b_x}{\partial t} = B_o \frac{\partial V_x}{\partial z},$$
$$\frac{\partial b_y}{\partial t} = B_o \frac{\partial V_y}{\partial z},$$
$$b_z = 0\ .$$

$$(3.5)\quad \frac{\partial V_x}{\partial t} = \frac{B_o}{4\pi\rho} \frac{\partial b_x}{\partial z},$$
$$\frac{\partial V_y}{\partial t} = \frac{B_o}{4\pi\rho} \frac{\partial b_y}{\partial z},$$
$$\frac{\partial V_z}{\partial t} = 0\ .$$

Using equations (3.4) and (3.5), we solve equations (3.2) and (3.3) and get the resultant equation as

$$(3.6)\quad -\frac{b_y}{4\pi\rho}\frac{\partial b_y}{\partial z} - \frac{b_x}{4\pi\rho}\frac{\partial b_x}{\partial z} - \frac{1}{\rho}\frac{\partial \rho}{\partial z} = -\frac{1}{\rho}\frac{\partial}{\partial z}\left(\frac{bx^2}{8\pi} + \frac{by^2}{8\pi} + P\right) = 0.$$

Since the magnetic field is perpendicular to the direction of propagation, the variation in the field is only along x and y direction and hence the z component of magnetic field is zero (equation 3.5).

From the equation (3.6) it follows that:-

$$(3.7)\quad \frac{b_x^2}{8\pi} + \frac{b_y^2}{8\pi} + P = Const.$$

Equation (3.7) depicts that the sum of the pressures of the field induced (b_x and b_y) and the plasma pressure (P) is independent of the coordinates. From the above equation, it is clear that the sum of plasma pressure and magnetic pressure in solar atmosphere is constant. The pressure is independent of coordinates (***Fig.* 3.1**) indicating that the total plasma pressure is uniformly distributed in solar atmosphere while the plasma pressure is dependent on the magnetic field or magnetic pressure in solar atmosphere. This three dimensional variation of plasma pressure P with magnetic field components (b_x and b_y) can be seen in (***Fig.* 3.2**) Arregui et al. (2004) ([2]), studied the potential magnetic arcades under the condition when the magnetic pressure is dominant over the plasma pressure.

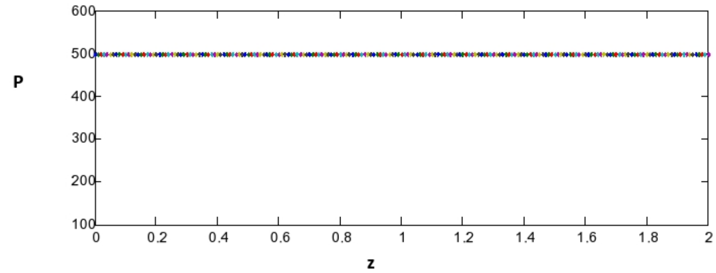

***Figure* 3.1:** Plot depicting that plasma pressure P is independent of coordinates

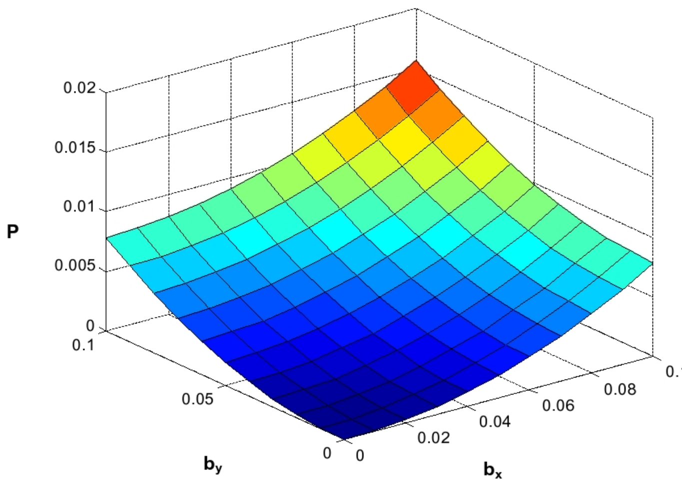

***Figure* 3.2:** Variation of plasma pressure P with magnetic field

The three dimensional variation of plasma pressure with components of magnetic field is parabolic in nature along *x*-axis and *y*-axis. A two dimensional plot of plasma pressure with different components of magnetic field is shown in ***Figs.*** **3.3(a)** and **3.3(b)**.

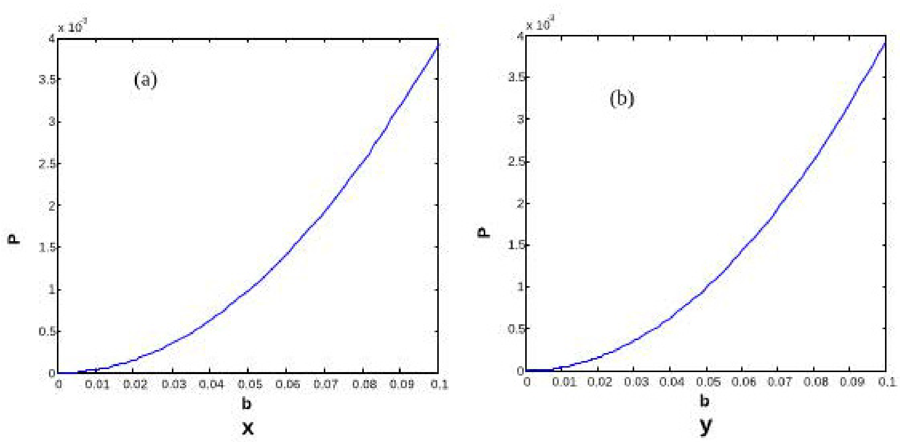

***Figure* 3.3:** Two dimensional plot between plasma pressure and different components of magnetic field (a) Plasma pressure and *x*-component of magnetic field (b) Plasma pressure and *y*-component of magnetic field.

Hence, if we consider a reference point in solar atmosphere (photosphere, chromosphere etc.)the plasma pressure will show an increase with increase of magnetic pressure along a particular direction. ***Fig.* 3.2** and ***Fig.* 3.3** indicate that the propagating Alfvén wavefollows a parabolic trajectory with magnetic field.

On differentiating equations (3.4) and (3.5) further w.r.t. t and z, we obtain two wave equations for b_y and V_y. In order to obtain solutions from equation (3.4), we have eliminated the terms $\frac{\partial^2 V_y}{\partial z \partial t}$ and $\frac{\partial^2 b_y}{\partial z \partial t}$. The resultant equations are

$$(3.8)\quad \frac{\partial^2 B_y}{\partial t^2} - \frac{B_o^2}{4\pi\rho}\frac{\partial^2 b_y}{\partial z^2},$$
$$\frac{\partial^2 V_y}{\partial t^2} - \frac{B_o^2}{4\pi\rho}\frac{\partial^2 V_y}{\partial z^2} = 0.$$

Equations (3.8) satisfies functions of argument, $t \pm \frac{z}{V_A}$, where $V_A = \frac{B_o}{\sqrt{4\pi\rho}}$, Since the plasma particles are moving with a velocity V along the direction of magnetic field, hence they vibrate due to the magnetic pressure while the Alfvén wave propagating with velocity $V_A$varies with respect to height z. Therefore, these two velocities are independent of each other.

Due to a small perturbation in magnetic field, Alfvén wave and magnetic plane wave(generated due to the vibration of plasma particles) propagate in solar atmosphere. Hence the solutions of equation(3.8) are sinusoidal periodic waves given by the equation (3.9) and equation (3.10) as below:

$$(3.9)\quad b_y = a \sin\omega\left(t - \frac{z}{V_A}\right),$$

$$(3.10)\quad V_y = -\frac{a}{\sqrt{4\pi\rho}} \sin\omega\left(t - \frac{z}{V_A}\right).$$

Soler et al. [13], studied the three dimensional propagation of *MHD* waves in solar coronal arcades and concluded that the measure of trapped wave energy depends on the wavelength of perturbations in the perpendicular direction. From equation (3.9) and (3.10), amplitude of magnetic plane wave is equal to a. Therefore, plasma pressure = $\frac{1}{8\pi}by^2 = \frac{1}{8\pi}a^2$.

Also, amplitude of particle wave is equal to $\frac{a}{\sqrt{4\pi\rho}}$; hence, kinetic energy of plasma particle

$$(3.11)\quad = \frac{1}{2}\rho V_y^2 = \frac{1}{2}\rho\frac{a^2}{4\pi\rho} = \frac{1}{8\pi}a^2.$$

It indicates, $= \frac{1}{2}\rho V_y^2 = \frac{1}{8\pi}b_y^2$. Hence, we conclude that the kinetic energy density of wave motion and the energy density of the field induced by the perturbation come out to be equal in both phase and amplitude.

Further, on using the equations (3.9) and (3.10), the expressions for the terms P, J_x and E_x are as follow:

$$(3.12)\quad P = P_o - \frac{b_y^2}{8\pi},$$

$$(3.13)\quad J_x = \frac{c}{4\pi}\nabla \times \vec{B} = -\frac{c}{4\pi}\frac{\partial B_y}{\partial z} = \frac{ac\omega}{4\pi V_A}\cos\omega\left(t - \frac{z}{V_A}\right),$$

$$(3.14)\quad E_X = -\frac{1}{c}(\vec{V} \times \vec{B}) = \frac{aV_A}{c} = \sin\omega\left(t - \frac{z}{V_A}\right).$$

Initially, if the disturbances emerge in a manner such that the plasma in a layer orthogonal to B_o starts moving on its own with the velocity equal to $2\vec{V}$, then it emanates two waves travelling with velocity $\vec{V}$and field $\vec{b}$. Hence, the total energy (comprising of kinetic and magnetic energy) is equal to the energy imparted to the initial perturbation.

On determining V_A, we conclude that $\frac{1}{2}\rho V_A^2 = \frac{1}{8\pi}B_o^2$.

If we compare it with the analogous equality for V and b, we find that $\frac{V}{V_A} = \frac{b}{b_o}$. In case V is sufficiently large, then $b >> B_o$. This clearly indicates that if the Alfvén wave is suitably large, the perturbation in the field surpasses the original field B_o.

We have dealt with three velocities i.e., fluid motion velocity V, Alfvén wave velocity V_A and the velocity of sound c. Alfvén waves are responsible for the perturbation of uniform magnetic field B_o to b and carry it to some good distance. Hence Alfvén waves need not to be strictly transverse. Thus, any state of motion within the acceptable limits of compressibility is found to travel along $(\vec{B_o})$ with velocity (V_A). Moreover, Alfvén waves satisfy the condition of frozen field([3]). This is made clear by the equation (3.15)

$$(3.15)\quad \frac{dy}{dz} = \frac{By}{Bz} = \frac{a}{B_o}\sin\omega\left(t - \frac{z}{V_A}\right).$$

The solution of the above equation is given as

(3.16) $y = y_o + \dfrac{a}{\omega\sqrt{\pi\rho}} \cos\omega\left(t - \dfrac{z}{V_A}\right).$

Each point on the line of force progress alongy-axis in direction parallel to velocity:-

(3.17) $\dfrac{dy}{dt} = -\dfrac{a}{4\pi\rho} \sin\omega\left(t - \dfrac{z}{V_A}\right),$

which comes out to be same as (V_y) in equation (3.10).

The waves are undamped so the amplitude of the wave a has been taken as constant with respect to time. Here, the Joule dissipation is ignored taking ($\lambda = \infty$), and viscosity is neglected. Due to this fact, phases of J_x and E_x exhibit a phase difference of $\pi/2$. Therefore, the work done is

(3.18) $\int_0^{2\pi/\omega} J_x E_x dt = 0.$

If we consider large and infinite conductivity and small Joule losses, the average energy density of the waves becomes

(3.19) $U_W = \dfrac{\omega}{2\pi} \int_0^{2\pi/\omega} \omega\left(\frac{1}{8\pi}b^2 + \frac{1}{2}\rho V^2\right)dt = \frac{a^2}{8\pi},$

and the average rate of decrease of the energy density is:-

(3.20) $\dfrac{dU_W}{dt} = -\dfrac{\omega}{2\pi} \int_0^{2\pi/\omega} \frac{J^2}{\lambda} dt = \frac{\omega}{2\pi} \frac{a^2}{\lambda} \frac{c^2\omega^2}{B_o^2} \frac{\rho}{4\pi} \int_0^{2\pi/\omega} \cos^2 \omega t \, dt.$

Here, the total time derivative is considered, suggesting that the point at which U_w is calculated, moves with the wave and not with the fluid. Moreover, U_W is the average over a wavelength.

The magnitude of the integral is equal to $\frac{\pi}{\omega}$. Solving with respect to $z(dt = V_A^{-1} dz)$ and replacing $\frac{a^2}{8\pi}$ with U_W, we obtain:

(3.21) $\dfrac{dlnU_W}{dz} = -2\alpha,$

$$\alpha = \frac{\sqrt{\pi}c^2\omega^2\rho^{8/2}}{B_o^8\lambda} = \frac{c^2\omega^2}{8\pi V_A^8\lambda}.$$

From this we get:

(3.22) $U_W = U_{W_o} e^{-2\alpha z},$

$$b_y = ae^{-2\alpha z} \sin\omega\left(t - \frac{z}{V_A}\right).$$

In the solar atmosphere, plasma pressure rises due to the loss of magnetic energy. As a result, damping in the amplitude of magnetic field takes place with respect to height. This damping is shown by equation (3.23).

(3.23) $b_y = ae^{-2\alpha z} \sin\omega\left(t - \dfrac{z}{V_A}\right).$

Magnetic field varies as a sinusoidal wave in 3-D with respect to height z and time t which is shown in ***Fig.* 3.4**. It is evident that the waves remain periodic but the amplitude of the waves decreases continuously at a slow pace.

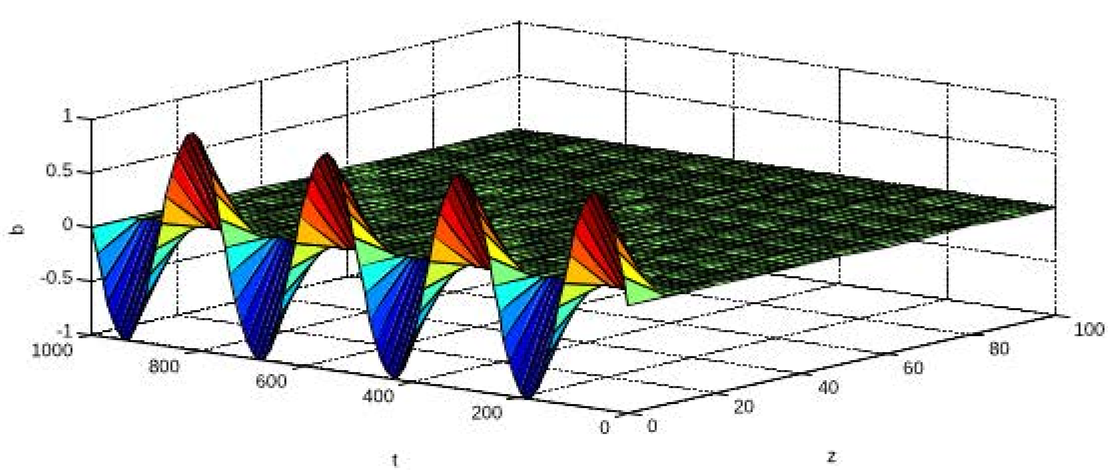

***Figure* 3.4:** Plot showing the variation of magnetic field with height z and time t

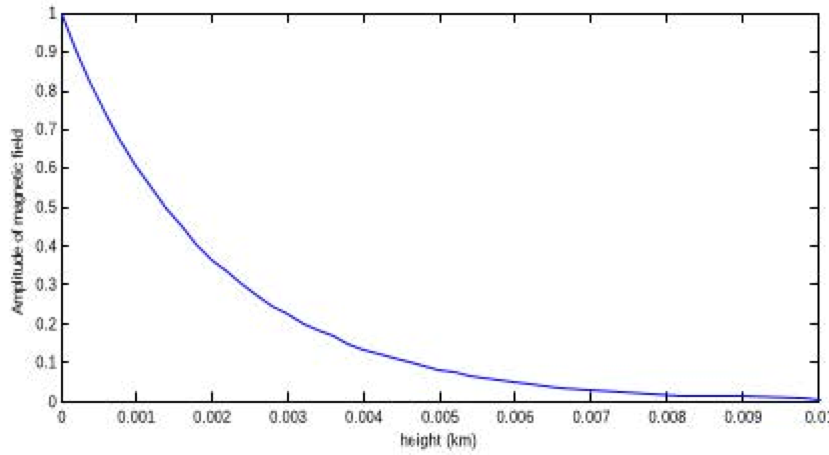

***Figure* 3.5:** Plot showing the variation of amplitude of magnetic field with height z.

Terradas et al. [14], have studied the properties of low-β *MHD* waves and successfully explained the effect of a physical dissipation on Alfvén modes leading to the damping of oscillations. Ryu et al. [12], studied the propagation and damping of Alfvén waves using a 2D *MHD* simulation code and observed effective damping. Carbonell et al. [4], observed time and spatial damping of linear non-adiabatic *MHD* waves in flowing partially ionized plasma.To analyze the variation of amplitude of magnetic field with height, we have plotted ***Fig.* 3.5** which reveals that the amplitude of magnetic field ($ae^{-\alpha z}$ decreases exponentially with height.

From damped amplitude of magnetic field $ae^{-\alpha z}$ we define damping factor α in terms of height.

If

(3.24) $\alpha = 1/z$,

then, $b_y = ae^{\frac{-1\times z}{z}} = a/e$.

Hence α factor is defined as the reciprocal of that height at which the amplitude of magnetic field becomes $\frac{1}{e}$ times its initial value and the time taken in this process is called the characteristic time t_o.

Now as $t_o = \frac{z}{A}$ and $z = \frac{1}{\alpha}$ [equation (3.24)]; therefore,

(3.25) $$t_o = \frac{1}{\alpha V_A}.$$

The characteristic time for change in b and V is given by

(3.26) $$t_o = \frac{1}{\alpha V_A} = \frac{8\pi\lambda V_A^2}{c^2\omega^2} = \frac{4\pi\lambda R_{me^2}}{c^2} = \frac{R_{me^2}}{\nu}.$$

It is evident that for oscillations of higher frequency, the characteristic time is smaller and thus damping is found to be larger.

4 Results and Discussions

The manifestation of different solar activities is due to the interaction of solar plasma with the magnetic field of the Sun as well as its convective motion and differential rotation. These interactions are studied by taking into consideration the dynamics of the plasma and its variation with the magnetic field. For analyzing this, aquantitative formulation of magneto hydrodynamic Alfvén waves has been attempted in this paper. On considering the motion in ordinary fluid, it is observed that the propagation of low frequency waves is only in the form of pressure waves. Here, viscosity is considered negligible, conductivity infinite and the fluid is assumed to be at rest initially. The sum of the pressures of the field induced (b_x b_y) and the plasma pressure (P) is found to be independent of the coordinates.

We have considered plane motion of the fluid wherein the plasma particles travel along ($\vec{B_o}$ with velocity V. It is independent of Alfvén wave velocity ($\vec{V_A}$) as the dependence of ($\vec{V_A}$) on z does not change the fluid velocity V. A small perturbation in magnetic field led to the propagation of Alfvén wave and magnetic plane wave. We conclude that the kinetic energy density of wave motion and the energy density of the field induced by the perturbation come out to be equal in both phase and amplitude. Hence, the total energy (comprising of kinetic and magnetic energy) is equal to the energy imparted to the initial perturbation.

In case V is sufficiently large, then $b >> B_o(B_o$ denotes uniform magnetic field and b is a small change in the value of magnetic field). It clearly indicates that, if the Alfvén wave is suitably large, the perturbation in the field surpasses the original field B_o.

The waves under consideration are undamped so the amplitude of wave "a" has been taken to be constant with respect to time. When we have ignored Joule dissipation and neglected viscosity, a phase difference of $\pi/2$ appeared between J_x and E_x. With theintroduction of large and finite conductivity with small Joule losses , the waves still remainedperiodic; however, the amplitude of the waves decreased exponentially with height. The damping factor α and the characteristic time t_o were obtained as $\alpha = \frac{1}{z}$ and $t_o = \frac{R_{me2}}{\nu}$. Thus, we have concluded that larger damping exists for oscillations of higher frequency.

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(Dedicated to Honor Dr. R. C. Singh Chandel on His 75^{th} Birth Anniversary Celebrations)

A COMPARATIVE STUDY OF PURCHASING *EOQ* (ECONOMIC ORDER QUANTITY) MODELS FOR NON-DETERIORATION AND DETERIORATING ITEMS UNDER STOCK-LINKED AND EXPONENTIAL DEMAND

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Abstract

Demand plays an important role for smooth running of any type of business. Most of the inventory modelers considered steady demand rate in their models. While in actual practice demand rate is in fluctuating state. In this study, a purchasing *EOQ* models for non-deteriorating and deteriorating items with stock-linked and exponential demand is considered. Three models are measured. In ***Model* I**, a purchasing *EOQ* model for non-deteriorating item with stock-dependent demand is assumed. In the second model an *EOQ* model for deteriorating item with stock-sensitive demand is considered. In the third model an *EOQ* model for decay item with exponential demand is considered. The mathematical models are established of all these three cases. Optimality conditions are also taken in account. Numerical examples and sensitivity analysis is also conversed. Taylor's series approximation is used for finding numerical results.

2010 Mathematics Subject Classifications: 90B05.
Keywords and phrases: Inventory, Stock-linked and exponential demand, Deterioration, Optimality.

1 Introduction

Inventory is an essential part for running profitable business. Demand plays a crucial and important role in making *EOQ* strategy. Several research papers published by researchers considering variable demand. Large pile of goods in the warehouse is also attracting customers to purchase more items. Dave and Patel [7] developed an *EOQ* for liner demand changeable demand with deterioration. Xu and Wang [28] designed an inventory model for decaying commodities with time changeable demand and limited shortage cost. Guchhait et al. [11] presented an *EPQ* (Economic Production Quantity) models for breakable items with variable demand, being dependent on time or on-hand stock. Tripathi and Pandey [21] analyzed an *EOQ* model for decaying products with Weibull distribution time-sensitive demand under trade credits. Some notable research papers with stock-sensitive and variable demand are published by Tripathi [22], Baker and Urban [2], Pal et al. [16], Teng and Chang [23], Soni and Shah [18], Musa and Sani [14], Tripathi et al. [24].

The problem of framing *EOQ* models for deteriorating items has received considerable attention in recent years. Most of the researchers assumed constant deterioration rate. Mandal and Phujdar [15] established an economic production quantity (*EPQ*) model for decaying commodities with steady production rate and stock-associated demand. Ghiami and Williams [17] presented EPQ model when a manufacturer delivered a deteriorating products to retailers. Sicilia et al. [19] designed a deterministic *EOQ* system for commodities with constant decay rate .Lee and Kim [12] developed the optimal ordering strategy considering both deteriorating and defecting items in an integrated production distribution model for a single –vendor, single – buyer supply chain. Tripathi [25] considered an *EOQ* model of deteriorating products with stock – sensitive demand under inflation. Dye [8] presented an *EOQ* model over a finite time for non-instantaneous deteriorating items using preservation technology. Researchers including Benkherouf and Balkhi [3], Chakrabarty et al. [5], Dye [9], Sarkar [20], Taleizadeh [26]. Wu and Sarkar [28], Yang et al. [29], Wang et al. [27], Liao [13], Chang et al. [6], Dye and Hsieh [10], Atici et al. [1], Bakker et al. [4] developed *EOQ* models that focused on deterioration rate.

The remainder of the study is prepared as follows. ***Section* 2** covers notations and assumptions of the model. In ***Section* 3**, the models are formulated with optimal solution, numerical examples and sensitivity studies are conversed. In this section the condition of total cost is minimized is obtained. ***Section* 4** , provides the comparative study of models I, II and III. Finally, some conclusions and future research lines are given in ***Section* 5**.

2 Notations and Assumptions

2.1 Notations:

The following notations are used in this study:

D	Demand rate
Q	Order quantity
C_h	Carrying cost / unit time
C_0	Ordering cost/ order
T	Cycle time
θ	Deterioration rate, $0 \leq \theta < 1$
OC and HC	Ordering cost and holding cost
Z	Total cost
OC^*, HC^* and Z^*	Optimal OC, HC, and $Z(T)$ respectively.

2.2 Assumptions

The following assumptions are used to build up the model:
(i).The demand rate is stock- dependent for models I and II (i.e. $D = \alpha + \beta I(t)$, $\alpha > 0$, $0 < \beta < 1$) , and exponential demand for model III (i.e. $D = \alpha e^{\beta t}$).
(ii).Models are considered for non- deteriorating in model I and deteriorating items in models II and III.
(iii). Shortages are not allowed.
(iv).There is no renovate or replenishment of the deteriorating commodities.
(v).The replenishment takes place immediately at an endless rate.

3 Mathematical Formulation

In this study three models are considered. In the first model demand rate is considered stock dependent. In the second model deterioration and stock- dependent demand both are considered. In the third model deterioration and exponential demand is considered.

3.1 *Model* I: Stock- dependent demand

In this model, it is assumed that demand rate for the item is stock- dependent. The inventory of commodities, decreases due to purchases and stock- linked demand during $[0, T]$. Therefore, the differential equation of the state is given by:

(3.1) $\dfrac{dI(t)}{dt} = -\{\alpha + \beta I(t)\},\ 0 \leq t \leq T,$

with

(3.2) $I(0) = Q$ and $I(T) = 0$.

Solution of (3.1) using (3.2) is:

(3.3) $I(t) = \dfrac{\alpha}{\beta}\{e^{(T-t)\beta} - 1\}$

and

(3.4) $Q = I(0) = \dfrac{\alpha}{\beta}(e^{\beta T} - 1) = \alpha T \left(1 + \dfrac{\beta T}{2}\right)$ (approx.).

The Total cost consists of OC and HC:

(3.5) $OC = \dfrac{C_0}{T}$

(3.6) $HC = \dfrac{C_h}{T} \int_0^T I(t)\ dt = \frac{C_h \alpha T}{2} \left(1 + \frac{\beta T}{3}\right)$, (approx.).

Therefore

(3.7) $Z = \dfrac{C_0}{T} + \dfrac{C_h \alpha T}{2} \left(1 + \dfrac{\beta T}{3}\right).$

Optimality Condition

Differentiating (3.7) w.r.t. T, we get

$$\frac{dZ}{dT} = -\frac{C_0}{T^2} + \frac{C_h\alpha}{2}\left(1 + \frac{2\beta T}{3}\right)$$

and

$$\frac{d^2Z}{dT^2} = \frac{2C_0}{T^3} + \frac{\alpha\beta C_h}{3} > 0.$$

It is seen that Z is a convex function in T. We can also the condition of minimization by graph shown below:

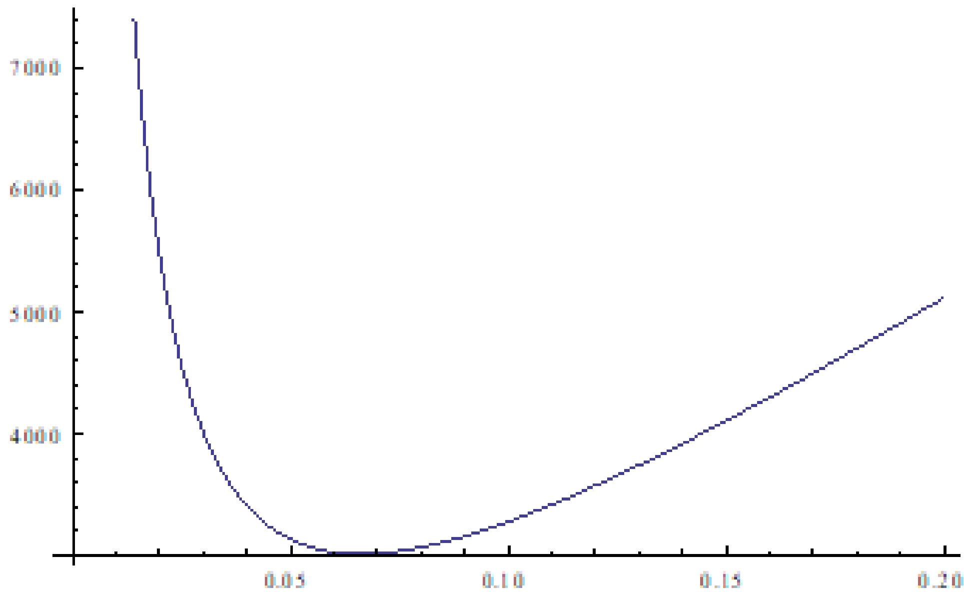

***Figure* 3.1:** Between T (0.00 – 2.00) and Z

T^* is calculated by solving

$$(3.8)\quad \frac{dZ}{dT} = 0 \Rightarrow \ \alpha C_h (2\beta T + 3)\, T^2 - 6C_0 = 0.$$

***Example* 3.1** *Let us consider the cost parameters:* $\alpha = 4500$, $\beta = 0.4$, $C_h = 10$, $C_0 = 100$ *in appropriate units. Substituting theses values in* (3.8)*, and solving for T, we get,* $T^* = 0.0660869$ *yrs, corresponding* $Q^* = 301.322$ *units,* $OC^* = 1513.159$*,* $HC^* = 1500.061$ *and* $Z^* = \$3013.22$*.*

Sensitivity Analysis

It is reasonable to study the sensitivity study with respect to constraints over a known optimum solution. It is imperative to get the belongings on dissimilar scheme parameters, such as holding cost, ordering cost, etc. In the following ***Table* 1**, keeping all parameters same, discussed in numerical ***Example* 3.1**, varying one parameter at a time.

***Table* 3.1:** The effect of parameters on T^*, Q^* and OC^*, HC^* and Z^*

Parameters		Optimal values				
		T^*	Q^*	OC^*	HC^*	Z^*
C_0	80	0.0591636	269.386	1352.183	1341.677	2693.86
	90	0.0627232	285.795	1434.876	1423.074	2857.95
	110	0.0692835	316.090	1587.680	1573.280	3160.96
	120	0.0723354	330.218	1658.939	1643.241	3302.18
	130	0.0752602	343.768	1727.341	1710.349	3437.69
C_h	12	0.0603740	274.964	1659.342	1643.218	3299.56
	14	0.0559281	254.492	1788.010	1774.870	3562.88
	16	0.0523406	237.998	1910.563	1897.407	3807.97
	18	0.0493665	224.343	2025.665	2012.505	4038.17
	20	0.0468487	212.794	2134.531	2121.359	4255.89
α	4600	0.0653707	304.637	1529.737	1516.633	3046.37
	4700	0.0646774	307.916	1546.135	1533.025	3079.16
	4800	0.0640058	311.161	1562.358	1555.632	3117.99
	4900	0.0633547	314.372	1578.415	1565.305	3143.72
	5000	0.0627232	317.550	1594.306	1581.194	3175.50
β	0.45	0.0660162	300.995	1514.780	1500.070	3014.85
	0.50	0.0659458	300.670	1516.397	1500.093	3016.49
	0.55	0.0658758	300.347	1518.008	1500.102	3018.11
	0.60	0.0658062	300.025	1519.614	1500.126	3019.74
	0.65	0.0657369	299.705	1521.216	1500.144	3021.36

3.2 *Model* II: Stock linked Demand under deterioration

In this model stock- dependent demand and deterioration both are measured. The majority of items in the universe deteriorate over time. Daily usable products like, bread, milk, green vegetable etc. deteriorate over time. The differential equation of this situation is

(3.9) $\dfrac{d}{dt}I(t) + \theta I(t) = -\{\alpha + \beta I(t)\};\ 0 < t < T,$

with $I(0) = Q$ and $I(T) = 0$.

The solution of (3.9) with above condition is:

(3.10) $I(t) = \dfrac{\alpha}{\theta+\beta}\left\{e^{(T-t)(\theta+\beta)} - 1\right\}$

and

(3.11) $Q = I(0) = \dfrac{\alpha}{\theta+\beta}\left\{e^{(\theta+\beta)T} - 1\right\} = \alpha T\left\{1 + \dfrac{(\theta+\beta)T}{2}\right\}$ (approx.).

Total Cost

Total cost is given as follows:

(3.12) $OC = \dfrac{C_0}{T}$

(3.13) $HC = \dfrac{C_h}{T}\int_0^T I(t)\,dt = \frac{\alpha C_h T}{2}\left\{1 + \frac{(\theta+\beta)T}{3}\right\},$

and

(3.14) $Z = \dfrac{C_0}{T} + \dfrac{\alpha C_h T}{2}\left\{1 + \dfrac{(\theta+\beta)T}{3}\right\}.$

Optimality Condition

Differentiating (3.14) w.r.t. 'T', we get

$$\frac{dZ}{dT} = -\frac{C_0}{T^2} + \frac{\alpha C_h}{2}\left\{1 + \frac{2(\theta+\beta)T}{3}\right\},$$

and

$$(3.15)\quad \frac{d^2Z}{dT^2} = \frac{2C_0}{T^3} + \frac{\alpha C_h(\theta+\beta)}{3} > 0.$$

It can be easily seen that Z is a convex function in T. We can also the condition of minimization by graph shown below:

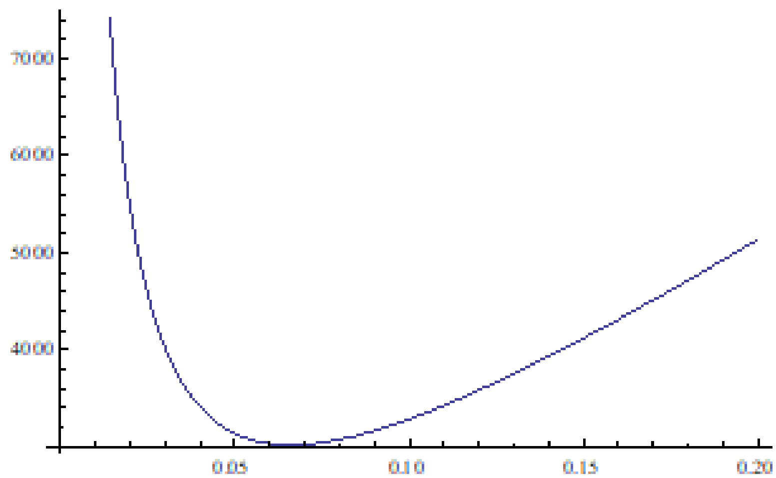

***Figure* 3.2:** Between $T(0.00 - -0.20)$ and Z

Hence, Optimal cycle time T can be calculated by solving

$$(3.16)\quad \frac{dZ}{dT} = 0 \Rightarrow \alpha C_h \{2(\theta+\beta)T + 3\}\, T^2 - 6C_0 = 0.$$

***Example* 3.2** *Let us consider the cost parameters:* $\alpha = 4500$, $\beta = 0.4$, $C_h = 10$, $C_0 = 100$, $\theta = 0.05$, *in appropriate units. Substituting theses values in* (3.16)*, and solving for* T*, we get,* $T^* = 0.0660162$ *yrs,* $Q^* = 301.486$ *units,* $OC^* = 1514.780$, $HC^* = 1500.07$ *and* $Z^* = \$3014.85$.

Sensitivity Analysis

It is reasonable to study the sensitivity with respect to constraints over an agreed best possible solution. It is important to obtain the belongings on unlike structure parameters, such as holding cost, ordering cost, etc. In the following ***Table* 3.2**, keeping all parameters same, discussed in numerical ***Example* 3.1**, varying one parameter at a time.

***Table* 3.2:** The effect of parameters on T^*, Q^* and OC^*, HC^* and Z^*

Parameters		Optimal values				
		T^*	Q^*	OC^*	HC^*	Z^*
C_0	80	0.0591067	269.517	1353.484	1341.696	2695.18
	90	0.0626594	285.943	1436.337	1423.083	2859.42
	110	0.0692059	316.276	1589.460	1573.300	3162.76
	120	0.0722508	330.414	1660.881	1643.259	3304.14
	130	0.0751689	343.981	1729.439	1710.371	3439.81
C_h	12	0.0603148	275.099	1657.768	1643.432	3301.20
	14	0.0558772	254.609	1769.639	1774.881	3564.52
	16	0.0522960	238.101	1912.192	1897.428	3809.62
	18	0.0493268	224.434	2027.296	2012.514	4039.81
	20	0.0468129	212.877	2136.163	2121.377	4257.54
α	4600	0.0653015	304.800	1531.358	1516.642	3048.00
	4700	0.0646097	308.080	1547.755	1533.045	3080.80
	4800	0.0639394	311.324	1563.981	1549.259	3113.24
	4900	0.0632894	314.535	1580.038	1565.312	3145.35
	5000	0.0626594	317.140	1595.930	1581.210	3177.14
β	0.45	0.0659458	301.649	1516.397	1500.093	3016.49
	0.50	0.0658758	301.811	1518.008	1500.102	3018.11
	0.55	0.0658062	301.974	1519.614	1500.126	3019.74
	0.60	0.0657369	302.136	1521.216	1500.144	3021.36
	0.65	0.0656681	302.298	1522.809	1500.171	3022.98
θ	0.06	0.0660021	301.518	1515.103	1500.077	3015.18
	0.07	0.0659880	301.551	1515.427	1500.083	3015.51
	0.08	0.0659739	301.583	1515.751	1500.079	3015.83
	0.09	0.0659598	301.616	1516.075	1500.085	3016.16
	0.10	0.0659458	301.686	1516.397	1500.093	3016.49

3.3 *Model* III: Exponential demand under deterioration

In most of EOQ model demand is considered invariable. While in real situation demand is always in dynamic state. The model developed for deteriorating inventory in which demand is a exponential function of time. The differential equation is:

(3.17) $\dfrac{dI(t)}{dt} + \theta I(t) = -\alpha e^{\beta t},$

with, $I(0) = Q$ and $I(T) = 0$.

Solution of (3.17) with the above condition is

(3.18) $I(t) = \dfrac{\alpha}{\theta+\beta}\left\{e^{\beta T} e^{\theta(T-t)} - e^{\beta t}\right\}.$

also,

$$Q = I(0) = \frac{\alpha}{\theta+\beta}\left\{e^{(\theta+\beta)T} - 1\right\} = \alpha T\left\{1 + \frac{(\theta+\beta)T}{2}\right\}, \quad \text{(approx.).}$$

Total Cost

Total cost consists of ordering cost and holding cost.

(3.19) $OC = \dfrac{C_0}{T},$

(3.20) $HC = \dfrac{C_h}{T}\int_0^T I(t)\,dt = \frac{C_h}{T}\int_0^T \frac{\alpha}{\theta+\beta}\left\{e^{\beta T} e^{\theta(T-t)} - e^{\beta t}\right\}dt$

$$= \frac{\alpha C_h}{(\theta+\beta)T}\left\{\frac{e^{(\theta+\beta)T} - e^{\beta T}}{\theta} - \frac{e^{\beta T}-1}{\beta}\right\} = \frac{\alpha C_h T}{2}\left\{1 + \frac{(\theta+2\beta)T}{3}\right\}, (approx.),$$

and

(3.21) $Z = \dfrac{C_0}{T} + = \dfrac{\alpha C_h T}{2}\left\{1 + \dfrac{(\theta+2\beta)T}{3}\right\}.$

Optimal Condition

Differentiating (3.21) w.r.t. 'T', we get

(3.22) $$\frac{dZ}{dT} = -\frac{C_0}{T^2} + \frac{\alpha C_h}{2}\left\{1 + \frac{2(\theta + 2\beta)T}{3}\right\}$$

and

$$\frac{d^2Z}{dT^2} = \frac{2C_0}{T^3} + \frac{\alpha C_h(\theta + 2\beta)}{3} > 0.$$

Since the second derivative of Z is positive. This shows that Z gives the minimum value at T^*. We can also the condition of minimization by graph shown below:

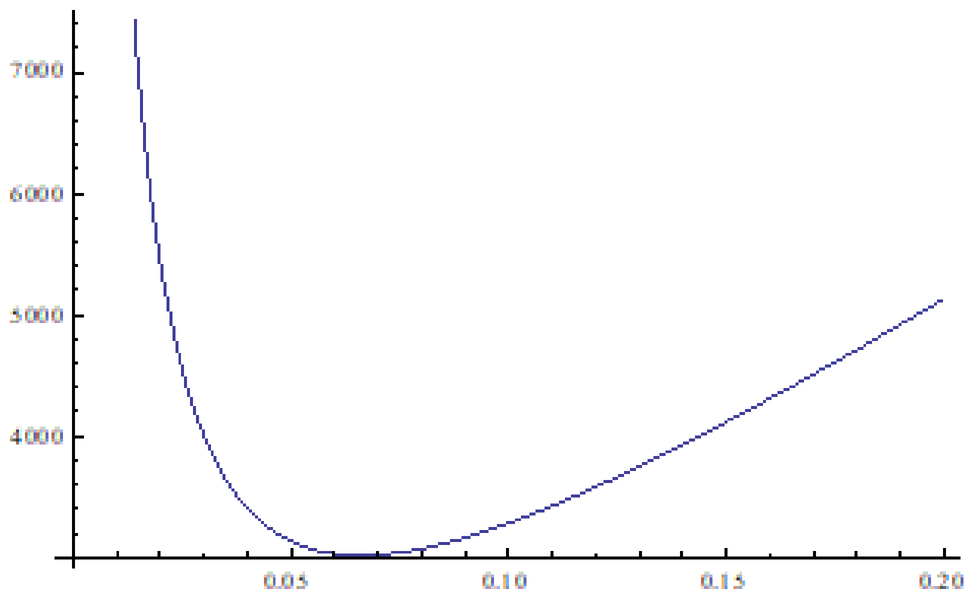

***Figure* 3.3:** Between $T(0.00 - -0.20)$ and Z

Thus, an Optimal cycle time T is obtained by putting

(3.23) $$\frac{dZ}{dT} = 0 \Rightarrow \alpha C_h\{2(\theta + 2\beta)T + 3\}T^2 - 6C_0 = 0.$$

Equations (3.8), (3.16) and (3.23) are cubic in T. The solution of these equations provides three roots. It is seen that these equations having only one change in sign. By Descartes' rule there exists only one positive root.

***Example* 3.3** *Let us consider the cost parameters:* $\alpha = 4500$, $\beta = 0.4$, $C_h = 10$, $C_0 = 100$, $\theta = 0.05$, *in appropriate units. Substituting theses values in* (3.23)*, and solving for* T*, we get,* $T^* = 0.0654635$ *yrs, corresponding* $Q^* = 298.925$ *units,* $OC^* = 1527.569$, $HC^* = 1500.251$ *and* $Z^* = 3027.82$.

Sensitivity Analysis

It is reasonable to study the sensitivity study with respect to model parameters over a given optimum solution. It is important to get the effects on different system parameters, such as holding cost, ordering cost, etc. In the following ***Table* 3.3**, keeping all parameters same, discussed in numerical ***Example* 1**, varying one parameter at a time.

Table **3.3:** Effect of several parameters on optimal values

Parameters		Optimal values				
		T^*	Q^*	OC^*	HC^*	Z^*
C_0	80	0.0586615	267.461	1363.756	1341.824	2705.58
	90	0.0621603	283.634	1447.869	1423.241	2871.11
	110	0.0685999	313.464	1603.501	1573.499	3177.00
	120	0.0715919	327.353	1676.167	1643.493	3319.66
	130	0.0744572	340.671	1745.969	1710.631	3456.60
C_h	12	0.0598516	272.959	1670.799	1643.401	3314.20
	14	0.0554783	252.769	1802.506	1775.034	3577.54
	16	0.0519456	236.487	1925.091	1897.569	3822.66
	18	0.0490144	222.997	2040.217	2012.653	4052.87
	20	0.0465310	211.582	2149.105	2121.506	4270.61
A	4600	0.0647605	302.239	1544.151	1516.819	3060.97
	4700	0.0640798	305.517	1560.554	1533.216	3093.77
	4800	0.0634202	308.761	1576.785	1549.435	3126.22
	4900	0.0627807	311.971	1592.846	1565.484	3158.33
	5000	0.0621603	315.148	1608.744	1581.376	3190.12
β	0.45	0.0653289	298.781	1530.716	1500.304	3031.02
	0.50	0.0651956	298.640	1533.846	1500.374	3034.22
	0.55	0.0650637	298.502	1536.955	1500.445	3037.40
	0.60	0.0649330	298.365	1540.049	1500.521	3040.57
	0.65	0.0648037	298.231	1543.122	1500.608	3043.73
θ	0.06	0.0654500	298.959	1527.884	1500.256	3028.14
	0.07	0.0654365	298.992	1528.199	1500.261	3028.46
	0.08	0.0654230	299.026	1528.514	1500.266	3028.78
	0.09	0.0654095	299.060	1528.830	1500.270	3029.10
	0.10	0.0653960	300.056	1529.146	1500.274	3029.42

On comparing all the three models, the sensitivity outcomes show that
(i). On increasing C_0, T^*, Q^*, OC^*, HC^* and Z^* increases. It shows that all optimal values move in the same track with ordering cost/ order.
(ii).On raising C_h; T^* and Q^* diminishes, while OC^*, HC^* and Z^* increases. It means that T^* and Q^* moves in the opposite direction with C_h, while OC^*, HC^* and Z^* moves in the same direction with inventory carrying cost/ unit time.
(iii).On improving 'α'; T^* decreases, while Q^*, OC^*, HC^* and Z^* increases.
(iv).On raising 'θ' and 'β', all the optimal values varies insignificantly (approximately).

4 Comparisons of Optimal Lot- Size and Costs

Table **4.1:** A comparative study is conversed between ***Models*** **I**, **II** and **III**.

	Q^*	OC^*	HC^*	Z^*
Model I	301.322	1513.159	1500.061	3013.22
Model II	301.486	1514.780	1500.070	3014.85
Model III	298.925	1527.569	1500.251	3027.82

From the above ***Table*** **4.1**, it is easily seen that the order quantity obtained from ***Model*** **II** is superior to ***Models*** **I** and **III**. It is also seen that optimal setup cost, holding cost and total cost rises from ***Model*** **I** to **III**. It means that all optimal cost (OCj, HCj, and Zj) in ***Model*** **III** is better to compare to the OCj, HCj and Zj in ***Models*** **I** and **II**.

5 Conclusion

In this study, a purchasing EOQ models is established to obtain cycle time using second order and third order approximation in exponential terms. Three dissimilar models are established. In ***Model*** **I**, purchasing EOQ ***Model*** with stock- linked demand, in ***Model*** **II** an *EOQ* model with inventory- sensitive demand under deterioration and in ***Model*** **III** an *EOQ* model with exponential demand under decay have been discussed. Mathematical models have been developed for all three models and optimal cycle times are obtained which minimize the total cost. It has been also shown that the conditions of minimization are satisfied in all models. Numerical examples and sensitivity analyses have been conversed to validate the projected model. On comparing all three models, it is easily seen that variations are quite sensitive except 'θ' and 'β'.

Some possible extensions of the model that can be future research topics are: (i) variable deterioration and Weibull deterioration (ii) to suppose a non- linear holding cost (iii) to incorporate discounts in the purchasing cost/ unit; and (iv) to study the case of inflation and time value of money.

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(Dedicated to Honor Dr. R. C. Singh Chandel on His 75^{th} *Birth Anniversary Celebrations)*

SLIP FLOW AND HEAT TRANSFER OF MHD POWER LAW FLUID OVER A POROUS SHEET IN THE PRESENCE OF JOULE HEATING AND HEAT SOURCE/SINK

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Abstract

An analysis has been carried out to study the *MHD* flow and heat transfer of a non-Newtonian power-law fluid over a porous sheet in the presence of partial slip and Joule heating. Here, thermal conductivity is assumed as a function of temperature. An external heat source is also applied. The governing equations are reduced into the system of non-linear ordinary differential equations by using similarity transformations. This system along with appropriate boundary conditions are solved numerically using the shooting technique with Runge-Kutta fourth order iteration scheme. The effects of suction/injection parameter, Eckert number and heat source parameter on the velocity and temperature profiles are studied. It is observed that the temperature profile increases with decreasing value of heat source parameter, Eckert number and suction/injection parameter. The influence of different parameters on the velocity and temperature profile is presented through graphs. The effect of all physical parameters on skin friction and local Nusselt number is displayed through tables.
2010 Mathematics Subject Classifications: 76A05, 76S05, 76W05, 65L06.
Keywords and phrases: Power law fluid, joule heating effects, MHD, porous sheet, porous medium, partial slip, heat source/sink, thermal radiation.

1 Introduction

The power-law fluids are used in a wide range of industrial applications. The power-law model is useful from chemical and petrochemical processes to food industries and biotechnology. Some examples include blood flow simulation, polymer solutions, bio colloids, etc. The power-law model is one of the models which has been used by researchers for simulating the non-Newtonian fluids. One reason for this popularity is the ability to simulate a wide range of non-Newtonian fluids by this model. In the pioneering work of Schowalter [21], the applicability of boundary layer theory to the two and three-dimensional flow of pseudoplastic power-law fluids was provided. In his work, particular emphasis has been given to the formulation of boundary-layer equations which provide similar solutions. Sarpkaya [19] carried out a study on the flow of non-Newtonian fluids under the effect of a magnetic field. Sarpkaya studied the laminar flow of Bingham plastics and dilatant substances between two parallel planes under the influence of a constant pressure gradient and a steady magnetic field perpendicular to the direction of motion. Throughout this study the generalized magnetic Reynolds number has been assumed to be sufficiently small. Non-Newtonian flow through porous media has been described by Savins [20]. Gupta and Gupta [13] investigated heat and mass transfer on a stretching sheet with suction or blowing. Numerical solution of the laminar boundary layer equation for power-law fluids was described by Andersson and Toften [3]. Alberta et al. [8] investigated some similarity solutions to shear flows of non-Newtonian power-law fluids. They described the flow resistance to the vertical motion of a slender circular body in an unbounded power law fluid, which constitutes a Stokes' first problem for a circular rod. Sharma and Mathur [18] have analyzed steady laminar free convection flow of an electrically conducting fluid along a porous hot vertical plate in the presence of heat source/sink. *MHD* flow of a power-law fluid over a rotating disk was discussed by Andersson and Korte [4]. Andersson et al. [5] carried out a study on slip flow past a stretching surface. Free convection flow with thermal radiation and mass transfer past a moving vertical porous plate has been investigated by Makinde [16]. Yurusoy [22] considered unsteady boundary layer flow of power-law fluid on stretching sheet surface. Effects of magnetic field and suction/injection on convection heat transfer of non-Newtonian power-law fluids past a power-law stretched sheet with surface heat flux was investigated by Chen [10]. Abel et al. [2] has analysed flow and heat transfer in a power-law fluid over a stretching sheet with variable thermal conductivity and non-uniform heat source. Mahmoud [15] studied slip velocity effect on a non-Newtonian power-law fluid over a moving permeable surface with heat generation. Mukhopadhyay [17] presented heat transfer analysis of the unsteady flow of a Maxwell fluid over a stretching surface in the presence of a heat source/sink. Abdel-Rahman [1] analyzed effect of variable viscosity and thermal conductivity on unsteady *MHD* flow of non-Newtonian fluid over a stretching porous sheet. Steady boundary layer slip flow along with heat and mass transfer over a flat porous plate embedded in a porous medium was discussed by Aziz et al. [7]. Heat transfer analysis for stationary boundary layer slip flow of a power-law fluid in a Darcy porous medium with plate suction/injection was done by Aziz et al. [6]. In their studies, they extended the work to steady boundary layer slip flow and heat transfer over a flat plate embedded in a porous media and investigated the slip effect on boundary layer flow of a power-law fluid including heat transfer over a porous flat sheet embedded in a porous medium. Kiyasatfar and Pourmahmoud [14] considered laminar *MHD* flow and heat transfer of power-law fluids in square micro channels.

The present study focuses on the Joule heating effect with non-Newtonian power law fluid over a porous sheet in the presence of heat source/sink. The governing system of partial differential equations has been adapted using appropriate similarity transformation to nonlinear ordinary differential equations which are then solved using fourth-order Runge kutta method with the shooting technique. The effect of various physical parameters on the dimensionless velocity and temperature profiles are plotted and discussed in the result section.

2 Mathematical Formulation

Consider the steady *2-D* laminar flow of an incompressible power-law fluid under the influence of magnetic field and thermal radiation, over a semi-infinite porous sheet in a porous medium. The applied magnetic field (B) is acting normal to the sheet and induced magnetic field is considered negligible as compared to applied magnetic field, because the Reynolds number for the flow is taken to be small. The surface of the sheet is insulated and admits partial slip condition. The origin of the coordinates is at the leading edge of the sheet, the x-axis along with the sheet and y-axis normal to it. The temperature of the sheet is T_w. The velocity and temperature far away from the sheet are U_∞ and T_∞, respectively. Using boundary layer approximation the governing equations for the problem along with the slip boundary conditions are as follows

$$(2.1)\quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,$$

$$(2.2)\quad u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{1}{\rho}\frac{\partial \tau_{xy}}{\partial y} - \frac{1}{\rho A}(u - U_\infty) - \frac{\sigma B^2}{\rho}(u - U_\infty),$$

$$(2.3)\quad u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \frac{1}{\rho C_p}\left\{\frac{\partial}{\partial y}\left[\kappa(T)\frac{\partial T}{\partial y}\right] - \frac{\partial q_r}{\partial y}\right\} + \frac{\sigma B^2}{\rho C_p}u^2 + \frac{Q}{\rho C_p}(T - T_\infty),$$

$$(2.4)\quad \begin{cases} u = L_1\frac{\partial u}{\partial y}, v = v_w, T = T_w + D_1\frac{\partial T}{\partial y}, \text{ at } y = 0, \\ u \to U_\infty, T \to T_\infty \text{ as } y \to \infty, \end{cases}$$

where u and v are the velocity components in x and y directions, ρ is the fluid density, τ_{xy} is the shear stress tensor, A is the permeability, σ is the electrical conductivity, B is the applied magnetic field, T is the temperature, C_p is the specific heat, κ is the variable thermal conductivity, q_r is the radiative heat flux, L_1 is the velocity slip factor, D_1 is the thermal slip factor, Q is the heat source parameter and v_w describes suction/blowing through the porous sheet.

The shear stress component τ_{xy} for the power-law fluid model, as derived by Bird et al. [9] is

$$(2.5)\quad \tau_{xy} = K|\frac{\partial u}{\partial y}|^{n-1}\frac{\partial u}{\partial y},$$

where K is the consistency coefficient and n is the power-law index. In equation (5) $n = 1$ represents the Newtonian behaviour of the fluid. In the case of $n < 1$ behaviour of the fluid is known as shear-thinning, which is categorized by an apparent viscosity which decreases with increase in shear rate. In the case of $n > 1$ fluid behaviour is called shear-thickening and characterized by an apparent viscosity which increases with an increase in shear rate Chhabra et al. [11]. Therefore, a single parameter n describes the nature of fluid behaviour. Substitution of equation (2.5) in equation (2.2) gives

$$(2.6)\quad u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{1}{\rho}\frac{\partial}{\partial y}\left(K\left|\frac{\partial u}{\partial y}\right|^{n-1}\frac{\partial u}{\partial y}\right) - \frac{1}{\rho A}(u - U_\infty) - \frac{\sigma B^2}{\rho}(u - U_\infty).$$

Following Das [12], we consider the temperature dependent thermal conductivity and radiative heat flux of the form

$$(2.7)\quad \kappa_T = \kappa_\infty\left(1 + \epsilon\frac{T - T_\infty}{\Delta T}\right), q_r = -\frac{16T_\infty^3\sigma}{3k}\frac{\partial T}{\partial y},$$

where ϵ is the thermal conductivity parameter, κ_∞ is the thermal conductivity at ambient temperature and $\Delta T = T_w - T_\infty$, σ is the Stefan-Boltzmann constant and k is the mean absorption coefficient. Substitution of equation (2.7) into equation (2.3) gives

$$(2.8)\quad u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \frac{1}{\rho C_p}\left\{\frac{\partial}{\partial y}\left[k_\infty\left(1 + \epsilon\frac{T - T_\infty}{\Delta T}\right)\frac{\partial T}{\partial y}\right] + \frac{16T_\infty^3\sigma}{3k}\frac{\partial^2 T}{\partial y^2}\right\} + \frac{\sigma B^2}{\rho C_p}u^2 + \frac{Q}{\rho C_p}(T - T_\infty).$$

3 Method of solution

Now we will transform the system of equations (2.1), (2.6) and (2.8) along with the boundary conditions (2.4) into a dimensionless form. For this purpose, we choose a stream function $\Psi(x, y)$ such that

$$(3.1)\quad u = \frac{\partial \Psi}{\partial y}, v = -\frac{\partial \Psi}{\partial x}.$$

Equation (3.1) identically satisfies the continuity equation (2.1). Using equation (3.1), equation (2.6) and (2.8) become

$$(3.2)\quad \frac{\partial \Psi}{\partial y}\frac{\partial^2 \Psi}{\partial x \partial y} - \frac{\partial \Psi}{\partial x}\frac{\partial^2 \Psi}{\partial y^2} = \frac{1}{\rho}\left\{\frac{\partial}{\partial y}\left[K|\frac{\partial^2 \Psi}{\partial y^2}|^{n-1}\frac{\partial^2 \Psi}{\partial y^2}\right]\right\} - \left(\frac{1}{\rho \bar{A}} + \frac{\sigma B^2}{\rho}\right)\left(\frac{\partial \Psi}{\partial y} - U_\infty\right),$$

$$(3.3)\quad \frac{\partial \Psi}{\partial y}\frac{\partial T}{\partial x} - \frac{\partial \Psi}{\partial x}\frac{\partial T}{\partial y} = \frac{1}{\rho C_p}\left\{\frac{\partial}{\partial y}\left[\kappa_\infty\left(1 + \epsilon\frac{T - T_\infty}{\Delta T}\right)\frac{\partial T}{\partial y}\right] + \frac{16T_\infty^3\sigma}{3\kappa}\frac{\partial^2 T}{\partial y^2}\right\} + \frac{\sigma B^2}{\rho C_p}\left(\frac{\partial \Psi}{\partial y}\right)^2 + \frac{Q}{\rho C_p}(T - T_\infty).$$

The boundary conditions are transformed into

$$(3.4)\quad \begin{cases} \frac{\partial \Psi}{\partial y} = L_1\frac{\partial^2 \Psi}{\partial y^2}, \frac{\partial \Psi}{\partial x} = -v_w, T = T_w + D_1\frac{\partial T}{\partial y} \text{ at } y = 0, \\ \frac{\partial \Psi}{\partial y} \to U_\infty, T \to T_\infty \text{ as } y \to \infty, \end{cases}$$

where

$$L_1 = L\frac{U_\infty \rho}{K}\left(\frac{Kx}{\rho U_\infty^{2-n}}\right)^{\frac{1}{n+1}}$$

is the velocity slip factor and

$$D_1 = D\frac{U_\infty \rho}{K}\left(\frac{Kx}{\rho U_\infty^{2-n}}\right)^{\frac{1}{n+1}}$$

is the thermal slip factor. Here L and D are the initial values of velocity and thermal slip parameters, respectively.

We introduce the following dimensionless similarity variable (η), dimensionless stream function $\Psi(\eta)$ and dimensionless temperature $\theta(\eta)$ of the form

$$(3.5)\quad \eta = (\frac{Re}{\frac{x}{L}})^{\frac{1}{n+1}}\frac{y}{L}, \Psi(x, y) = LU_\infty(\frac{\frac{x}{L}}{Re})^{\frac{1}{n+1}}f(\eta), \theta(\eta) = \frac{T - T_\infty}{\Delta T},$$

where $Re = \frac{\rho U_\infty^{2-n}L^n}{K}$ is the generalized Reynolds number. Using equation (3.5) into equation (3.2) and (3.3), we obtain the following system of ODE

$$(3.6)\quad n|f''|^{n-1}f''' + \frac{1}{n+1}ff'' - (k + M)(f' - 1) = 0,$$

$$(3.7)\quad \theta'' + \frac{Pr_\infty}{(n + 1)(1 + \epsilon\theta + Nr)}f\theta' + \frac{\epsilon}{(1 + \epsilon\theta + Nr)}\theta'^2 + \frac{MPr_\infty Ec}{(1 + \epsilon\theta + Nr)}f'^2 + \frac{Pr_\infty Q}{(1 + \epsilon\theta + Nr)}\theta = 0.$$

The corresponding boundary conditions are

$$(3.8)\quad \begin{cases} f(\eta) = S, f'(\eta) = \delta f''(\eta), \theta(\eta) = 1 + \beta\theta'(\eta) \text{ at } \eta = 0, \\ f'(\eta) \to 1, \theta(\eta) \to 0, \text{ as } \eta \to \infty, \end{cases}$$

where k is the permeability parameter, M is the magnetic parameter, Pr_∞ is the local Prandtl number, Nr is the thermal radiation parameter, S is the suction/blowing parameter, which corresponds to suction when $S > 0$, and corresponds to blowing when $S < 0$, Ec is Eckert number, Q is heat source parameter. In boundary conditions δ and β are the dimensionless velocity and thermal slip parameters, respectively. These parameters are further defined as

$$(3.9)\quad k = \frac{\mu x}{U_\infty \rho A}, M = \frac{x\sigma B^2}{U_\infty \rho}, S = -v_w\frac{x(n + 1)}{U_\infty}(\frac{\rho U_\infty^{2-n}}{K})^{\frac{1}{n+1}}, \delta = L\frac{U_\infty \rho}{K}, \beta = D\frac{U_\infty \rho}{K},$$

$$Pr_\infty = \frac{C_p}{\kappa_\infty}K^{\frac{2}{n+1}}(\frac{U_\infty^3 \rho}{x})^{\frac{n-1}{n+1}} = (1 + \epsilon\theta)Pr, Ec = \frac{U_\infty^2}{C_{p\Delta T}},$$

where $Pr = \frac{C_p}{\kappa}K^{\frac{2}{n+1}}(\frac{U_\infty^3 \rho}{x})^{\frac{n-1}{n+1}}$ is the Prandtl number.

4 Relevant physical measures

The shear stress is given by

$$(4.1)\quad \tau_{xy} = [K|\frac{\partial u}{\partial y}|^{n-1}\frac{\partial u}{\partial y}]_{y=0} = \rho U_\infty^2 Re_x^{-\frac{1}{(n+1)}}|f''(0)|^{n-1}f''(0).$$

The skin-friction coefficient is defined as

$$(4.2)\quad C_f = \frac{\tau_{xy}}{\rho u_\infty^2/2} = 2Re_x^{-\frac{1}{(n+1)}}|f''(0)|^{n-1}f''(0).$$

The local Nusselt number is given by
$Nu_x = -\frac{q(x)x}{\kappa(T-T_\infty)}$,
where $q(x) = -\kappa(\frac{\partial T}{\partial y})_{y=0}$is the heat flux at the surface.

$$(4.3)\quad Nu_x = -\frac{\lambda}{T_w - T_\infty}(\frac{\partial T}{\partial y})_{y=0} = -Re_x^{\frac{1}{(n+1)}}\theta'(0),$$

where λ is characteristic length.

The nonlinear coupled boundary-layer equations (3.6) and (3.7) together with the boundary conditions (3.8) are solved numerically using Runge-kutta method along with shooting technique. At first, higher-order nonlinear differential equations (3.6) and (3.7) are converted into the simultaneous nonlinear differential equations of first order and they are further transformed into initial value problem by applying the shooting technique.

Let $f = f_1, f' = f_2, f'' = f_3, \theta = f_4$, $\theta' = f_5$ then we get following system of first order differential equations:
$f_1' = f_2,$
$f_2' = f_3,$

$$(4.4)\quad f_3' = \frac{(K+M)}{n}\frac{(f_2-1)}{f_3^{(n-1)}} - \frac{f_1 f_3^{(2-n)}}{n(n+1)}, f_4' = f_5,$$

$$(4.5)\quad -f_5' = \frac{Pr}{(n+1)}\frac{(1+\epsilon f_4)}{(1+\epsilon f_4+Nr)}f_1 f_5 + \frac{\epsilon}{(1+\epsilon f_4+Nr)}f_5^2 + \frac{MPrEc(1+\epsilon f_4)}{(1+\epsilon f_4+Nr)}f_2^2 + \frac{PrQ(1+\epsilon f_4)}{(1+\epsilon f_4+Nr)}f_4$$

subject to the following boundary conditions

$$(4.6)\quad \begin{cases} f_1 = S, f_2 = \delta f_3\ f_4 = 1 + \beta f_5 \text{ at } \eta = 0, \\ f_2 = 1, f_4 = 0 \text{ at } \eta = \infty, \end{cases}$$

Now, to convert above system into initial value problems, we take initial guesses of f_3 and f_5 at $\eta = 0$. The resultant initial value problems are solved by employing Runge-kutta fourth order method. The step size $\Delta\eta = 0.016$ is used to obtain the numerical solution with five decimal place accuracy as the criterion of convergence.

5 Results and discussion

Now, we shall analyse the results obtained through numerical computation for variables like power law index (n), velocity slip parameter (δ), magnetic parameter *(M)*, permeability parameter (k), suction/injection (S), thermal radiation *(Nr)*, variable thermal conductivity (ϵ), thermal slip parameter (β), Prandtl number (Pr), Eckert number (Ec) and heat source parameter (Q). The default values of the parameters are set as $M = 0.2$, $S = 0.3$, $Nr = 0.3$, $Pr = 1.2$, $Ec = 0.01$, $k = 0.3$, $\delta = 0.4, \beta = 0.4$, unless otherwise specified. Their results are presented through graphs.

***Figure*.5.1** shows the effect of velocity slip parameter on the velocity profile of shear thinning and shear thickening fluids. The comparison of curves with same power-law index shows that velocity slip at boundary and the fluid velocity within the boundary are proportionally related due to positive value of the fluid viscosity adjacent to the surface. Moreover, increase in slip parameter leads to increased flow through the boundary layer. ***Figure* 5.2(a)** and ***Figure* 5.2(b)** elucidate the effect of slip parameter on the temperature profile for $n = 0.8$ and $n = 1.3$, respectively.

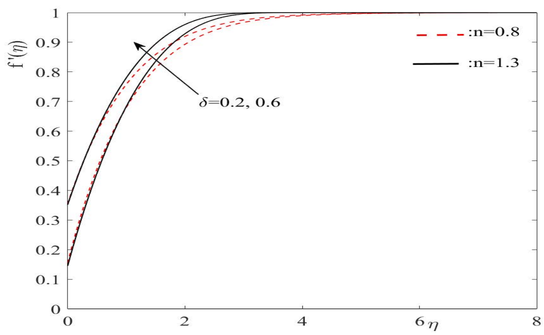

***Figure* 5.1:** Velocity profiles for different values of velocity slip parameter.

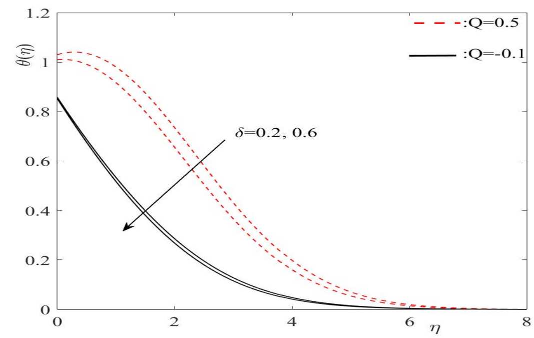

***Figure* 5.2(a):** Velocity profiles for different values of velocity slip parameter.

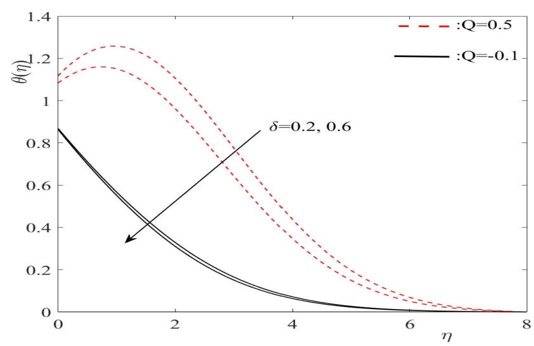

***Figure* 5.2(b):** Temperature profiles for different values of velocity slip parameter for n = 1.3.

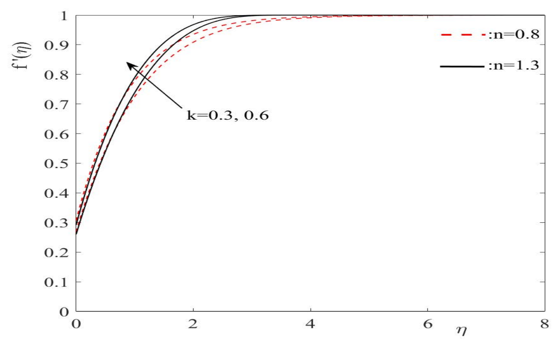

***Figure* 5.3:** Velocity profiles for different values of permeability parameter.

It is evident from ***Figure* 5.2(a)** and ***Figure* 5.2(b)** that an increase in the velocity slip parameter leads to a decrease in the temperature profile. The effect of permeability parameter on fluid velocity profile is shown in ***Figure* 5.3.** It is noticed that an increase in porosity, the magnitude of Darcian body force reduces, which increases the velocity of the fluid in the boundary layer. It is clear from the ***Figure* 5.3**. that initially, shear-thinning fluid increases faster than shear-thickening fluid. This is due to the smallest effective viscosity of shear-thinning fluids. While the opposite trend is seen in later time as the viscosity of the shear thickening fluid will reduce.

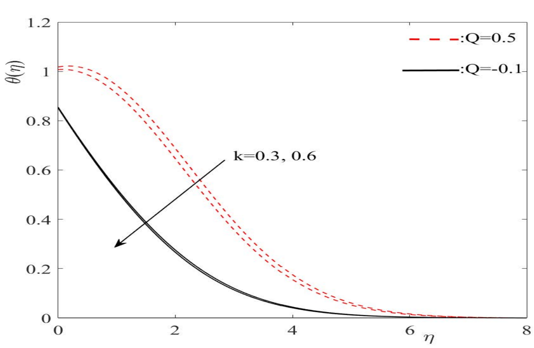

***Figure* 5.4(a):** Temperature profiles for different values of permeability parameter for n= 0.8.

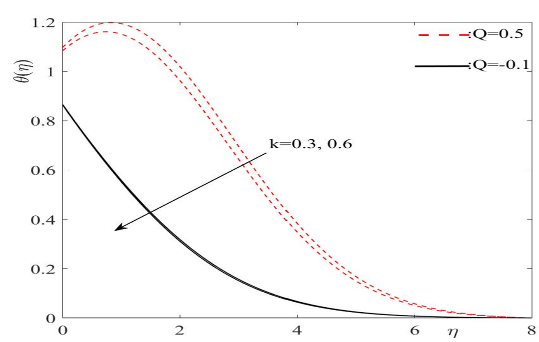

***Figure* 5.4(b):** Temperature profiles for different values of permeability parameter for n = 1.3.

Figure* 5.4(a)** and ***Figure* 5.4(b)** elucidate the effect of permeability parameter on the temperature profile for $n = 0.8$ and $n = 1.3$, respectively. It is observed that the increase in permeability parameter leads to an increase in the heat transfer rate and a decrease in the thickness of the thermal boundary layer. ***Figure* 5.5** exhibits the effect of magnetic parameter on the velocity profile. It is clear from the figure that the increase in magnetic field causes an increase in the velocity profile. ***Figure **5.6(a)** and ***Figure*** **5.6(b)** display the influence of magnetic parameter on the temperature profile for $n = 0.8$ and $n = 1.3$, respectively. In the case of heat source, the temperature profile decreases with increasing value of magnetic parameter.

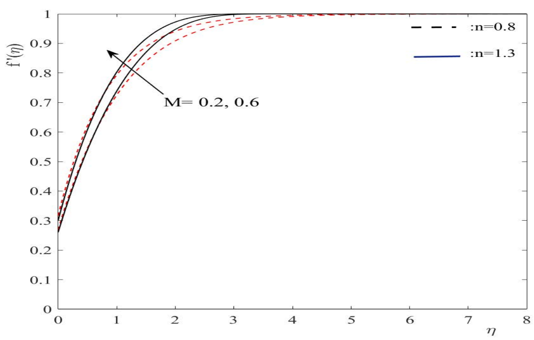

Figure **5.5:** Velocity profiles for different values of Magnetic parameter.

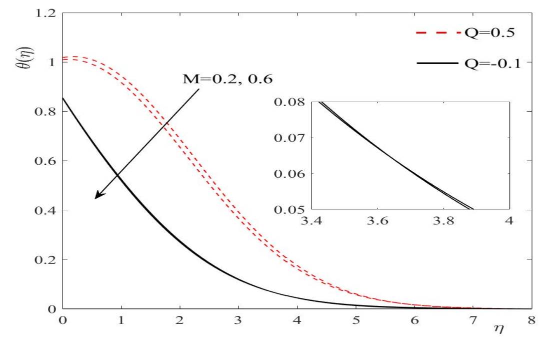

Figure **5.6(a):** Temperature profiles for different values of Magnetic parameter for n= 0.8.

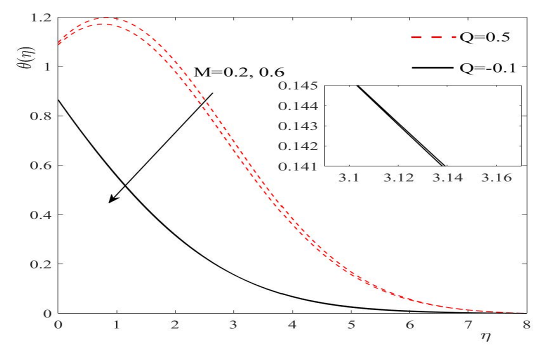

Figure **5.6(b):** Temperature profiles for different values of Magnetic parameter for $n = 1.3$.

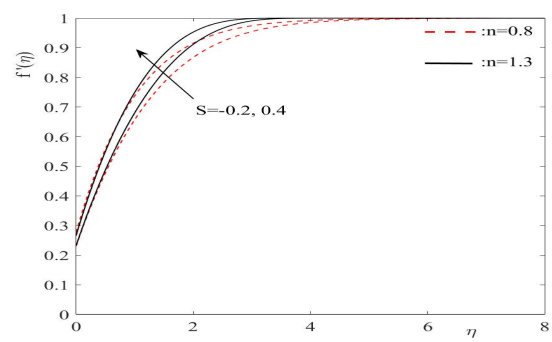

Figure **5.7:** Velocity profiles for different values of suction/injection parameter.

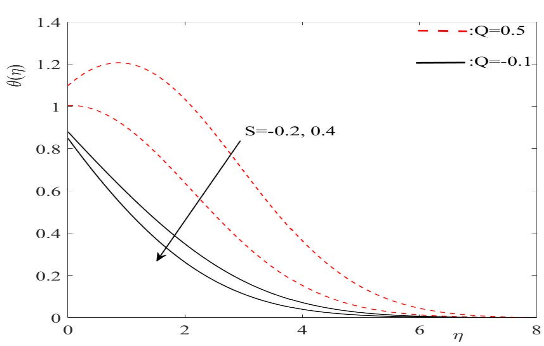

Figure **5.8(a):** Temperature profiles for different values of suction/injection parameter for $n = 0.8$.

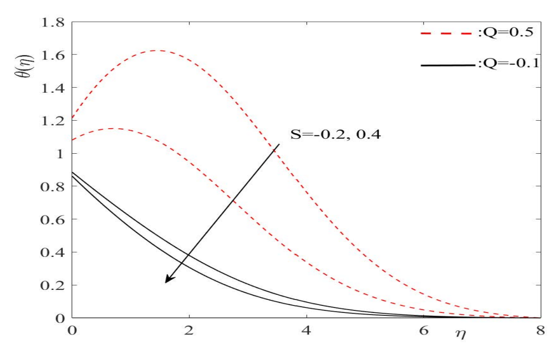

Figure **5.8(b):** Temperature profiles for different values of suction/injection parameter for $n = 1.3$.

It is clear from ***Figure*** **5.6(a)** that in the case of the heat sink, initially, temperature profile decreases with increasing value of magnetic parameter but far from the plate for $\eta > 3.6$, the temperature profile increase with increasing value of the magnetic parameter. The same effect in the temperature profile is seen in ***Figure*** **5.6(b)** for $\eta > 3.12$. ***Figure***

5.7 presents the effect of the suction/injection parameter on the velocity profile for porous sheet in the presence of slip conditions and magnetic fields in a porous medium. It is evident from the ***Figure* 5.7**, that velocity profile increases with the increasing value of suction/injection parameter. ***Figure* 5.8(a)** and ***Figure* 5.8(b)** are drawn to discuss the effect of suction/injection parameter on the temperature profile for $n = 0.8$ and $n = 1.3$, respectively. From ***Figure* 5.8(a)** and ***Figure* 5.8(b)**, it is seen that temperature profile decreases as the value of S increases which results in the increased rate of heat transfer through the boundary layer. ***Figure* 5.9(a)** and ***Figure* 5.9(b)** describe the impact of Prandtl number on the temperature profile for $n = 0.8$ and $n = 1.3$, respectively. The thermal conductivity of the fluid declines by enhancing the Prandtl number. Thus transfer of the heat slows which fall down the temperature of flow distribution. ***Figure* 5.9(a)** and ***Figure* 5.9(b)** validates the above results i.e. the temperature of the flow distribution falls when Prandtl number increases. In the case of heat source, near the plate temperature profile increases due to slip effect and far from the plate the temperature profile decreases with increasing value of Prandtl number. In the case of the heat sink, the temperature profile decreases with the increasing value of the Prandtl number.

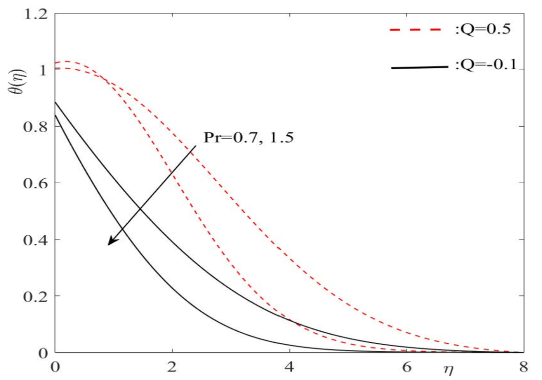

***Figure* 5.9(a):** Temperature profiles for different values of Prandtl number for $n = 0.8$.

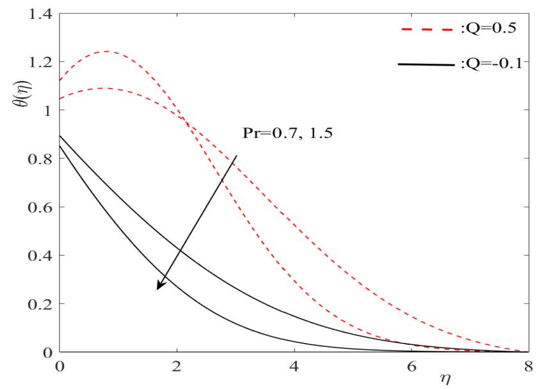

***Figure* 5.9(b):** Temperature profiles for different values of Prandtl number for $n = 1.3$.

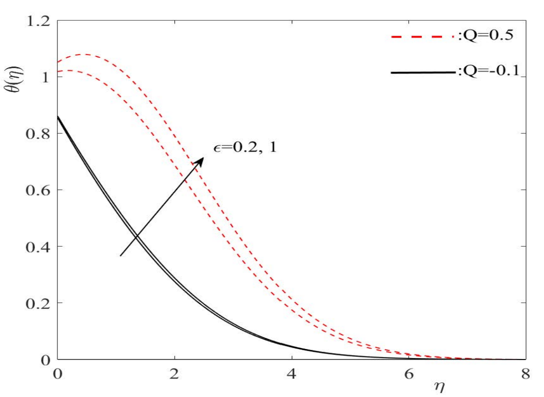

***Figure* 5.10(a):** Temperature profiles for different values of thermal conductivity parameter for $n = 0.8$.

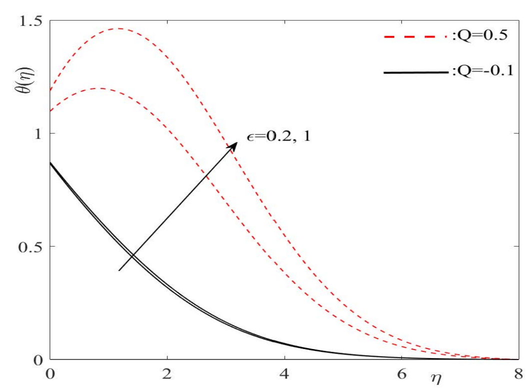

***Figure* 5.10(b):** Temperature profiles for different values of thermal conductivity parameter for $n = 1.3$.

***Figure* 5.10(a)** and ***Figure* 5.10(b)** depict the effect of the thermal conductivity parameter on the temperature profile for $n = 0.8$ and $n = 1.3$, respectively. The results show that with the increase in the thermal conductivity parameter, the temperature profile increases ***Figure* 5.11(a)** and ***Figure* 5.11(b)** reveals the usual effect of the thermal slip parameter on the temperature profile for $n = 0.8$ and $n = 1.3$, respectively. It is evident from the **Fig.5.11(a)** and ***Figure* 5.11(b)** that in the case of $Q > 0$, the temperature increases with increasing values of thermal slip parameter. The reverse happens in the case of the $Q < 0$. ***Figure* 5.12(a)** and ***Figure* 5.12(b)** show the effect of the thermal radiation parameter on the temperature profile for $n = 0.8$ and $n = 1.3$, respectively.

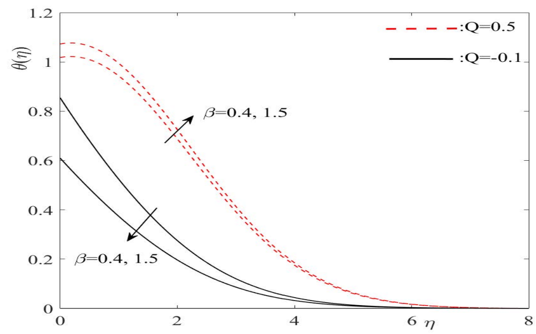

***Figure* 5.11(a):** Temperature profiles for different values of thermal slip parameter for n = 0.8.

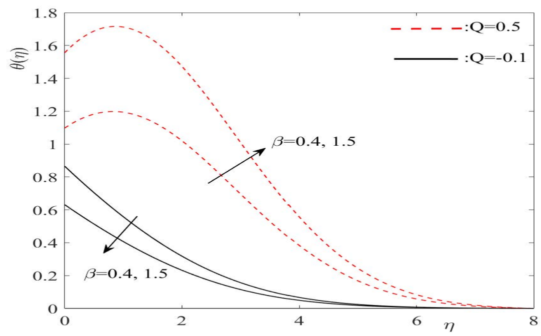

***Figure* 5.11(b):** Temperature profiles for different values of thermal slip parameter for n = 1.3.

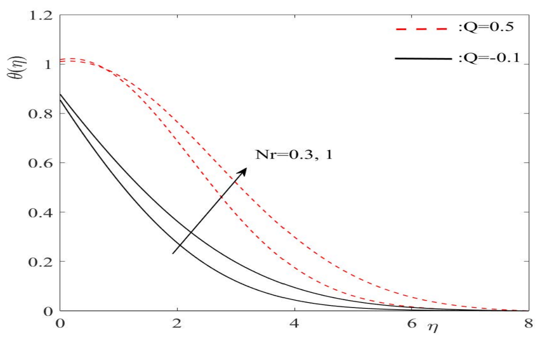

***Figure* 5.12(a):** Temperature profiles for different values of thermal radiation parameter for n = 0.8.

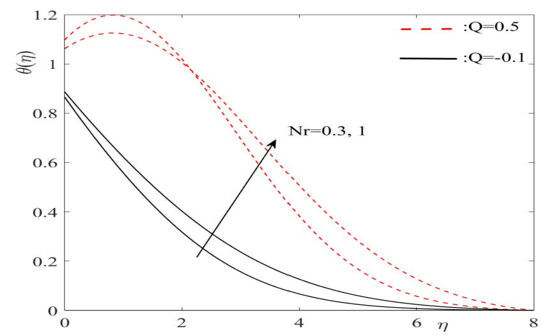

***Figure* 5.12(b):** Temperature profiles for different values of thermal radiation parameter for n = 1.3.

It is depicted from the ***Figure* 5.12(a)** and ***Figure* 5.12(b)** that in the case of heat source, near the plate temperature profile decrease with increasing value of thermal radiation parameter and far from the plate the temperature profile increases with increasing value of thermal radiation parameter. In the case of the heat sink, the temperature profile increases with the increasing value of the thermal radiation parameter.

***Figure* 5.13(a)** and ***Figure* 5.13(b)** demonstrate the temperature distribution for the different values of Eckert number for n = 0.8 and n = 1.3, respectively. Eckert number expresses the relationship between flow kinetic energy to the boundry layer enthalpy difference. So an increase in Eckert number causes an enlargement in kinetic energy, hence temprature of the fluid rises. It is observed from ***Figure* 5.13(a)** and ***Figure* 5.13(b)** that temperature profile increases with the increasing values of Eckert number.

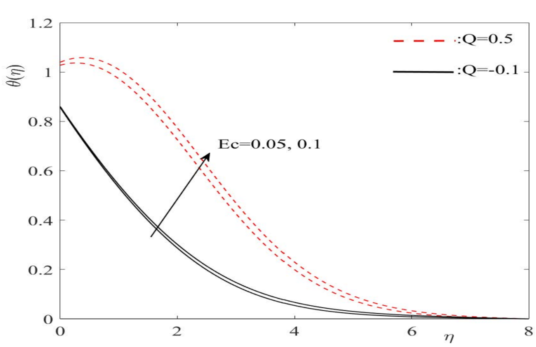

***Figure* 5.13(a):** TTemperature profiles for different values of Eckert number for n = 0.8.

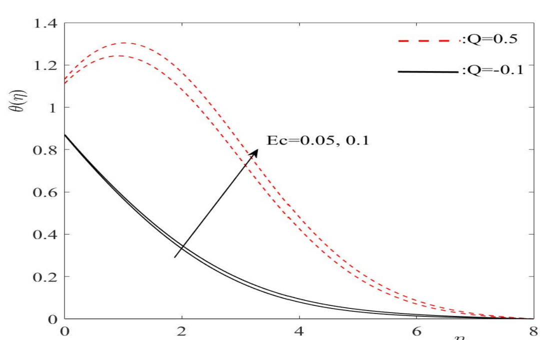

***Figure* 5.13(b):** Temperature profiles for different values of Eckert number for n = 1.3.

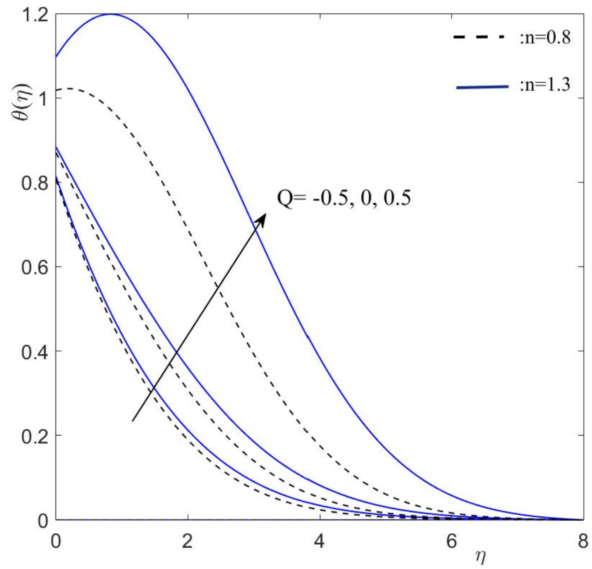

***Figure* 5.14:** Temperature profiles for different values of heat source/sink parameter.

***Figure* 5.14** depicts the effect of heat source/sink parameter on temperature profile. It is observed that the temperature profile increases with the increasing values of heat source/sink parameter.

***Table* 5.1** shows the nature of the skin friction coefficient for various physical parameters. It is observed that the shear-thinning fluid has greater values of skin friction coefficient as compared to shear thickening fluid. The skin friction coefficient increases with an increase in magnetic parameter, permeability parameter and suction/injection parameter and the skin friction coefficient decreasing with an increase in slip parameter.

***Table* 5.1:** Comparison of $f''(0)$ for the different values of parameters k, M, δ and S for n= 0.8 (Shear thinning) and n=1.3 (Shear thickening).

n	k	M	S	δ	$f''(0)$
0.8 1.3	0.3	0.2	0.3	0.4	0.672091 0.650778
0.8 1.3	0.6	0.2	0.3	0.4	0.76586 0.729205
0.8 1.3	0.3	0.6	0.3	0.4	0.79212 0.751132
0.8 1.3	0.3	0.2	0.3	0.6	0.59158 0.586712
0.8 1.3	0.3	0.2	-0.2 0.4 -0.2 0.4	0.4	0.575742 0.691872 0.580588 0.66491

***Table* 5.2** shows the values of Nusselt number for different physical parameters concerning power law index 0.8 and 1.3 respectively. It is observed that the Nusselt number decreases with an increase in power law index. It is noticed from the table that the Nusselt number increases with an increase in suction/injection parameter, permeability parameter, magnetic parameter and slip parameter while the reverse effect is seen in the case of thermal slip parameter, Eckert number, thermal conductivity parameter and heat source/sink parameter. In the case of the heat source, the Nusselt number decreases with the increasing values of the Prandtl number. In the case of the heat sink, the Nusselt number increases with the increasing values of the Prandtl number: in the case of heat source, the Nusselt number increases with the increasing values of the thermal radiation parameter while Nusselt number decreases with the increasing values of the thermal radiation parameter in the case of heat sink.

***Table* 5.2:** Comparison of $-\theta'(0)$ for the different values of parameters k, M, δ, S, β, Pr, Nr, Ec, Q and ϵ for $n = 0.8$ (Shear thinning) and $n = 1.3$ (Shear thickening).

S	k	M	δ	β	Pr	Nr	Ec	ε	Q	$-\theta'(0)$ **for n= 0.8**	$-\theta'(0)$ **for n= 1.3**
-0.2	0.3	0.2	0.4	0.4	1.2	0.3	0.01	0.2	0.5	-0.24471	-0.53755
									-0.1	0.299094	0.286279
0.4									0.5	-0.01121	-0.2007
									-0.1	0.376295	0.34617
0.3	0.6	0.2	0.4	0.4	1.2	0.3	0.01	0.2	0.5	-0.01863	-0.20963
									-0.1	0.369373	0.340504
0.3	0.3	0.2	0.4	0.4	1.2	0.3	0.01	0.2	0.5	-0.044945	-0.24344
									-0.1	0.36323	0.336027
		0.6							0.5	-0.02397	-0.21995
									-0.1	0.367622	0.338097
0.3	0.3	0.2	0.2	0.4	1.2	0.3	0.01	0.2	0.5	-0.07463	-0.29373
									-0.1	0.355587	0.329055
			0.6						0.5	-0.02416	-0.2089
									-0.1	0.368908	0.34139
0.3	0.3	0.2	0.4	0.4	1.2	0.3	0.01	0.2	0.5	-0.044928	-0.24344
									-0.1	0.363236	0.336027
				1.5					0.5	-0.048414	-0.37053
									-0.1	0.260026	0.245742
0.3	0.3	0.2	0.4	0.4	0.7	0.3	0.01	0.2	0.5	-0.012404	-0.11564
									-0.1	0.28659	0.265737
					1.5				0.5	-0.058841	-0.30329
									-0.1	0.400579	0.370406
0.3	0.3	0.2	0.4	0.4	1.2	1	0.01	0.2	0.5	-0.02521	-0.15452
									-0.1	0.305735	0.283336
0.3	0.3	0.2	0.4	0.4	1.2	0.3	0.05	0.2	0.5	-0.0688	-0.282
									-0.1	0.35695	0.32923
							0.1		0.5	-0.099	-0.3318
									-0.1	0.34911	0.32075
0.3	0.3	0.2	0.4	0.4	1.2	0.3	0.01	1	0.5	-0.12761	-0.47219
									-0.1	0.347678	0.321791
0.3	0.3	0.2	0.4	0.4	1.2	0.3	0.01	0.2	0.5	-0.045	-0.2435
									0	0.32373	0.2915
									-0.5	0.4833	0.46618

6 Conclusions

In this paper, we have discussed the MHD flow and heat transfer of a non-Newtonian power-law fluid over a porous sheet in the presence of heat source/sink, joule heating and partial slip conditions. The magnetic parameter, Prandtl number, Eckert number, heat source/sink parameter are varied in the ranges 0.2 to 0.6, 0.7 to 1.5, 0.05 to 0.1, and -0.5 to 0.5, respectively. According to the study the following conclusions are made:

1. The increase in magnetic field causes to increase in the velocity profile and decrease in the temperature profile.
2. The temperature profile decreases with increasing values of the Prandtl number under slip condition.
3. The temperature profile increases with increasing values of Eckert number under slip condition.
4. The temperature profile increases with increasing values of heat source/sink parameter under slip condition.
5. The skin friction coefficient increases with an increase in section/injection.
6. The Nusselt number increses with an increase in sectiom/injection parameter, permeability parameter or thermal radiation parameter when $Q > 0$.

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ON A PSEUDO FIBONACCI SEQUENCE

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Abstract

This article deals with a pseudo Fibonacci sequence and its properties. Some well known identities are obtained in terms of the identities of generalised Fibonacci sequence. Modular properties different from those of Fibonacci sequence are reported.
2010 Mathematics Subject Classifications: 11B39, 11B50.
Keywords and phrases: Pseudo Fibonacci Sequence, Generalised Fibonacci sequence, modulo properties, period of the sequence.

1 Introduction

The Fibonacci sequence $\{F_n\}$ is defined by the recurrence relation

(1.1) $F_{n+2} = F_{n+1} + F_n, n \geq 0,$

with $F_0 = 0$ and $F_1 = 1$ [3, 10]. This sequence has been extended in many ways [See [2, 8] and references therein]. In [1], generalised Fibonacci sequence called B- Fibonacci sequence, defined by

(1.2) ${}^fB_{n+2} = a\,{}^fB_{n+1} + b\,{}^fB_n,$

with ${}^fB_0 = 0$, ${}^fB_1 = 1$, is discussed. In [4], Phadte - Pethe has introduced pseudo Fibonacci sequence $\{g_n\}$, defined by the non-homogeneous recurrence relation,

(1.3) $g_{n+2} = g_{n+1} + g_n + At^n,\ n \geq 0$

with $g_0 = 0$ and $g_1 = 1$. Here $A \neq 0$ is a constant and t is a real number such that $t \neq 0,\ \lambda_1,\ \lambda_2$ where λ_1, λ_2 are roots of the equation $\lambda^2 - \lambda - 1 = 0$. g_n is called the n^{th} pseudo Fibonacci number. First few pseudo Fibonacci numbers are:
$g_0 = 0,\ \ g_1 = 1,\ \ g_2 = 1 + A,\ \ g_3 = 2 + A + At$ and $g_4 = 3 + 2A + At + At^2$.

Observe that each pseudo Fibonacci number is such that its first term is a Fibonacci number and the remaining terms form a polynomial in t whose coefficients are A times Fibonacci numbers. More literature on pseudo Fibonacci sequence and its extensions can be seen in [5, 6, 7].

In this paper we shall consider pseudo Fibonacci sequence $\{G_n\}$ defined by the non-homogeneous recurrence relation

(1.4) $G_{n+2} = aG_{n+1} + b\,G_n + A(-1)^n,\ n \geq 0,$

with $G_0 = \omega$, $G_1 = 1 - \omega$ and study its properties. We assume that $a, b \in \mathbb{Z}$ and A be a constant such that $\omega = \dfrac{A}{1 + a - b} \in \mathbb{Z}$. Following is immediate.

***Theorem* 1.1** *The n^{th} term G_n of* (1.4) *is given by*

(1.5) $G_n = {}^f B_n + \omega(-1)^n,$

where ${}^fB_n$ is defined by (1.2).

We list below some identities for the sequence G_n. These idenetities can be obtained by using corresponding identities for ${}^fB_n$.[1]

***Theorem* 1.2** *G_n satisfies following identities*

i) $G_{n+1}G_{n-1} - G_n^2 = (-1)^n\ b^{n-1} - \omega(-1)^n(G_{n-1} + 2G_n + G_{n+1})$

ii) $\sum_{r=0}^{n} G_r = \dfrac{bG_n + G_{n+1} - \omega(-1)^n(b-1) - 1}{a+b-1} + \omega\epsilon_n$
where

$$\epsilon_n = \begin{cases} 0, if\ n\ is\ odd, \\ 1, if\ n\ is\ even. \end{cases}$$

iii) $G_{n+1}G_m - G_nG_{m+1}$
$= (-b)^n G_{m-n} + \omega\{(G_{n+1} + G_n)(-1)^m + (G_{m+1} - G_m)(-1)^n - ((-b)^n + 2(-1)^{m+n})\}.$

iv) $G_n^2 - G_{n+r}G_{n-r} = (-b)^{n-r}G_r^2 + \omega[2G_n - (-1)^{-r}G_{n+r} - (-1)^r G_{n-r}](-1)^n +$
$(-b)^{n-r}\omega^2 - 2\omega(-b)^{n-r}(-1)^r G_r.$

2 Modulo Properties

In this section we study some modulo properties of the sequence $\{G_n\}$. We have the following result.

***Theorem* 2.1** *Let $\pi(m)$ be the period of G_n modulo m. Let $e \geq 1$ be given. Then*

i) For odd prime p, $\pi(p^e) = p^{e-e'}\pi(p)$, where $1 \leq e' \leq e$ is maximal so that $\pi(p^{e'}) = \pi(p)$.
ii) For $p = 2$ and $e \geq 2$, $\pi(2^e) = 2^{e-e'}\pi(4)$, where $2 \leq e' \leq e$ is maximal so that $\pi(2^{e'}) = \pi(4)$.

Proof. Let $\pi'(m)$ be the period of $\{^f B_n\}$ modulo m. $\pi'(m)$ is always even.
Now $G_0 =^f B_0 + \omega = \omega$ and $G_1 =^f B_1 - \omega = 1 - \omega$.
Hence $G_{\pi'(m)} =^f B_{\pi'(m)} + \omega(-1)^{\pi'(m)} \equiv \omega(\mod m)$ and
$G_{\pi'(m)+1} =^f B_{\pi'(m)+1} + \omega(-1)^{\pi'(m)+1} \equiv 1 - \omega(\mod m)$ so that the period
$\pi'(m)$ of $^f B_n$ and $\pi(m)$ of G_n are same. Now the result follows from Theorem 2 of [9].

***Remark* 2.1** *Note that if three consecutive values of G_n modulo m are same, then the remaining values repeat. This is different from Fibonacci sequence where two consecutive values of F_n modulo m are same then the remaining values repeat.*

We now consider a particular case of $\{G_n\}$ with $a = 1$, $b = 2$, and $A = 1$. For this, ***Table* 2.1** below gives $G_n(mod\ n)$.

Using ***Table* 2.1**, we can state the following results.

***Proposition* 2.1**

$$G(n) = \begin{cases} 0 \mod 3\ \ if\ n \equiv 0,5,6 \mod 8, \\ 1 \mod 3\ \ if\ n \equiv 1 \mod 8, \\ 2 \mod 3\ \ if\ n \equiv 2,3,4,7 \mod 8. \end{cases}$$

***Proposition* 2.2**

$$G(n) = n \mod 4.$$

***Proposition* 2.3**

$$G(n) = \begin{cases} 0 \mod 5\ \ if\ n \equiv 0,8,17,21,22 \mod 24, \\ 1 \mod 5\ \ if\ n \equiv 1,4,6,7,13,14,19 \mod 24, \\ 2 \mod 5\ \ if\ n \equiv 2,5,9,16,18 \mod 24, \\ 3 \mod 5\ \ if\ n \equiv 10,11,12,15,20 \mod 24, \\ 4 \mod 5\ \ if\ n \equiv 3,23 \mod 24. \end{cases}$$

***Proposition* 2.4**

$$G(n) = \begin{cases} 0 \mod 6\ \ if\ n \equiv 0,6 \mod 8, \\ 1 \mod 6\ \ if\ n \equiv 1 \mod 8, \\ 2 \mod 6\ \ if\ n \equiv 2,4 \mod 8, \\ 3 \mod 6\ \ if\ n \equiv 5 \mod 8, \\ 5 \mod 6\ \ if\ n \equiv 3,7 \mod 8. \end{cases}$$

***Table* 2.1:** G_n(mod n) for $a = 1, b = 2$ and $A = 1$

n	G_n	mod 3	mod 4	mod 5	mod 6	mod 7	mod 8	mod 9	mod10	mod15
0	0	0	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2	2	2	2
3	-1	2	3	4	5	6	7	8	9	14
4	-4	2	0	1	2	3	4	5	6	11
5	-3	0	1	2	3	4	5	6	74	12
6	6	0	2	1	0	6	6	6	6	6
7	11	2	3	1	5	4	3	2	1	11
8	0	0	0	0	0	0	0	0	0	0
9	-23	1	1	2	1	5	1	4	7	7
10	-22	2	2	3	2	6	2	5	8	8
11	23	2	3	3	5	2	7	5	3	8
12	68	2	0	3	2	5	4	5	8	8
13	21	0	1	1	3	2	5	3	1	6
14	-114	0	2	1	0	5	6	3	6	6
15	-157	2	3	3	5	4	3	5	3	8
16	72	0	0	2	0	2	0	0	2	12
17	385	1	1	0	1	0	1	7	5	10
18	242	2	2	2	2	4	2	8	2	2
19	-529	2	3	1	5	3	7	2	1	11
20	-1012	2	0	3	2	3	4	5	8	8
21	45	0	1	0	3	3	5	5	0	0
22	2070	0	2	0	0	5	6	0	0	0
23	1979	2	3	4	5	5	3	8	9	14
24	-2160	0	0	0	0	3	0	0	0	0
25	-6119	1	1	1	1	6	1	1	1	1

***Proposition* 2.5**

$$G(n) = \begin{cases} n \mod 8 \;\; if\; n \not\equiv 3,7 \mod 8, \\ 3 \mod 8 \;\; if\; n \equiv 7 \mod 8, \\ 7 \mod 8 \;\; if\; n \equiv 3 \mod 8. \end{cases}$$

***Proposition* 2.6**

$$G(n) = \begin{cases} 0 \mod 9 \;\; if\; n \equiv 0,8,16 \mod 24, \\ 1 \mod 9 \;\; if\; n \equiv 1 \mod 24, \\ 2 \mod 9 \;\; if\; n \equiv 2,7,19 \mod 24, \\ 3 \mod 9 \;\; if\; n \equiv 13,14 \mod 24, \\ 4 \mod 9 \;\; if\; n \equiv 9 \mod 24, \\ 5 \mod 9 \;\; if\; n \equiv 4,10,11,12,15,20 \mod 24, \\ 6 \mod 9 \;\; if\; n \equiv 5,6 \mod 24, \\ 7 \mod 9 \;\; if\; n \equiv 17 \mod 24, \\ 8 \mod 9 \;\; if\; n \equiv 3,18,23 \mod 24. \end{cases}$$

***Proposition* 2.7**

$$G(n) = \begin{cases} 0 \mod 10 & if\ n \equiv 0, 8, 22 \mod 24, \\ 1 \mod 10 & if\ n \equiv 1, 7, 13, 19 \mod 24, \\ 2 \mod 10 & if\ n \equiv 2, 16, 18 \mod 24, \\ 3 \mod 10 & if\ n \equiv 11, 15 \mod 24, \\ 5 \mod 10 & if\ n \equiv 17, 21 \mod 24, \\ 6 \mod 10 & if\ n \equiv 4, 6, 14 \mod 24, \\ 7 \mod 10 & if\ n \equiv 5, 9 \mod 24, \\ 8 \mod 10 & if\ n \equiv 10, 12, 20 \mod 24, \\ 9 \mod 10 & if\ n \equiv 3, 23 \mod 24. \end{cases}$$

***Proposition* 2.8**

$$G(n) = \begin{cases} 0 \mod 15 & if\ n \equiv 0, 8, 21, 22 \mod 24, \\ 1 \mod 15 & if\ n \equiv 1 \mod 24, \\ 2 \mod 15 & if\ n \equiv 2, 18 \mod 24, \\ 6 \mod 15 & if\ n \equiv 6, 13, 14 \mod 24, \\ 7 \mod 15 & if\ n \equiv 9 \mod 24, \\ 8 \mod 15 & if\ n \equiv 10, 11, 12, 15, 20 \mod 24, \\ 10 \mod 15 & if\ n \equiv 17 \mod 24, \\ 11 \mod 15 & if\ n \equiv 4, 7, 19 \mod 24, \\ 12 \mod 15 & if\ n \equiv 5, 16 \mod 24, \\ 14 \mod 15 & if\ n \equiv 3, 23 \mod 24. \end{cases}$$

3 Conclusion

A new pseudo Fibonacci sequence is studied whose modular properties are different from those of Fibonacci sequence.

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CERTAIN QUADRUPLE SERIES EQUATIONS INVOLVING LAGUERRE POLYNOMIALS

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Abstract

Srivastava ([13], [15]) has solved dual series equations involving Bateman-k functions and Jacobi polynomials. Srivastava [16] has obtained more results for the Konhauser-biorhogonal set. Lowndes ([3], [4]), Srivastava [12], Lowndes and Srivastava [5], Srivastava[14], Srivastava and Panda [17] have obtained the solution of dual series equations involving Jacobi and Laguerre polynomials and also solved triple series equations involving Laguerre polynomials. Singh, Rokne and Dhaliwal [10] have find out the solution of triple series equations involving Laguerre polynomials in a closed form. Kuldeep Narain ([7], [8]), Rajnesh Krishnan Mudaliar and Kuldeep Narain [6] have solved Certain dual and quadruple series equations involving generalized Laguerre polynomials and Jacobi polynomials as kernels. In the present paper, an exact solution has been obtained for the quadruple series equations involving Laguerre polynomials by Noble [9] modified multiplying factor technique.
2010 Mathematics Subject Classifications: 45XX, 33C45, 33D45, 34BXX
Keywords and phrases: Laguerre polynomials, Basic orthogonal polynomials and functions, Boundary value problems.

1 Introduction

Earlier Srivastava ([13], [15]) has solved dual series equations involving Bateman-k functions and Jacobi polynomials. Srivastava [16] has obtained more results like generating functions, bilinear generating functions, recurrence relations, some expansions of functions for the Konhauser-biorhogonal set and general result for the dual series equations involving generalized Laguerre polynomials by putting $k = 1$ in (3.10) and (3.11) in [15,p.645]. Lowndes ([3], [4]), Srivastava [12], Lowndes and Srivastava [5], Srivastava[14], Srivastava and Panda [17] have obtained the solution of dual series equations involving Jacobi and Laguerre polynomials and also solved triple series equations involving Laguerre polynomials. Singh, Rokne and Dhaliwal [10] have find out the solution of triple series equations involving Laguerre polynomials in a closed form. Kuldeep Narain ([7], [8]), Rajnesh Krishnan Mudaliar and Kuldeep Narain [6] have solved Certain dual and quadruple series equations involving generalized Laguerre polynomials and Jacobi polynomials as kernels. In this paper, we have obtained the solution of the following quadruple series equations:

(1.1) $\sum_{n=0}^{\infty} \frac{A_n}{\Gamma(\alpha+n+p+1)} L_{n+p}^{\alpha}(x) = \phi_1(x), 0 \leq x < a,$

(1.2) $\sum_{n=0}^{\infty} \frac{A_n}{\Gamma(\alpha+\beta+n+p)} L_{n+p}^{\alpha}(x) = \phi_2(x), a < x < b,$

(1.3) $\sum_{n=0}^{\infty} \frac{A_n}{\Gamma(\alpha+n+p+1)} L_{n+p}^{\alpha}(x) = \phi_3(x), b < x < c,$

(1.4) $\sum_{n=0}^{\infty} \frac{A_n}{\Gamma(\alpha+\beta+n+p)} L_{n+p}^{\alpha}(x) = \phi_4(x), c < x < \infty,$

where $0 < \beta + m,\ 0 < \alpha + \beta < \alpha + 1,\ p$ and m are non-negative integer.

(1.5) $L_{n+p}^{\alpha}(x) = \binom{\alpha+n+p}{n+p},\ F_1[-n-p;\ \alpha+1;\ x]$

is the Laguerre Polynomial, $\phi_1(x), \phi_2(x), \phi_3$ and $\phi_4(x)$ are prescribed functions.

The solution presented in this paper is obtained by employing a multiplying factor technique similar to that used by Noble [9] or Lowndes ([3],[4]).

2 Preliminaries

The results, which will be required in the course of analysis, are given below for ready reference. From Erdélyi [2], it can be deduced that

(2.1) $\int_0^y x^\alpha (y-x)^{\beta+m-1} L_{n+p}^\alpha(x)dx = \frac{\Gamma(\beta+m)\Gamma(\alpha+n+p+1)}{\Gamma(\alpha+\beta+m+n+p+2)} y^{\alpha+\beta+m} L_{n+p}^{\alpha+\beta+m}(y)$

where $0 < y < d,\ -1 < k,\ 0 < \beta + m$, and

(2.2) $\int_y^\infty e^{-x}(x-y)^{-\beta} L_{n+p}^\alpha(x)dx = \Gamma(1-\beta)e^{-y} L_{n+p}^{\alpha+\beta-1}(x)$

where $d < y < \infty,\ \alpha + 1 > \alpha + \beta > 0$.

From Erdélyi [2], we derive the following orthogonality relation for the Laguerre polynomial:

(2.3) $\int_y^\infty e^{-x} x^\alpha L_m^\alpha(x) L_n^{(\alpha)}(x)dx = \frac{\Gamma(\alpha+n+1)}{n!}\delta_{m,n}$

where $\alpha > -1$ and $\delta_{m,n}$ is the Kronecker delta.

The differentiation formula:

(2.4) $\dfrac{d^{m+1}}{dx^{m+1}}\{x^{\alpha+m+1} L_n^{\alpha+m+1}(x)\} = \dfrac{\Gamma(\alpha+m+n+2)}{\Gamma(\alpha+n+1)} x^\alpha L_n^\alpha(x)$

follows from Erdélyi [1].

The analysis here is formal and no attempt has been made to justify the various limiting process.

Making an appeal to the results due to Lowndes ([3], p.123, p.126 eqns (5), (20)), he easily derived ([4], p.168, eqn. (10))

(2.5) $S(r,x) = (r,x)^\alpha \sum_{n=0}^\infty \frac{\Gamma(n+1)\Gamma(\alpha+\beta+n)}{\{\Gamma(\alpha+1+n)\}^2} L_n(\alpha,x) L_n(\alpha,r)$,

(2.6) $= \dfrac{1}{\{\Gamma(1-\beta)\}^2} \int_0^l n(y)(r-y)^{-\beta}(x-y)^{-\beta}dy$,

(2.7) $= -\dfrac{1}{\{\Gamma(1-\beta)\}^2} S_t(r,x)$,

where $\beta < 1,\ \alpha + \beta > 0,\ n(y) = e^y\ \ y^{\alpha+\beta-1}$ and $t = \min(r,x)$ of $f(x)$ and $f'(x)$ are continuous in $a \le x \le b$ and if $0 < \sigma < 1$, then the solutions of the Abel integral equations

(2.8) $f(x) = \int_0^x \frac{F(y)}{(x-y)^\sigma}dy$,

(2.9) $f(x) = \int_x^b \frac{F(y)}{(y-x)^\sigma}dy$,

are given by

(2.10) $F(y) = \dfrac{\sin\sigma\pi}{\pi}\dfrac{d}{dy}\int_a^y \frac{f(x)}{(x-y)^{1-\sigma}}dx$

and

(2.11) $F(y) = \dfrac{\sin\sigma\pi}{\pi}\dfrac{d}{dy}\int_y^b \frac{f(x)}{(x-y)^{1-\sigma}}dx$

respectively.

3 Solution of the problem

Multiply equation (1.1) by $x^\alpha(y-x)^{\beta+m-1}$, integrate with respect to x over $(0,y)$ and then use (2.1) to obtain

(3.1) $\sum_{n=0}^\infty \frac{A_n}{\Gamma(\alpha+\beta+m+n+p+2)} y^{\alpha+\beta+m} L_{n+p}^{\alpha+\beta+m} = \frac{1}{\Gamma(\beta+m)}\int_0^y x^\alpha(y-x)^{\beta+m-1}\phi_1(x)dx$,

where $0 < y < a,\ -1 < \alpha,\ 0 < \beta + m$ and m is a non-negative integer.

Differentiate (3.1) $(m+1)$ times with respect to y and use (2.4) to find

(3.2) $\sum_{n=0}^\infty \frac{A_n}{\Gamma(\alpha+\beta+n+p)} L_{n+p}^{\alpha+\beta-1}(y) = \frac{y^{1-n-\beta}}{\Gamma(\beta+m)}\frac{d^{m+1}}{dy^{m+1}}\int_0^y x^\alpha(y-x)^{\beta+m-1}\phi_1(x)dx$,

where $0 < y < a,\ -1 < \alpha,\ 0 < \beta + m$ and m is a non-negative integer. Again multiply (1.2) by $e^{-x}(x-y)^{-\beta}$, integrate with respect to x over (y,∞), then use (2.2) to find

(3.3) $\sum_{n=0}^\infty \frac{A_n}{\Gamma(\alpha+\beta+n+p)} L_{n+p}^{\alpha+\beta-1}(y) = \frac{e^{-y}}{\Gamma(1-\beta)}\int_y^b (x-y)^{-\beta}e^{-x}\phi_2(x)dx$,

where $a < y < b,\ \beta < 1$ and $0 < \alpha + \beta$.

Now, multiply equation (1.3) by $x^\alpha(y-x)^{\beta+m-1}$, integrate with respect to x over (a,y), then use (2.1) to get

(3.4) $\sum_{n=0}^\infty \frac{A_n}{\Gamma(\alpha+\beta+m+n+p+2)} y^{\alpha+\beta+m} L_{n+p}^{\alpha+\beta+m}(y) = \frac{1}{\Gamma(beta+m)}\int_b^y x^\alpha(y-x)^{\beta+m-1}\phi_3(x)dx$, where $b < y < c, \ -1 < \alpha, \ 0 < \beta + m$ and m is a non-negative integer.

Differentiate (3.4), $(m+1)$ times with respect to y and use (2.4) to find

(3.5) $\sum_{n=0}^{\infty} \frac{A_n}{\Gamma(\alpha+\beta+n+p)} L_{n+p}^{\alpha+\beta-1}(y) = \frac{y^{1-\alpha-\beta}}{\Gamma(beta+m)} \frac{d^{m+1}}{dy^{m+1}} \int_b^y x^{\alpha}(y-x)^{\beta+m-1}\phi_3(x)dx,$

where $b < y < c, \ -1 < \alpha, \ 0 < \beta + m$ and m is a non-negative integer.

Now multiply (1.4) by $e^{-x}(x-y)^{-\beta}$, integrate with respect to x over (y, ∞), then use (2.2) to get

(3.6) $\sum_{n=0}^{\infty} \frac{A_n}{\Gamma(\alpha+\beta+n+p)} L_{n+p}^{\alpha+\beta-1}(y) = \frac{e^{-y}}{\Gamma(1-\beta)} \int_y^{\infty} (x-y)^{-\beta} e^{-x}\phi_4(x),$

where $c < y < \infty$, $\beta < 1$ and $0 < \alpha + \beta$.

The left hand sides of equations (3.2), (3.3), (3.5), (3.6) are now identical and hence on using orthogonality relation (2.3), we obtain the solution of equations (1.1), (1.2), (1.3) and (1.4) in the form:

$$(3.7)\ \ A_n = \frac{(n+p)!}{\Gamma(\beta+m)} \left[\int_0^a e^{-y} L_{n+p}^{\alpha+\beta-1}(y) F_1(y)dy + \int_b^c e^{-y} L_{n+p}^{\alpha+\beta-1}(y) F_3(y)dy \right]$$
$$+ \frac{(n+p)!}{\Gamma(1-\beta)} \left[\int_a^b y^{\alpha+\beta-1} L_{n+p}^{\alpha+\beta-1}(y) F_2(y)dy + \int_c^{\infty} e^{-y} L_{n+p}^{\alpha+\beta-1}(y) F_4(y)dy \right],$$

where

$$(3.8)\ \ F_1(y) = \frac{d^{m+1}}{dy^{m+1}} \int_0^y x^{\alpha}(y-x)^{\beta+m-1}\phi_1(x)dx,$$

$$(3.9)\ \ F_2(y) = \int_y^b (x-y)^{-\beta} e^{-x}\phi_2(x)dx,$$

$$(3.10)\ \ F_3(y) = \frac{d^{m=1}}{dy^{m+1}} \int_b^y x^{\alpha}(y-x)^{\beta+m-1}\phi_3(x)dx,$$

$$(3.11)\ \ F_4(y) = \int_y^{\infty} (x-y)^{-\beta} e^{-x}\phi_4(x)dx.$$

The solution of Lowndes equations

(3.12) $\sum_{n=0}^{\infty} C_n \Gamma(\alpha+\beta+n) L_n^{\alpha}(x) = \phi_1(x), \ 0 \le x < a,$

(3.13) $\sum_{n=0}^{\infty} C_n \Gamma(\alpha+1+n) L_n^{\alpha}(x) = \phi_2(x), \ a < x < b,$

(3.14) $\sum_{n=0}^{\infty} C_n \Gamma(\alpha+\beta+n) L_n^{\alpha}(x) = \phi_3(x), \ b < x < c,$

(3.15) $\sum_{n=0}^{\infty} C_n \Gamma(\alpha+1+n) L_n^{\alpha}(x) = \phi_4(x), \ c < x < \infty,$

can be obtained by putting $A_n = C_n \Gamma(\alpha+n+1)\Gamma(\alpha+\beta+n)$ and $p = 0$ in the solution (3.7).

4 Conclusion

The Laguerre polynomials have been applied by many authors like Lowndes ([3],[4]), Srivastava [12], Srivastava and Panda [17], Lowndes and Srivastava [5], Singh, Rokne and Dhaliwal [10], Kuldeep Narain ([7], [8]), Mudaliar and Kuldeep Narain [6] to solve dual, triple and quadruple series equations. The solution presented in this paper is obtained by employing a multiplying factor technique similar to that used by Noble [9] or Lowndes ([3], [4]).Thus we have obtained an exact solution for the quadruple series equations involving Laguerre polynomials by modified multiplying factor technique Noble [9] .

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Dedication This paper is dedicated to Prof. *R. C. Singh Chandel* on his 75^{th}Birth Anniversary Celebrations for his noteworthy contribution to Mathematical Sciences, *Jñānābha* and VPI continuously since 1971.

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(Dedicated to Honor Dr. R. C. Singh Chandel on His 75^{th} Birth Anniversary Celebrations)

A STUDY ON THE EFFICIENCY OF PSEUDO RANDOM NUMBERS BY ANALYZING THE ERROR PROPAGATION WITH REFERENCE TO THE MONTE CARLO METHOD FOR NUMERICAL INTEGRATION

By

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Abstract

Monte Carlo method is a powerful method for computing the value of complex integrals using probabilistic techniques and estimates the integrals or other quantities that can be expressed as an expectation by averaging the results of a high number of statistical trials.Its convergence rate$O(\sqrt{N})$, is independent of dimension and hence it is preferred for a wide range of high dimensional problems. In this article efficiency of random numbers is being analysed by comparing the error in the evaluation of bi-variate integral corresponding to different size of random numbers, and equi-spaced points by Monte Carlo method. Also analysed and discussed the propagation of error in every case.

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1 Introduction

1.1 Need for study

The Monte Carlo method for numerical integration is believed to rely absolutely on the pseudo random numbers in the context of their randomness. So far great analysis has been done to establish the fact that by assuring the randomness of pseudo random numbers we may have more accurate results i.e. the computational error may be minimized but does this error also depends upon the size of random numbers. Such analysis takes an attention towards the size of random numbers and will always play a key role to get the value of the integral more accurately.

1.2 Pseudo random numbers

Randomness generated by any deterministic pattern with the help of the system is called Pseudo- randomness and the numbers attaining such type of randomness are known as pseudo random numbers. A process that appears to be random but is not is said to be pseudorandom process. Statistical randomness is a typical exhibition of pseudorandom sequences while it is generated by an entirely deterministic process. Most computer programming languages include functions or library routines that purport to be random number generators. They are often designed to provide a random byte or word, or a floating point number uniformly distributed between 0 and 1. Such library functions often have poor statistical properties and some will repeat patterns after only tens of thousands of trials. They all fall in the category of pseudo random numbers.

1.3 Monte Carlo method for numerical integration

The Monte Carlo method is a method for solving problems using random variables. This method was first introduced by Stanislaw Ulam for simulations in physics and other fields that require solutions for problems that are impractical or impossible to solve by traditional analytical or numerical methods. Monte Carlo method has become very popular in recent years, especially in those cases where the number of factors included in the problem are large in numbers and an analytical solution is impossible (for example numerical integration for higher orders).

The central idea behind the Monte Carlo method is either to construct a stochastic model which is in agreement with the actual problem analytically or to simulate the whole problem directly. In both the cases the element of randomness has to be introduced according to well defined rules. After that a large number of trials are performed and the results are observed and finally a statistical analysis is undertaken in the usual way. The advantages of the method are, above everything is that even very difficult problems can often be treated quite easily and desired modifications can be applied without too much trouble.

For a bi-variate integral like

(1.1) $I = \int_c^d \int_a^b f(x, y)dxdy.$

The Monte Carlo method consists of the following steps.

1.3.1 (Using Random Numbers)

***Step* 1.** Pick up n randomly generated points
$(x_1, y_1), (x_2, y_2), (x_3, y_3), \ldots, (x_n, y_n)$ in the rectangle $[a, b] \times [c, d]$.

***Step* 2.** Determine the average value of the function which is given by

(1.2) $\hat{f} = \dfrac{1}{n} \sum_{i=1}^{n} f(x_i, y_i).$

***Step* 3.** Compute the approximation to the integral

(1.3) $\int_c^d \int_a^b f(x, y)dxdy \approx (b - a) \times (d - c)\hat{f}.$

The error for this estimation is given by

(1.4) Error $= (b - a) \times (d - c)\sqrt{\dfrac{f^2 - (\hat{f})^2}{n}},$

where

(1.5) $\hat{f^2} = \dfrac{1}{n} \sum_{i=1}^{n} f^2(x_i, y_i).$

1.3.2 (Using Equispaced Points)

***Step* 1.** Divide the x-range and y-range in n equal parts to get
$x_i = a + ih$ where $i = 1, 2, 3, \ldots, n$
$y_j = c + jk$ where $j = 1, 2, 3, \ldots, n$
where $h = \frac{1}{n}(b - a)$ and $k = \frac{1}{n}(d - c)$.
Our equispaced nodes are (x_i, y_i) $\forall i, j$ in the rectangle $[a, b] \times [c, d]$.

***Step* 2.** Determine the average value of the function which is given by

(1.6) $\hat{f} = \dfrac{1}{n \times n} \sum_{i=1}^{n} \sum_{j=1}^{n} f(x_i, y_j).$

***Step* 3.** Compute the approximation to the integral

(1.7) $\int_c^d \int_a^b f(x, y)dxdy \approx (b - a) \times (d - c)\hat{f}.$

The error for this estimation is given by

(1.8) Error $= (b - a) \times (d - c)\sqrt{\dfrac{f^2 - (\hat{f})^2}{n \times n}},$

where

(1.9) $\hat{f^2} = \dfrac{1}{n \times n} \sum_{i=1,j=1}^{n} f^2(x_i, y_j).$

2 Objective

The focal point of this investigation is primarily to analyze the error obtained in the evaluation of a bi-variate integral by Monte Carlo method using different size of random numbers and equi-spaced points. The basic objective of this investigation is to explore and analyze the error propagation and the efficiency of random numbers for Monte Carlo integration with reference to the error produced as the size of numbers is increased.

3 Review of literature

The detailed study regarding the beginning of Monte Carlo method and its origination is discussed in a scholarly article by Metropolis[11] then after Eckhardt[4] in his scholarly article mentioned the problem, as a solution of which this technique was discovered. In this article the letters showing the personal conversation of Stan Ulam and John Von Neumann regarding Monte Carlo method were also published. A comprehensive review of literature concerning Monte Carlo method may be found in the book by Kalos et al. [10]. A chapter in encyclopedia of Biostatistics by Smyth[14] throws light on all the techniques available to solve one or two dimensional integration including Monte Carlo technique. In 2001 Gould, Tobochnik and Christian[8] in their book studied all the techniques of numerical integration and mentioned their error analysis. In 2002 Yang[15] discussed all other methods and techniques of numerical integration and stated the benefits of Monte Carlo integration over the other methods available in an extended essay in mathematics during his IB diploma program. In parallel to the same we can also find the practical limitations of numerical integration discussed by Evans[6].

Since Monte Carlo Integration was based on random numbers therefore equal attention of scientists were attracted towards the random numbers. In early times a great work has accomplished regarding random numbers, randomness and their use in which a great role was played by Chaitin[3] to discuss the same. A detailed study of random numbers and various techniques (Random Number Generators) to produce random numbers may be found in the book by Gentle[7]. The types of random numbers and randomness are clearly defined and illustrated in an article by Eddelbuettel[5]. In a research paper by Hellekalek[9] he mentioned that what should be the properties of good random number generators.

Early studies on Monte Carlo integration were mainly concerned with the problem of improving the randomness of numbers used. The non-parametric tests to check the randomness of numbers may be referred from Bhar[1]. Just to avoid the inherent errors of random numbers Park and Miller[12] suggested to follow the minimal standards for random number generators. To get rid of this situation of ambiguity that whether the numbers in use are true random or not, concept of quasi random numbers was coined and discussed by Caflisch[2]. The question on the reliability of random numbers with respect to one dimensional Monte Carlo Integration was raised by Saxena and Saxena[13].

4 Methodology

For the proposed objective first random numbers were collected and tested for their randomness in the sense of their independence and uniformity. Two bi-variate integrals are evaluated by Monte Carlo Method using the different size of these random numbers and equi-spaced points. Error in every case was recorded and analyzed to state the findings and conclusion.

The random number's data files of different size numbers of 1000, 2000, 3000, 4000, 5000through two different sources, online and computer generated, are collected and saved with following nomenclature

***Table* 4.1:** Online generated random numbers

File Name	Online Source	Size
olrr1.dat	Research Randomizer	1000
olrr1.dat	Research Randomizer	2000
olrr1.dat	Research Randomizer	3000
olrr1.dat	Research Randomizer	4000
olrr1.dat	Research Randomizer	5000
olrorg1.dat	Random.Org	1000
olrorg2.dat	Random.Org	2000
olrorg3.dat	Random.Org	3000
olrorg4.dat	Random.Org	4000
olrorg5.dat	Random.Org	5000
olgp1.dat	Graph Pad	1000
olgp2.dat	Graph Pad	2000
olgp3.dat	Graph Pad	3000
olgp4.dat	Graph Pad	4000
olgp5.dat	Graph Pad	5000

***Table* 4.2:** Computer generated random numbers

File Name	Source	Size
Int_1_1	Computer generated(RNG)	1000
Int_1_2	Computer generated (RNG)	2000
Int_1_3	Computer generated (RNG)	3000
Int_1_4	Computer generated (RNG)	4000
Int_1_5	Computer generated (RNG)	5000
Int_2_1	Computer generated (RNG)	1000
Int_2_2	Computer generated (RNG)	2000
Int_2_3	Computer generated (RNG)	3000
Int_2_4	Computer generated (RNG)	4000
Int_2_5	Computer generated (RNG)	5000
Int_3_1	Computer generated (RNG)	1000
Int_3_2	Computer generated (RNG)	2000
Int_3_3	Computer generated (RNG)	3000
Int_3_4	Computer generated (RNG)	4000
Int_3_5	Computer generated (RNG)	5000

Then to assure the randomness of these numbers with reference to their independence and uniformity these numbers have been tested by four of the important statistical tests namely Poker Test, Run Test, Frequency Test and Frequency Monobit Test. The detailed study of the tests applied may be obtained from the web address*:
https://drive.google.com/file/d/1ja22gYjVOxliXDI3vmaqiGFsyeC6wJ4A/view?usp=sharing

4.1 Error Evaluation

Although many bi-variate integrals were evaluated but in the present work, only two bi-variate integral is considered to justify the efficiency of random numbers and to discuss the error analysis, generated by Monte Carlo method using random points (computer generated & online generated) as well as equispaced points.

4.1.1 First Integral

The first integral under investigation is following of which the exact value is 1.71828

$$I_1 = \int_0^1 \int_0^{\frac{\pi}{2}} (e^y \cos x)dxdy.$$

4.1.1.1 (Using Random Nodes)

Out of the six data files for 1000 data size there can be 15 different combinations and 15 more combinations when data files for x-series and y-series are interchanged. These 30 pairs of codes for 30 file combinations are

(1,2), (1,3), (1,4), (1,5), (1,6), (2,3), (2,4), (2,5), (2,6), (3,4), (3,5), (3,6) (4,5), (4,6), (5,6), (2,1), (3,1), (4,1), (5,1), (6,1), (3,2), (4,2), (5,2), (6,2), (4,3), (5,3), (6,3), (5,4), (6,4), (6,5)

The same number of combinations may be obtained for 2000, 3000,4000 and 5000 data size.

We shall now use the following program to get the error in the evaluation of the above integral corresponding to all the 30 combinations in each case of different data size of 1000, 2000, 3000, 4000 and 5000.

```
10   CLS:KEY OFF:DIM F$(30)
20   LOCATE 10,5: INPUT "Give drive letter of data files";D$:CLS
30   FOR I = 1 TO 30 :READ F$(I) :NEXT I
40   DATA "INT_1_1.DAT", "INT_2_1.DAT","INT_3_1.DAT","olrr1.DAT", "olrorg1.DAT","olgp1.DAT"
50   DATA "INT_1_2.DAT", "INT_2_2.DAT","INT_3_2.DAT","olrr2.DAT", "olrorg2.DAT", "olgp2.DAT"
60   DATA "INT_1_3.DAT", "INT_2_3.DAT","INT_3_3.DAT","olrr3.DAT", "olrorg3.DAT", "olgp3.DAT"
70   DATA "INT_1_4.DAT", "INT_2_4.DAT","INT_3_4.DAT","olrr4.DAT", "olrorg4.DAT","olgp4.DAT"
80   DATA "INT_1_5.DAT", "INT_2_5.DAT","INT_3_5.DAT", "olrr5.DAT","olrorg5.DAT","olgp5.DAT"
90   FOR I = 1 TO 30: F$(I)= D$ +":"+ F$(I): NEXT I
100  LOCATE 10,2:PRINT "Choose your DATA FILE  size": LOCATE 16,1
110  FOR I=1 TO 5: PRINT I;".  "; I*1000,:NEXT I
120  LOCATE 18,2: INPUT "Select choice Number";C :CLS
130  IF C = 1 THEN LOI = 1:IF C =1 THEN UPI = 6
140  IF C= 2 THEN LOI =7:IF C = 2 THEN UPI =12
150  IF C= 3 THEN LOI =13:IF C=3 THEN UPI =18
160  IF C= 4 THEN LOI =19:IF C=4 THEN UPI =24
170  IF C= 5 THEN LOI =25:IF C=5 THEN UPI =30
180  DS = C*1000
190  LOCATE 10,5: PRINT "Select the Data file":LOCATE 12,1
200  FOR I= LOI TO UPI : PRINT K+1;". "+F$(I),:K=K+1:NEXT I
210  LOCATE 15,1: INPUT "For x-range type your File Number";C1
220  IF C1 =< 0 OR C1 > 6 THEN 230 ELSE 240
230  LOCATE 15,1:PRINT"   " : GOTO 210
240  LOCATE 17,1: INPUT "For y-range type your File Number";C2
250  IF C2 =< 0 OR C2 > 6 THEN 260 ELSE 270
260  LOCATE 17,1:PRINT "   " : GOTO 240
270  IF C1 = C2 THEN 280 ELSE 330
280  LOCATE 18,1: PRINT"choose distinct files"
290  ANS$=INKEY$:IF ANS$="" THEN 290 ELSE 300
300  LOCATE 18,1:PRINT "   " : GOTO 210
310  IF C=1 THEN C1=C1 :IF C=1 THEN C2 = C2
320  IF C=2 THEN C1=C1+6 :IF C=2 THEN C2=C2+6
330  IF C=3 THEN C1=C1+12 :IF C=3 THEN C2=C2+12
340  IF C=4 THEN C1=C1+18 : IF C=4 THEN C2=C2+18
350  IF C=5 THEN C1=C1+24 : IF C=5 THEN C2=C2+24
360  F1$=F$(C1):F2$=F$(C2):CLS
370  DEF FNI(A,B)=  EXP(B)*(COS(A))
380  XLO = 0: XUP = 1.57: YLO = 0: YUP = 1:CLS
390  H=(XUP-XLO)/DS : K=(YUP-YLO)/DS
400  OPEN F1$ FOR INPUT AS #1
410  OPEN F2$ FOR INPUT AS #2
420  INPUT #1,X : X=X*(XUP-XLO)
430  INPUT #2,Y : Y=Y*(YUP-YLO)
440  Y = YLO +Y:X = XLO+X:SUM = SUM + FNI(X,Y)
450  IF NOT EOF(1) THEN 470
460  SUM=SUM/DS :SUM=SUM*(XUP-XLO)*(YUP-YLO)
470  LOCATE 8,10:PRINT "By Monte-Carlo Integration :-"
480  LOCATE 10,10 :  PRINT "Using  Random data files: ";F1$; " and ";F2$
490  LOCATE 12,10:PRINT"Value of integral ";SUM
500   LOSE:END
```

By repeated execution of the program to evaluate the integral for the above 30 combinations of files for *x* and *y* series for different data size, each time we get the different values of the integral given in the following ***Table* 4.3**

Table **4.3:** Error in the evaluation of the first integral corresponding to different data size of random numbers

S.No.	File Combination	No of Random Points				
		1000	**2000**	**3000**	**4000**	**5000**
1	(1,2)	0.067408	0.048285	0.017989	0.003063	0.00491
2	(1,3)	0.041154	-0.062	-0.00728	0.00838	-0.00991
3	(1,4)	0.032333	0.034495	-0.00917	-0.00541	-0.0055
4	(1,5)	0.039267	0.021702	0.01348	-0.00322	-0.00505
5	(1,6)	0.057623	0.034826	0.002838	0.01335	-0.00996
6	(2,1)	-0.05087	-0.05682	-0.02459	0.000674	0.00351
7	(2,3)	-0.05254	-0.03479	-0.02618	0.001821	-0.0156
8	(2,4)	-0.04025	-0.03009	-0.02576	-0.00819	-0.00431
9	(2,5)	-0.01698	-0.04166	-0.00776	0.001848	-0.007
10	(2,6)	-0.01698	-0.03773	-0.01538	0.01783	-0.01571
11	(3,1)	-0.00833	-0.02658	0.012672	0.002917	0.020358
12	(3,2)	0.018618	0.005832	0.033779	-0.00075	0.015812
13	(3,4)	-0.01177	0.003089	0.011078	-0.00438	0.012958
14	(3,5)	0.00941	-0.01308	0.019395	-0.00127	0.025447
15	(3,6)	0.037444	-0.00405	0.023443	0.004379	0.011069
16	(4,1)	-0.0157	-0.04233	-0.00815	0.014344	0.00533
17	(4,2)	0.032217	-0.00112	0.017967	0.013179	0.00734
18	(4,3)	-0.01004	-0.01062	-0.00607	0.020466	-0.00116
19	(4,5)	0.012551	-0.02935	0.015483	0.017253	0.004658
20	(4,6)	0.030893	-0.01538	0.015429	0.019	-0.00493
21	(5,1)	-0.04	-0.01764	-0.00935	-0.00618	-0.00655
22	(5,2)	0.004684	0.022436	0.010774	0.000235	-0.00731
23	(5,3)	-0.0211	0.010683	-0.02219	0.00123	-0.0064
24	(5,4)	-0.0203	0.008032	-0.00927	-0.00472	-0.00734
25	(5,6)	0.000663	0.011038	0.001451	0.020623	-0.01201
26	(6,1)	-0.04784	-0.0254	-0.01884	-0.01361	0.011057
27	(6,2)	-0.00151	0.006768	0.005076	-0.00695	0.007871
28	(6,3)	-0.02007	-0.00191	-0.01608	-0.01724	0.002574
29	(6,4)	-0.02801	0.001943	-0.00855	-0.0275	0.006288
30	(6,5)	-0.02269	-0.01021	0.003333	-0.00292	0.009838

The following ***Figure* 4.1** displays the values of our first integral corresponding to the same combination using different size of random numbers.

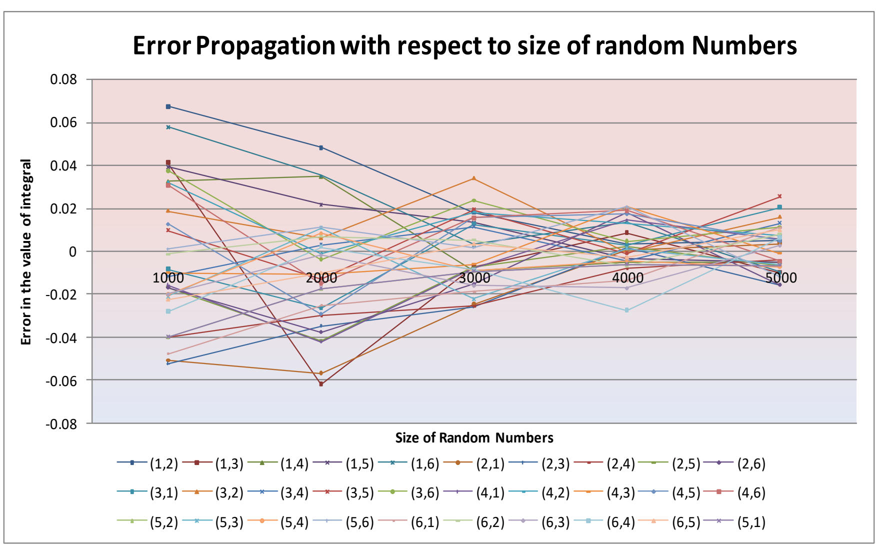

Figure **4.1:** Value of first Integral corresponding to different combination of random numbers

4.1.1.2 (Using Equispaced Nodes)

In order to evaluate the integral using equispaced nodes, we now present a tiny program

```
10    REM "First Integral (2-D)-Equi-spaced Nodes"
20    DEF FNI(A,B)=  EXP(B)*(COS(A))
30    CLS:XLO = 0:XUP = 1.57:YLO = 0:YUP = 1
40    LOCATE 10,10: INPUT "No of Equi-spaced
      Nodes ";N
50    DIM A(N): DIM B(N)
60    A(0)= XLO :B(0)= YLO
70    H=(XUP - XLO)/N:K =(YUP - YLO)/N
80    FOR I= 1 TO N
90    A(I)= A(I-1)+H:B(I)= B(I-1)+K
100   NEXT I
110   FOR I = 1 TO N
120   FOR J = 1 TO N
130   ESUM = ESUM + FNI(A(I),B(J))
140   NEXT J
150   NEXT I
160   ESUM = ESUM*H*K
170   LOCATE 15,10: PRINT "Divisions = "; N,
      "Value = ";ESUM
180   END
```

Here we are making equal numbers of divisions in x & y range. Corresponding to different numbers of division from 20 up to 400 with step size of 20 we get the following observations.

***Table* 4.4:** Error in the evaluation of the first integral corresponding to equi-spaced points

S. No.	No. of Equi-spaced Nodes	Error
1	20	-0.02668
2	40	-0.01277
3	60	-0.00839
4	80	-0.00625
5	100	-0.00497
6	120	-0.00414
7	140	-0.00353
8	160	-0.00308
9	180	-0.00274
10	200	-0.00247
11	220	-0.00225
12	240	-0.00206
13	260	-0.00189
14	280	-0.00176
15	300	-0.00164
16	320	-0.00153
17	340	-0.00145
18	360	-0.00137
19	380	-0.00129
20	400	-0.00124

The following ***Fig* 4.2** displays the values of first integral using different no. of equi-spaced points

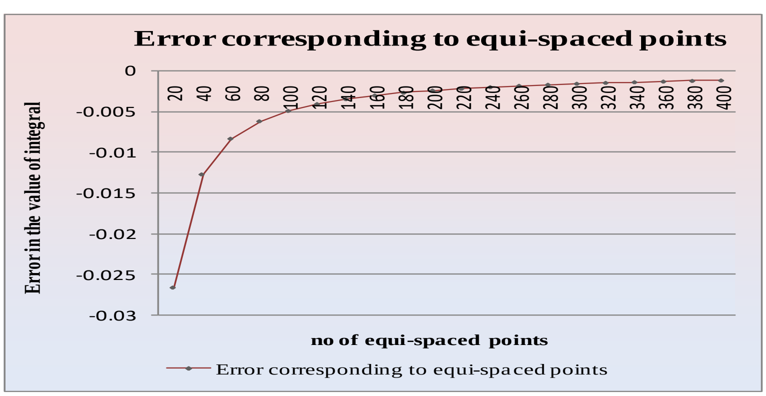

***Figure* 4.2:** Value of first Integral using Equi-spaced points

4.1.2 Second Integral

The second integral under investigation is following, of which the exact value is **30.75**

$$I_2 = \int_0^3 \int_1^2 xy(1 + x + y)dydx$$

4.1.2.1 (Using Random Nodes)

For the evaluation of our second integral with random nodes, we shall make use of the same program as in case of first integral with a change in line 390 which is corresponding to the integrand and limit of our integral. The modified form of this line should be

370 *DEF FNI(A,B)= A*B **(1 + *A* + *B*).

By repeated execution of the program to evaluate the integral for the above 30 combinations of files for x and y series for different data size, each time we get the different values of the integral given in the following ***Table* 4.5**

***Table* 4.5:** Error in the evaluation of the first integral corresponding to different data size of random numbers

S.No.	File Combination	No of Random Points				
		1000	**2000**	**3000**	**4000**	**5000**
1	(1,2)	0.68739	0.8731	0.50389	-0.21843	0.17533
2	(1,3)	-0.38242	0.17063	-0.48562	-0.26221	-0.23931
3	(1,4)	-0.22649	0.51525	0.06432	-0.55092	0.08877
4	(1,5)	0.33795	-0.10327	0.1723	-0.04857	0.37194
5	(1,6)	0.71704	0.12801	0.35889	0.14425	-0.05125
6	(2,1)	-0.57884	-0.25659	0.05084	-0.24966	0.16622
7	(2,3)	0.56632	0.89652	-0.35583	-0.14421	-0.14014
8	(2,4)	0.22057	1.05289	0.08923	-0.50922	0.05638
9	(2,5)	0.85234	0.44431	0.34344	-0.17728	0.40211
10	(2,6)	1.21182	0.877	0.41709	0.0478	0.01805
11	(3,1)	-0.93095	-0.52355	-0.27571	-0.32399	0.08473
12	(3,2)	1.33098	1.31994	0.27125	-0.17266	0.19092
13	(3,4)	0.20432	0.73865	-0.28985	-0.6144	-0.08092
14	(3,5)	0.50104	0.18175	0.16808	-0.13837	-0.03861
15	(3,6)	0.63055	0.51737	-0.02228	0.30537	-0.28115
16	(4,1)	-0.74456	-0.31848	0.07094	-0.33188	0.20884
17	(4,2)	0.98992	1.34504	0.56246	-0.27532	0.17807
18	(4,3)	0.23626	0.58867	-0.45599	-0.33575	-0.29005
19	(4,5)	0.42065	0.33956	0.1582	-0.33416	0.21872
20	(4,6)	0.80417	0.58671	0.12106	0.20341	-0.1379
21	(5,1)	-0.51242	-0.51602	-0.06132	-0.07299	0.35125
22	(5,2)	1.2658	1.1318	0.54243	-0.18941	0.39019
23	(5,3)	0.1884	0.44252	-0.26213	-0.10559	-0.38221
24	(5,4)	0.07553	0.75337	-0.0983	-0.56874	0.09071
25	(5,6)	1.08072	0.34093	0.24161	-0.01607	-0.06091
26	(6,1)	-0.41571	-0.52064	0.12649	-0.1454	0.17736
27	(6,2)	1.392	1.35566	0.63354	-0.21886	0.27853
28	(6,3)	0.02216	0.53968	-0.43079	0.0716	-0.3643
29	(6,4)	0.1783	0.78782	-0.13822	-0.30138	-0.00375
30	(6,5)	0.85284	0.10392	0.25903	-0.26995	0.17493

The following ***Fig* 4.3** displays the values of our first integral corresponding to the same combination using different size of random numbers.

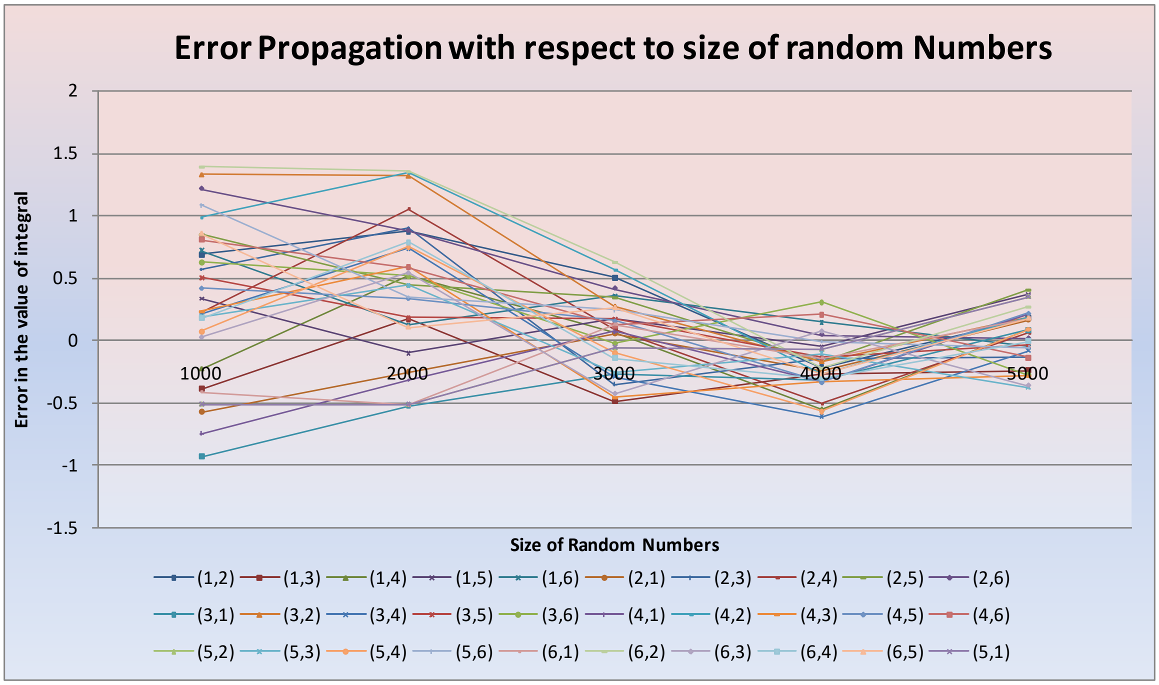

***Figure* 4.3:** Value of second Integral corresponding to different combination of random numbers

4.1.2.2 (Using Equi-spaced Nodes)

In order to evaluate the value of the second integral by using equi-spaced nodes, we will make use of the same program with following modification in line 20 and 30

20 $DEF\ FNI(A, B) = A * B * (1 + A + B).$

30 $CLS\ : XLO = 1 : XUP = 2 : YLO = 0 : YUP = 3.$

Here we are making equal numbers of divisions in x and y range. Corresponding to different numbers of division starting from 20 up to 400 with step size of 20, we get the following observations.

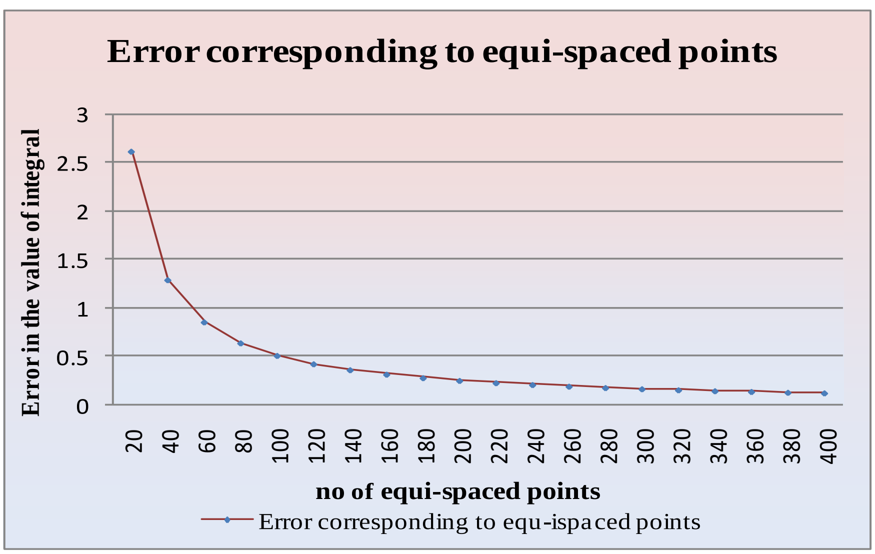

***Figure* 4.4:** Value of second Integral using Equi-spaced points

***Table* 4.6:** Error in the evaluation of the first integral corresponding to equi-spaced points

S. No.	No. of Equi-spaced Nodes	Error
1	20	2.60849
2	40	1.2896
3	60	0.85644
4	80	0.64122
5	100	0.51226
6	120	0.42656
7	140	0.36555
8	160	0.31975
9	180	0.28382
10	200	0.25549
11	220	0.2323
12	240	0.21317
13	260	0.1966
14	280	0.18232
15	300	0.17027
16	320	0.15947
17	340	0.14991
18	360	0.14209
19	380	0.13413
20	400	0.12804

The following ***Fig* 4.4** displays the values of second integral using different size of equi-spaced points.

5 Observations

In the evaluation of the integrals we observe that whatever the combination of files we take the length of the interval of the error decreases as we increase the size of the random numbers [see ***Tables* 4.3** and **4.5**].The supporting results are shown in the following table.

***Table* 5.1:** Length of the interval of error corresponding to different size of random numbers

Integral	**Size of the random numbers**				
	1000	2000	3000	4000	5000
First	.119945	.110287	.059540	.048118	.041154
Second	2.32295	1.87921	1.06433	.92013	.766410

Also the path of error corresponding to each combination while increasing the random numbers seems to be zigzag i.e. the error in the value of both the double integrals corresponding to different size of random numbers, don't follow any pattern and also seems to be random in natural though as per the theory of interpolation a fourth degree polynomial (since the number of data points are five in this case) may be obtained as a trend line to analyze the path of error but it will not be giving satisfactory results as we are dealing with random numbers.

In case of equi-spaced points if we take n points then we are actually dealing with $n \times n$ points i.e. by taking only $n = 100$ function needs to be evaluated for $100 \times 100 = 10,000$ points. Although error follows a smooth pattern and accuracy is assured [see ***Figs.* 4.2** and **4.4**] but here we observe that whatever the accuracy in the value of the integral we obtained corresponding to (say) $400 \times 400 = 1,60,000$ points, same or better accuracy may be achieved by using only 5000 or less random numbers.

6 Conclusion

If the randomness of random numbers is justified with reference to their uniformity and independence then convergence of approximations of a bi-variate integral by Monte Carlo method is assured with the increment in the size of random numbers i.e. if we increase the random numbers used in this process error gets decreased and the propagation of error will be random in nature.

Also we conclude that random numbers plays a better role in comparison to equi-spaced points with reference to the accuracy and computational work involved in the evaluation of the integral and as a result Monte Carlo integration is considered to be time efficient as well. Hence by assuring the randomness of numbers and increment in the size of the numbers,

" Efficiency of random numbers may be accepted in case of bi-variate Monte Carlo integration."

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