ISSN 0304-9892 (Print) ISSN 2455-7463 (Online)

Jñānābha ज्ञानाभ

(HALF YEARLY JOURNAL OF MATHEMATICS)
[Included : UGC-CARE List]

VOLUME 50

NUMBER 1

JUNE 2020

Dedicated to
Honor Professor H. M. Srivastava
on His 80th Birth Anniversary Celebrations

Published by :

The Vijñāna Parishad of India
[Society for Applications of Mathematics]
DAYANAND VEDIC POSTGRADUATE COLLEGE
(Bundelkhand University)
ORAI-285001, U.P., INDIA
www.vijnanaparishadofindia.org/jnanabha

ISSN 0304-9892 (Print) ISSN 2455-7463 (Online)

Jñānābha

Jñānābha, Vol. 50(1) (2020), 1-13
(Dedicated to Honor Professor H.M. Srivastava on His 80th Birth Anniversary Celebrations)

This Special Volume of
JÑĀNĀBHA
is Being Dedicated to Honor
PROFESSOR H. M. SRIVASTAVA
on His 80th Birth Anniversary Celebrations

DOI: https://doi.org/10.58250/jnanabha.2020.50100

PROFESSOR HARI MOHAN SRIVASTAVA

(Born : July 05, 1940)

Jñānābha, Vol. 50(1) (2020), 1-13
(Dedicated to Honor Professor H.M. Srivastava on His 80th Birth Anniversary Celebrations)

PROFESSOR HARI MOHAN SRIVASTAVA : A TOWERING AND TOPMOST LEADING MATHEMATICIAN

By

R.C. Singh Chandel
Executive Editor: Jñānābha , Founder Secretary: Vijñāna Parishad of India
D.V. Postgraduate College, Orai-285001, Uttar Pradesh, India
Email: rc_chandel@yahoo.com

DOI: https://doi.org/10.58250/jnanabha.2020.50100

On behalf of **Vijñāna Parishad of India** and **Jñānābha** Family, we ourselve feel great honored to publish Special Issue of *Jñānābha* , Vol. **50(1)** 2020 (Dedicated to Honor Professor H.M. Srivastava on His 80th Birth Anniversary Celebrations).

Professor Hari Mohan Srivastava is an amazing man, towering and leading mathematician, well known topmost eminent figure of Special Functions and Allied Topics Mathematical Analysis. He is topmost Researcher, Supervisor of several *Ph.D.* and *D.Sc.* theses, well reputed University Teacher, Editor or Member on Editorial Boards of various International Journals, Reviewer of various Reviews, Elected Fellows of various International Societies, Recipient of International Prizes, Awards, Honors, Author of various Internationally prescribed Text Books having Special Dedication Volumes/ Dedication Issues (and/ or Dedication Messages) of International Scientific Research Journals.

He has credit to be associated initially with *Jñānābha* , Vol.1, 1971 as an active member on its Editorial Board with me (as Editor). Since 1972, he is continuously giving his Dedicated Services as Foreign Secretary of Vijñāna Parishad of India and gracing the Chair of Chief Editor : Jñānābha with me (as Founder Secretary of VPI and Executive Editor : *Jñānābha*).

He was elected and Honored as one of the first two Honorary Fellows of Vijñāna Parishad of India *(FVPI)* with Professor J. N. Kanpur during Silver Jubilee Conference of VPI held at Parishad Head Quarters: D. V. Postgraduate College, Orai-285001, UP, India (May 10-11, 1996).

We published Jñānābha , Vol. **31/32**, 2002 Dedicated to Honor Professor H.M. Srivastava on his 62nd Birthday Celebrations. While Jñānābha , Vol. 45,2015 was Dedicated to Honor Professor H. M. Srivastava on His Platinum Jubilee Celebrations.

18th Annual Cum 1st International Conference of VPI was Dedicated to Honor Professor H. M. Srivastava on His Platinum Jubilee Celebrations held at MANIT, Bhopal. MP, India on December 11-14, 2015.

Professor H. M. Srivastava was also Honored by **LIFE-LONG ACHIVEMENTS AWARD, the Highest Prestigious** Award of VPI for his Outstanding Contribution to His Subject and Life-Time Distinguished Services Dedicated to **Vijñāna Parishad of India**, its Journal **JÑĀNĀBHA and/or to Nation/ World Development** at the Occasion of 20th Annual Conference of VPI held at Manipal University, Jaipur, India on November 24-26, 2017 in his absentia.

The Mathematics community has been very privileged to have Professor Srivastava as its guiding force, leader and a great mentor. He has been a role model and an inspiration to every mathematician and countless people, whose life Professor Srivastava has touched. His work has taken Mathematics to new heights and helped researchers accomplish goals that could have never been dreamed of before.

On this great occasion of Professor Srivastava's Birth Anniversary Celebrations, we wish him a happy, healthy, and long joyful life. May he continue to guide, encourage, and enlighten the global Mathematics community for decades to come.

At a Glance
Professor Hari Mohan Srivastava (H. M. Srivastava)
Ph.D., D.Sc. (h.c), D.Sc. (h.c.)
2006-*Present : Professor Emeritus*
1974–2006: Full Professor
1969–1974: Associate Professor
Department of Mathematics and Statistics,
University of Victoria,
Victoria, British Columbia, V8W 2Y2 Canada
1959–1969: Assistant Professor/Reader/Lecturer in Universities in India *and* U.S.A.
E-Mail: harimsri@math.uvic.ca
Date of Birth: July 5, 1940
Place of Birth: Karon (District Ballia), Uttar Pradesh, India

Education:
B.Sc. 1957 University of Allahabad, India
M.Sc. 1959 University of Allahabad, India
Ph.D. 1965 J. N. Vyas University of Jodhpur, India
D.Sc. (*Honoris Causa*) 2006 Chung Yuan Christian University, Taiwan, Republic of China
D.Sc. (*Honoris Causa*) 2007 "1 Decembrie 1918" University of Alba Iulia, Romania

Professional Qualifications and Recognitions:
F.R.A.S. 1968 Royal Astronomical Society (London, U.K.)
F.N.A.Sc. 1969 National Academy of Sciences (India)
F.I.M.A. 1975 Institute of Mathematics and Its Applications (U.K.)
F.M.R.A.S. 1991 The Royal Academy of Sciences, Literature and Fine Arts (Belgium)
C.Math. 1991 Institute of Mathematics and Its Applications (U.K.)
F.V.P.I. 1996 Vijñāna Parishad (Science Academy) of India
F.A.A.A.S. 1996 American Association for the Advancement of Science (U.S.A.)
F.A.A.C. 1998 La Academia Canaria de Ciencias (Spain)
F.F.A. 1999 Forum d'Analystes (India)
C.Sc. 2005 Institute of Mathematics and Its Applications (U.K.)
D.Sc. (*h.c.*) 2006 Chung Yuan Christian University (Taiwan, Republic of China)
F.M.A.S.A. 2007 The Macedonian Academy of Sciences and Arts (Macedonia)
D.Sc. (*h.c.*) 2007 "1 Decembrie 1918" University of Alba Iulia (Romania)
F.F.R.A.S. 2016 The Royal Academy of Sciences (Spain)

2004: NSERC 25-Year Award (Canada)

2004: Nishiwaki Prize (Japan)

2012: Listed in the Second Place among *Canada's Top Researchers* in the discipline of Mathematics and Statistics in Terms of Productivity and Impact Based Upon a Measure of Citations to Their Published Works (*The Globe and Mail*, Toronto, March 27, 2012, Page B7 *et seq.*)

***2020: Obada Prize for Distinguished Researcher* (Egypt)**

2015–Present: Thomson Reuters Highly Cited Researcher

Special Volumes and Special Issues of (and/or Dedication Messages in) International Scientific Research Journals Dedicated to his 60th, 62nd, 70th, 75th and 80th Birth Anniversaries:

These include (*to quote only a few*) ***Fractional Calculus and Applied Analysis*** **(Volume 3, Number 3, 2000; Volume 13, Number 3, 2010; Volume 13, Number 4, 2010),** ***Applied Mathematics and Computation*** **(Volume 187, Number 1, 2007; Volume 218, Number 3, 2011)**, and so on. Moreover, the following **880-page** Springer volume may be cited here:

***Analytic Number Theory*, *Approximation Theory*, *and Special Functions*:**
In Honor of Hari M. Srivastava
(xi + 880 pp.; ISBN 978-1-4939-0257-1) (Gradimir V. Milovanović and Michael Th. Rassias, Editors), Springer, Berlin, Heidelberg and New York, 2014.

Professor Srivastava began his **university-level** teaching career in 1959 itself at the age of 19 years. Currently, Professor Srivastava holds the position of a **Professor Emeritus** in the Department of Mathematics and Statistics at the University of Victoria in Canada. He joined the faculty there in 1969 [first as Associate Professor (1969–1974) and then as Full Professor (1974–2006)]. Professor Srivastava has held (and/or currently holding) numerous positions of *Visiting Professor* **and** *Chair Professor* including (for example) those at West Virginia University in U. S. A. (1967–1969), Université Laval in Canada (1975), and the University of Glasgow in U. K. (1975–1976), and indeed also at many other universities and research institutes in different parts of the world.

Professor Srivastava has published 33 books, monographs and edited volumes, 36 book (and encyclopedia) chapters, 48 papers in international conference proceedings, and **more than 1,300** scientific research journal articles on various topics of mathematical analysis and applicable mathematics. In addition, he has written Forewords to several books by other authors and to several special issues of scientific journals. He has also edited (and contributed to) many volumes which are dedicated to the memories of famous mathematical scientists. Citations of his research contributions can be found in many books and monographs, *Ph.D.* and *D.Sc.* theses, and scientific journal articles, much too numerous to be recorded here. Currently, he is actively associated editorially (that is, as an Editor, Honorary Editor, Editor-in-Chief, Senior Editor, Associate Editor or Editorial Board Member) with **over 200** international scientific research journals. His biographical sketches (many of which are illustrated with his photograph) have appeared in various issues of more than 50 international biographies, directories, and ***Who's Who's***.

Professor Srivastava's **over 60-year career** as a **university-level** teacher and as a remarkably prolific researcher in many different areas of the mathematical, physical, and statistical sciences is highlighted by (among other things) the fact that he has collaborated and published joint papers with as many as **650** mathematicians, physicists, statisticians, chemists, astrophysicists, geochemists, as well as information and business management scientists, who are scattered throughout the world, thereby qualifying for his **Erdös number** 2, implying that at least one of Professor Srivastava's co-authors is a co-author of the famous Hungarian mathematician, Paul Erdös (1913–1996). Professor Srivastava's collaboration distances with other famous scientists include his **Einstein number** 3, **Pólya number** 3, **von Neumann number** 3, **Wiles number** 3, and so on.

Outline of Research Contributions:

Many mathematical entities and objects are attributed to (and named after) him. These entities and objects include (among other items) Srivastava's polynomials and functions, Carlitz-Srivastava polynomials, Srivastava-Buschman polynomials, Srivastava-Singhal polynomials, Chan-Chyan-Srivastava polynomials, Erkuş-Srivastava polynomials, Srivastava-Daoust multivariable hypergeometric function, Srivastava-Panda multivariable H-function, Singhal-Srivastava generating function, Srivastava-Agarwal *basic* (or q-) generating function, and Wu-Srivastava inequality in the field of Higher Transcendental Functions; Srivastava-Owa, Choi-Saigo-Srivastava, Jung-Kim-Srivastava, Liu-Srivastava, Cho-Kwon-Srivastava, Dziok-Srivastava, Srivastava-Attiya, Srivastava-Wright and Srivastava-Gaboury operators in the field of Geometric Function Theory in Complex Analysis; Srivastava-Gupta operator in the field of Approximation Theory; Srivastava, Adamchik-Srivastava and Choi-Srivastava constants and methods in the field of Analytic Number Theory; **and so on.**

Professor Srivastava has supervised (and is currently supervising) a number of post-graduate students working toward their Master's, *Ph.D.* and/or *D.Sc.* degrees in different parts of the world. Besides, many post-doctoral fellows and research associates have worked with him at West Virginia University in U.S.A. and at the University of Victoria in Canada.

Some of the significant and remarkable contributions by Professor Srivastava are being listed below under each of the **main** topics of his **current** research interests:

(i) Real and Complex Analysis: A unified theory of numerous potentially useful function classes, and of various integral and convolution operators using hypergeometric functions, especially in Geometric Function Theory in Complex Analysis, and several classes of analytic and geometric inequalities in the field of Real Analysis.

(ii) Fractional Calculus and Its Applications: Generalizations of such classical fractional-calculus operators as the Riemann-Liouville and Weyl operators together with their fruitful applications to numerous families of differential, integral, and integro-differential equations, especially some general classes of fractional kinetic equations and also to some Volterra-type integro-differential equations which emerge from the unsaturated behavior of the free electron laser.

(iii) Integral Equations and Transforms: Explicit solutions of several general families of dual series and integral equations occurring in Potential Theory; Unified theory of many known generalizations of the classical Laplace transform (such as the Meijer and Varma transforms) and of other multiple integral transforms by means of the Whittaker $W_{\kappa,\mu}$-function and the (Srivastava-Panda) multivariable H-function in their kernels.

(iv) Higher Transcendental Functions and Their Applications: Discovery, introduction, and systematic (and unified) investigation of a set of **205** triple Gaussian hypergeometric series, especially the triple hypergeometric functions H_A, H_B and H_C added to the 14-member set conjectured and defined in 1893 by Giuseppe Lauricella (1867–1913). Unified theory and applications of the multivariable extensions of the celebrated higher transcendental (Ψ- and H-) functions of Charles Fox (1897–1977) and Edward Maitland Wright (1906–2005), and also of the Mittag-Leffler E-functions which are named after Gustav Mittag-Leffler (1846–1927). Mention should be made also of his applications of some of these Higher Transcendental Functions in Quantum and Fluid Mechanics, Astrophysics, Probability Distribution Theory, Queuing Theory and other related Stochastic Processes, and so on.

(v) *q*-Series and *q*-Polynomials: Basic theory of general q-polynomial expansions for functions of several complex variables, extensions of several celebrated q-identities of Srinivasa Ramanujan (1887–1920), and systematic introduction and investigation of multivariable basic (or q-) hypergeometric series.

(vi) Analytic Number Theory: Presentation of several computationally-friendly and rapidly-converging series representations for Riemann's Zeta function, Dirichlet's L-series, introduction and application of some novel techniques for closed-form evaluations of series involving a wide variety of sequences and functions of analytic number theory, and so on. His applications of (especially) the Hurwitz-Lerch Zeta function in Geometric Function Theory in Complex Analysis and in Probability Distribution Theory and related topics of Statistical Sciences deserve to be recorded here.

(vii) Analytic and Geometric Inequalities: Unified presentations and generalizations of a number of analytic and geometric inequalities.

(viii) Probability and Statistics: Probabilistic derivations of generating functions and statistical applications of various special functions and orthogonal polynomials.

(ix) Inventory Modelling and Optimization: Systematic analytical investigation of many potentially useful problems in supply chain management.

Professor Srivastava's publications have been reviewed by (among others) **Mathematical Reviews** (U.S.A.), **Referativnyi Zhurnal Matematika** (Russia), **Zentralblatt für Mathematik** (Germany), and **Applied Mechanics Reviews** (U.S.A.) under various **2010 *Mathematical Subject Classifications*** (**MathSciNet**) including (for example) the following general classifications:

00 General
01 History and Biography
05 Combinatorics
11 Number Theory
15 Linear and Multilinear Algebra; Matrix Theory
26 Real Functions
30 Functions of a Complex Variable
31 Potential Theory
33 Special Functions
34 Ordinary Differential Equations
35 Partial Differential Equations
39 Difference and Functional Equations
40 Sequences, Series, Summability
41 Approximations and Expansions
42 Fourier Analysis
44 Integral Transforms, Operational Calculus
45 Integral Equations
46 Functional Analysis
47 Operator Theory
51 Geometry
58 General Global Analysis, Analysis on Manifolds

60 Probability Theory and Stochastic Processes
62 Statistics
65 Numerical Analysis
70 Mechanics of Particles and Systems
76 Fluid Mechanics
81 Quantum Theory
85 Astronomy and Astrophysics
90 Operations Research, Mathematical Programming
91 Game Theory, Economics, Social and Behavioral Sciences

Publications (Selected and the Most Recent):

• H. M. Srivastava, Operators of basic (or q-) calculus and fractional q-calculus and their applications in geometric function theory of complex analysis, Iran. J. Sci. Technol. Trans. A: Sci. 44 (2020), 327–344.
• H. Tang, H. M. Srivastava, H.-S. Li and G.-T. Deng, Majorization results for subclasses of starlike functions based on the sine and cosine functions, Bull. Iranian Math. Soc. 46 (2020), 381-388; see also Correction, Bull. Iranian Math. Soc. 46 (2020), 389–391.
• H. M. Srivastava, H. I. Abdel-Gawad and K. M. Saad, Stability of traveling waves based upon the Evans function and Legendre polynomials, Appl. Sci. 10 (2020), Article ID 846, 1–16.
• H. M. Srivastava, M. P. Chaudhary and S. Chaudhary, A family of theta-function identities related to Jacobi's triple-product identity, Russian J. Math. Phys. 27 (2020), 139–144.
• H. M. Srivastava, S. Khan, S. Araci, M. Acikgoz and M. Riyasat, A general class of the three-variable unified Apostol-type q-polynomials and multiple power q-sums, Bull. Iranian Math. Soc. 46 (2020), 519–542.
• H. Ozden, F. E. Zihni, F. O. Erdogan, I. N. Cangul, G. Srivastava and H. M. Srivastava, Independence number of graphs and line graphs of trees by means of omega invariant, Rev. Real Acad. Cienc. Exactas Fís. Natur. Ser. A Mat. (RACSAM) 114 (2020), Article ID 91, 1–14. • M. Masjed-Jamei, Z. Moalemi, H. M. Srivastava and I. Area, Some modified Adams-Bashforth methods based upon the weighted Hermite quadrature rules, Math. Methods Appl. Sci. 43 (2020), 1380–1398.
• H. M. Srivastava, M. I. Qureshi and S. Jabee, Some general series identities and summation theorems for the Gauss hypergeometric function with negative integer numerator and denominator parameters, J. Nonlinear Convex Anal. 21 (2020), 463–478.
• H. M. Srivastava, M. S. Chauhan and S. K. Upadhyay, L_p-boundedness of the pseudo-differential operators associated with the Kontorovich-Lebedev transform, Rev. Real Acad. Cienc. Exactas Fís. Natur. Ser. A Mat. (RACSAM) 114 (2020), Article ID 80, 1–18.
• K.-J. Chung, J.-J. Liao, S.-D. Lin, S.-T. Chuang and H. M. Srivastava, Manufacturer's optimal pricing and lot-sizing policies under trade-credit financing, Math. Methods Appl. Sci. 43 (2020), 3099–3116.
• H. M. Srivastava and K. M. Saad, New approximate solution of the time-fractional Nagumo equation involving fractional integrals without singular kernel, Appl. Math. Inform. Sci. 14 (2020), 1–8.
• H. M. Srivastava, A. Motamednezhad and E. A. Adegani, Faber polynomial coefficient estimates for bi-univalent functions defined by using differential subordination and a certain fractional derivative operator, Mathematics 8 (2020), Article ID 172, 1–12.

• Y. J. Sim, O. S. Kwon, N. E. Cho and H. M. Srivastava, Bounds for the real parts and arguments of normalized analytic functions defined by the Srivastava-Attiya operator, J. Comput. Anal. Appl. 28 (2020), 628–645.
• H. M. Srivastava, M. Riyasat, S. Khan, S. Araci and M. Acikgoz, A new approach to Legendre-truncated-exponential-based Sheffer sequences via Riordan arrays, Appl. Math. Comput. 369 (2020), Article ID 124683, 1–22.
• J.-J. Liao, K.-N. Huang, K.-J. Chung, S.-D. Lin, S.-T. Chuang and H. M. Srivastava, Optimal ordering policy in an economic order quantity (EOQ) model for non-instantaneous deteriorating items with defective quality and permissible delay in payments, Rev. Real Acad. Cienc. Exactas Fís. Natur. Ser. A Mat. (RACSAM) 114 (2020), Article ID 41, 1–26.
• H. Singh, D. Baleanu, H. M. Srivastava, H. Dutta and N. K. Jha, Solution of multi-dimensional Fredholm equations using Legendre scaling functions, Appl. Numer. Math. 150 (2020), 313–324.
•M. S. U. Rehman, Q. Z. Ahmad, H. M. Srivastava, B. Khan and N. Khan, Partial sums of generalized q-Mittag-Leffler functions, AIMS Mathematics, 5 (2020), 408–420.
• K.-J. Chung, J.-J. Liao, S.-D. Lin, S.-T. Chuang and H. M. Srivastava, Mathematical analytic techniques and the complete squares method for solving an inventory modelling problem with a mixture of backorders and lost sales, Rev. Real Acad. Cienc. Exactas Fís. Natur. Ser. A Mat. (RACSAM) 114 (2020), Article ID 28, 1-10.
• K. M. Saad, H. M. Srivastava and J. F. Gómez-Aguilar, A fractional quadratic autocatalysis associated with chemical clock reactions involving linear inhibition, Chaos Solitons Fractals 132 (2020), Article ID 109557, 1–9.
• H. M. Srivastava, Some general families of the Hurwitz-Lerch Zeta functions and their applications: Recent developments and directions for further researches, Proc. Inst. Math. Mech. Nat. Acad. Sci. Azerbaijan 45 (2019), 234–269.
• W. Kumam, H. M. Srivastava, S. A. Wani, S. Araci and P. Kumam, Truncated-exponential-based Frobenius-Euler polynomials, Adv. Differ. Equ. 2019 (2019), Article ID 530, 1–12.
• H. M. Srivastava and C. KizilateÅ, A parametric kind of the Fubini-type polynomials, Rev. Real Acad. Cienc. Exactas Fís. Natur. Ser. A Mat. (RACSAM) 113 (2019), 3253–3267.
• S. S. Bhoosnurmath, B. Chakraborty and H. M. Srivastava, A note on the value distribution of differential polynomials, Commun. Korean Math. Soc. 34 (2019), 1145–1155.
• H. M. Srivastava, M.Tahir, B. Khan, Q. Z. Ahmad and N. Khan, Some general families of q-starlike functions associated with the Janowski functions, Filomat 33 (2019), 2613–2626.
• H. M. Srivastava, G. Yasmin and A. Muhyi, Lie algebra representations and 2-index 4-variable 1-parameter Hermite polynomials, Note di Mat. 39 (1) (2019), 65–87.
• H. M. Srivastava and P. Baliarsingh, The Leibniz and chain rules for fractional derivatives, Appl. Anal. Optim. 3 (2019), 343–357.
• H. M. Srivastava, G. Ícoz and B. Çekim, Approximation properties of an extended family of the Szász-Mirakjan Beta-type operators, Axioms 8 (2019), Article ID 111, 1–13.
• V. Kumar, N. E. Cho, V. Ravichandran and H. M. Srivastava, Sharp coefficient bounds for starlike functions associated with the Bell numbers, Math. Slovaca 69 (2019), 1053–1064.
• H. M. Srivastava, B. Y. YaÅar and M. A. Özarslan, A class of big (p, q)-Appell polynomials and their associated difference equations, Filomat 33 (2019), 3085–3121.
• C.-L. Shiue, H.-H. Chiang, M.-M. Wong and H. M. Srivastava, Optimal t-pebbling in cycles, Utilitas Math. 111 (2019), 49–66.
• H. M. Srivastava, M. K. Aouf and A. O. Mostafa, Some properties of analytic functions

associated with fractional q-calculus operators, Miskolc Math. Notes 20 (2019), 1245–1260.
• H. M. Srivastava, H. Günerhan and B. Ghanbari, Exact traveling wave solutions for resonance nonlinear Schrdinger equation with intermodal dispersions and the Kerr law nonlinearity, Math. Methods Appl. Sci. 42 (2019), 7210–7221.
• A. Das, B. Hazarika, H. M. Srivastava, M. Rabbani and R. Arab, Solvability of infinite systems of nonlinear integral equations in two variables by using semi-analytic method, Filomat 33 (2019), 5377–5388.
• H. M. Srivastava, B. Khan, N. Khan, Q. Z. Ahmad and M. Tahir, A generalized conic domain and its applications to certain subclasses of analytic functions, Rocky Mountain J. Math. 49 (2019), 2325–2346.
• H. M. Srivastava, Q. Z. Ahmad, M. Darus, N. Khan, B. Khan, N. Zaman and H. H. Shah, Upper bound of the third Hankel determinant for a subclass of close-to-convex functions associated with the lemniscate of Bernoulli, Mathematics 7 (2019), Article ID 848, 1–10.
• H. M. Srivastava and F. A. Shah, AB-Wavelet frames in $L_2(\mathbb{R}^n)$, Filomat 33 (2019), 3587–3597.
• Z.-H. Zhang and H. M. Srivastava, Some characteristic properties of the weighted particular Schur polynomial mean, Math. Methods Appl. Sci. 42 (2019), 6459–6474.
• H. M. Srivastava, W. A. Khan, and H. Haroon, Some expansions for a class of generalized Humbert matrix polynomials, Rev. Real Acad. Cienc. Exactas Fís. Natur. Ser. A Mat. (RACSAM) 113 (2019), 3619–3634.
• H. M. Srivastava, G. Yasmin, A. Muhyi and S. Araci, Certain results for the twice-iterated 2D q-Appell polynomials, Symmetry 11 (2019), Article ID 1307, 1–23.
• H. M. Srivastava, S. Arjika and A. S. Kelil, Some homogeneous q-difference operators and the associated generalized Hahn polynomials, Appl. Set-Valued Anal. Optim. 1 (2019), 187–201.
• H. M. Srivastava, N. Khan, M. Darus, M. T. Rahim, Q. Z. Ahmad and Y. Zeb, Properties of spiral-like close-to-convex functions associated with conic domains, Mathematics 7 (2019), Article ID 706, 1–12.
• H. M. Srivastava, A. Tassaddiq, G. Rahman, K. S. Nisar and I. Khan, A new extension of the τ-Gauss hypergeometric function and its associated properties, Mathematics 7 (2019), Article ID 996, 1–9.
• H. M. Srivastava, N. Raza, E. S. A. AbuJarad, G. Srivastava and M. H. AbuJarad, Fekete-Szegö inequality for classes of (p,q)-starlike and (p,q)-convex functions, Rev. Real Acad. Cienc. Exactas Fís. Natur. Ser. A Mat. (RACSAM) 113 (2019), 3563–3584.
• H. M. Srivastava, G. Rahman and K. S. Nisar, Some extensions of the Pochhammer symbol and the associated hypergeometric functions, Iran. J. Sci. Technol. Trans. A: Sci. 43 (2019), 2601–2606.
• U. Kadak, H. M. Srivastava and M. Mursaleen, Relatively uniform weighted summability based on fractional-order difference operator, Bull. Malays. Math. Sci. Soc. 42 (2019), 2453–2480.
• H. M. Srivastava, R. S. Dubey and M. Jain, A study of the fractional-order mathematical model of diabetes and its resulting complications, Math. Methods Appl. Sci. 42 (2019), 4570–4583.
• H. M. Srivastava, A. Das, B. Hazarika and S. A. Mohiuddine, Existence of solution for non-linear functional integral equations of two variables in Banach algebra, Symmetry 11 (2019), Article ID 674, 1–16.
• H. M. Srivastava, M. Masjed-Jamei and M. R. Beyki, Some new generalizations and applications of the Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials, Rocky Mountain J. Math. 49 (2019), 681–697.

• H. M. Srivastava and A. K. Wanas, Strong differential sandwich results of λ-pseudo-starlike functions with respect to symmetrical points, Math. Moravica 23 (2) (2019), 45–58.

• H. M. Srivastava, Å. Altinkaya and S. Yalin, Certain subclasses of bi-univalent functions associated with the Horadam polynomials, Iran. J. Sci. Technol. Trans. A: Sci. 43 (2019), 1873–1879.

• K.-J. Chung, J.-J. Liao, S.-D. Lin, S.-T. Chuang and H. M. Srivastava, The inventory model for deteriorating items under conditions involving cash discount and trade credit, Mathematics 7 (2019), Article ID 596, 1–20.

• H. M. Srivastava, B. Kurt and V. Kurt, Identities and relations involving the modified degenerate Hermite-based Apostol-Bernoulli and Apostol-Euler polynomials, Rev. Real Acad. Cienc. Exactas Fís. Natur. Ser. A Mat. (RACSAM) 113 (2019), 1299–1313.

• H. M. Srivastava and A. K. Wanas, Initial Maclaurin coefficient bounds for new subclasses of analytic and m-fold symmetric bi-univalent functions defined by a linear combination, Kyungpook Math. J. 59 (2019), 493–503.

• G. Dattoli, S. Licciardi, E. Sabia and H. M. Srivastava, Some properties and generating functions of generalized harmonic numbers, Mathematics 7 (2019), Article ID 577, 1–12.

• S. Mahmood, H. M. Srivastava, M. Arif, F. Ghani and E. S. A. Abujarad, A criterion for subfamilies of multivalent functions of reciprocal order with respect to symmetric points, Fractal Fract. 3 (2019), Article ID 35, 1–9.

• H. M. Srivastava, K.-J. Chung, J.-J. Liao, S.-D. Lin and S.-T. Chuang, Some modified mathematical analytic derivations of the annual total relevant cost of the inventory model with two levels of trade credit in the supply chain system, Math. Methods Appl. Sci. 42 (2019), 3967–3977.

• H. M. Srivastava, M. Mursaleen and Md. Nasiruzzaman, Approximation by a class of q-beta operators of the second kind via the Dunkl-type generalization on weighted spaces, Complex Anal. Oper. Theory 13 (2019), 1537–1556.

• H. M. Srivastava, A. Fernandez and D. Baleanu, Some new fractional-calculus connections between Mittag-Leffler functions, Mathematics 7 (2019), Article ID 485, 1–10.

• C. M. A. Pinto, A. R. M. Carvalho, D. Baleanu and H. M. Srivastava, Efficacy of the post-exposure prophylaxis and of the HIV latent reservoir in HIV infection, Mathematics 7 (2019), Article ID 515, 1–17.

• H. M. Srivastava, A. O. Mostafa, M. K. Aouf and H. M. Zayed, Basic and fractional q-calculus and associated Fekete-Szegö problem for p-valently q-starlike functions and p-valently q-convex functions of complex order, Miskolc Math. Notes 20 (2019), 489–509.

• H. M. Srivastava, N. Tuglu and M. etin, Some results on the q-analogues of the incomplete Fibonacci and Lucas polynomials, Miskolc Math. Notes 20 (2019), 511–524.

• H. Singh and H. M. Srivastava, Jacobi collocation method for the approximate solution of some fractional-order Riccati differential equations with variable coefficients, Physica A: Statist. Mech. Appl. 523 (2019), 1130–1149.

• S. Mahmood, G. Srivastava, H. M. Srivastava, E. S. A. Abujarad, M. Arif and F. Ghani, Sufficiency criterion for a subfamily of meromorphic multivalent functions of reciprocal order with respect to symmetric points, Symmetry 11 (2019), Article ID 764, 1–7.

• S. Mahmood, N. Raza, E. S. A. Abujarad, G. Srivastava, H. M. Srivastava and S. N. Malik, Geometric properties of certain classes of analytic functions associated with a q-integral operator, Symmetry 11 (2019), Article ID 719, 1–14.

• B. Ahmad, M. Alghanmi, A. Alsaedi, H. M. Srivastava and S. K. Ntouyas, The Langevin equation in terms of generalized Liouville-Caputo derivatives with nonlocal boundary conditions involving a generalized fractional integral, Mathematics 7 (2019), Article ID 533, 1–10.
• C.-L. Shiue, H.-H. Chiang, M.-M. Wong and H. M. Srivastava, The optimal t-pebbling number of a certain complete m-ary tree, Rev. Real Acad. Cienc. Exactas Fís. Natur. Ser. A Mat. (RACSAM) 113 (2019), 2889–2910; see also Correction, Rev. Real Acad. Cienc. Exactas Fís. Natur. Ser. A Mat. (RACSAM) 113 (2019), 2911–2911.
• H. M. Srivastava, B. B. Jena and S. K. Paikray, Deferred Cesàro statistical probability convergence and its applications to approximation theorems, J. Nonlinear Convex Anal. 20 (2019), 1777–1792.
• H. M. Srivastava, M. A. Özarslan and B. Y. YaÅar, Difference equations for a class of twice-iterated Δ_h-Appell sequences of polynomials, Rev. Real Acad. Cienc. Exactas Fís. Natur. Ser. A Mat. (RACSAM) 113 (2019), 1851–1871.
• H. M. Srivastava, N. Jolly, M. K. Bansal and R. Jain, A new integral transform associated with the λ-extended Hurwitz-Lerch zeta function, Rev. Real Acad. Cienc. Exactas Fís. Natur. Ser. A Mat. (RACSAM) 113 (2019), 1679–1692.
• H. M. Srivastava, P. E. Ricci and P. Natalini, A family of complex Appell polynomial sets, Rev. Real Acad. Cienc. Exactas Fís. Natur. Ser. A Mat. (RACSAM) 113 (2019), 2359–2371.
• H. M. Srivastava, B. Khan, N. Khan and Q. Z. Ahmad, Coefficient inequalities for q-starlike functions associated with the Janowski functions, Hokkaido Math. J. 48 (2019), 407–425.
• H. M. Srivastava, K. Khatterwani and S. K. Upadhyay, A certain family of fractional wavelet transformations, Math. Methods Appl. Sci. 42 (2019), 3103–3122.
• L. Shi, H. M. Srivastava, M. Arif, S. Hussain and H. Khan, An investigation of the third Hankel determinant problem for certain subfamilies of univalent functions involving the exponential function, Symmetry 11 (2019), Article ID 598, 1–14.
• M. Masjed-Jamei, Z. Moalemi, W. Koepf and H. M. Srivastava, An extension of the Taylor series expansion by using the Bell polynomials, Rev. Real Acad. Cienc. Exactas Fís. Natur. Ser. A Mat. (RACSAM) 113 (2019), 1445–1461.
• H. Ö. Güney, G. Murugusundaramoorthy and H. M. Srivastava, The second Hankel determinant for a certain class of bi-close-to-convex functions, Results Math. 74 (2019), Article ID 93, 1–13.
• H. M. Srivastava, The zeta and related functions: Recent developments, J. Adv. Engrg. Comput. 3 (2019), 329–354.
• S. Mahmood, H. M. Srivastava, S. N. Malik, M. Raza, N. Shahzadi S. Zainab, A certain family of integral operators associated with the Struve functions, Symmetry 11 (2019), Article ID 463, 1–16.
• M. R. Ali, A. R. Hadhoud and H. M. Srivastava, Solution of fractional Volterra-Fredholm integro-differential equations under mixed boundary conditions by using the HOBW method, Adv. Difference Equations 2019 (2019), Article ID 115, 1–14.
• M. Arif, H. M. Srivastava and S. Umar, Some applications of a q-analogue of the Ruscheweyh type operator for multivalent functions, Rev. Real Acad. Cienc. Exactas Fís. Natur. Ser. A Mat. (RACSAM) 113 (2019), 1211–1221.
• I. Kucukoğlu, Y. Simsek and H. M. Srivastava, A new family of Lerch-type zeta functions interpolating a certain class of higher-order Apostol-type numbers and Apostol-type polynomials, Quaestiones Math. 42 (2019), 465–478.
• H. M. Srivastava, S. Araci, W. A. Khan and M. Acikgöz, A note on the truncated-exponential

based Apostol-type polynomials, Symmetry 11 (2019), Article ID 538, 1–20.
• S. Mahmood, J. Sokól, H. M. Srivastava and S. N. Malik, Some reciprocal classes of close-to-convex and quasi-convex analytic functions, Mathematics 7 (2019), Article ID 309, 1–13.
• H. M. Srivastava, Y. Vyas and K. Fatawat, Extensions of the classical theorems for very well-poised hypergeometric functions, Rev. Real Acad. Cienc. Exactas Fís. Natur. Ser. A Mat. (RACSAM) 113 (2019), 367–397.
• H. M. Srivastava, F. A. Shah and R. Abass, An application of the Gegenbauer wavelet method for the numerical solution of the fractional Bagley-Torvik equation, Russian J. Math. Phys. 26 (2019), 77–93.
• S. Mahmood, Q. Z. Ahmad, H. M. Srivastava, N. Khan, B. Khan and M. Tahir, A certain subclass of meromorphically q-starlike functions associated with the Janowski functions, J. Inequal. Appl. 2019 (2019), Article ID 88, 1–11.
• A. Fernandez, D. Baleanu, and H. M. Srivastava, Series representations for fractional-calculus operators involving generalised Mittag-Leffler functions, Commun. Nonlinear Sci. Numer. Simulat. 67 (2019), 517–527; see also Corrigendum, Commun. Nonlinear Sci. Numer. Simulat. 82 (2020), Article ID 104963, 1–1.
• H. M. Srivastava, Q. Z. Ahmad, N. Khan and B. Khan, Hankel and Toeplitz determinants for a subclass of q-starlike functions associated with a general conic domain, Mathematics 7 (2019), Article ID 181, 1–15.
• H. Singh, R. K. Pandey and H. M. Srivastava, Solving non-linear fractional variational problems using Jacobi polynomials, Mathematics 7 (2019), Article ID 224, 1–24.
• S. Mahmood, H. M. Srivastava and S. N. Malik, Some subclasses of uniformly univalent functions with respect to symmetric points, Symmetry 11 (2019), Article ID 287, 1–14.
• H. M. Srivastava and H. Günerhan, Analytical and approximate solutions of fractional-order susceptible-infected-recovered epidemic model of childhood disease, Math. Methods Appl. Sci. 42 (2019), 935–941.
• H. M. Srivastava, B. B. Jena, S. K. Paikray and U. K. Misra, Statistically and relatively modular deferred-weighted summability and Korovkin-type approximation theorems, Symmetry 11 (2019), Article ID 448, 1–20.
• A. Alsaedi, M. Alsulami, H. M. Srivastava, B. Ahmad and S. K. Ntouyas, Existence theory for nonlinear third-order ordinary differential equations with nonlocal multi-point and multi-strip boundary conditions, Symmetry 11 (2019), Article ID 281, 1–18.
• S. Mahmood, I. Khan, H. M. Srivastava and S. N. Malik, Inclusion relations for certain families of integral operators associated with conic regions, J. Inequal. Appl. 2019 (2019), Article ID 59, 1–11.
• H. M. Srivastava, M. Tahir, B. Khan, Q. Z. Ahmad and N. Khan, Some general classes of q-starlike functions associated with the Janowski functions, Symmetry 11 (2019), Article ID 292, 1–14.
• H. M. Srivastava, S. Khan and M. Riyasat, q-Difference equations for the 2-iterated q-Appell and mixed type q-Appell polynomials, Arabian J. Math. 8 (2019), 63–77.
• S. Mahmood, H. M. Srivastava, N. Khan, Q. Z. Ahmad, B. Khan and I. Ali, Upper bound of the third Hankel determinant for a subclass of q-starlike functions, Symmetry 11 (2019), Article ID 347, 1–13.
• H. M. Srivastava, F. Özger and S. A. Mohiuddine, Construction of Stancu-type Bernstein operators based on Bézier bases with shape parameter λ, Symmetry 11 (2019), Article ID 316,

1–22.
• L. Mahto, S. Abbas, M. Hafayed and H. M. Srivastava, Approximate controllability of sub-diffusion equation with impulsive condition, Mathematics 7 (2019), Article ID 190, 1–16.
• N. E. Cho, S. Kumar, V. Kumar, V. Ravichandran and H. M. Srivastava, Starlike functions related to the Bell numbers, Symmetry 11 (2019), Article ID 219, 1–17.
• J. N. Pandey, J. S. Maurya, S. K. Upadhyay and H. M. Srivastava, Continuous wavelet transform of Schwartz tempered distributions in $S'(\mathbb{R}^n)$, Symmetry 11 (2019), Article ID 235, 1–8.

Profile Details: Further details about Professor Srivastava's professional achievements and accomplishments [**as well as** the lists of his **most recent publications** including Refereed Journal Articles; Books, Monographs and Edited Volumes; Book Chapters; Encyclopedia Chapters; Papers in Conference Proceedings; Forewords to Books and Journals (including Preface, Editorial, *et cetera*)] can be found at the following Web Site:

URL: http://www.math.uvic.ca/~harimsri/
Telephone: 1-250-472-5313 (Office); 1-250-477-6960 (Home)
FAX: 1-250-721-8962
E-Mail: harimsri@math.uvic.ca

Acknowledgements
Our sincere thanks are due to Professor H.M. Srivastava and Professor Rekha Srivastava (Retd. Adjunct Professor, Department of Mathematics, University of Victoria, B.C., Canada), the wife of Professor H. M. Srivastava for their kind help to collect the bibliography and many other details presented here.

Bibliography
[1] R. C. S. Chandel, Professor H. M. Srivastava: An Amazing Mathematician, Jñānābha , **31/32** (2002), 1-24.
(Dedicated to Honor Professor H. M. Srivastava on his 62nd Birthday Celebrations)
[2] R. C. S. Chandel, Professor H. M. Srivastava: Man and Mathematician, Jñānābha , **45** (2015), 1-12.
(Dedicated to Honor Professor H. M. Srivastava on His Platinum Jubilee Celebrations)
[3] URL:http://www.math.uvic.ca/ harimsri
[4] *Clarivate Analytics [Thomson-Reuters] (Web of Science) Highly Cited Researcher*
[5] *h-index (Google Scholar): 88*
[6] *ORCID ID: https://orcid.org/0000-0002-9277-8092*
[7] *Research ID: http://www.researcherid.com/rid/N-9532-2013*
[8] *Scopus Author ID: 23152241800*
[9] ResearchGate: https://www.researchgate.net/profile/Hari_Srivastava
[10] Google Scholar: https://scholar.google.com/citations?user=cdRbfo8AAAAJ&hl=en
[11] *Facebook Entry by Springer Math*
[12] https://www.facebook.com/SpringerMath/posts/1433670693349498
[13] URL (with *Photograph*): http://www.math.uvic.ca/ harimsri
[14] http://www.springer.com/mathematics/numbers/book/978-1-4939-0257-6

Jñānābha, Vol. 50(1) (2020), 14-19
(Dedicated to Honor Professor H.M. Srivastava on His 80th Birth Anniversary Celebrations)

ENERGY OF SOME GRAPHS OF PRIME GRAPH OF A RING

By

Sandeep S. Joshi
Department of Mathematics,
D.N.C.V.P's Shirish Madhukarrao Chaudhari College,
Jalgaon - 425 001, Maharashtra, India
sandeep.s.joshi07@gmail.com
Kishor F. Pawar
Department of Mathematics, School of Mathematical Sciences,
Kavayitri Bahinabai Chaudhari North Maharashtra University,
Jalgaon - 425 001, Maharashtra, India
kfpawar@nmu.ac.in

(*Received : September* 10, 2019 ; *Revised: March* 04, 2020)
DOI: https://doi.org/10.58250/jnanabha.2020.50101

Abstract

Let R be a commutative ring and $PG(R)$ is a graph whose vertices are all the elements of ring R and two vertices are adjacent if their product is zero. In this article, we study the energy of 1-Quasitotal and 2-Quasitotal Prime Graph of a Ring $\mathbb{Z}_p$ and also find the energy of $PG_1(\mathbb{Z}_p)$ and $PG_2(\mathbb{Z}_p)$, p prime. A General SCILAB Software code for our calculation is also presented.

2010 Mathematics Subject Classifications: 05C25, 05C15, 13E15.
Keywords and phrases: Ring, Prime graph of a Ring $PG(R)$, Quasi-total graph, Energy.

1 Introduction

The study of graph theory for a commutative ring began when Beck in [3] introduced the notion of zero divisor of the graph. The graphs $\Gamma_1(R)$ and $\Gamma_2(R)$ are defined by R. Sen Gupta et al. in [4]. Another graph structure associated to a ring called prime graph was introduced by Satyanarayana et al. [2]. Prime graph is defined as a graph whose vertices are all elements of the ring and any two distinct vertices $x, y \in R$ are adjacent if and only if $xRy = 0$ or $yRx = 0$. This graph is denoted by $PG(R)$. Pawar and Joshi in [10] gave a simple formulation for finding the degrees of vertices of prime graph $PG(R)$ as well as it's complement $(PG(R))^c$. Also the number of triangles in $PG(R)$ and $(PG(R))^c$ have been calculated using simple combinatorial approach. We have introduced the prime graphs $PG_1(R)$ in [9] and $PG_2(R)$ in [8] of a ring and discussed all the results related to degree of vertices, Eulerianity, planarity and girth.

In third section of this paper we give definition and some examples of 1-Quasitotal and 2-Quasitotal Prime Graph of a Ring $\mathbb{Z}_n$. In last four sections we find the energy of 1-Quasitotal and 2-Quasitotal Prime Graph of a Ring $\mathbb{Z}_p$ and also find the energy of $PG_1(\mathbb{Z}_p)$ and $PG_2(\mathbb{Z}_p)$, where p is prime and give a general SCILAB software code for finding the energy of any Graph.

2 Preliminary Definitions

Here we are listing some preliminary definitions. For basic terminology and definitions the reader is referred to [2], [5].

***Definition* 2.1.** [4] *For a ring R, a simple undirected graph $G = (V, E)$ is said to be a graph $\Gamma_1(R)$ if all the nonzero elements of R as vertices, and two distinct vertices a and b are adjacent if and only if either $a \cdot b = 0$ or $b \cdot a = 0$ or $a + b$ is a unit.*

***Definition* 2.2.** [4] *For a ring R, a simple undirected graph $G = (V, E)$ is said to be a graph $\Gamma_2(R)$ if all the nonzero elements of R as vertices, and two distinct vertices a and b are adjacent if and only if either $a \cdot b = 0$ or $b \cdot a = 0$ or $a + b$ is a zero divisor (including zero).*

***Definition* 2.3.** [9] *The prime graph $PG_1(R)$ is a graph with all the elements of a ring R as vertices, and any two distinct vertices x, y are adjacent if and only if $x \cdot y = 0$ or $y \cdot x = 0$ or $x + y \in U(R)$, the set of all units of R.*

***Definition* 2.4.** [8] *The prime graph $PG_2(R)$ is a graph with all the elements of a ring R as vertices, and any two distinct vertices x, y are adjacent if and only if $x \cdot y = 0$ or $y \cdot x = 0$ or $x + y \in Z(R)$, the set of all zero divisors of R.*

***Definition* 2.5.** [6] *The Energy of the prime graph of a ring $PG(\mathbb{Z}_n)$ is defined as the sum of the absolute values of all the eigen values of its adjacency matrix $M(PG(R))$. i.e. if $\lambda_1, \lambda_2, ..., \lambda_n$ are n eigen values of $M(PG(R))$, then the energy of $PG(\mathbb{Z}_n)$ is -*

$$E(PG(R)) = \sum_{i=1}^{n} |\lambda_i| .$$

3 1-Quasitotal and 2-Quasitotal Prime graph of a Ring

From the definitions of satyanarayana Bhavanari and his co-authors in [1], we have define here Quasitotal graphs of prime graph of a ring.

***Definition* 3.1.** *Let $PG(R)$ be a prime graph of a ring with vertex set $V(PG(R))$ and edge set $E(PG(R))$. The 1-Quasitotal graph of prime graph of a ring, (denoted by $Q_1(PG(R))$) and is defined as follows:*

The vertex set of $Q_1(PG(R))$, that is $V(Q_1(PG(R))) = V(PG(R)) \bigcup E(PG(R))$. Two vertices a, b in $V(Q_1(PG(R)))$ are adjacent if they satisfy one of the following conditions:

1. *a, b are in $V(PG(R))$ and $ab \in E(PG(R))$*
2. *a, b are in $E(PG(R))$ and a, b are incident in $PG(R)$.*

***Example* 3.1.** *Consider $\mathbb{Z}_n$, the ring of integers modulo n.*

Let $R = \mathbb{Z}_3$. The vertex set $V(PG(R)) = \{0, 1, 2\}$. Since, $0R1 = 0, 0R2 = 0$ and edge set $E(PG(R)) = \{01, 02\}$. So, the vertex set $V(Q_1(PG(R))) = \{v_1, v_2, v_3, e_1, e_2\}$ and edge set $E(Q_1(PG(R))) = \{v_1v_2, v_1v_3, e_1e_2\}$ and the graph is as shown in figure below-

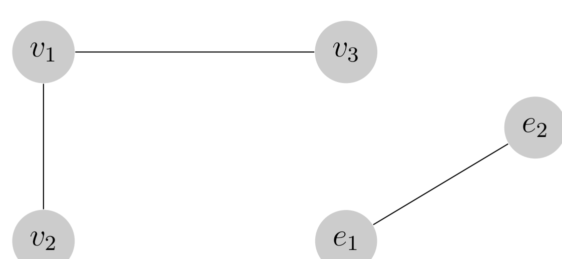


***Figure* 3.1:** $Q_1(PG(\mathbb{Z}_3))$

1. $Q_1(PG(R))$ is a graph without loops and multiple edges, i.e. the graph is simple.
2. The graph of $Q_1(PG(\mathbb{Z}_p))$, p prime, is a disconnected graph containing two components - the first component is itself $PG(\mathbb{Z}_p)$ and the other component is a complete graph K_{p-1} on $p-1$ vertices.

***Definition* 3.2.** *Let $PG(R)$ be a prime graph of a ring with vertex set $V(PG(R))$ and edge set $E(PG(R))$. The 2-Quasitotal graph of prime graph of a ring, (denoted by $Q_2(PG(R))$) and is defined as follows:*

The vertex set of $Q_2(PG(R))$, that is $V(Q_2(PG(R))) = V(PG(R)) \bigcup E(PG(R))$. Two vertices a, b in $V(Q_2(PG(R)))$ are adjacent in $Q_2(PG(R))$ in case one of the following holds:

1. *a, b are in $V(PG(R))$ and $ab \in E(PG(R))$*
2. *a is in $V(PG(R))$; b is in $E(PG(R))$; and a, b are incident in $PG(R)$.*

***Example* 3.2.** *Consider $\mathbb{Z}_n$, the ring of integers modulo n.*

Let $R = \mathbb{Z}_3$. So, the vertex set $V(Q_2(PG(R))) = \{v_1, v_2, v_3, e_1, e_2\}$ and edge set $E(Q_2(PG(R))) = \{v_1v_2, v_1v_3, v_1e_1, v_1e_2, v_2e_1, v_3e_2\}$ and the graph is as shown in figure below-

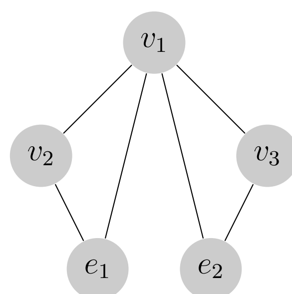


***Figure* 3.2:** $Q_2(PG(\mathbb{Z}_3))$

1. $Q_2(PG(R))$ is a simple graph, i.e without multiple edges and loops.
2. The graph of $Q_2(PG(\mathbb{Z}_p))$, p prime, is a connected graph containing $p-1$ number of triangles having the vertex zero is a common vertex.

4 Energy of $Q_1(PG(\mathbb{Z}_p))$

***Example* 4.1.** *For $p = 2$, the adjacency matrix of $Q_1(PG(\mathbb{Z}_2))$ is*

$$M(Q_1(PG(\mathbb{Z}_2))) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The eigen values are $-1, 0, 1$. Therefore, $E(Q_1(PG(\mathbb{Z}_2))) = 2$.

***Example* 4.2.** *For $p = 3$, the adjacency matrix of $Q_1(PG(\mathbb{Z}_3))$ is*

$$M(Q_1(PG(\mathbb{Z}_3))) = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Therefore, $E(Q_1(PG(\mathbb{Z}_3))) = 4.8284$.

From the SCILAB Software we found here some values of Energy of $Q_1(PG(\mathbb{Z}_p))$ given in

***Table* 4.1**

Sr.No.	n	Graph	Energy
1	2	$Q_1(PG(\mathbb{Z}_2))$	2
2	3	$Q_1(PG(\mathbb{Z}_3))$	4.8284
3	5	$Q_1(PG(\mathbb{Z}_5))$	10
4	7	$Q_1(PG(\mathbb{Z}_7))$	14.8989
5	11	$Q_1(PG(\mathbb{Z}_{11}))$	24.3245
6	13	$Q_1(PG(\mathbb{Z}_{13}))$	28.9282

.

As per the above discussion we conclude the following Theorem -

***Theorem* 4.1.** *If p is a prime number then energy of* $Q_1(PG(\mathbb{Z}_p))$ *is* $(2p-4)+2\sqrt{p-1}$.

5 Energy of $Q_2(PG(\mathbb{Z}_p))$

***Example* 5.1.** *For* $p = 2$*, the adjacency matrix of* $Q_2(PG(\mathbb{Z}_2))$ *is*

$$M(Q_2(PG(\mathbb{Z}_2))) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

The eigen values are $-1, -1, 2$. Therefore, $E(Q_2(PG(\mathbb{Z}_2))) = 4$.

***Example* 5.2.** *For* $p = 3$*, the adjacency matrix of* $Q_2(PG(\mathbb{Z}_3))$ *is*

$$M(Q_2(PG(\mathbb{Z}_3))) = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

Therefore, $E(Q_2(PG(\mathbb{Z}_3))) = 7.1231$.

From the SCILAB Software we found here some values of Energy of $Q_2(PG(\mathbb{Z}_p))$ given in the

***Table* 5.1**

Sr.No.	n	Graph	Energy
1	2	$Q_2(PG(\mathbb{Z}_2))$	4
2	3	$Q_2(PG(\mathbb{Z}_3))$	7.1231
3	5	$Q_2(PG(\mathbb{Z}_5))$	12.7445
4	7	$Q_2(PG(\mathbb{Z}_7))$	18
5	11	$Q_2(PG(\mathbb{Z}_{11}))$	28
6	13	$Q_2(PG(\mathbb{Z}_{13}))$	32.8488

.

As per the above discussion we conclude the following Theorem -

***Theorem* 5.1.** *If p is a prime number then energy of* $Q_2(PG(\mathbb{Z}_p))$ *is* $(2p-3)+\sqrt{7p+(p-7)}$.

6 Energy of $PG_1(\mathbb{Z}_p)$

***Example* 6.1.** *For $p = 2$, the adjacency matrix of $PG_1(\mathbb{Z}_2)$ is*

$$M(PG_1(\mathbb{Z}_2)) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

The eigen values are $-1, 1$. Therefore, $E(PG_1(\mathbb{Z}_2)) = 2$.

***Example* 6.2.** *For $p = 3$, the adjacency matrix of $PG_1(\mathbb{Z}_3)$ is*

$$M(PG_1(\mathbb{Z}_3)) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Therefore, $E(PG_1(\mathbb{Z}_3)) = 2.8284$.

From the SCILAB Software we found here some values of Energy of $PG_1(\mathbb{Z}_p)$ given in

***Table* 6.1**

Sr.No.	n	Graph	Energy
1	2	$PG_1(\mathbb{Z}_2)$	2
2	3	$PG_1(\mathbb{Z}_3)$	2.8284
3	5	$PG_1(\mathbb{Z}_5)$	6.4721
4	7	$PG_1(\mathbb{Z}_7)$	10.3245
5	11	$PG_1(\mathbb{Z}_{11})$	18.1980
6	13	$PG_1(\mathbb{Z}_{13})$	22.1655

.

As per the above discussion we conclude the following Theorem -

***Theorem* 6.1.** *If p is an odd prime number then energy of $PG_1(\mathbb{Z}_p)$ is $(p-3) + \sqrt{(p-1)^2 + 4}$.*

7 Energy of $PG_2(\mathbb{Z}_p)$

***Example* 7.1.** *For $p = 2$, the adjacency matrix of $PG_2(\mathbb{Z}_2)$ is*

$$M(PG_2(\mathbb{Z}_2)) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

The eigen values are $-1, 1$. Therefore, $E(PG_2(\mathbb{Z}_2)) = 2$.

***Example* 7.2.** *For $p = 3$, the adjacency matrix of $PG_2(\mathbb{Z}_3)$ is*

$$M(PG_2(\mathbb{Z}_3)) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

Therefore, $E(PG_2(\mathbb{Z}_3)) = 4$.

From the SCILAB Software we found here some values of Energy of $PG_2(\mathbb{Z}_p)$ given in the

***Table* 7.1**

Sr.No.	n	Graph	Energy
1	2	$PG_2(\mathbb{Z}_2)$	2
2	3	$PG_2(\mathbb{Z}_3)$	4
3	5	$PG_2(\mathbb{Z}_5)$	7.1231
4	7	$PG_2(\mathbb{Z}_7)$	10
5	11	$PG_2(\mathbb{Z}_{11})$	15.4031
6	13	$PG_2(\mathbb{Z}_{13})$	18

.

As per the above discussion we conclude the following ***Table* 7.1**.

***Theorem* 7.1.** *If p is an odd prime number then energy of $PG_2(\mathbb{Z}_p)$ is $(p-2)+\sqrt{3p+(p-3)}$.*

General Scilab software code to find Energy of a Graph:

(1) $A = [...;...;...;...]$: To create a matrix that has multiple rows, separate, the rows with semicolons.
(2) $poly(A, x)$: Gives the polynomial of matrix A in variable x.
(3) $spec(A)$: Gives the Eigen Values of matrix A.
(4) $abs(spec(A))$: Gives absolute values of Eigen values of matrix A.
(5) $sum(abs(spec(A)))$: Gives the Energy of a Graph.

Acknowledgments
The authors would like to express their sincere thanks to the Editors and referee of *Jñānābha* for their valuable suggestions to bring the paper in its present form and also to the parents and family for their inseparable support and prayers.

References

[1] S. Bhavanari, S. Devanaboina, S. P. Kuncham, Line Graphs and Quasi-total Graphs, *International Journal of Computer Applications*, **105**(3)(2014), 0975-8887.
[2] S. Bhavanari, S. P. Kuncham, Nagaraju Dasari, Prime Graph of a Ring, *J. of Combinatorics, Information & System Sciences*, **35**(1-2)(2010), 27-42.
[3] Beck I., Coloring of Commutative Rings, *Journal of Algebra*, **116**(1)(1988), 208-226.
[4] R. Sen Gupta, The Graph $\Gamma_2(R)$ over a ring R, *International Journal of Pure and Applied Mathematics*, **86**(6)(2013), 893-904.
[5] F. Harary, Graph Theory Third ed., *Addison-Wesley*, Reading, MA, 1972.
[6] Sandeep S. Joshi and Kishor F. Pawar, Energy, Wiener index and Line graph of Prime graph of a Ring, *International Journal of Mathematical Combinatorics*, **3**(2018), 74-80.
[7] S. S. Joshi and K. F. Pawar, Coloring of Prime Graph $PG_1(R)$ and $PG_2(R)$ of a Ring, *Palestine Journal of Mathematics*, **9**(1)(2020), 97-104.
[8] S. S. Joshi and K. F. Pawar, On Prime Graph $PG_2(R)$ of a Ring, *International Journal of Mathematical Combinatorics*, **3**(2019), 11-22.
[9] Kishor F. Pawar and Sandeep S. Joshi, The Prime Graph $PG_1(R)$ of a ring, *Palestine Journal of Mathematics*, **6**(1)(2017), 153-158.
[10] Kishor F. Pawar and Sandeep Joshi, Study of Prime Graph of a Ring, *Thai Journal of Mathematics*, **17**(2)(2019), 369-377.

Jñānābha, Vol. 50(1) (2020), 20-26
(Dedicated to Honor Professor H.M. Srivastava on His 80th Birth Anniversary Celebrations)

SOME FIXED POINT THEOREMS FOR WEAK CONTRACTION MAPPINGS IN S-METRIC SPACES

By

G. S. Saluja
Department of Mathematics
Government Kaktiya Postgraduate College Jagdalpur,
Jagdalpur - 494001 (Chhattisgarh), India.
Email:saluja1963@gmail.com

(*Received : September* 30, 2019 ; *Revised: April* 26, 2020)
DOI: https://doi.org/10.58250/jnanabha.2020.50102

Abstract

In this article, we establish some fixed point theorems for weak contraction mappings in the setting of complete S-metric spaces. Our results extend, generalize and unify several results from the existing literature regarding S-metric space.

2010 Mathematics Subject Classifications: 54H25.
Keywords and phrases: Fixed point, weak contraction, S-metric space.

1 Introduction and Preliminaries

Fixed point theory is one of the famous and traditional theories in mathematics and has a broad set of applications. In this theory, contraction is one of the main tools to prove the existence and uniqueness of a fixed point. Banach's contraction principle which gives an answer to the existence and uniqueness of a solution of an operator equation $Tx = x$, is the most widely used fixed point theorem in all of analysis. This principle is constructive in nature and is one of the most useful techniques in the study of nonlinear equations. The statement of the Banach contraction principle is as follows.

***Theorem* 1.1.** *(Banach Contraction Principle) Let (X, d) be a complete metric space, $c \in [0, 1)$ and $f : X \to X$ a mapping such that for each $x, y \in X$,*

$$d(f(x), f(y)) \le c\, d(x, y). \tag{1.1}$$

Then f has a unique fixed point $p \in X$, such that for each $x \in X$, $\lim_{n\to\infty} f^n x = p$. Inequality (1.1) implies the continuity of f.

The Banach contraction principle has been generalized in many ways over the years. In some generalizations, the contractive notion of the map is weakened, see [2, 3, 4, 5, 6, 7, 10, 11] and others.

There are many generalizations of the Banach contraction principle for different metric spaces that exist in the literature of metric fixed point theory.

Recently, Sedghi et al. [8] introduced the notion of S-metric space which is a generalization of a G-metric space and D^*-metric space. In [8] the authors proved some properties of S-metric spaces. Also, they obtained some fixed point theorems in S-metric space for a self-map.

The definition and properties of S-metric spaces are as follows (see, [8]).

***Definition* 1.1.** ([8]), *A S-metric on a non-empty set X is a function $S : X^3 \to \mathbb{R}^+$ satisfying the following conditions:*

(SM_1) $S(x,y,z) = 0$ *if and only if* $x = y = z$*;*

(SM_2) $S(x,y,z) \leq S(x,x,t) + S(y,y,t) + S(z,z,t)$*;*

for all x, y, z, $t \in X$, where $\mathbb{R}^+ = [0,\infty)$. Then the function S is called an S-metric on X and the pair (X,S) is called an S-metric space or simply SMS.

***Example* 1.1.** ([8]), *Let $X = \mathbb{R}^n$ and $\|.\|$ a norm on X, then $S(x,y,z) = \|y + z - 2x\| + \|y - z\|$ is an S-metric on X.*

***Example* 1.2.** ([8]), *Let $X = \mathbb{R}^n$ and $\|.\|$ a norm on X, then $S(x,y,z) = \|x - z\| + \|y - z\|$ is an S-metric on X.*

***Example* 1.3.** ([9]), *Let $X = \mathbb{R}$ be the real line. Then $S(x,y,z) = |x - z| + |y - z|$ for all $x,y,z \in \mathbb{R}$ is an S-metric on X. This S-metric on X is called the usual S-metric on X.*

***Lemma* 1.1.** ([8], ***Lemma* 2.5**) *If (X,S) is an S-metric space, then we have $S(x,x,y) = S(y,y,x)$ for all $x,y \in X$.*

***Lemma* 1.2.** ([8], ***Lemma* 2.12)** *Let (X,S) be an S-metric space. If $\{x_n\}$ and $\{y_n\}$ are sequences in X converging to x and y respectively, that is, $x_n \to x$ and $y_n \to y$ as $n \to \infty$, then $S(x_n,x_n,y_n) \to S(x,x,y)$ as $n \to \infty$.*

***Lemma* 1.3.** ([8], ***Lemma* 2.10**) *Let (X,S) be an S-metric space. If the sequence $\{x_n\}$ in X converges to x, then the limit x is unique.*

***Lemma* 1.4.** ([8], ***Lemma* 2.11**) *Let (X,S) be an S-metric space. If the sequence $\{x_n\}$ in X converges to x, then $\{x_n\}$ is a Cauchy sequence.*

***Definition* 1.2.** ([8]) *Let (X,S) be an S-metric space.*

(1) *A sequence $\{x_n\}$ in X converges to $x \in X$ if $S(x_n,x_n,x) \to 0$ as $n \to \infty$; that is, for each $\varepsilon > 0$, there exists an $m_0 \in \mathbb{N}$ such that for all $n \geq m_0$ we have $S(x_n,x_n,x) < \varepsilon$. We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.*

(2) *A sequence $\{x_n\}$ in X is called a Cauchy sequence if $S(x_n,x_n,x_m) \to 0$ as $n,m \to \infty$; that is, for each $\varepsilon > 0$, there exists an $m_0 \in \mathbb{N}$ such that for all $n,m \geq m_0$ we have $S(x_n,x_n,x_m) < \varepsilon$.*

(3) *The S-metric space (X,S) is called complete if every Cauchy sequence in (X,S) is convergent in (X,S).*

***Definition* 1.3.** *Let T be a self mapping on an S-metric space (X,S). Then T is said to be continuous at $x \in X$ if for any sequence $\{x_n\}$ in X with $x_n \to x$ implies that $Tx_n \to Tx$ as $n \to \infty$.*

***Definition* 1.4.** ([8]) *Let (X,S) be an S-metric space. A mapping $T : X \to X$ is said to be a contraction if there exists a constant $0 \leq L < 1$ such that*

$$S(Tx,Tx,Ty) \leq L\,S(x,x,y), \tag{1.2}$$

for all $x,y \in X$. If the S-metric space (X,S) is complete then the mapping defined as above has a unique fixed point.

***Definition* 1.5.** ([1]) *(Weak Contraction Mapping) Let (X, d) be a complete metric space. A mapping $T: X \to X$ is said to be weakly contractive if*

$$d(T(x), T(y)) \leq d(x, y) - \psi(d(x, y)), \tag{1.3}$$

where $x, y \in X$, $\psi: [0, \infty) \to [0, \infty)$ is continuous and non-decreasing, $\psi(x) = 0$ if and only if $x = 0$ and $\lim_{x\to\infty} \psi(x) = \infty$.

If we take $\psi(x) = cx$ where $0 < c < 1$ then (1.3) reduces to (1.1).

Now, we introduce the notion of weak contraction in S-metric space as follows.

***Definition* 1.6.** *Let (X, S) be an S-metric space. A mapping $T: X \to X$ is said to be a weak contraction on X if there exists a function $\psi: [0, \infty) \to [0, \infty)$ with $\psi(t) = 0$ if and only if $t = 0$ and satisfying the following condition*

$$S(Tx, Tx, Ty) \leq S(x, x, y) - \delta\,\psi(S(x, x, y)), \tag{1.4}$$

for all $x, y \in X$, where $0 \leq \delta < 1$.

If we take $\psi(x) = x$ and $\delta = L$, then (1.4) reduces to (1.2).

***Example* 1.4.** *Let $X = \mathbb{R}$ and defined $S: X^3 \to \mathbb{R}^+$ by*

$$S(x, x, y) = \begin{cases} 4x^2 + y^2 & \text{if } x \neq y, \\ 0 & \text{if } x = y, \end{cases}$$

for all $x, y \in X$. Then S is an S-metric on X and (X, S) is a S-metric space. Let $T: X \to X$ defined by $T(x) = \frac{x}{4}$ and $\psi(t) = 15t$ for all $t \geq 0$, where $\psi: [0, \infty) \to [0, \infty)$ is continuous and non-decreasing function. Then

$$\begin{aligned} S(T(x), T(x), T(y)) &= S\left(\frac{x}{4}, \frac{x}{4}, \frac{y}{4}\right) = \frac{x^2}{4} + \frac{y^2}{16} \\ &= 4x^2 + y^2 - \frac{15}{16}(4x^2 + y^2) \\ &= S(x, x, y) - \frac{1}{16}\psi(S(x, x, y)). \end{aligned}$$

Thus T is a weak contraction on X.

The purpose of this paper is to prove some fixed point theorems under a weak contraction condition in the setting of S-metric spaces. Our results extend, generalize and improve several results from the existing literature in S-metric spaces.

2 Main Results

In this section, we shall prove some fixed point theorem in a complete S-metric space for weak contraction mapping.

***Theorem* 2.1.** *Let (X, S) be a complete S-metric space. Let $T: X \to X$ be a mapping satisfying the condition:*

$$\begin{aligned} S(Tx, Tx, Ty) \leq{}& \min\{S(x, x, Tx), S(y, y, Ty)\} \\ &- h\,\phi\Big(\max\{S(x, x, Ty), S(y, y, Tx)\}\Big), \end{aligned} \tag{2.1}$$

for all $x, y \in X$ where $h > 0$ and $\phi: [0, \infty) \to [0, \infty)$ is a continuous function with $\phi(t) = 0$ if and only if $t = 0$. Then T has a unique fixed point in X.

Proof. Let $x_0 \in X$ and $\{x_n\}$ be a sequence defined by $x_{n+1} = Tx_n$ for $n = 0, 1, 2, \ldots$. If $x_n = x_{n+1} = Tx_n$, then x_n is a fixed point of T. So we assume that $x_n \neq x_{n+1}$. It follows from (2.1), (SM_2) and ***Lemma* 1.1** that

$$\begin{aligned} S(x_n, x_n, x_{n+1}) &= S(Tx_{n-1}, Tx_{n-1}, Tx_n) \\ &\leq \min\{S(x_{n-1}, x_{n-1}, Tx_{n-1}), S(x_n, x_n, Tx_n)\} \\ &\quad -h\,\phi\Big(\max\{S(x_{n-1}, x_{n-1}, Tx_n), S(x_n, x_n, Tx_{n-1})\}\Big) \\ &= \min\{S(x_{n-1}, x_{n-1}, x_n), S(x_n, x_n, x_{n+1})\} \\ &\quad -h\,\phi\Big(\max\{S(x_{n-1}, x_{n-1}, x_{n+1}), S(x_n, x_n, x_n)\}\Big) \\ &= \min\{S(x_{n-1}, x_{n-1}, x_n), S(x_n, x_n, x_{n+1})\} \\ &\quad -h\,\phi\Big(\max\{S(x_{n-1}, x_{n-1}, x_{n+1}), 0\}\Big) \\ &= S(x_{n-1}, x_{n-1}, x_n) - h\,\phi\Big(S(x_{n-1}, x_{n-1}, x_{n+1})\Big). \end{aligned} \tag{2.2}$$

Since $\phi \geq 0$, then we obtain from equation (2.2) that

$$S(x_n, x_n, x_{n+1}) \leq S(x_{n-1}, x_{n-1}, x_n). \tag{2.3}$$

Thus we have a non-negative and non-increasing sequence $\{S(x_n, x_n, x_{n+1})\}$. Therefore, there exists an $L \geq 0$ such that

$$\lim_{n\to\infty} S(x_n, x_n, x_{n+1}) = L. \tag{2.4}$$

Since ϕ is continuous on $[0, \infty)$, using (2.2), (SM_2), *Lemma* 1.1 and taking the limit as $n \to \infty$, we obtain

$$\begin{aligned} L &\leq L - h \lim_{n\to\infty} \phi\Big(2S(x_{n-1}, x_{n-1}, x_n) + S(x_n, x_n, x_{n+1})\Big) \\ &= L - h\,\phi\Big(\lim_{n\to\infty} 2S(x_{n-1}, x_{n-1}, x_n) + \lim_{n\to\infty} S(x_n, x_n, x_{n+1})\Big) \\ &= L - h\,\phi(3L). \end{aligned} \tag{2.5}$$

Since $h > 0$ and $\phi(3L) \geq 0$, then equation (2.5) is possible only if $\phi(3L) = 0$. Thus, we get $L = 0$. Hence, we obtain that

$$\lim_{n\to\infty} S(x_n, x_n, x_{n+1}) = L = 0. \tag{2.6}$$

This proves that $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, then there exists an element $z \in X$ such that $\lim_{n\to\infty} x_n = z$, that is, $x_n \to z$ as $n \to \infty$. To show that z is a fixed point of T. Using (2.1), we have

$$\begin{aligned} S(x_{n+1}, x_{n+1}, Tz) &= S(Tx_n, Tx_n, Tz) \\ &\leq \min\{S(x_n, x_n, Tx_n), S(z, z, Tz)\} \\ &\quad -h\,\phi\Big(\max\{S(x_n, x_n, Tz), S(z, z, Tx_n)\}\Big) \\ &= \min\{S(x_n, x_n, x_{n+1}), S(z, z, Tz)\} \\ &\quad -h\,\phi\Big(\max\{S(x_n, x_n, Tz), S(z, z, x_{n+1})\}\Big). \end{aligned} \tag{2.7}$$

Since $\lim_{n\to\infty} x_n = z$ and $\lim_{n\to\infty} S(x_n, x_n, x_{n+1}) = 0$, then from equation (2.7) and taking the limit as $n \to \infty$, we get

$$\begin{aligned} S(z, z, Tz) &\leq \min\{S(z.z.Tz), S(z, z, Tz)\} \\ &\quad -h\,\phi\Big(\max\{S(z, z, Tz), S(z, z, z)\}\Big) \\ &= \min\{0, S(z, z, Tz)\} \end{aligned} \tag{2.8}$$

$$-h\,\phi\Big(\max\{S(z,z,Tz),0\}\Big)$$
$$= \quad 0 - h\,\phi\Big(S(z,z,Tz)\Big).$$

The above inequality (2.8) is possible only if $S(z,z,Tz) = 0$. Thus $z = Tz$. This shows that z is a fixed point of T. To prove the uniqueness of fixed point of T, assume that v be another fixed point of T such that $v = Tv$ with $v \neq z$. Using (2.1) and ***Lemma* 1.1**, we have

$$\begin{aligned}
S(z,z,v) &= S(Tz,Tz,Tv) \\
&\leq \min\{S(z.z.Tz), S(v,v,Tv)\} \\
&\quad -h\,\phi\Big(\max\{S(z,z,Tv), S(v,v,Tz)\}\Big) \\
&= \min\{S(z,z,z), S(v,v,v)\} \\
&\quad -h\,\phi\Big(\max\{S(z,z,v), S(v,v,z)\}\Big) \\
&= \min\{0,0\} \\
&\quad -h\,\phi\Big(\max\{S(z,z,v), S(z,z,v)\}\Big) \\
&= 0 - h\,\phi\Big(S(z,z,v)\Big).
\end{aligned} \tag{2.9}$$

Inequality (2.9) is possible only if $S(z,z,v) = 0$. Hence $z = v$. This shows that the fixed point of T is unique. This completes the proof.

***Theorem* 2.2.** *Let (X,S) be a complete S-metric space. Let $T\colon X \to X$ be a continuous mapping satisfying the condition:*

$$\Psi\Big(S(Tx,Tx,Ty), S(Ty,Ty,Tx)\Big) \leq q\,\Psi\Big(S(x,x,y), S(y,y,x)\Big), \tag{2.10}$$

for all $x, y \in X$, where $0 < q < 1$ and $\Psi\colon [0,\infty)^2 \to [0,\infty)^2$ is a continuous function on $[0,\infty)^2$ with $\Psi(a,b) = 0$ if and only if $a = 0 = b$. Then T has a unique fixed point in X.

Proof. Let $x_0 \in X$ and $\{x_n\}$ be a sequence defined by $x_{n+1} = Tx_n$ for $n = 0,1,2,\ldots$. If $x_n = x_{n+1} = Tx_n$, then x_n is a fixed point of T. So, we assume that $x_n \neq x_{n+1}$. It follows from (2.10), (SM_2) and ***Lemma* 1.1** that

$$\begin{aligned}
&\Psi\Big(S(x_{n+1},x_{n+1},x_{n+2}), S(x_{n+2},x_{n+2},x_{n+1})\Big) \\
&= \Psi\Big(S(Tx_n,Tx_n,Tx_{n+1}), S(Tx_{n+1},Tx_{n+1},Tx_n)\Big) \\
&\leq q\,\Psi\Big(S(x_n,x_n,x_{n+1}), S(x_{n+1},x_{n+1},x_n)\Big) \\
&\ \ \vdots \\
&\leq q^{n+1}\,\Psi\Big(S(x_0,x_0,x_1), S(x_1,x_1,x_0)\Big).
\end{aligned} \tag{2.11}$$

Since $0 < q < 1$ and, for $n \to \infty$, we get

$$\Psi\Big(S(x_{n+1},x_{n+1},x_{n+2}), S(x_{n+2},x_{n+2},x_{n+1})\Big) \to 0. \tag{2.12}$$

Since Ψ is a continuous function,

$$\begin{aligned}
0 &= \lim_{n\to\infty} \Psi\Big(S(x_{n+1},x_{n+1},x_{n+2}), S(x_{n+2},x_{n+2},x_{n+1})\Big) \\
&= \Psi\Big(\lim_{n\to\infty} S(x_{n+1},x_{n+1},x_{n+2}), \lim_{n\to\infty} S(x_{n+2},x_{n+2},x_{n+1})\Big).
\end{aligned}$$

Thus, by the property of Ψ,

$$\lim_{n\to\infty} S(x_{n+1},x_{n+1},x_{n+2}) = \lim_{n\to\infty} S(x_{n+2},x_{n+2},x_{n+1}) = 0.$$

Hence by ***Lemma* 1.1** and Definition 1.2(2), $\{x_n\}$ is a Cauchy sequence in a complete S-metric space X. Therefore there exists an $u \in X$ such that $\lim_{n\to\infty} x_n = u$. Since T is continuous, we have

$$Tu = T(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_{n+1} = u,$$

and u is a fixed point of T. Now, we shall show that the uniqueness u is unique. For this, suppose that v is another fixed point of the mapping T such that $v = Tv$ with $v \neq u$. Using equation (2.10), we have

$$\begin{aligned} \Psi\Big(S(u,u,v), S(v,v,u)\Big) &= \Psi\Big(S(Tu,Tu,Tv), S(Tv,Tv,Tu)\Big) \\ &\leq q\Psi\Big(S(u,u,v), S(v,v,u)\Big). \end{aligned}$$

Since $0 < q < 1$, then we get $\Psi\Big(S(u,u,v), S(v,v,u)\Big) = 0$, which by the property of Ψ, implies that $S(u,u,v) = S(v,v,u) = 0$. Thus we obtain $u = v$. This shows that the fixed point of T is unique. This completes the proof.

If we take $\Psi(x,y) = x + y$, $q = L$ and using ***Lemma* 1.1** in ***Theorem* 2.2**, then we have the following result as corollary.

***Corollary* 2.1.** *Let (X,S) be a complete S-metric space. Suppose that $T\colon X \to X$ be a mapping satisfying the condition:*

$$S(Tx,Tx,Ty) \leq L\,S(x,x,y),$$

for all $x, y \in X$ and $0 < L < 1$ is a constant. Then T has a unique fixed point in X.

***Remark* 2.1. *Corollary* 2.1** *extends the well known Banach contraction principle from complete metric space to that in the setting of a complete S-metric space considered in this paper.*

***Example* 2.1.** *Let $X = \mathbb{R}$ be the real line and S be the usual S-metric on X defined as $S(x,y,z) = |x-z| + |y-z|$ for all $x, y, z \in \mathbb{R}$. Then (X,S) is called an S-metric space. Consider the mapping $T\colon X \to X$ defined by $T(x) = \frac{x}{7}$ for all $x \in [0,1]$. Then, we have*

$$\begin{aligned} S(Tx,Tx,Ty) &= S\Big(\frac{x}{7}, \frac{x}{7}, \frac{y}{7}\Big) \\ &= \Big|\frac{x}{7} - \frac{y}{7}\Big| + \Big|\frac{x}{7} - \frac{y}{7}\Big| \\ &= 2\Big|\frac{x}{7} - \frac{y}{7}\Big| \\ &= \frac{2}{7}|x-y| \\ &\leq |x-y| \\ &= \frac{1}{2}\Big(2|x-y|\Big) \\ &= L\,S(x,x,y), \end{aligned}$$

where $L = \frac{1}{2} < 1$. Thus T satisfies all the conditions of ***Corollary* 2.1**. *Hence, by applying* ***Corollary* 2.1**, *T has a unique fixed point in X. It is seen that $0 \in X$ is the unique fixed point of T.*

3 Conclusion

In this article, we have established some unique fixed point theorems under a weak contractive condition in the framework of complete S-metric spaces. Our results extend, generalize and unify some recent results from the existing literature.

4 Acknowledgement

The author would like to thank the anonymous referee and Editor for their careful reading and valuable suggestions to improve the manuscript.

References

[1] Ya. I. Alber and S. Guerre-Delabrieer, Principles of weakly contractive maps in Hilbert spaces, new results in operator theory, Advances and Appl. (ed. by I. Gohberg and Yu Lyubich), *Birkhauser Verlag*, Basel 98 (1997), 7-22.

[2] D. W. Boyd and J. S. W. Wong, On nonlinear contractions, *Proc. Amer. Math. Soc.,* **20** (1969), 458-464.

[3] L. B. Ciric, A generalization of Banach's contraction principle, *Proc. Amer. Math. Soc.,* **45(2)** (1974), 267-273.

[4] W. A. Kirk, Fixed points of asymptotic contractions, *J. Math. Anal. Appl.,* **277** (2003), 645-650.

[5] A. Meir and E. Keeler, A theorem on contraction mappings, *J. Math. Anal. Appl.,* **28** (1969), 326-329.

[6] E. Rakotch, A note on contractive mappings, *Proc. Amer. Math. Soc.,* **13** (1962), 459-465.

[7] S. Reich, Fixed points of contractive functions, *Boll. Unione Mat. Ital.,* **5** (1972), 26-42.

[8] S. Sedghi, N. Shobe and A. Aliouche, A generalization of fixed point theorems in S-metric spaces, *Mat. Vesnik,* **64(3)** (2012), 258-266.

[9] S. Sedghi and N. V. Dung, Fixed point theorems on S-metric spaces, *Mat. Vesnik,* **66(1)** (2014), 113-124.

[10] T. Suzuki, Fixed point theorem for asymptotic contractionsof Meir-Keeler type in complete metric spaces, *Nonlinear Anal.,* **64** (2006), 971-978.

[11] D. Wordowski, Fixed points of a new type of contractive mappings in complete metric spaces, *Fixed Point Theory Appl.,* **2012** (2012), Article ID 94.

Jñānābha, Vol. 50(1) (2020), 27-34
(Dedicated to Honor Professor H.M. Srivastava on His 80th Birth Anniversary Celebrations)

CERTAIN LAURENT TYPE LINEAR AND BILATERAL GENERATING RELATIONS

By
Mohammad Idris Qureshi, Mahvish Ali⋆ and Dilshad Ahamad
Department of Applied Sciences and Humanities
Faculty of Engineering and Technology
Jamia Millia Islamia (A Central University)
New Delhi-110025, India
Email:miqureshi_delhi@yahoo.co.in, mahvishali37@gmail.com,
dlshdhmd4@gmail.com
(*Received : October* 31, 2019 *; Revised: January* 23, 2020)
DOI: https://doi.org/10.58250/jnanabha.2020.50103

Abstract

In this article, certain Laurent type linear and bilateral hypergeometric generating relations are derived by using series rearrangement technique, summation theorems of Pfaff-Saalchütz, Chu-Vandermonde and some reduction formulas.
2010 Mathematics Subject Classifications: 33B15, 33C10, 33C20.
Keywords and phrases: Hypergeometric functions; Series rearrangement technique; Chu-Vandermonde theorem; Pfaff-Saalschütz theorem.

1 Introduction and preliminaries

The generalized hypergeometric series is defined as:

$$(1.1)\qquad {}_pF_q\left[\begin{matrix}(\alpha_p);\\ \\ (\beta_q);\end{matrix}\quad z\right] = {}_pF_q\left[\begin{matrix}\alpha_1,\alpha_2,\ldots,\alpha_p;\\ \\ \beta_1,\beta_2,\ldots,\beta_q;\end{matrix}\quad z\right] = \sum_{n=0}^{\infty}\frac{(\alpha_1)_n(\alpha_2)_n\ldots(\alpha_p)_n}{(\beta_1)_n(\beta_2)_n\ldots(\beta_q)_n}\frac{z^n}{n!}.$$

We recall here the definition of a more general double hypergeometric function (than the one defined by Kampé de Fériet) in a slightly modified notation [7, p.423, Eq.(26)]:

$$(1.2)\qquad F^{p:\,q;\,k}_{\ell:\,m;\,n}\left[\begin{matrix}(a_p)\ :\ (b_q)\ ;\ (c_k)\ ;\\ \\ (\alpha_\ell)\ :\ (\beta_m)\ ;\ (\gamma_n)\ ;\end{matrix}\quad x,\ y\right] = \sum_{r,s=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_{r+s}\prod_{j=1}^{q}(b_j)_r\prod_{j=1}^{k}(c_j)_s}{\prod_{j=1}^{\ell}(\alpha_j)_{r+s}\prod_{j=1}^{m}(\beta_j)_r\prod_{j=1}^{n}(\gamma_j)_s}\frac{x^r}{r!}\frac{y^s}{s!},$$

where, for convergence,

$$(1.3)\qquad (i)\quad p+q<\ell+m+1,\quad p+k<\ell+n+1,\quad |x|<\infty,\quad |y|<\infty,\ \text{or}$$

$$(1.4)\qquad (ii)\quad p+q=\ell+m+1,\quad p+k=\ell+n+1\ \text{and}$$

$$(1.5)\qquad \begin{cases}|x|^{1/(p-\ell)}+|y|^{1/(p-\ell)}<1, & \text{if } p>\ell\\ \max\{|x|,|y|\}<1, & \text{if } p\le\ell.\end{cases}$$

Series rearrangement technique is based upon certain interchanges of the order of a double (or multiple) summation. Several hypergeometric generating relations have been established using series rearrangement technique.

Here, we consider some well known results.

Cauchy's double series identity [6, p.100]:

$$\sum_{m,n=0}^{\infty} \Phi(m,n) = \sum_{m=0}^{\infty}\sum_{n=0}^{m} \Phi(m-n,n), \tag{1.6}$$

provided that the associated double series are absolutely convergent.

Chu-Vandermonde theorem [2, p.69, Q.No. 4]:

$$ {}_2F_1\left[\begin{array}{ccc} -N,\ G & ; & \\ & & 1 \\ H & ; & \end{array}\right] = \frac{(H-G)_N}{(H)_N}; \quad N = 0,1,2,\cdots, \tag{1.7}$$

such that ratio of Pochhammer symbols in r.h.s. is well defined and $H, G \in \mathbb{C}\setminus\mathbb{Z}_0^-$.

Pfaff-Saalschütz theorem [2, p.87, Theorem 29]:

If n is a non-negative integer, then

$$ {}_3F_2\left[\begin{array}{ccc} -n,\ a,\ b & ; & \\ & & 1 \\ c,\ a+b-c-n+1 & ; & \end{array}\right] = \frac{(c-a)_n\,(c-b)_n}{(c)_n\,(c-a-b)_n}, \tag{1.8}$$

such that ratios of Pochhammer symbols in r.h.s. are well defined and $a, b, c, 1+a+b-c-n \in \mathbb{C}\setminus\mathbb{Z}_0^-$.

Srivastava's multiple series identity [5, p.4, Eqn(12)]:

$$\sum_{m=0}^{\infty} f(m)\frac{(x_1+x_2+\cdots+x_n)^m}{m!} = \sum_{m_1,m_2,\cdots,m_n=0}^{\infty} f(m_1+m_2+\cdots+m_n)\frac{{x_1}^{m_1}}{m_1!}\frac{{x_2}^{m_2}}{m_2!}\cdots\frac{{x_n}^{m_n}}{m_n!}, \tag{1.9}$$

provided that the multiple series involved are absolutely convergent.

Motivated by the work on generating functions and generating relations recorded in beautiful monographs of Rainville [2, Chapter 8], Srivastava-Manocha [6] and recent investigations including [3, 1, 4], in this article, we derive certain Laurent type generating relations.

The paper is organized as: In Section 2, some auxiliary results are derived by using series rearrangement technique which are used in our main results. In Section 3, some hypergeometric generating relations are established with the help of the auxiliary results obtained in Section 2.

2 Some reduction formulae

Here, we prove the following auxiliary results:

***Lemma* 2.1.** *The following result holds true:*

$$F^{1:1;0}_{1:1;0}\left[\begin{array}{ccccccc} A & : & C & ; & — & ; & \\ & & & & & & X,\ -X \\ B & : & D & ; & — & ; & \end{array}\right] = {}_2F_2\left[\begin{array}{ccc} A,\ D-C & ; & \\ & & -X \\ B,\ D & ; & \end{array}\right], \tag{2.1}$$

where $B, D \in \mathbb{C}\setminus\mathbb{Z}_0^-$ and for all finite values of X.

***Lemma* 2.2.** *The following result holds true:*

$$F^{0:1;2}_{1:0;1}\left[\begin{array}{ccccccc} — & : & G & ; & H,\ J & ; & \\ & & & & & & X,\ X \\ E & : & — & ; & K & ; & \end{array}\right] = \sum_{m=0}^{\infty}\frac{(G)_m X^m}{(E)_m\, m!}\, {}_3F_2\left[\begin{array}{ccc} -m,\ H,\ J & ; & \\ & & 1 \\ K,\ 1-G-m & ; & \end{array}\right], \tag{2.2}$$

where $E, K \in \mathbb{C}\setminus\mathbb{Z}_0^-$ and for all finite values of X.

The ${}_3F_2(1)$ in the r.h.s. of equation (2.2) can be summed with the help of hypergeometric summation theorems of Dixon, Whipple, Watson and Pfaff-Saalschütz and other theorems for terminating Clausen series.

***Lemma* 2.3.** *The following result holds true:*

$$(2.3)\qquad F^{0:1;2}_{1:0;1}\left[\begin{array}{ccccc} — & : & -\nu+n & ; & 1+\alpha,\ -p-n\ ; \\ & & & & & X,\ X \\ 1+\alpha-\nu+n & : & — & ; & 1-\nu+\alpha-p\ ; \end{array}\right]$$

$$= {}_1F_1\left[\begin{array}{cc} -\nu-p\ ; & \\ & X \\ 1-\nu+\alpha-p\ ; & \end{array}\right],$$

where $1+\alpha-\nu+n,\ 1-\nu+\alpha-p \in \mathbb{C}\setminus\mathbb{Z}_0^-$ *and for all finite values of* X.

Proof of Lemma 2.1:

Suppose the power series form of l.h.s. of equation (2.1) is denoted by Π. Then, we have

$$(2.4)\qquad \Pi = \sum_{m,n=0}^{\infty} \frac{(A)_{m+n}(C)_m X^m(-X)^n}{(B)_{m+n}(D)_m\ m!\ n!}.$$

Replacing m by $m-n$ in equation (2.4) and using Cauchy's double series identity (1.6), we get

$$(2.5)\qquad \Pi = \sum_{m=0}^{\infty} \frac{(A)_m(C)_m X^m}{(B)_m(D)_m\ m!} \sum_{n=0}^{m} \frac{(1-D-m)_n(-m)_n}{(1-C-m)_n\ n!}$$

$$= \sum_{m=0}^{\infty} \frac{(A)_m(C)_m X^m}{(B)_m(D)_m\ m!}\ {}_2F_1\left[\begin{array}{cc} -m,\ 1-D-m\ ; & \\ & 1 \\ 1-C-m\ ; & \end{array}\right].$$

Now, applying Chu-Vandermonde summation ***Theorem*** **1.7** in equation (2.5), we get

$$\Pi = \sum_{m=0}^{\infty} \frac{(A)_m(C)_m X^m}{(B)_m(D)_m\ m!} \frac{(D-C)_m}{(1-C-m)_m}.$$

Simplifying above equation, we get equation (2.1).

Proof of Lemma 2.2:

Suppose the power series form of l.h.s. of equation (2.2) is denoted by Φ. Then, we have

$$(2.6)\qquad \Phi = \sum_{m,n=0}^{\infty} \frac{(G)_m(H)_n(J)_n X^{m+n}}{(E)_{m+n}(K)_n\ m!\ n!}$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{m} \frac{(G)_{m-n}(H)_n(J)_n X^m(-m)_n}{(E)_m(K)_n\ m!(-1)^n\ n!},\ \text{(on replacing } m \text{ by } m-n)$$

$$= \sum_{m=0}^{\infty} \frac{(G)_m\ X^m}{(E)_m\ m!} \sum_{n=0}^{m} \frac{(G+m)_{-n}(H)_n(J)_n(-m)_n}{(K)_n(-1)^n\ n!}.$$

Simplifying equation (2.6), we get equation (2.2).

Proof of Lemma 2.3:

If we choose $G=-\nu+n$, $E=1+\alpha-\nu+n$, $H=1+\alpha$, $J=-p-n$, $K=1-\nu+\alpha-p$ in ***Lemma*** **2.2**, we find

$$(2.7)\qquad F^{0:1;2}_{1:0;1}\left[\begin{array}{ccccc} — & : & -\nu+n & ; & 1+\alpha,\ -p-n\ ; \\ & & & & & X,\ X \\ 1+\alpha-\nu+n & : & — & ; & 1-\nu+\alpha-p\ ; \end{array}\right]$$

$$= \sum_{m=0}^{\infty} \frac{(-\nu+n)_m X^m}{(1+\alpha-\nu+n)_m\ m!}\ {}_3F_2\left[\begin{array}{cc} -m,\ 1+\alpha,\ -p-n\ ; & \\ & 1 \\ 1-\nu+\alpha-p,\ 1+\nu-n-m\ ; & \end{array}\right].$$

On using Pfaff-Saalchütz summation ***Theorem*** **1.8** in equation (2.7) and applying some algebraic properties of Pochhammer symbols, we obtain ***Lemma*** **2.3**.

3 Laurent type generating relations

In this section, we prove the following Laurent type hypergeometric generating relations:

***Theorem* 3.1.** *The following Laurent type bilateral generating relation for hypergeometric functions ${}_2F_2(-ab)$ or ${}_1F_1(x)$ holds true for all finite values of a, b, x, t:*

$$\exp(axt)(1-at)^{\alpha-\nu}{}_1F_1\left[\begin{array}{ll} -\nu & ; \\ & x-ab-axt+\frac{b}{t} \\ 1+\alpha-\nu & ; \end{array}\right] \tag{3.1}$$

$$= \sum_{p=-\infty}^{\infty} \frac{(\nu-\alpha)_p(-\nu)_{p^\star}(-ab)^{p^\star}}{(p+p^\star)!(p^\star)!}\, {}_2F_2\left[\begin{array}{ll} -\nu+p^\star,\ 1 & ; \\ & -ab \\ 1+p+p^\star,\ 1+p^\star & ; \end{array}\right]\times$$

$$\times {}_1F_1\left[\begin{array}{ll} -\nu-p & ; \\ & x \\ 1-\nu+\alpha-p & ; \end{array}\right](at)^p, \quad 0<|at|<1; \quad t\neq 0,$$

where

$$p^\star = \max\{0,-p\} = \begin{cases} -p, & \text{when } p=\cdots,-3,-2,-1 \\ 0, & \text{when } p=0,1,2,\cdots \end{cases} \tag{3.2}$$

and numerator, denominator parameters are neither zero nor negative integers in each hypergeometric function.

***Theorem* 3.2.** *The following Laurent type linear generating relation for the hypergeometric function ${}_3F_2\left(\frac{a}{b}\right)$ holds true:*

$$\left(a+\frac{c}{t}\right)^{\alpha-\gamma}(bt+d)^{-\alpha}\,{}_2F_1\left[\begin{array}{ll} \alpha,\ \lambda & ; \\ & \frac{at+c}{bt+d} \\ \mu & ; \end{array}\right] \tag{3.3}$$

$$= \frac{a^{\alpha-\gamma}}{d^\alpha}\sum_{p=-\infty}^{\infty}(\alpha)_p\left(\frac{-b}{d}\right)^p\sum_{\ell=0}^{\infty}\frac{(\gamma-\alpha)_{\ell+p^\star}(\alpha+p)_{\ell+p^\star}\left(\frac{bc}{ad}\right)^{\ell+p^\star}}{(\ell+p^\star)!\,(\ell+p+p^\star)!}\times$$

$$\times {}_3F_2\left[\begin{array}{ll} \lambda,\ 1-\gamma+\alpha,\ -\ell-p-p^\star & ; \\ & \frac{a}{b} \\ \mu,\ 1-\gamma+\alpha-\ell-p^\star & ; \end{array}\right]t^p$$

$$\left(0<\left|\frac{c}{at}\right|<1, \quad 0<\left|\frac{bt}{d}\right|<1, \quad 0<\left|\frac{at+c}{bt+d}\right|<1; \quad t\neq 0\right),$$

where $p^\star$ is defined by equation (3.2) *and numerator, denominator parameters are neither zero nor negative integers in each hypergeometric function.*

Proof of Theorem 3.1

Suppose the power series form of l.h.s. of equation (3.1) is denoted by Ω. Then, we have

$$\Omega = \exp(axt)\,{}_1F_0\left[\begin{array}{ll} \nu-\alpha & ; \\ & at \\ \text{——} & ; \end{array}\right]\sum_{N=0}^{\infty}\frac{(-\nu)_N\left(x-ab-axt+\frac{b}{t}\right)^N}{(1+\alpha-\nu)_N\,N!}, \tag{3.4}$$

where $\nu-\alpha, -\nu, 1+\alpha-\nu \in \mathbb{C}\setminus\mathbb{Z}_0^-$.

Using Srivastava's multiple series identity (1.9) in the r.h.s. of equation (3.4), we obtain

$$(3.5)\qquad \Omega = \sum_{\ell=0}^{\infty}\frac{(axt)^{\ell}}{\ell!}\sum_{m=0}^{\infty}\frac{(\nu-\alpha)_m(at)^m}{m!}\sum_{r,s,q,n=0}^{\infty}\frac{(-\nu)_{r+s+q+n}\ x^r(-ab)^s(-axt)^q\left(\frac{b}{t}\right)^n}{(1+\alpha-\nu)_{r+s+q+n}\ r!\ s!\ q!\ n!}$$

$$= \sum_{m=0}^{\infty}\sum_{r,s,n=0}^{\infty}\frac{(\nu-\alpha)_m(at)^m(-\nu)_{r+s+n}\ x^r(-ab)^s\left(\frac{b}{t}\right)^n}{m!(1+\alpha-\nu)_{r+s+n}\ r!\ s!\ n!}\times$$

$$\times\sum_{\ell=0}^{\infty}\sum_{q=0}^{\infty}\frac{(-\nu+r+s+n)_q(-1)^q(axt)^{\ell+q}}{(1+\alpha-\nu+r+s+n)_q\ \ell!\ q!}.$$

On replacing ℓ by $\ell-q$ in equation (3.5), it follows that

$$(3.6)\qquad \Omega = \sum_{m,r,s,n=0}^{\infty}\frac{(\nu-\alpha)_m(-\nu)_{r+s+n}(at)^m\ x^r(-ab)^s\left(\frac{b}{t}\right)^n}{(1+\alpha-\nu)_{r+s+n}\ m!\ r!\ s!\ n!}\sum_{\ell=0}^{\infty}\frac{(axt)^{\ell}}{\ell!}\times$$

$$\times{}_2F_1\left[\begin{array}{lc} -\ell,\ -\nu+r+s+n\ ; & \\ & 1 \\ 1+\alpha-\nu+r+s+n\ ; & \end{array}\right].$$

Now, applying Chu-Vandermonde summation ***Theorem* 1.7** in equation (3.6), we get

$$(3.7)\qquad \Omega = \sum_{m,r,s,n=0}^{\infty}\frac{(\nu-\alpha)_m(-\nu)_{r+s+n}(at)^m\ x^r(-ab)^s\left(\frac{b}{t}\right)^n}{(1+\alpha-\nu)_{r+s+n}\ m!\ r!\ s!\ n!}$$

$$\times\sum_{\ell=0}^{\infty}\frac{(axt)^{\ell}}{\ell!}\frac{(1+\alpha)_{\ell}}{(1+\alpha-\nu+r+s+n)_{\ell}}$$

$$= \sum_{m,r,s,n,\ell=0}^{\infty}\frac{(\nu-\alpha)_m(-\nu)_{r+s+n}(1+\alpha)_{\ell}}{(1+\alpha-\nu)_{r+s+n+\ell}}\frac{(-1)^s a^{m+s+\ell}b^{s+n}x^{r+\ell}t^{m-n+\ell}}{m!\ r!\ s!\ n!\ l!}.$$

Further, putting $m-n+\ell=p$, equation (3.7) becomes

$$(3.8)\qquad \Omega = \sum_{p=-\infty}^{\infty}\sum_{r,s,n,\ell=0}^{\infty}\frac{(\nu-\alpha)_{p+n-\ell}(-\nu)_{r+s+n}(1+\alpha)_{\ell}}{(1+\alpha-\nu)_{r+s+n+\ell}}\frac{(-1)^s a^{p+n+s}b^{s+n}x^{r+\ell}t^p}{(p+n-\ell)!\ r!\ s!\ n!\ \ell!}$$

$$= \sum_{p=-\infty}^{\infty}\sum_{r,\ell=0}^{\infty}\frac{(\nu-\alpha)_{p-\ell}(-\nu)_r(1+\alpha)_{\ell}\ x^r x^{\ell}}{(1+\alpha-\nu)_{r+\ell}\ (1)_{p-\ell}\ r!\ \ell!}\times$$

$$\times\sum_{s,n=0}^{\infty}\frac{(\nu-\alpha+p-\ell)_n(-\nu+r)_{s+n}(-ab)^s(ab)^n}{(1+\alpha-\nu+r+\ell)_{s+n}(1+p-\ell)_n\ s!\ n!}(at)^p$$

$$= \sum_{p=-\infty}^{\infty}\sum_{r,\ell=0}^{\infty}\frac{(\nu-\alpha)_{p-\ell}(-\nu)_r(1+\alpha)_{\ell}\ x^r x^{\ell}}{(1+\alpha-\nu)_{r+\ell}\ (1)_{p-\ell}\ r!\ \ell!}\times$$

$$\times F^{1:1;0}_{1:1;0}\left[\begin{array}{ccccc} -\nu+r & : & \nu-\alpha+p-\ell\ ; & —\ ; & \\ & & & & ab,\ -ab \\ 1+\alpha-\nu+r+\ell & : & 1+p-\ell\ ; & —\ ; & \end{array}\right](at)^p.$$

Now, applying ***Lemma* 2.1** in equation (3.8), we obtain

$$(3.9)\qquad \Omega = \sum_{p=-\infty}^{\infty}\sum_{r,\ell=0}^{\infty}\frac{(\nu-\alpha)_{p-\ell}(-\nu)_r(1+\alpha)_{\ell}\ x^r x^{\ell}}{(1+\alpha-\nu)_{r+\ell}\ (1)_{p-\ell}\ r!\ \ell!}\times$$

$$\times {}_2F_2\left[\begin{array}{ccc} -\nu+r,\ 1-\nu+\alpha & ; & \\ & & -ab \\ 1+\alpha-\nu+r+\ell,\ 1+p-\ell & ; & \end{array}\right](at)^p$$

$$= \sum_{p=-\infty}^{\infty} \sum_{r,\ell,n=0}^{\infty} \frac{(\nu-\alpha)_{p-\ell}(-\nu)_{r+n}(1+\alpha)_\ell\ (1-\nu+\alpha)_n\ x^r x^\ell (-ab)^n}{(1+\alpha-\nu)_{r+\ell+n}\ (1)_{p-\ell+n}\ r!\ \ell!\ n!}(at)^p$$

$$= \sum_{p=-\infty}^{\infty} \sum_{n=0}^{\infty} \frac{(\nu-\alpha)_p(-\nu)_n(-ab)^n(at)^p}{(1)_{p+n}\ n!} \times$$

$$\times \sum_{r,\ell=0}^{\infty} \frac{(-\nu+n)_r(1+\alpha)_\ell\ (-p-n)_\ell\ x^r x^\ell}{(1+\alpha-\nu+n)_{r+\ell}\ (1-\nu+\alpha-p)_\ell\ r!\ \ell!}$$

$$= \sum_{p=-\infty}^{\infty} \sum_{n=0}^{\infty} \frac{(\nu-\alpha)_p(-\nu)_n(-ab)^n(at)^p}{(1)_{p+n}\ n!} \times$$

$$\times F^{0:1;2}_{1:0;1}\left[\begin{array}{ccccc} \text{——} & : & -\nu+n & ; & 1+\alpha,\ -p-n\ ; \\ & & & & & x,\ x \\ 1+\alpha-\nu+n & : & \text{——} & ; & 1-\nu+\alpha-p\ ; \end{array}\right].$$

Further, using ***Lemma* 2.3** in equation (3.9), we find

$$\text{(3.10)} \qquad \Omega = \sum_{p=-\infty}^{\infty} (\nu-\alpha)_p \sum_{n=0}^{\infty} \frac{(-\nu)_n(-ab)^n}{(n+p)!\ n!}\, {}_1F_1\left[\begin{array}{cc} -\nu-p\ ; & \\ & x \\ 1-\nu+\alpha-p\ ; & \end{array}\right](at)^p$$

$$= \sum_{p=-\infty}^{-1} (\nu-\alpha)_p \sum_{n=0}^{\infty} \frac{(-\nu)_n(-ab)^n}{(n+p)!)\ n!}\, {}_1F_1\left[\begin{array}{cc} -\nu-p\ ; & \\ & x \\ 1-\nu+\alpha-p\ ; & \end{array}\right](at)^p +$$

$$+ \sum_{p=0}^{\infty} (\nu-\alpha)_p \sum_{n=0}^{\infty} \frac{(-\nu)_n(-ab)^n}{(n+p)!\ n!}\, {}_1F_1\left[\begin{array}{cc} -\nu-p\ ; & \\ & x \\ 1-\nu+\alpha-p\ ; & \end{array}\right](at)^p$$

$$= \sum_{p=1}^{\infty} (\nu-\alpha)_{-p} \sum_{n=p}^{\infty} \frac{(-\nu)_n(-ab)^n}{(n-p)!\ n!}\, {}_1F_1\left[\begin{array}{cc} -\nu+p\ ; & \\ & x \\ 1-\nu+\alpha+p\ ; & \end{array}\right](at)^{-p}$$

$$+ \sum_{p=0}^{\infty} (\nu-\alpha)_p \sum_{n=0}^{\infty} \frac{(-\nu)_n(-ab)^n}{(n+p)!\ n!}\, {}_1F_1\left[\begin{array}{cc} -\nu-p\ ; & \\ & x \\ 1-\nu+\alpha-p\ ; & \end{array}\right](at)^p$$

$$= \sum_{p=-\infty}^{-1} (\nu-\alpha)_p \sum_{n=-p}^{\infty} \frac{(-\nu)_n(-ab)^n}{(n+p)!\ n!}\, {}_1F_1\left[\begin{array}{cc} -\nu-p\ ; & \\ & x \\ 1-\nu+\alpha-p\ ; & \end{array}\right](at)^p$$

$$+ \sum_{p=0}^{\infty} (\nu-\alpha)_p \sum_{n=0}^{\infty} \frac{(-\nu)_n(-ab)^n}{(n+p)!\ n!}\, {}_1F_1\left[\begin{array}{cc} -\nu-p\ ; & \\ & x \\ 1-\nu+\alpha-p\ ; & \end{array}\right](at)^p$$

$$= \sum_{p=-\infty}^{\infty} (\nu-\alpha)_p \sum_{n=p^\star}^{\infty} \frac{(-\nu)_n(-ab)^n}{(n+p)!\ n!}\, {}_1F_1\left[\begin{array}{cc} -\nu-p\ ; & \\ & x \\ 1-\nu+\alpha-p\ ; & \end{array}\right](at)^p,$$

where $p^\star$ is defined by equation (3.2).

Replacing n by $n+p^\star$ in equation (3.10), we get

$$(3.11)\qquad \Omega = \sum_{p=-\infty}^{\infty}(\nu-\alpha)_p\sum_{n=0}^{\infty}\frac{(-\nu)_{n+p^\star}(-ab)^{n+p^\star}}{(n+p+p^\star)!\,(n+p^\star)!}\,{}_1F_1\left[\begin{array}{ccc}-\nu-p & ; & \\ & & x\\ 1-\nu+\alpha-p & ; & \end{array}\right](at)^p$$

$$= \sum_{p=-\infty}^{\infty}\frac{(\nu-\alpha)_p(-\nu)_{p^\star}(-ab)^{p^\star}}{(p+p^\star)!\,(p^\star)!}\sum_{n=0}^{\infty}\frac{(-\nu+p^\star)_n(1)_n(-ab)^n}{(1+p+p^\star)_n(1+p^\star)_n\,n!}\times$$

$$\times{}_1F_1\left[\begin{array}{ccc}-\nu-p & ; & \\ & & x\\ 1-\nu+\alpha-p & ; & \end{array}\right](at)^p.$$

Using definition of ${}_2F_2$ in the r.h.s. of equation (3.11), assertion (3.1) follows.

Proof of Theorem 3.2

Suppose the l.h.s of equation (3.3) is denoted by Δ. Then we have

$$(3.12)\qquad \Delta = \frac{a^{\alpha-\gamma}}{d^\alpha}\sum_{k=0}^{\infty}\frac{(\alpha)_k\,(\lambda)_k\,(at)^k}{(\mu)_k\,d^k\,k!}\left(1+\frac{c}{at}\right)^{k+\alpha-\gamma}\left(1+\frac{bt}{d}\right)^{-(k+\alpha)}$$

$$= \frac{a^{\alpha-\gamma}}{d^\alpha}\sum_{k=0}^{\infty}\frac{(\alpha)_k\,(\lambda)_k\,(at)^k}{(\mu)_k\,d^k\,k!}\,{}_1F_0\left[\begin{array}{ccc}\gamma-\alpha-k & ; & \\ & & -\frac{c}{at}\\ \text{---} & ; & \end{array}\right]{}_1F_0\left[\begin{array}{ccc}\alpha+k & ; & \\ & & -\frac{bt}{d}\\ \text{---} & ; & \end{array}\right]$$

$$= \frac{a^{\alpha-\gamma}}{d^\alpha}\sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty}\sum_{m=0}^{\infty}\frac{(\alpha)_k\,(\lambda)_k\,(at)^k}{(\mu)_k\,d^k\,k!}\,\frac{(\gamma-\alpha-k)_\ell}{\ell!}\left(\frac{-c}{at}\right)^\ell\frac{(\alpha+k)_m}{m!}\left(\frac{-bt}{d}\right)^m$$

$$= \frac{a^{\alpha-\gamma}}{d^\alpha}\sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty}\sum_{m=0}^{\infty}\frac{(\alpha)_{k+m}(\lambda)_k\left(\frac{a}{d}\right)^k(\gamma-\alpha-k)_\ell\left(\frac{-c}{a}\right)^\ell\left(\frac{-b}{d}\right)^m}{(\mu)_k\,k!\,\ell!\,m!}t^{k-\ell+m}.$$

Replacing m by $m-k$, we get

(3.13)

$$\Delta = \frac{a^{\alpha-\gamma}}{d^\alpha}\sum_{\ell=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{m}\frac{(\alpha)_m(\lambda)_k\left(\frac{a}{d}\right)^k(\gamma-\alpha-k)_\ell\left(\frac{-c}{a}\right)^\ell\left(\frac{-b}{d}\right)^{m-k}}{(\mu)_k\,k!\,\ell!\,(m-k)!}\,t^{m-\ell}$$

$$= \frac{a^{\alpha-\gamma}}{d^\alpha}\sum_{\ell=0}^{\infty}\frac{(\gamma-\alpha)_\ell\left(\frac{-c}{a}\right)^\ell}{\ell!}\sum_{m=0}^{\infty}\frac{(\alpha)_m\left(\frac{-b}{d}\right)^m}{m!}\sum_{k=0}^{m}(\lambda)_k\frac{(\gamma-\alpha+\ell)_{-k}}{(\gamma-\alpha)_{-k}}\,\frac{\left(\frac{-b}{d}\right)^{-k}\left(\frac{a}{d}\right)^k}{(\mu)_k\,k!\,(1+m)_{-k}}\,t^{m-\ell}$$

$$= \frac{a^{\alpha-\gamma}}{d^\alpha}\sum_{\ell,m=0}^{\infty}\frac{(\gamma-\alpha)_\ell(\alpha)_m\left(\frac{-c}{a}\right)^\ell\left(\frac{-b}{d}\right)^m}{\ell!\,m!}\sum_{k=0}^{m}\frac{(\lambda)_k\,(1-\gamma+\alpha)_k\,(-m)_k}{(\mu)_k\,(1-\gamma+\alpha-\ell)_k}\frac{\left(\frac{a}{b}\right)^k}{k!}t^{m-\ell}$$

$$= \frac{a^{\alpha-\gamma}}{d^\alpha}\sum_{\ell,m=0}^{\infty}\frac{(\gamma-\alpha)_\ell(\alpha)_m\left(\frac{-c}{a}\right)^\ell\left(\frac{-b}{d}\right)^m}{\ell!\,m!}\,{}_3F_2\left[\begin{array}{ccc}\lambda,\ 1-\gamma+\alpha,\ -m & ; & \\ & & \frac{a}{b}\\ \mu,\ 1-\gamma+\alpha-\ell & ; & \end{array}\right]t^{m-\ell}.$$

Now, putting $m-\ell=p$ or $m=p+\ell$, we get

(3.14)

$$\Delta = \frac{a^{\alpha-\gamma}}{d^\alpha}\sum_{p=-\infty}^{\infty}\sum_{\ell=0}^{\infty}\frac{(\gamma-\alpha)_\ell(\alpha)_{\ell+p}\left(\frac{-c}{a}\right)^\ell\left(\frac{-b}{d}\right)^{\ell+p}}{\ell!\,(\ell+p)!}\,{}_3F_2\left[\begin{array}{ccc}\lambda,\ 1-\gamma+\alpha,\ -\ell-p & ; & \\ & & \frac{a}{b}\\ \mu,\ 1-\gamma+\alpha-\ell & ; & \end{array}\right]t^p$$

$$= \frac{a^{\alpha-\gamma}}{d^{\alpha}} \sum_{p=-\infty}^{\infty} (\alpha)_p \left(\frac{-b}{d}\right)^p \sum_{\ell=0}^{\infty} \frac{(\gamma-\alpha)_\ell(\alpha+p)_\ell \left(\frac{bc}{ad}\right)^\ell}{\ell!\,(\ell+p)!} {}_3F_2\left[\begin{array}{ll} \lambda,\ 1-\gamma+\alpha,\ -\ell-p & ; \\ & \frac{a}{b} \\ \mu,\ 1-\gamma+\alpha-\ell & ; \end{array}\right] t^p$$

$$= \frac{a^{\alpha-\gamma}}{d^{\alpha}} \left\{ \sum_{p=-\infty}^{-1} (\alpha)_p \left(\frac{-b}{d}\right)^p \sum_{\ell=0}^{\infty} \frac{(\gamma-\alpha)_\ell(\alpha+p)_\ell \left(\frac{bc}{ad}\right)^\ell}{\ell!\,(\ell+p)!} {}_3F_2\left[\begin{array}{ll} \lambda,\ 1-\gamma+\alpha,\ -\ell-p & ; \\ & \frac{a}{b} \\ \mu,\ 1-\gamma+\alpha-\ell & ; \end{array}\right] t^p + \right.$$

$$\left. + \sum_{p=0}^{\infty} (\alpha)_p \left(\frac{-b}{d}\right)^p \sum_{\ell=0}^{\infty} \frac{(\gamma-\alpha)_\ell(\alpha+p)_\ell \left(\frac{bc}{ad}\right)^\ell}{\ell!\,(\ell+p)!} {}_3F_2\left[\begin{array}{ll} \lambda,\ 1-\gamma+\alpha,\ -\ell-p & ; \\ & \frac{a}{b} \\ \mu,\ 1-\gamma+\alpha-\ell & ; \end{array}\right] t^p \right\}$$

$$= \frac{a^{\alpha-\gamma}}{d^{\alpha}} \left\{ \sum_{p=1}^{\infty} (\alpha)_{-p} \left(\frac{-b}{d}\right)^{-p} \sum_{\ell=p}^{\infty} \frac{(\gamma-\alpha)_\ell(\alpha-p)_\ell \left(\frac{bc}{ad}\right)^\ell}{\ell!\,(\ell-p)!} {}_3F_2\left[\begin{array}{ll} \lambda,\ 1-\gamma+\alpha,\ -\ell+p & ; \\ & \frac{a}{b} \\ \mu,\ 1-\gamma+\alpha-\ell & ; \end{array}\right] t^{-p} + \right.$$

$$\left. + \sum_{p=0}^{\infty} (\alpha)_p \left(\frac{-b}{d}\right)^p \sum_{\ell=0}^{\infty} \frac{(\gamma-\alpha)_\ell(\alpha+p)_\ell \left(\frac{bc}{ad}\right)^\ell}{\ell!\,(\ell+p)!} {}_3F_2\left[\begin{array}{ll} \lambda,\ 1-\gamma+\alpha,\ -\ell-p & ; \\ & \frac{a}{b} \\ \mu,\ 1-\gamma+\alpha-\ell & ; \end{array}\right] t^p \right\}$$

$$= \frac{a^{\alpha-\gamma}}{d^{\alpha}} \sum_{p=-\infty}^{\infty} (\alpha)_p \left(\frac{-b}{d}\right)^p \sum_{\ell=p^\star}^{\infty} \frac{(\gamma-\alpha)_\ell(\alpha+p)_\ell \left(\frac{bc}{ad}\right)^\ell}{\ell!\,(\ell+p)!} {}_3F_2\left[\begin{array}{ll} \lambda,\ 1-\gamma+\alpha,\ -\ell-p & ; \\ & \frac{a}{b} \\ \mu,\ 1-\gamma+\alpha-\ell & ; \end{array}\right] t^p,$$

where $p^\star = \max\{0, -p\}$.

On replacing ℓ by $\ell + p^\star$ in equation (3.14), we obtain assertion (3.3).

Several other bilinear (multilinear) and bilateral (multilateral) hypergeometric generating relations may also be derived by using series rearrangement technique.

Acknowledgements: This work has been sponsored by Dr. D. S. Kothari Post Doctoral Fellowship (Award letter No. F.4-2/2006(BSR)/MA/17-18/0025) awarded to **Dr. Mahvish Ali** by the University Grants Commission, Government of India, New Delhi.

The authors are also thankful to the referee for his valuable suggestions.

References

[1] K. Bhandari, General Class of Generating Functions and its Application, *Jñānābha*, **47**(1) (2017), 103-106.

[2] E. D. Rainville, *Special Functions*, Reprint of 1960 first edition. Chelsea Publishig Co., Bronx, New York, 1971.

[3] P. E. Ricci, P. Natalini and G. Bretti, Sheffer and Brenke Polynomials Associated with Generalized Bell Numbers, *Jñānābha* , **47**(2) (2017), 337-352.

[4] Y. Simsek, Generating Functions for Finite Sums Involving Higher Powers of Binomial Coefficients: Analysis of Hypergeometric Functions Including New Families of Polynomials and Numbers, *J. Math. Anal. Appl.*, **477**(2) (2019), 1328-1352.

[5] H. M. Srivastava; Certain Double Integrals Involving Hypergeometric Functions , *Jñānābha Sect. A*, **1** (1971), 1-10.

[6] H. M. Srivastava and H. L. Manocha, *A Treatise on Generating Functions*, Halsted Press, New York, 1984.

[7] H. M. Srivastava and R. Panda; An Integral Representation for the Product of Two Jacobi Polynomials, *J. London Math. Soc.*, **12**(2) (1976), 419-425.

Jñānābha, Vol. 50(1) (2020), 35-42
(Dedicated to Honor Professor H.M. Srivastava on His 80th Birth Anniversary Celebrations)

FIXED POINT OF SUZUKI-TYPE GENERALIZED MULTIVALUED CONTRACTION MAPPINGS ON WEAK PARTIAL METRIC SPACES

By
Smita Negi and U. C. Gairola
Department of Mathematics
H.N.B. Garhwal University, BGR Campus, Pauri Garhwal-246001, Uttrakhand, India
Email:smitanegi.sn@gmail.com, ucgairola@rediffmail.com
(*Received : November* 08, 2019 *; Revised: January* 25, 2020)
DOI: https://doi.org/10.58250/jnanabha.2020.50104

Abstract

Motivated by the recent result of Aydi et al.[4], we establish a fixed point theorem of Suzuki-type generalized multivalued contraction mappings in the framework of weak partial metric space. An example is also given to show the significance of our result.
2010 Mathematics Subject Classifications: 47H10, 54H25.
Keywords and phrases: Weak partial metric space, H^+-type Hausdorff metric, generalized Suzuki type contraction.

1 Introduction

The notion of partial metric space was introduced in 1994 by Matthews[7] as a generalization of the metric space. Such spaces are useful for modeling the problems occurring in computer science. He also proved the famous Banach Contraction Principle in partial metric space. Later on many fixed point results in partial metric space have been proved (see, for instance [1], [2], [6] and references therein). Further Aydi et al.[3] obtained an analogue of Nadler's fixed point theorem in partial metric space using the concept of a partial Hausdorff metric. Recently, Beg and Pathak[5] introduced a weaker form of partial metric called weak partial metric and gave a fixed point theorem. This result extended and generalized by Negi and Gairola[8].

The aim of this paper is to generalize the result of Aydi et al. [4] by introducing generalized Suzuki type multivalued contraction mapping in weak partial metric space. Our result extend various known comparable results in the literature.

2 Preliminaries

The following definitions and results are followed by Beg and Pathak[5].

***Definition* 2.1.** *Let X be a nonempty set. A function $q : X \times X \to \mathbb{R}^+$ is called a weak partial metric on X if for all $x, y, z \in X$, the following conditions hold:*
(Q1) $q(x, x) = q(x, y) \Leftrightarrow x = y$;
(Q2) $q(x, x) \leq q(x, y)$;
(Q3) $q(x, y) = q(y, x)$;
(Q4) $q(x, y) \leq q(x, z) + q(z, y)$.
The pair (X, q) is called a weak partial metric space.

Let $CB^q(X)$ be the family of all non-empty, closed and bounded subsets of weak partial metric space (X, q). For $E, F \in CB^q(X)$ and $x \in X$, define $\delta_q : CB^q(X) \times CB^q(X) \to [0, \infty)$ and

$q(x, E) = \inf\{q(x, a) : a \in E\}$,
$\delta_q(E, F) = \sup\{q(a, F) : a \in E\}$,
$\delta_q(F, E) = \sup\{q(b, E) : b \in F\}$.

Each weak partial metric q on X generates a T_0 topology τ_q on X which has as a base the family of open q-balls $\{B_q(x, \epsilon) : x \in X,\ \epsilon > 0\}$, where $B_q(x, \epsilon) = \{y \in X : q(x, y) < q(x, x) + \epsilon\}$ for all $x \in X$ and $\epsilon > 0$.

***Remark* 2.1.** *Let $\phi \neq E$ be a set in (X, q). Then*
$a \in \overline{E}$ if and only if $q(a, E) = q(a, a)$,
where $\overline{E}$ denotes the closure of E with respect to the weak partial metric space.
Note that E is closed in (X, q) if and only if $E = \overline{E}$.

***Definition* 2.2.** *A sequence $\{x_n\}$ in (X, q) converges to a point $x \in X$ with respect to τ_q if and only if $q(x, x) = \lim_{n\to\infty} q(x, x_n)$.*

***Remark* 2.2.** *If q is a weak partial metric on X, the function $q^s : X \times X \to \mathbb{R}^+$ given by $q^s(x, y) = q(x, y) - \frac{1}{2}[q(x, x) + q(y, y)]$, defines a metric on X. Further, a sequence $\{x_n\}$ converges in (X, q^s) to a point $x \in X$ if and only if*

$$q(x, x) = \lim_{n\to\infty} q(x_n, x) = \lim_{n,m\to\infty} q(x_n, x_m). \tag{2.1}$$

***Proposition* 2.1.** *Let (X, q) be a weak partial metric space. For any $E, F, H \in CB^q(X)$, the following holds:*
(i) $\delta_q(E, E) = \sup\{q(a, a) : a \in E\}$;
(ii) $\delta_q(E, E) \leq \delta_q(E, F)$;
(iii) $\delta_q(E, F) = 0 \Rightarrow E \subseteq F$;
(iv) $\delta_q(E, F) \leq \delta_q(E, H) + \delta_q(H, F)$.

***Proposition* 2.2.** *Let (X, q) be a weak partial metric space. For all $E, F, H \in CB^q(X)$, we have*
(wp1) $H_q^+(E, E) \leq H_q^+(E, F)$;
(wp2) $H_q^+(E, F) = H_q^+(F, E)$;
(wp3) $H_q^+(E, F) \leq H_q^+(E, H) + H_q^+(H, F)$.

***Definition* 2.3.** *Let (X, q) be a weak partial metric space. For $E, F \in CB^q(X)$, define*

$$H_q^+(E, F) = \frac{1}{2}\{\delta_q(E, F) + \delta_q(F, E)\}.$$

The mapping $H_q^+ : CB^q(X) \times CB^q(X) \to [0, +\infty)$, is called H_q^+-type Hausdorff metric induced by q.

***Definition* 2.4.** *Let (X, q) be a complete weak partial metric space. A multi-valued map $T : X \to CB^q(X)$ is called H_q^+ -contraction if for every $x, y \in X$,*
(i) there exists r in $(0, 1)$ such that
$H_q^+(T(x) \setminus \{x\}, T(y) \setminus \{y\}) \leq r\, q(x, y)$,
(ii) for every x in X, y in $T(x)$ and $\epsilon > 0$, there exists z in $T(y)$ such that
$q(y, z) \leq H_q^+(T(y), T(x)) + \epsilon$.

Beg and Pathak[5] gave the following variant of Nadler's fixed point theorem.

***Theorem* 2.1.** [5] *Every H_q^+- type multivalued contraction on a complete weak partial metric space (X, q) has a fixed point.*

Recently Aydi et al.[4] introduced H_q^+- type Suzuki multivalued contraction mappings and prove the following theorem.

***Theorem* 2.2.** [4] *Let (X, q) be a complete weak partial metric space, and let $T : X \to CB^q(X)$ be a multivalued mapping. Let $\psi : [0, 1) \to (0, 1]$ be the non increasing function defined by*

$$\psi(r) = \begin{cases} 1 & if\ 0 \le r < \frac{1}{2} \\ 1 - r & if\ \frac{1}{2} \le r < 1. \end{cases} \tag{2.2}$$

Suppose that there exists $0 \le r < 1$ such that T satisfies the condition $\psi(r)q(x, Tx) \le q(x, y)$ implies

$$H_q^+(Tx \setminus \{x\}, Ty \setminus \{y\}) \le rq(x, y),$$

for all $x, y \in X$.

Suppose also that, for all x in X, y in Tx , and $t > 1$, there exists z in Ty such that

$$q(y, z) \le tH_q^+(Ty, Tx).$$

Then T has a fixed point.

3 Main result

Now we state our main result.

***Theorem* 3.1.** *Let (X, q) be a complete weak partial metric space and let $T : X \to CB^q(X)$ be a multivalued mapping and $\psi : [0, 1) \to (0, 1]$ be the non-increasing function defined by (2.2). If there exists $0 \le r < 1$ such that T satisfies the condition*

$$\psi(r)q(x, Tx) \le q(x, y) \text{ implies } H_q^+(Tx \setminus \{x\}, Ty \setminus \{y\}) \le rM(x, y), \tag{3.1}$$

where $M(x, y) = \max\{q(x, y), q(x, Tx), q(y, Ty)\}$ for all $x, y \in X$.

Suppose also that for every $x \in X, y \in Tx$ and $t > 1, \exists\ z \in Ty$ such that

$$q(y, z) \le tH_q^+(Ty, Tx). \tag{3.2}$$

Then T has a fixed point.

Proof. Let r_1 be a real number such that $0 \le r \le r_1 < 1$ and $w_0 \in X$. Since Tw_0 is nonempty, it follows that if $w_0 \in Tw_0$, then proof is finished. Let $w_1 \in Tw_0$ be such that $w_0 \neq w_1$. Similarly $\exists\ w_2 \in Tw_1$ such that $w_1 \neq w_2$. From (3.2) we have

$$q(w_1, w_2) \le \frac{1}{\sqrt{r_1}} H_q^+(Tw_0, Tw_1). \tag{3.3}$$

Since $\psi(r) \le 1$, we have

$$\psi(r)q(w_1, Tw_1) \le q(w_1, Tw_1) \le q(w_1, w_2).$$

Using (3.1) in (3.3), we have

$$\begin{aligned} q(w_1, w_2) &\le \frac{1}{\sqrt{r_1}} H_q^+(Tw_0, Tw_1) \\ &\le \frac{1}{\sqrt{r_1}} H_q^+(Tw_0 \setminus \{w_0\}, Tw_1 \setminus \{w_1\}) \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{r}{\sqrt{r_1}} M(w_0, w_1) < \sqrt{r_1} M(w_0, w_1) \\
&= \sqrt{r_1} \max\{q(w_0, w_1), q(w_0, Tw_0), q(w_1, Tw_1)\} \\
&\leq \sqrt{r_1} \max\{q(w_0, w_1), q(w_0, w_1), q(w_1, w_2)\}.
\end{aligned}
$$

If $q(w_0, w_1) \leq q(w_1, w_2)$ then

$$q(w_1, w_2) \leq \sqrt{r_1} q(w_1, w_2)$$

but $\sqrt{r_1} < 1$ then we get a contradiction.

Thus, we have

$$q(w_1, w_2) \leq \sqrt{r_1} q(w_0, w_1).$$

Continuing this process, we obtain a sequence $\{w_n\}$ in X such that

$$q(w_n, w_{n+1}) \leq (\sqrt{r_1})^n q(w_0, w_1).$$

Now, we prove that $\{w_n\}$ is a Cauchy sequence in (X, q^s).

For all $k \in \mathbb{N}$, we have

$$
\begin{aligned}
q^s(w_n, w_{n+k}) &\leq q(w_n, w_{n+k}) \\
&\leq q(w_n, w_{n+1}) + q(w_{n+1}, w_{n+2}) + ... + q(w_{n+k-1}, w_{n+k}) \\
&\leq [(\sqrt{r_1})^n + (\sqrt{r_1})^{n+1} + ... + (\sqrt{r_1})^{n+k-1}] q(w_0, w_1) \\
&\leq \frac{(\sqrt{r_1})^n}{1 - \sqrt{r_1}} q(w_0, w_1)
\end{aligned}
$$

$$\longrightarrow 0 \text{ as } n \to \infty.$$

Hence, $\lim_{n\to\infty} q^s(w_n, w_{n+k}) = 0$.

This implies that $\{w_n\}$ is a Cauchy sequence in (X, q^s). Since (X, q) is complete, therefore (X, q^s) is also complete metric space. It follows that there exists $u \in X$ such that $\lim_{n\to\infty} w_n = u$ in (X, q^s).

Therefore, $\lim_{n\to\infty} q^s(w_n, u) = 0$.

From (2.1), we have

$$q(u, u) = \lim_{n\to\infty} q(w_n, u) = \lim_{n,k\to\infty} q(w_n, w_k) = 0.$$

Now, from triangle inequality

$$q(u, Tx) \leq q(u, w_{n+1}) + q(w_{n+1}, Tx)$$

and

$$q(w_{n+1}, Tx) \leq q(w_{n+1}, w_n) + q(w_n, u) + q(u, Tx).$$

Taking limit in the above inequalities, we get

$$q(u, Tx) = \lim_{n\to\infty} q(w_{n+1}, Tx). \tag{3.4}$$

We claim that

$$q(u, Tx) \leq 2r \max\{q(u, x), q(x, Tx)\}, \ \forall\, x \in X \setminus \{u\}.$$

Since, $\lim_{n\to\infty} q(w_n, u) = 0$, $\exists\, n_0 \in \mathbb{N}$ such that $q(w_n, u) \leq \frac{1}{3} q(u, x)$, $\forall\, n \geq n_0$.

As $w_{n+1} \in Tw_n$ then we have

$$
\begin{aligned}
\psi(r) q(w_n, Tw_n) \leq q(w_n, Tw_n) &\leq q(w_n, w_{n+1}) \\
&\leq q(w_n, u) + q(u, w_{n+1}) \\
&\leq \frac{1}{3} q(u, x) + \frac{1}{3} q(u, x) = \frac{2}{3} q(u, x)
\end{aligned}
$$

$$
\begin{aligned}
&\le q(u,x) - \frac{1}{3} q(u,x)\\
&\le q(u,x) - q(w_n,u)\\
&\le q(x,w_n).
\end{aligned}
$$

Hence, for any $n \ge n_0$ we get

$$\psi(r) q(w_n, Tw_n) \le q(x, w_n).$$

From (3.1) we have

$$H_q^+(Tw_n, Tx) \le rM(w_n, x).$$

Since

$$
\begin{aligned}
q(w_{n+1}, Tx) &\le \delta_q(Tw_n, Tx)\\
&\le 2H_q^+(Tw_n, Tx)\\
&\le 2rM(w_n, x) = 2r \max\{q(w_n, x), q(w_n, Tw_n), q(x, Tx)\}\\
&\le 2r \max\{q(w_n, u) + q(u, x), q(w_n, w_{n+1}), q(x, Tx)\}.
\end{aligned}
$$

Taking limit as $n \to \infty$, we get

$$\lim_{n\to\infty} q(w_{n+1}, Tx) \le 2r \max\{q(u,x), q(x,Tx)\}.$$

From (3.4) we get

$$q(u, Tx) \le 2r \max\{q(u,x), q(x,Tx)\}, \ \forall\, x \in X \setminus \{u\}. \tag{3.5}$$

Now, we claim that

$$H_q^+(Tx, Tu) \le r \max\{q(x,u), q(x,Tx), q(u,Tu)\} \tag{3.6}$$

for all $x \in X$ such that $x \ne u$.

For each $n \in \mathbb{N}$, $\exists\, y_n \in Tx$ such that

$$q(u, y_n) \le q(u, Tx) + \frac{1}{n} q(u, x).$$

Therefore

$$
\begin{aligned}
q(x, Tx) &\le q(x, y_n)\\
&\le q(x,u) + q(u, y_n)\\
&\le q(x,u) + q(u,Tx) + \frac{1}{n} q(u,x)\\
&\le q(x,u) + 2r\max\{q(u,x), q(x,Tx)\} + \frac{1}{n} q(u,x).
\end{aligned} \tag{3.7}
$$

Suppose $\max\{q(u,x), q(x,Tx)\} = q(u,x)$ then

$$q(x,Tx) \le (1 + 2r + \frac{1}{n}) q(u,x),$$

which implies

$$\frac{1}{1 + 2r + \frac{1}{n}} q(x, Tx) \le q(u,x).$$

This further implies that

$$H_q^+(Tu, Tx) \le rM(x,u),$$

and equation (3.6) holds.
Now if $\max\{q(x,u), q(x,Tx)\} = q(x,Tx)$ then from (3.7), we get

$$q(x,Tx) \leq q(x,u) + 2rq(x,Tx) + \frac{1}{n}q(x,u)$$
$$\frac{(1-2r)}{(1+\frac{1}{n})}q(x,Tx) \leq q(x,u).$$

This also implies that
$H_q^+(Tx,Tu) \leq rM(x,u)$ and equation (3.6) holds.

Finally, let $b \in Tu$ then

(3.8) $$q(b,Tw_n) \leq \delta_q(Tu,Tw_n).$$

Also, we know that

$$\begin{aligned} q(u,Tu) &\leq q(u,Tw_n) + q(Tw_n,Tu) \\ &\leq q(u,w_{n+1}) + q(b,Tw_n). \end{aligned}$$

From (3.8), we have

$$q(u,Tu) \leq q(u,w_{n+1}) + \delta_q(Tu,Tw_n).$$

Taking limit, we get

(3.9) $$q(u,Tu) \leq \lim_{n\to\infty} \delta_q(Tu,Tw_n).$$

Also we know that

$$q(w_{n+1},Tu) \leq \delta_q(Tw_n,Tu).$$

Taking limit, we have

(3.10) $$q(u,Tu) \leq \lim_{n\to\infty} \delta_q(Tw_n,Tu).$$

From the definition(2.3), we know that

$$\frac{1}{2}[\delta_q(Tw_n,Tu) + \delta_q(Tu,Tw_n)] = H_q^+(Tw_n,Tu).$$

Taking limit in the above expression and using (3.9) and (3.10) we get

$$\begin{aligned} \frac{1}{2}[q(u,Tu) + q(u,Tu)] &\leq \lim_{n\to\infty} \frac{1}{2}[\delta_q(Tw_n,Tu) + \delta_q(Tu,Tw_n)] \\ &= \lim_{n\to\infty} H_q^+(Tw_n,Tu) \\ &= \lim_{n\to\infty} H_q^+(Tw_n \setminus \{w_n\}, Tu \setminus \{u\}) \\ &\leq \lim_{n\to\infty} r\max\{q(w_n,u), q(w_n,Tw_n), q(u,Tu)\} \\ &= rq(u,Tu) \end{aligned}$$

which implies that

$$q(u,Tu) \leq rq(u,Tu).$$

Since $r < 1$ then we get a contradiction.
Hence, we have $q(u,Tu) = 0 = q(u,u)$. Since Tu is closed then $u \in \overline{Tu} = Tu$.
Now we give an example to verify our result.

***Example* 3.1.** *Let* $X = \{0, \frac{1}{3}, 1\}$ *and define a weak partial metric* $q : X \times X \to [0, \infty)$ *as follows:* $q(0,0) = 0,\ q(\frac{1}{3}, \frac{1}{3}) = \frac{1}{5},\ q(1,1) = \frac{1}{3},\ q(0,1) = q(1,0) = \frac{2}{5},\ q(1, \frac{1}{3}) = q(\frac{1}{3}, 1) = \frac{3}{5},\ q(0, \frac{1}{3}) = q(\frac{1}{3}, 0) = \frac{1}{4}.$

Define a mapping $T : X \to CB^q(X)$ *by*

$$Tx = \begin{cases} \{0\} & \text{if } x = 0 \\ \{1\} & \text{if } x = \frac{1}{3} \\ \{0, \frac{1}{3}\} & \text{if } x = 1. \end{cases}$$

Clearly (X, q) is a weak partial metric space.

Choose $r = 0.9$. From the definition of ψ we have $\psi(0.9) = 0.1$. To investigate the contraction condition (3.1) holds for all $x, y \in X$, we assume the following cases:

Case (I) When $x = 0$, we have

$$\psi(r)q(0, T(0)) = 0 \leq q(0, y),\ \forall\ y \in X.$$

For $y = 0$, we have

$$H_q^+(T(0) \setminus \{0\}, T(0) \setminus \{0\}) = H_q^+(\phi, \phi) = 0 \leq rM(0, 0).$$

For $y = \frac{1}{3}$, we have

$$H_q^+(T(0) \setminus \{0\}, T(\tfrac{1}{3}) \setminus \{\tfrac{1}{3}\}) = H_q^+(\phi, \{1\}) = 0 \leq rM(0, \tfrac{1}{3}).$$

For $y = 1$, we have

$$H_q^+(T(0) \setminus \{0\}, T(1) \setminus \{1\}) = H_q^+\Big(\phi, \Big\{0, \tfrac{1}{3}\Big\}\Big) = 0 \leq rM(0, 1).$$

Case (II) When $x = \frac{1}{3}$, we have

$$\psi(r)q\Big(\frac{1}{3}, T\Big(\frac{1}{3}\Big)\Big) = (0.1)q(\frac{1}{3}, 1) = 0.06 \leq q(\frac{1}{3}, y),\ \forall\ y \in X.$$

For $y = 0$, we have

$$H_q^+(T(\tfrac{1}{3}) \setminus \{\tfrac{1}{3}\}, T(0) \setminus \{0\}) = H_q^+(\{1\}, \phi) = 0 \leq rM(\tfrac{1}{3}, 0).$$

For $y = \frac{1}{3}$, we have

$$H_q^+(T(\tfrac{1}{3}) \setminus \{\tfrac{1}{3}\}, T(\tfrac{1}{3}) \setminus \{\tfrac{1}{3}\}) = H_q^+(\{1\}, \{1\}) = \tfrac{1}{3} \leq rM(\tfrac{1}{3}, \tfrac{1}{3}) = r\tfrac{3}{5}.$$

For $y = 1$, we have

$$H_q^+(T(\tfrac{1}{3}) \setminus \{\tfrac{1}{3}\}, T(1) \setminus \{1\}) = H_q^+\Big(\ \{1\}, \Big\{0, \tfrac{1}{3}\Big\}\Big) = \tfrac{1}{2} \leq rM(\tfrac{1}{3}, 1) = r\tfrac{3}{5}.$$

Case (III) When $x = 1$, we have

$$\psi(r)q(1, T(1)) = (0.1)q\Big(1, \Big\{0, \frac{1}{3}\Big\}\Big) = 0.04 \leq q(1, y),\ \forall\ y \in X.$$

For $y = 0$, we have

$$H_q^+(T(0) \setminus \{0\}, T(1) \setminus \{1\}) = H_q^+\Big(\phi, \Big\{0, \tfrac{1}{3}\Big\}\Big) = 0 \leq rM(1, 0).$$

For $y = \frac{1}{3}$, we have

$$H_q^+(T(0) \setminus \{0\}, T(\tfrac{1}{3}) \setminus \{\tfrac{1}{3}\}) = H_q^+(\phi, \{1\}) = 0 \leq rM(0, \tfrac{1}{3}) = r\tfrac{3}{5}.$$

For $y = 1$, we have

$$H_q^+(T(1) \setminus \{1\}, T(1) \setminus \{1\}) = H_q^+\Big(\Big\{0, \tfrac{1}{3}\Big\}, \Big\{0, \tfrac{1}{3}\Big\}\Big) = \tfrac{1}{5} \leq rM(1, 1) = r\tfrac{2}{5}.$$

Finally, we will enquire the condition (3.2) with $t = 2$. For this, we discuss the following situations:

(i) If $x = 0$, then $y \in T(0) = \{0\}$, so $\exists\, z \in T(y) = \{0\}$ such that

$$0 = q(y, z) \leq 2H_q^+(T(y), T(x)).$$

(ii) If $x = \frac{1}{3}$, then $y \in T(\frac{1}{3}) = \{1\}$, so $\exists\, z$(say $z = 0$) $\in T(1) = \{0, \frac{1}{3}\}$ such that

$$\frac{2}{5} = q(y, z) \leq 2H_q^+\Big(T(1), T\Big(\frac{1}{3}\Big)\Big) = 1.$$

(iii) If $x = 1$, then $y \in T(1) = \{0, \frac{1}{3}\}$. If $y = 0$, then $z = 0$, and condition holds.

Also if $y = \frac{1}{3}$, then $\exists\, z \in T(\frac{1}{3}) = \{1\}$ such that

$$\frac{3}{5} = q(y, z) \leq 2H_q^+\Big(\{1\}, \Big\{0, \frac{1}{3}\Big\}\Big) = 1.$$

Hence all the conditions of ***Theorem*** **3.1** are satisfied. Here $x = 0$ is fixed point of T.

On the other hand the result of Aydi et al.[4] is not applicable. We see that

$$H_q^+(T(\tfrac{1}{3}) \setminus \{\tfrac{1}{3}\}, T(\tfrac{1}{3}) \setminus \{\tfrac{1}{3}\}) = H_q^+(\{1\}, \{1\}) = \tfrac{1}{3} \leq r\tfrac{1}{5} = rq(\tfrac{1}{3}, \tfrac{1}{3})$$

is not satisfied for any $r \in (0, 1)$.

Acknowledgements. We are very much thankful to the worthy referee for his valuable suggestions to bring the paper in its present form.

References

[1] M. Abbas, B. Ali and C. Vetro, A Suzuki type fixed point theorem for a generalized multivalued mapping on partial Hausdorff metric spaces. *Topol. Appl.*, **160** (2013), 553-563.

[2] I. Altun, F. Sola and H. Simsek, Generalized contraction on partial metric spaces. *Topol. Appl.*, **157** (2010), 2778-2785.

[3] H. Aydi, M. Abbas and C. Vetro, Partial Hausdorff metric and Nadler's fixed point theorem on partial metric spaces. *Topol. Appl.*, **159** (2012), 3234-3242.

[4] H. Aydi, M. A. Barakat, Z. D. Mitrovic and V. S. Cavic, A Suzuki-type multivalued contraction on weak partial metric spaces and applications. *J. Inequal. Appl.*, **2018**, 270 (2018), 1-14.

[5] I. Beg and H. K. Pathak, A variant of Nadler's theorem on weak partial metric spaces with application to a homotopy result. *Vietnam J. Math.*, **46**(2018), 693-706.

[6] L. Ciric, B. Samet, H. Aydi and C. Vetro, Common fixed points of generalized contraction on partial metric spaces and an application. *Appl. Math. Comp.*, **218**(2011), 2398-2406.

[7] S. G. Matthews, Partial metric topology. *Ann. N.Y.Acad. Sci.*, **728**(1994), 183-197.

[8] S. Negi and U. C. Gairola, Common Fixed Points for Generalized Multivalued Contraction Mappings on Weak Partial Metric Spaces. *Jñānābha.*, **49(2)**(2019), 34-44.

Jñānābha, Vol. 50(1) (2020), 43-48
(Dedicated to Honor Professor H.M. Srivastava on His 80th Birth Anniversary Celebrations)

CARISTI-BANACH TYPE CONTRACTION VIA SIMULATION FUNCTION

By

Swati Antal and U. C. Gairola
Department of Mathematics
H. N. B. Garhwal University, BGR Campus Pauri Garhwal-246001, Uttarakhand, India.
Email:antalswati11@gmail.com, ucgairola@rediffmail.com

(*Received : February* 09, 2020 *; Revised: February* 22, 2020)
DOI: https://doi.org/10.58250/jnanabha.2020.50105

Abstract

In this paper we introduce the notion of a Caristi-Banach type $\mathcal{Z}^b_{\mathcal{R}}$- contraction in the framework of b-metric space endowed with a transitive relation that combine the ideas of Caristi type contraction and Banach contraction with a help of simulation function. We present an example to clarify the statement of the given result.

2010 Mathematics Subject Classifications: 47H10, 54H25.
Keywords and phrases: Caristi-Banach type $\mathcal{Z}^b_{\mathcal{R}}$- contraction, Simulation function, transitive relation.

1 Introduction and preliminaries

In 1922, Polish mathematician Stefan Banach [2] gave a fixed point theorem. It is also known as the Banach Contraction mapping theorem or principle (BCP). It is an important tool in the metric fixed point theory. It confirms the existence and uniqueness of fixed point of certain self maps of metric spaces and provides a constructive method to find fixed points. There are so many extension, generalizations of BCP in different settings and there applications. Among them, In 1976, Caristi [4] proved a fixed point theorem and applied to derive a generalization of the Contraction Mapping Principle in a complete metric space. Recently, In 2019, E. Karapinar et al., [9] give a new fixed point theorem in b-metric space which is inspired from both Caristi and Banach. b-metric space introduced by Czerwik [5] to generalize the concept of metric space by introducing a real number $s \geq 1$ in the triangle inequality of metric space.

Inspired by E. Karapinar et al., [9] we introduce the notion of a Caristi-Banach type $\mathcal{Z}^b_{\mathcal{R}}$-contraction in the framework of b-metric space endowed with a transitive relation that combine the ideas of Caristi type contraction and Banach contraction with a help of simulation function. We present an example also to clarify the statement of the given result.

***Definition* 1.1.** [5] *Let M be a non-empty set and $s \geq 1$ be a given real number. A function $d : M \times M \to [0, \infty)$ is said to be a b-metric space if, for all $\sigma, \rho, w \in M$, the following conditions are satisfied:*

(i) $d(\sigma, \rho) = 0$ *iff* $\sigma = \rho$;

(ii) $d(\sigma, \rho) = d(\rho, \sigma)$;

(iii) $d(\sigma, w) \leq s[d(\sigma, \rho) + d(\rho, w)]$.

The triple (M, d, s) is called a b-metric space.

It should be noted that, every metric space is a b-metric space with $s = 1$ and hence the class of b-metric spaces is larger than the class of metric spaces. But a metric space need not be b-metric space (see example 1.4 [14]).

***Definition* 1.2.** [3] *Let (M, d, s) be a b-metric space.*

(i) A sequence (σ_n) in M is called b-convergent if and only if there exist $\sigma \in M$ such that $d(\sigma_n, u) \to 0$, as $n \to \infty$. In this case, we write $\lim_{n\to\infty} \sigma_n \to u$.
(ii) (σ_n) in M is said to be b-Cauchy if and only if $d(\sigma_n, \sigma_m) \to 0$, as $n, m \to \infty$.
(iii) The b-metric space (M, d, s) is said to be b-complete if every b-Cauchy sequence (σ_n) in M is convergent.

Recently, in 2015, Khojasteh et al. [10] introduced the notion of simulation function with a view to consider a new class of contractions, called $\mathcal{Z}$-contraction with respect to a simulation function.

***Definition* 1.3.** [10] *A mapping $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ is a simulation function if:*

(ζ_1) $\zeta(0, 0) = 0$*;*
(ζ_2) $\zeta(\mathrm{t}, \mathfrak{s}) < \mathfrak{s} - \mathrm{t}$*,* $\mathfrak{s}, \mathrm{t} > 0$*;*
(ζ_3) (t_n) *and* $(\mathfrak{s}_n)$ *are sequences in* $(0, \infty)$ *satisfying* $\lim_{n\to\infty} \mathrm{t}_n = \lim_{n\to\infty} \mathfrak{s}_n > 0$*, then* $\limsup_{n\to\infty} \zeta(\mathrm{t}_n, \mathfrak{s}_n) < 0$.

Set of all simulation functions is denoted by $\mathcal{Z}$. For examples of simulation function we may refer to ([7], [8], [10]).

In what follows $(\mathcal{M}, d)$, $\mathcal{R}$, $\mathbb{N}$ and $\mathbb{N}_0$ respectively, stand for a metric space, a non-empty binary relation defined on a non-empty set $\mathcal{M}$, the set of natural numbers and the set of whole numbers.

***Definition* 1.4.** [12] *A binary relation $\mathcal{R}$ on a non-empty set $\mathcal{M}$ is defined as a subset of $\mathcal{M} \times \mathcal{M}$. We say that "$\sigma$ is $\mathcal{R}$-related to ρ" iff $(\sigma, \rho) \in \mathcal{R}$.*

***Definition* 1.5.** [13] *A binary relation $\mathcal{R}$ is complete if either $(\sigma, \rho) \in \mathcal{R}$ or $(\rho, \sigma) \in \mathcal{R}$ (i.e. $[\rho, \sigma] \in \mathcal{R}$), $\forall\ \sigma, \rho \in \mathcal{M}$.*

***Definition* 1.6.** [1] *Let $\mathcal{F}$ be a self-mapping defined on a non-empty set $\mathcal{M}$. Then binary relation $\mathcal{R}$ is $\mathcal{F}$-closed if*

$$(\sigma, \rho) \in \mathcal{R} \Rightarrow (\mathcal{F}\sigma, \mathcal{F}\rho) \in \mathcal{R}, \quad \sigma, \rho \in \mathcal{M}.$$

***Definition* 1.7.** [12] *The symmetric closure $\mathcal{R}^s$ is the smallest symmetric relation containing $\mathcal{R}$, i.e., $\mathcal{R}^s = \mathcal{R} \cup \mathcal{R}^{-1}$.*

***Proposition* 1.1.** [1] *If $\mathcal{R}$ is $\mathcal{F}$-closed, then $\mathcal{R}^s$ is also $\mathcal{F}$-closed.*

***Definition* 1.8.** [1] *A sequence (σ_n) in $\mathcal{M}$ is $\mathcal{R}$-preserving if*

$$(\sigma_n, \sigma_{n+1}) \in \mathcal{R}, \quad n \in \mathbb{N}_0.$$

***Definition* 1.9.** *Let $X, d)$ be a d-metric space. Then a binary relation $\mathcal{R}$ is transitive if $(\sigma, \rho) \in \mathcal{R}$ and $(\rho, \eta) \in \mathcal{R}$ implies that $(\sigma, \eta) \in \mathcal{R}$*

***Definition* 1.10.** [16] *Let $(\mathcal{M}, d, s)$ be a b-metric space. A binary relation $\mathcal{R}$ on $\mathcal{M}$ is (b-d)-self-closed if (σ_n) is an $\mathcal{R}$-preserving sequence and*

$$\sigma_n \to \sigma \ \ as \ \ n \to \infty,$$

then there exists a subsequences (σ_{n_k}) of (σ_n) with $[\sigma_{n_k}, \sigma] \in \mathcal{R}$, $k \in \mathbb{N}$.

***Definition* 1.11.** [15] *A subset D of $\mathcal{M}$ is $\mathcal{R}$-directed if for each pair of points $\sigma, \rho \in D$, there exists $\eta \in \mathcal{M}$ satisfying $(\sigma, \eta) \in \mathcal{R}$ and $(\rho, \eta) \in \mathcal{R}$.*

***Definition* 1.12.** [11] *For $\sigma, \rho \in \mathcal{M}$, a path of length k in $\mathcal{R}$ from σ to ρ is a finite sequence $(\eta_0, \eta_1, \eta_2, ..., \eta_k) \subset \mathcal{M}$ satisfying:*
(i) $\eta_0 = \sigma$ and $\eta_k = \rho$,
(ii) $(\eta_i, \eta_{i+1}) \in \mathcal{R}$ for each i $(0 \leq i \leq k-1)$ (k is a natural number).
Clearly a path of length k necessitate $k+1$ elements of $\mathcal{M}$, which are not essentially distinct.

In the following
$\mathcal{M}(\mathcal{F}; \mathcal{R}) := \{\sigma \in \mathcal{M} : (\sigma, \mathcal{F}\sigma) \in \mathcal{R}\}$, where $\mathcal{F} : \mathcal{M} \to \mathcal{M}$ and $\gamma(\sigma, \rho, \mathcal{R})$ is the class of all paths in $\mathcal{R}$ from σ to ρ.

2 Main Result

In this section, the notation $\mathcal{Z}^b_{\mathcal{R}}$ denotes the set of simulation function in b-metric space endowed with an given binary relation $\mathcal{R}$.

***Definition* 2.1.** *Let $\mathcal{F}$ be a self mapping on a b-metric space $(\mathcal{M}, d, s)$ equipped with a binary relation $\mathcal{R}$. If there exist $\zeta \in \mathcal{Z}^b_{\mathcal{R}}$ and $\phi : \mathcal{M} \to [0, \infty)$ such that*

$$d(\sigma, \mathcal{F}\sigma) > 0 \Rightarrow \zeta(sd(\mathcal{F}\sigma, \mathcal{F}\rho), (\phi(\sigma) - \phi(\mathcal{F}\sigma))d(\sigma, \rho)) \geq 0, \tag{2.1}$$

$\forall\ \sigma, \rho \in \mathcal{M}$, $(\sigma, \rho) \in \mathcal{R}$, then $\mathcal{F}$ is called Caristi-Banach type $\mathcal{Z}^b_{\mathcal{R}}$-contraction.

***Theorem* 2.1.** *Let $(\mathcal{M}, d, s)$ be a complete b-metric space equipped with a binary relation $\mathcal{R}$ and $\mathcal{F}$ be a self mapping on $\mathcal{M}$. Let the following hypotheses holds:*
(i) $\mathcal{M}(\mathcal{F}; \mathcal{R})$ is non-empty;
(ii) $\mathcal{R}$ is $\mathcal{F}$-closed and transitive;
(iii) either $\mathcal{F}$ is $\mathcal{R}$- continuous or $\mathcal{R}$ is (b-d)-self-closed;
(iv) $\mathcal{F}$ is Caristi-Banach type $\mathcal{Z}^b_{\mathcal{R}}$-contraction with respect to $\zeta \in \mathcal{Z}$. Then $\mathcal{F}$ has a fixed point.

Proof. Let σ_0 be an arbitrary point in $\mathcal{M}(\mathcal{F}; \mathcal{R})$. Put $\sigma_n = \mathcal{F}\sigma_{n-1} = \mathcal{F}^n\sigma_0$ $\forall n \in \mathbb{N}$. Let $C_{n+1} = d(\sigma_n, \sigma_{n+1})$, if for some $n' \in \mathbb{N}_0$, $\sigma_{n'} = \sigma_{n'+1}$, then $\sigma_{n'}$ is a fixed point of $\mathcal{F}$ and so the proof is complete. Thus, we let $\sigma_n \neq \sigma_{n+1}$ $\forall\ n \in \mathbb{N}_0$ i.e., $C_{n+1} > 0$. Since $(\sigma_0, \mathcal{F}\sigma_0) \in \mathcal{R}$, using the $\mathcal{F}$-closedness of $\mathcal{R}$, we obtain

$$(\mathcal{F}\sigma_0, \mathcal{F}^2\sigma_0), (\mathcal{F}^2\sigma_0, \mathcal{F}^3\sigma_0), ..., (\mathcal{F}^n\sigma_0, \mathcal{F}^{n+1}\sigma_0), ... \in \mathcal{R}.$$

Thus

$$(\sigma_n, \sigma_{n+1}) \in \mathcal{R}, \tag{2.2}$$

and the sequence (σ_n) is $\mathcal{R}$- preserving. Since $\mathcal{F}$ is Caristi-Banach type $\mathcal{Z}^b_{\mathcal{R}}$-contraction, we have

$$\begin{aligned} 0 &\leq \zeta\big(sd(\mathcal{F}\sigma_{n-1}, \mathcal{F}\sigma_n), (\phi(\sigma_{n-1}) - \phi(\mathcal{F}\sigma_{n-1}))d(\sigma_{n-1}, \sigma_n)\big) \\ &< (\phi(\sigma_{n-1}) - \phi(\mathcal{F}\sigma_{n-1}))d(\sigma_{n-1}, \sigma_n) - sd(\mathcal{F}\sigma_{n-1}, \mathcal{F}\sigma_n), \end{aligned}$$

yields

$$\begin{aligned} C_{n+1} = d(\sigma_n, \sigma_{n+1}) &= d(\mathcal{F}\sigma_{n-1}, \mathcal{F}\sigma_n) \leq sd(\mathcal{F}\sigma_{n-1}, \mathcal{F}\sigma_n) \\ &< (\phi(\sigma_{n-1}) - \phi(\mathcal{F}\sigma_{n-1}))d(\sigma_{n-1}, \sigma_n) \\ &= (\phi(\sigma_{n-1}) - \phi(\sigma_n))C_n. \end{aligned}$$

So we have

$$0 < \tfrac{C_{n+1}}{C_n} \leq (\phi(\sigma_{n-1}) - \phi(\sigma_n)) \text{ for each } n \in \mathbb{N}.$$

Thus the sequence $(\phi(\sigma_n))$ is necessarily non-negative and decreasing. Hence, it converges to some $\mathfrak{a} \geq 0$. On the other hand, for each $n \in \mathbb{N}$, we have

$$\begin{aligned} \sum_{k=1}^{n} \frac{C_{k+1}}{C_k} &\leq \sum_{k=1}^{n} (\phi(\sigma_{k-1}) - \phi(\sigma_k)) \\ &= (\phi(\sigma_0) - \phi(\sigma_1)) + (\phi(\sigma_1) - \phi(\sigma_2)) + ... + (\phi(\sigma_{n-1}) - \phi(\sigma_n)) \\ &= (\phi(\sigma_0) - \phi(\sigma_n)) \to \phi(\sigma_0) - \mathfrak{a} < \infty, \ as\ n \to \infty. \end{aligned}$$

It means that

$$\sum_{n=1}^{\infty} \frac{C_{n+1}}{C_n} < \infty.$$

Accordingly, we have

$$\lim_{n\to\infty} \frac{C_{n+1}}{C_n} = 0. \tag{2.3}$$

On account of (2.3), for $\varrho \in (0,1)$, there exist $n_0 \in \mathbb{N}$ such that

$$\frac{C_{n+1}}{C_n} \leq \varrho, \ \forall\ n \geq n_0. \tag{2.4}$$

yields that

$$d(\sigma_n, \sigma_{n+1}) \leq \varrho d(\sigma_{n-1}, \sigma_n), \ \forall\ n \geq n_0. \tag{2.5}$$

Now using ***Lemma* 3.1** [17] we obtain that the sequence (σ_n) is Cauchy. Thus from the completeness of $\mathcal{M}$, there exist $\sigma \in \mathcal{M}$ such that $\sigma_n \to \sigma$ as $n \to \infty$. By (iii), if $\mathcal{F}$ is $\mathcal{R}$-continuous then $\mathcal{F}\sigma_n \to \mathcal{F}\sigma$ as $n \to \infty$.

Alternately, let us assume that $\mathcal{R}$ is (*b*-*d*)-self-closed. As (σ_n) is an $\mathcal{R}$ preserving sequence and $\sigma_n \to \sigma$ as $n \to \infty$. So there exist a subsequence (σ_{n_k}) of (σ_n) with $[\sigma_{n_k}, \sigma] \in \mathcal{R}$, $\forall\ k \in \mathbb{N}_0$. Notice that $[\sigma_{n_k}, \sigma] \in \mathcal{R}$, $\forall\ k \in \mathbb{N}_0$ implies that either $(\sigma_{n_k}, \sigma) \in \mathcal{R}$, $\forall\ k \in \mathbb{N}_0$ or $(\sigma, \sigma_{n_k}) \in \mathcal{R}$, $\forall\ k \in \mathbb{N}_0$. Applying condition (iv) to $(\sigma_{n_k}, \sigma) \in \mathcal{R}$, $\forall\ k \in \mathbb{N}_0$, we have

$$\begin{aligned} 0 &\leq \zeta(sd(\mathcal{F}\sigma_{n_k}, \mathcal{F}\sigma), (\phi(\sigma_{n_k}) - \phi(\mathcal{F}(\sigma_{n_k}))d(\sigma_{n_k}, \sigma)) \\ &< (\phi(\sigma_{n_k}) - \phi(\mathcal{F}(\sigma_{n_k}))d(\sigma_{n_k}, \sigma) - sd(\mathcal{F}\sigma_{n_k}, \mathcal{F}\sigma) \end{aligned}$$

$$\Longrightarrow \qquad sd(\mathcal{F}\sigma_{n_k}, \mathcal{F}\sigma) < (\phi(\sigma_{n_k}) - \phi(\mathcal{F}(\sigma_{n_k}))d(\sigma_{n_k}, \sigma).$$

Using the triangle inequality together with the inequality above, we derive that

$$\begin{aligned} d(\sigma, \mathcal{F}\sigma) &\leq s[d(\sigma, \sigma_{n_k+1}) + d(\sigma_{n_k+1}, \mathcal{F}\sigma)] \\ &= s[d(\sigma, \sigma_{n_k+1}) + d(\mathcal{F}\sigma_{n_k}, \mathcal{F}\sigma)] \\ &< sd(\sigma, \sigma_{n_k+1}) + (\phi(\sigma_{n_k}) - \phi(\mathcal{F}(\sigma_{n_k}))d(\sigma_{n_k}, \sigma), \end{aligned}$$

putting $n \to \infty$ and using $\sigma_{n_k} \to \sigma$, above inequality $\to 0$ as $n \to \infty$. Consequently, we obtain that $d(\sigma, \mathcal{F}\sigma) = 0$, i.e., $\mathcal{F}\sigma = \sigma$.

Similarly, if $(\sigma, \sigma_{n_k}) \in \mathcal{R}$, $\forall\ k \in \mathbb{N}_0$, we obtain $d(\mathcal{F}\sigma, \sigma) = 0$. So that $\mathcal{F}\sigma = \sigma$, i.e., σ is a fixed point of $\mathcal{F}$.

Theorem* 2.2.** *Under the conditions of* ***Theorem* 2.1, if* *(v)* $\gamma(\sigma, \rho, \mathcal{R}) \neq \phi$,

then $\mathcal{F}$ *has a unique fixed point.*

Proof. Let σ^*, ρ^* are two fixed point of $\mathcal{F}$ such that $\sigma^* \neq \rho^*$. Since $\gamma(\sigma^*, \rho^*, \mathcal{R}) \neq \phi$, there exists a path $(\eta_0, \eta_1, \eta_2, ..., \eta_k)$ of some finite length k in $\mathcal{R}$ from σ to ρ so that

$$\eta_0 = \sigma^*, \eta_k = \rho^*, (\eta_i, \eta_{i+1}) \in \mathcal{R}, i = 0, 1, 2, ..., k-1.$$

Since $\mathcal{R}$ is transitive,

$$(\eta_0, \eta_k) \in \mathcal{R}.$$

Therefore

$$\begin{aligned} 0 &\leq \zeta(sd(\mathcal{F}\eta_0, \mathcal{F}\eta_k), (\phi(\eta_0) - \phi(\mathcal{F}\eta_0))d(\eta_0, \eta_k)) \\ &< (\phi(\eta_0) - \phi(\mathcal{F}\eta_0))d(\eta_0, \eta_k) - sd(\mathcal{F}\eta_0, \mathcal{F}\eta_k) \\ &= (\phi(\sigma^*) - \phi(\mathcal{F}\sigma^*))d(\sigma^*, \rho^*) - sd(\mathcal{F}\sigma^*, \mathcal{F}\rho^*) < 0, \end{aligned}$$

which is a contradiction. Thus $\mathcal{F}$ has a unique fixed point.

***Example* 2.1.** *Let* $(\mathcal{M}, d) = [1, 4]$ *and* $d(\sigma, \rho) = (\sigma - \rho)^2$. *Then* $(\mathcal{M}, d, s)$ *be a complete b-metric space with coefficient* $s = 2$. *Define a binary relation*

$$\mathcal{R} = \{(1,1), (1,2), (1,3), (1,4), (2,1), (2,2), (2,3), (2,4), (3,1), (3,2), (3,3), (3,4)\}$$

on $\mathcal{M}$ *and the mapping* $\mathcal{F} : \mathcal{M} \to \mathcal{M}$ *by*

$$\mathcal{F}(\sigma) = \begin{cases} 1, & \text{if } 1 \leq \sigma \leq 2; \\ 2, & \text{if } 2 < \sigma \leq 3; \\ 3, & \text{if } 3 < \sigma \leq 4. \end{cases}$$

Let $\phi : \mathcal{M} \to [0, \infty)$ *defined by* $\phi(\sigma) = 3\sigma$, $\sigma \in [0, \infty)$. *Since for* $(\sigma, \rho) \in \mathcal{R}$ *we have* $(\mathcal{F}\sigma, \mathcal{F}\rho) \in \mathcal{R}$ *which implies that* $\mathcal{R}$ *is* $\mathcal{F}$ *- closed and transitive. For* $\sigma = 2, \mathcal{F}\sigma = 1$, $(\sigma, \mathcal{F}\sigma) \in \mathcal{R}$ *i.e.,* $\mathcal{M}(\mathcal{F}; \mathcal{R}) \neq \phi$. *If we take any* $\mathcal{R}$*- preserving sequence* (σ_n) *with* $\sigma_n \to \sigma$ *and* $(\sigma_n, \sigma_{n+1}) \in \mathcal{R}$, $\forall$ $n \in \mathbb{N}_0$, *this implies that there exists an integer* $N \in \mathbb{N}_0$ *such that* $\sigma_n = \sigma$ *for all* $n \geq N$. *So, we can take a subsequence* (σ_{n_k}) *of the sequence* (σ_n) *such that* $\sigma_{n_k} = \sigma$ *for all* $k \in \mathbb{N}_0$,. *which amounts to saying that* $[\sigma_{n_k}, \sigma] \in \mathcal{R}$, *for all* $k \in \mathbb{N}_0$. *Therefore,* $\mathcal{R}$ *is (b-d)-self-closed.*

Now, with a view to check that $\mathcal{F}$ *is Caristi-Banach type* $\mathcal{Z}^b_{\mathcal{R}}$*-contraction, let for all* $\sigma \in \mathcal{M}$ *such that* $d(\sigma, \mathcal{F}\sigma) > 0$ *and* $(\sigma, \rho) \in \mathcal{R}$, *(in this example* $\sigma \neq 1$*), we have*

$$0 \leq \zeta(sd(\mathcal{F}\sigma, \mathcal{F}\rho), (\phi(\sigma) - \phi(\mathcal{F}\sigma))d(\sigma, \rho)).$$

Thus all the hypotheses of ***Theorems* 2.1** *and* **2.2** *are verified. Hence* $\sigma = 1$ *is the unique fixed point of* $\mathcal{F}$.

***Remark* 2.1.** *It is interesting to see that in* ***Example* 2.1** *at* **(2.4)**, *it is not a b-simulation function* [6] *i.e.,*

$$\begin{aligned} 0 &\leq \zeta(2d(\mathcal{F}2, \mathcal{F}4), d(2, 4)) \\ &< d(2, 4) - 2d(1, 3) = -4 < 0, \end{aligned}$$

which is a contradiction and if we take the usual metric on place of $d(\sigma, \rho) = (\sigma - \rho)^2$, *then at the same point we notice that* $d(\mathcal{F}2, \mathcal{F}4) = d(2, 4)$. *Thus it is not satisfy the BCP* [2] *and also the contractive condition (iv) of* ***Theorem* 3.1** [1].

***Remark* 2.2.** *In* ***Example* 2.1**, *observe that the binary relation* $\mathcal{R}$ *is nonreflexive, nonirreflexive, nonsymmetric and nonantisymmetric. Therefore it is none of near-order, partial order, pre-order and tolerance. Thus, it is worth mentioning that* ***Theorem* 2.2** *is genuine extension and improvement of Alam and Imdad* [1] *in b-metric space.*

Acknowledgement. The authors are thankful to the referee for his valuable suggestions to bring the paper in its present form.

References

[1] A. Alam and M. Imdad, Relation-theoretic contraction principle, *J. Fixed Point Theory Appl.* **17** (4) (2015), 693–702.

[2] S. Banach, Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales, *Fundamenta Mathematicae*, **3** (1922), 133–181.

[3] M. Boriceanu, Strict fixed point theorems for multivalued operators in b-metric space, *Int. J. Modern math,* **4** (3) (2009), 285–301.

[4] J. Caristi, Fixed point theorem for mappings satisfying inwardness conditions. *Trans. Amer. Math. Soc.* **215** (1976), 241–251.

[5] S. Czerwik, Contraction mappings in b-metric spaces, *Acta Math. Inf. Univ. Ostrav.,* **1** (1993), 5–11.

[6] M. Demma, R. Saadati and P. Vetro, Fixed point results on b-metric space via Picard sequences and b-simulation functions, *Iranian Journal of Mathematical Sciences and Informatics,* **11** (1) (2016), 123–136.

[7] A. F. Roldán-López-de-Hierro, Antonio-Francisco, E. Karapinar, C. Roldán-López-de-Hierro and J. Martínez-Moreno, Coincidence point theorems on metric spaces via simulation functions, *Journal of Computational and Applied Mathematics,* **275** (2015), 345–355.

[8] E. Karapinar, Fixed points results via simulation functions. Filomat **30** (8) (2016), 2343–2350.

[9] E. Karapinar, F. Khojasteh and Z. D. Mitrović, A Proposal for Revisiting Banach and Caristi Type Theorems in b-Metric Spaces. *Mathematics*, **7** (4) (2019).

[10] F. Khojasteh, S. Shukla, S. Radenovic, A new approach to the study of fixed point theorems via simulation functions, *Filomat,* **29** (6) (2015), 1189–1194.

[11] B. Kolman, R. C. Busby and S. Ross, *Discrete Mathematical Structures*, 3rd ed., PHI Pvt. Ltd., New Delhi, 2000.

[12] S. Lipschutz, *Schaum's Outlines of Theory and Problems of Set Theory and Related Topics*, McGraw-Hill, New York, 1964.

[13] R. D. Maddux, Relation Algebras, *Studies in Logic and the Foundations of Mathematics*, Elsevier B. V., Amsterdam, 150, 2006.

[14] J. R. Roshan, V. Parvaneh and Z. Kadelberg, Common fixed point theorems for weakly isotone increasing mappings in orderd b-metric spaces, *J. Nonlinear Sci. Appl.,* **7** (2014), 229–245.

[15] B. Samet and M. Turinici, Fixed point theorems on a metric space endowed with an arbitrary binary relation and applications, *Commun. Math. Anal.,* **13** (2012), 82–97.

[16] K. Sawangsup, W. Sintunavarat, On solving nonlinear matrix equations in terms of b-simulation functions in b-metric spaces with numerical solutions. *Computational and Applied Mathematics,* **37** (5) (2018), 5829–5843.

[17] S. L. Singh, S. Czerwik, K. Król and A. Singh, Coincidence and fixed points of hybried contraction, *Tamsui Oxf. J. Math Sci.,* **24** (2008), 401–416.

Jñānābha, Vol. 50(1) (2020), 49-56
(Dedicated to Honor Professor H.M. Srivastava on His 80th Birth Anniversary Celebrations)

ON HYPERSURFACE OF THE FINSLER SPACE OBTAINED BY CONFORMAL $\beta-$ CHANGE

By

H. S. Shukla[1], Neelam Mishra[2] and Vivek Shukla[3]
[1&2]Department of Mathematics & Statistics
DDU Gorakhpur University,Gorakhpur-273009, Uttar Pradesh, India
[3]Department of Mathematics, K.D.R.T.P.G. College, Dumari Khas,
Gorakhpur-273202, U.P., India
Email:profhsshuklagkp@rediffmail.com, pneelammishra@gmail.com,
shukladrvivek13@gmail.com (corresponding author)
(*Received : February* 26, 2020 *; Revised: March* 09, 2020)
DOI: https://doi.org/10.58250/jnanabha.2020.50106

Abstract

The conformal $\beta-$ change of Finsler metric $L(x, y)$ is given by $L^*(x, y) = e^{\sigma(x)} f(L(x, y), \beta(x, y))$, where $\sigma(x)$ is a function of x, $\beta(x, y) = b_i(x)y^i$ is a one-form on the underlying manifold M^n, and $f(L(x, y),\beta(x, y))$ is a homogeneous function of degree one in L and β. Let F^n and F^{*n} be Finsler spaces with metric functions L and L^* respectively. In this paper we study the hypersurface of F^{*n} and find condition under which this hypersurfcae becomes a hyperplane of first kind, a hyperplane of second kind and a hyperplane of third kind. In this endeavour we connect quantities of F^{*n} with those of F^n. When the hypersurface of F^{*n} is a hyperplane of first kind, we investigate the conditions under which it becomes a Landsberg space, a Berwald space, or a locally Minkowskian space.
2010 Mathematics Subject Classifications: 53B40, 53C60.
Keywords and phrases: Finsler space, Hypersurface, Cartan-parallel, Hyperplane, Conformal $\beta-$change, Homothetic $\beta-$change.

1 Introduction

Let $F^n = (M^n, L)$ be an $n-$dimentional Finsler space on the differentiable manifold M^n equipped with the fundamental function $L(x, y)$. B. N. Prasad and Bindu Kumari [10] have studied the general $\beta-$change, i.e., $L'(x, y) = f(L,\beta)$, where f is positively homogeneous function of degree one in L and β, $\beta(x, y) = b_i(x)y^i$ is a one-form on M^n.

The conformal theory of Finsler space has been dealt by M. Hashiguchi [3], H. Izumi [4, 5], M. Kitayama [6], S. H. Abed [1, 2]. The conformal change is given by $L''(x, y) = e^{\sigma(x)}L(x, y)$, where $\sigma(x)$ is a function of position only. In 2009 and 2010, Nabil L. Youssef, S. H. Abed and S. G. Elgendi [14, 15] introduced the transformation $L''(x, y) = f(e^{\sigma}L,\beta)$, which is general $\beta-$change of conformally changed Finsler metric L.

H. S. Shukla and Neelam Mishra [11] have changed the order of combination of the above two changes as

$$L^*(x, y) = e^{\sigma(x)} f(L(x, y),\beta(x, y)), \tag{1.1}$$

where $\sigma(x)$ is a function of x only and $\beta(x, y) = b_i(x)y^i$ is a one-form on M^n. They have called this change as conformal $\beta-$change of Finsler metric and have studied its geometrical properties in [11] and [12]. When $\sigma = 0$, the change (1.1) reduces to general $\beta-$change. When $\sigma =$ constant, it becomes a homothetic β−change. Some properties of homothetic β−change with $b_i(x)$ as Cartan-parallel have been studied by H. S. Shukla and Neelam Mishra in [13].

In 1985 M. Matsumoto [7] studied the theory of Finsler hypersurfaces. In 2011, S. K. Narasimhamurthy et al. [9] have considered hypersurface of Finsler space with metric $L'''(x, y) = f(e^{\sigma}L, \beta)$ and studied its geometric properties.

In this paper we shall study the hypersurface of $F^{*n} = (M^n, L^*)$.

2 Hypersurface of F^n

The metric tensor $g_{ij}(x, y)$ and Cartan's C−tensor $C_{ijk}(x, y)$ of F^n are given by

$$g_{ij} = \frac{1}{2}\frac{\partial^2 L^2}{\partial y^i\, \partial y^j}, \qquad C_{ijk} = \frac{1}{2}\frac{\partial g_{ij}}{\partial y^k}$$

respectively. Let $C\Gamma = (F^i_{jk}, G^i_j, C^i_{jk})$ denote the Cartan's connection of F^n.

Let $(M^{n-1}, \overline{L})$ be a hypersurface of (M^n, L) given by

$$x^i = x^i(u^\alpha). \tag{2.1}$$

Let us suppose that the functions (2.1) are at least of class C^3 in u^α and the projection factor $B^j_\beta = \frac{\partial x^j}{\partial u^\beta}$ are such that their matrix has maximum rank $(n-1)$. We shall use the following notations:

$$B^i_{\alpha\beta} = \frac{\partial^2 x^i}{\partial u^\alpha \partial u^\beta}, \qquad B^i_{0\beta} = v^\alpha B^i_{\alpha\beta}, \qquad B^{ij}_{\alpha\beta} = B^i_\alpha B^j_\beta,$$

where v^α is the element of support for the hypersurface satisfying the relation $y^i = B^i_\alpha(u)v^\alpha$. The fundamental metric function of the hypersurface is given by

$$\overline{L}(u^\alpha, v^\alpha) = L(x^i(u^\alpha), B^i_\alpha v^\alpha).$$

At each point (u^α) of F^{n-1} the unit normal vector $N^i(u, v)$ is defined by

$$g_{ij}B^i_\alpha N^j = 0, \qquad g_{ij}\, N^i\, N^j = 1. \tag{2.2}$$

If (B^α_i, N_i) is the inverse matrix of (B^i_α, N^i), we have

$$B^i_\alpha B^\beta_i = \delta^\beta_\alpha, \quad B^i_\alpha N_i = 0, \quad N^i N_i = 1 \ \text{ and } \ B^i_\alpha B^\alpha_j + N^i N_j = \delta^i_j. \tag{2.3}$$

Making use of the inverse matrix $(g^{\alpha\beta})$ of $(g_{\alpha\beta})$, we get

$$B^\alpha_i = g^{\alpha\beta} g_{ij} B^j_\beta. \tag{2.4}$$

For the induced Cartan's connection $IC\Gamma = (F^\alpha_{\beta\gamma}, G^\alpha_\beta, C^\alpha_{\beta\gamma})$ of F^{n-1} induced from the Cartan's connection $C\Gamma = (F^i_{jk}, G^i_j, C^i_{jk})$ of F^n, the second fundamental h−tensor $H_{\alpha\beta}$ and the normal curvature vector H_β are respectively given by [8]:

$$H_{\alpha\beta} = N_i(B^i_{\alpha\beta} + F^i_{jk}B^j_\alpha B^k_\beta) + M_a H_\beta, \qquad H_\beta = N_i(B^i_{0\beta} + F^i_{0j}B^j_\beta), \tag{2.5}$$

where $M_\alpha = C_{ijk}B^i_\alpha N^j N^k$.

Contracting $H_{\alpha\beta}$ by v^α, we get $H_{0\beta} = H_{\alpha\beta}v^\alpha = H_\beta$. The second fundamental v−tensor $M_{\alpha\beta}$ is given by [8]:

$$M_{\alpha\beta} = C_{ijk}B^i_\alpha B^j_\beta N^k. \tag{2.6}$$

The Gauss characteristic equation with respect to $IC\Gamma$ is written as

$$R_{\alpha\beta\gamma\delta} = R_{ijkh}B^{ijkh}_{\alpha\beta\gamma\delta} + P_{ijkh}(B^h_\gamma H_\delta - B^h_\delta H_\gamma)B^{ij}_{\alpha\beta}H^k + H_{\alpha\gamma}H_{\beta\delta} - H_{\alpha\delta}H_{\beta\gamma}. \tag{2.7}$$

3 Fundamental quantities of F^{*n}

We shall denote the quantities corresponding to F^{*n} by putting star on the top of them. Differentiating equation (1.1) with respect to y^i, we have

$$l_i^* = e^{\sigma}(f_1 l_i + f_2 b_i), \tag{3.1}$$

where the subscripts 1 and 2 denote the partial derivatives with respect to L and β respectively.

Differentiating (3.1) with respect to y^j, we get

$$h_{ij}^* = e^{2\sigma}(ph_{ij} + q_0 m_i m_j), \tag{3.2}$$

where $m_i = b_i - \frac{\beta}{L}L_i, \quad p = \frac{ff_1}{L}, \quad q_0 = fL^2 w, \quad w = \frac{f_{11}}{\beta^2} = \frac{-f_{12}}{L\beta} = \frac{f_{22}}{L^2}$.

From (3.1) and (3.2), we get the following relation between metric tensors of F^n and F^{*n}:

$$g_{ij}^* = h_{ij}^* + l_i^* l_j^* = e^{2\sigma}\{pg_{ij} + p_0 b_i b_j + p_1(l_i b_j + l_j b_i) + p_2 l_i l_j\}, \tag{3.3}$$

where

$$p_0 = q_0 + f_2^2, \quad p_1 = ff_1 - fL\beta w, \quad p_2 = \frac{(ff_1 - fL\beta w)\beta}{L}. \tag{3.4}$$

From (3.3), we get the following relations between Cartan's $C-$tensors of F^n and F^{*n}:

$$C_{ijk}^* = e^{2\sigma}\{pC_{ijk} + \frac{p}{2L}(h_{ij}m_k + h_{jk}m_i + h_{ki}m_j) + \frac{qL^2}{2}m_i m_j m_k\}, \tag{3.5}$$

where $q = 3f_2 w + fw_2$;

$$\begin{aligned} C_{ij}^{*h} =& C_{ij}^h + \frac{p}{2ff_1}(h_{ij}m^h + h_j^h m_i + h_i^h m_j) + \frac{qL^3}{2ff_1}m_i m_j m^h \\ &- \frac{L}{ft}C_{.ij}n^h - \frac{pL\triangle}{2f^2 f_1 t}h_{ij}n^h - \frac{(2pL + qL^4\triangle)}{2f^2 f_1 t}m_i m_j n^h, \end{aligned} \tag{3.6}$$

where $C_{.jk} = C_{ijk}b^i$, $n^h = fL^2 wb^h + pl^h$, $h_j^i = g^{il}h_{lj}$.

The spray coefficient of F^{*n} is given by [13]:

$$G^{*i} = \frac{1}{2}\gamma_{jk}^{*i}y^j y^k = G^i + D^i, \tag{3.7}$$

where the vector D^i is given by

$$D^i = \frac{f_2 L}{f_1}s_0^i - \frac{L}{ff_1 t}(f_1 r_{00} - 2Lf_2 s_{r0}b^r)(py^i - L^2 wfb^i) + \sigma_0 y^i - \frac{1}{2}f^2\sigma^i, \tag{3.8}$$

with

$$2r_{ij} = b_{i|j} + b_{j|i}, \quad 2s_{ij} = b_{i|j} - b_{j|i}, \quad s_0^i = g^{ir}s_{rj}y^j, \quad \sigma_k = \frac{\partial\sigma}{\partial x^k}, \quad \sigma^i = g^{ij}\sigma_j. \tag{3.9}$$

and '0' standing for contradiction with y^k, viz., $\sigma_0 = \sigma_k y^k$, $s_{r0} = s_{rk}y^k$, etc.

The Cartan's non-linear connection of F^{*n} is given by [13]:

$$G_j^{*i} = G_j^i + D_j^i, \tag{3.10}$$

where the tensor $D_j^i = \dot{\partial}_j D^i$ is given by

$$\begin{aligned} D_j^i = & \frac{Le^{2\sigma}}{ff_1}A_j^i - Q^i A_{rj}b^r + \frac{pLf_2}{f^2 f_1^2 t}b_{0|j}\{-Lf_1 b^i + (f\beta - L^2\triangle f_2)y^i\} \\ & + \sigma_j y^i - f\sigma^i(f_1 f_j + f_2 b_j), \end{aligned} \tag{3.11}$$

in which

$$A_{ij} = \frac{1}{2}r_{00}B_{ij} + e^{2\sigma}ff_2 s_{ij} + s_{i0}Q_j - (\frac{e^{2\sigma}ff_1}{L}C_{imj} + V_{ijm})D^m,$$

$$A^i_j = g^{ir}A_{rj}, \quad V_{ijm} = g_{sj}V^s_{im}, \quad Q_i = e^{2\sigma}\{(p + fL^2 w)y_i + f_2^2 b_i\},$$
$$B_{jk} = \frac{1}{2}e^{2\sigma}(ph_{jk} + \frac{1}{2}qL^2 m_j m_k), \quad \dot{\partial}_k Q_j = \frac{1}{2}B_{jk}.$$

The Berwald's connection coefficient of F^{*n} is given by [13]:

$$G^{*i}_{jk} = G^i_{jk} + B^i_{jk}, \tag{3.11a}$$

where $B^i_{jk} = \dot{\partial}_k D^i_j$. The Cartan's connection coefficient of F^{*n} is given by [13]:

$$F^{*i}_{jk} = F^i_{jk} + D^i_{jk}, \tag{3.12}$$

where

$$\begin{aligned} D^i_{jk} =&[\frac{e^{-2\sigma}L}{ff_1}g^{is} - Q^i b^s + y^s \frac{e^{-2\sigma}pL}{f^3 f_1 t}\{-Lfb^i + (f\beta - \triangle L^2 f_2)y^i\}] \\ &(B_{sj}b_{0|k} + B_{sk}b_{0|j} - B_{kj}b_{0|s} + s_{sj}Q_k + s_{sk}Q_j + r_{kj}Q_s \\ &+ \frac{ff_1 e^{2\sigma}}{L}C_{jkr}D^r_s + V_{jkr}D^r_s - \frac{ff_1 e^{2\sigma}}{L}C_{skm}D^m_j - V_{sjm}D^m_k \\ &- \frac{ff_1 e^{2\sigma}}{L}C_{sjm}D^m_k - V_{skm}D^m_j) - e^{-2\sigma}\sigma^i g^*_{jk}. \end{aligned} \tag{3.13}$$

The tensor D^i_{jk}, called the difference tensor, has the following properties:

$$D^i_{j0} = B^i_{j0} = D^i_j, \qquad D^i_{00} = 2D^i. \tag{3.14}$$

The $(v)h-$torsion tensor of F^n is defined as [8]:

$$R^i_{jk} = \delta_k G^i_j - \delta_j G^i_k, \qquad \delta_k = \partial_k - G^r_k \dot{\partial}_r, \tag{3.15}$$

the $h-$curvature tensor of F^n is defined as [8]:

$$R^i_{hjk} = \theta(j,k)[\delta_k F^i_{hj} - F^m_{hj}F^i_{mk} + C^i_{hm}R^m_{jk}], \tag{3.16}$$

and the $(v)hv-$torsion tensor of F^n is defined as [8]:

$$P^i_{jk} = \dot{\partial}_k G^i_j - F^i_{jk}. \tag{3.17}$$

4 Hypersurface of F^{*n}

Let us consider a hypersurface $F^{n-1} = (M^{n-1}, \overline{L}(u,v))$ of F^n and another hypersurface $F^{*n-1} = (M^{n-1}, \overline{L}^*(u,v))$ of F^{*n}, both hypersurfaces being represented by the same equation (2.1). Let N^i be the unit normal vector at each point of F^{n-1}, which is invariant under the conformal $\beta-$change. The unit normal vector $N^{*i}(u,v)$ of F^{*n-1} is uniquely determined by

$$g^*_{ij}B^i_\alpha N^{*i} = 0, \qquad g^*_{ij}N^{*i}N^{*j} = 1. \tag{4.1}$$

Transvecting the first equation of (2.2) by v^α, we get

$$y_i N^i = 0. \tag{4.2}$$

Contracting (3.3) by $N^i N^j$ and using (2.2) and (4.2), we have

$$g^*_{ij}N^i N^j = e^{2\sigma}\{p + p_0(b_i N^i)^2\}. \tag{4.3}$$

This gives

$$g^*_{ij}[\pm\frac{N^i}{e^\sigma\sqrt{\{p + p_0(b_i N^i)^2\}}}][\pm\frac{N^j}{e^\sigma\sqrt{\{p + p_0(b_i N^i)^2\}}}] = 1.$$

Hence, we can put

$$N^{*i} = \frac{N^i}{e^{\sigma}\sqrt{\{p + p_0(b_iN^i)^2\}}}, \tag{4.4}$$

where we have chosen the positive sign of the radical in order to fix the orientation. Using equations (2.2) and (4.2), the first condition of (4.1) gives

$$(p_0b_iB^i_\alpha + p_1l_iB^i_\alpha)\frac{b_jN^j}{e^{\sigma}\sqrt{\{p + p_0(b_iN^i)^2\}}} = 0. \tag{4.5}$$

Suppose that $p_0b_iB^i_\alpha + p_1l_iB^i_\alpha = 0$. Then transvecting it by v^α, we get $p_0\beta + p_1L = 0$. But this is impossible as L and β are independent. Hence

$$b_jN^j = 0. \tag{4.6}$$

Therefore (4.4) is rewritten as

$$N^{*i} = \frac{1}{e^{\sigma}\sqrt{p}}N^i. \tag{4.7}$$

Thus, we have

***Proposition* 4.1.** *For a field of linear frame (B^i_α, N^i) of F^n, there exists a field of linear frame (B^i_α, N^{*i}) of F^{*n} such that the conditions* (4.1) *are satisfied along F^{*n-1} and b_i is tangential to both the hypersurfaces F^{n-1} and F^{*n-1}.*

The quantities $B^{*\alpha}_i$ are uniquely defined along F^{*n-1} by

$$B^{*\alpha}_i = g^{*\alpha\beta}g^*_{ij}B^j_\beta,$$

where $(g^{*\alpha\beta})$ is the inverse matrix of $(g^*_{\alpha\beta})$. Let $(B^{*\alpha}_i, N^*_i)$ be the inverse matrix of (B^i_α, N^{*i}). Then, we have

$$B^i_\alpha B^{*\beta}_i = \delta^\beta_\alpha, \qquad B^i_\alpha N^*_i = 0, \qquad N^{*i}N^*_i = 1, \qquad B^{*\alpha}_i N^{*i} = 0.$$

Also, $B^i_\alpha B^{*\alpha}_j + N^{*i}N^*_j = \delta^i_j$, where

$$N^*_i = e^{\sigma}\sqrt{p}\,N_i. \tag{4.8}$$

From (4.8) *and* (2.5)*, we get*

$$H^*_\alpha = e^{\sigma}\sqrt{p}\,N_i(B^i_{0\beta} + F^{*i}_{0j}B^j_\beta). \tag{4.9}$$

If each path of a hypersurface F^{n-1} with respect to the induced connection is also a path of the enveloping space F^n, then F^{n-1} is called a hyperplane of the first kind. A hyperplane of the first kind is characterized by $H_\alpha = 0$.

We shall use the following theorem which has been proved in [13]*:*

***Theorem* 4.1.** *Under the conformal $\beta-$change* (1.1) *consider the following two assertions:*

(1) *The covariant vector field $b_i(x)$ is Cartan-parallel.*
(2) *The difference tensor D^i_{jk} vanishes identically.*

Then we have:

(a) *If* (1) *and* (2) *hold, then σ is homothetic.*
(b) *If σ is homothetic, then* (1) *and* (2) *are equivalent.*

Let σ be homothetic and $b_i(x)$ be Cartan-parallel in F^n. Then from (4.9)*,* (2.5) *and* ***Theorem* 4.1.***, we get*

$$H^*_\alpha = e^\sigma \sqrt{p}\, H_\alpha. \tag{4.10}$$

From (4.10) we find that $H^*_\alpha = 0$ iff $H_\alpha = 0$. Thus we have the theorem:

***Theorem* 4.2.** *Let σ be homothetic and $b_i(x)$ be Cartan-parallel in F^n. Then the hypersurface F^{*n-1} is a hyperplane of the first kind iff the hypersurface F^{n-1} is a hyperplane of the first kind.*

Next, contracting (3.5) *by $B^i_\alpha N^{*j} N^{*k}$, making use of* (4.7)*, $M_\alpha = C_{ijk} B^i_\alpha N^j N^k$, $m_i N^i = 0$, $b_i N^i = 0$, $h_{jk} N^j N^k = 1$ and $h_{ij} B^i_\alpha N^j = 0$, we get*

$$M^*_\alpha = M_\alpha. \tag{4.11}$$

To compute $H^*_{\alpha\beta}$ we use (2.5)*,* (4.7)*,* (4.10)*,* (4.11) *and* ***Theorem* 4.1** *to get*

$$H^*_{\alpha\beta} = e^\sigma \sqrt{p}\, H_{\alpha\beta}. \tag{4.12}$$

If each $h-$path of a hypersurface F^{n-1} with respect to the induced connection is also an $h-$path of the enveloping space F^n, then F^{n-1} is called a hyperplane of the second kind. A hyperplane of the second kind is characterized by $H_\alpha = 0$, $H_{\alpha\beta} = 0$. From (4.12) we find that $H^*_{\alpha\beta} = 0$ iff $H_{\alpha\beta} = 0$. Therefore from (4.10) and (4.12) we have the theorem:

***Theorem* 4.3.** *Let σ be homothetic and $b_i(x)$ be Cartan-parallel in F^n. Then the hypersurface F^{*n-1} is a hyperplane of the second kind iff the hypersurface F^{n-1} is a hyperplane of the second kind.*

Finally, contracting (3.5) *by $B^i_\alpha B^j_\beta N^{*k}$ and making use of* (2.6)*,* (4.7)*, $m_i N^i = 0$, $h_{ij} B^i_\alpha N^j = 0$ and* ***Theorem* 4.1***, we have*

$$M^*_{\alpha\beta} = e^\sigma \sqrt{p}\, M_{\alpha\beta}. \tag{4.13}$$

If the unit normal vector of F^{n-1} is parallel along each curve of F^{n-1}, then F^{n-1} is called a hyperplane of the third kind. A hyperplane of the third kind is characterized by $H_\alpha = 0$, $H_{\alpha\beta} = 0$ and $M_{\alpha\beta} = 0$. Hence from (4.10), (4.12) and (4.13), we have the theorem:

***Theorem* 4.4.** *Let σ be homothetic and $b_i(x)$ be Cartan-parallel in F^n. Then the hypersurface F^{*n-1} is a hyperplane of the third kind iff the hypersurface F^{n-1} is a hyperplane of the third kind.*

For hyperplane of the first kind, the $(v)hv-$*torsion tensor is given by* [7]*:*

$$P^\alpha_{\beta\gamma} = B^\alpha_i K^i_{\beta\gamma}, \tag{4.14}$$

where

$$K^i_{\beta\gamma} = P^i_{\ jk} B^{jk}_{\beta\gamma}.$$

Using (4.14) *and the last relation of* (2.3)*, we get*

$$K^i_{\beta\gamma} = B^i_\delta P^\delta_{\beta\gamma} + N^i N_h K^h_{\beta\gamma}. \tag{4.15}$$

Under homothetic $\beta-$change with $b_i(x)$ as Cartan-parallel it has been proved in [13] *that*

(i) *a Landsberg space remains a Landsberg space,*
(ii) *a Berwald space remains a Berwald space,*
(iii) *a locally Minkowskian space remains a locally Minkowskian space.*

When σ is homothetic and $b_i(x)$ is Cartan-parallel, we have $K^{*i}_{\beta\gamma} = K^i_{\beta\gamma}$, and then it follows that

$$P^{*\alpha}_{\beta\gamma} = B^{*\alpha}_i K^i_{\beta\gamma}. \tag{4.16}$$

On substituting (4.15) *in* (4.16) *and using* (2.3), *we get*

$$P^{*\alpha}_{\beta\gamma} = P^\alpha_{\beta\gamma}. \tag{4.17}$$

Thus we have the theorem:

***Theorem* 4.5.** *Let σ be homothetic and $b_i(x)$ be Cartan-parallel in F^n. Then the hyperplane F^{*n-1} of the first kind is a Landsberg space iff the hyperplane F^{n-1} of the first kind is a Landsberg space.*

For hyperplane of the first kind, the Berwald connection coefficients $G^\alpha_{\beta\gamma}$ are given by [7]*:*

$$G^\alpha_{\beta\gamma} = B^\alpha_i A^i_{\beta\gamma}, \tag{4.18}$$

where

$$A^i_{\beta\gamma} = G^i_{jk} B^{jk}_{\beta\gamma} + B^i_{\beta\gamma}.$$

Using (4.18) *and the last relation of* (2.3), *we get*

$$A^i_{\beta\gamma} = B^i_\delta G^\delta_{\beta\gamma} + N^i N_h A^h_{\beta\gamma}. \tag{4.19}$$

When σ is homothetic and $b_i(x)$ is Cartan-parallel, we have $G^{*i}_{jk} = G^i_{jk}$. Then $A^{*i}_{\beta\gamma} = A^i_{\beta\gamma}$ and it follows that

$$G^{*\alpha}_{\beta\gamma} = B^{*\alpha}_i A^i_{\beta\gamma}. \tag{4.20}$$

On substituting (4.19) in (4.20) and using (2.3), we get

$$G^{*\alpha}_{\beta\gamma} = G^\alpha_{\beta\gamma}. \tag{4.21}$$

Then we have the theorem:

***Theorem* 4.6.** *Let σ be homothetic and $b_i(x)$ be Cartan-parallel in F^n. Then the hyperplane F^{*n-1} of the first kind is a Berwald space iff the hyperplane F^{n-1} of the first kind is a Berwald space.*

From (2.7) *the Gauss characteristic equation of hyperplane F^{n-1} of the first kind is written as*

$$R_{\alpha\beta\gamma\delta} = R_{ijkh} B^{ijkh}_{\alpha\beta\gamma\delta} + H_{\alpha\gamma} H_{\beta\delta} - H_{\alpha\delta} H_{\beta\gamma}. \tag{4.22}$$

The Gauss characteristic equation of hyperplane F^{*n-1} of the first kind is similarly written as

$$R^*_{\alpha\beta\gamma\delta} = R^*_{ijkh} B^{ijkh}_{\alpha\beta\gamma\delta} + H^*_{\alpha\gamma} H^*_{\beta\delta} - H^*_{\alpha\delta} H^*_{\beta\gamma}. \tag{4.23}$$

Making use of the equation (4.12)*, the above equation becomes*

$$R^*_{\alpha\beta\gamma\delta} = R^*_{ijkh} B^{ijkh}_{\alpha\beta\gamma\delta} + e^{2\sigma} p (H_{\alpha\gamma} H_{\beta\delta} - H_{\alpha\delta} H_{\beta\gamma}). \tag{4.24}$$

Equations (4.24) *and* (4.22) *together give*

$$R^*_{\alpha\beta\gamma\delta} = e^{2\sigma} p R_{\alpha\beta\gamma\delta} + (R^*_{ijkh} - e^{2\sigma} p R_{ijkh}) B^{ijkh}_{\alpha\beta\gamma\delta}. \tag{4.25}$$

We know that when σ is homothetic and $b_i(x)$ is Cartan-parallel, then if F^n is locally Minkowskian, so is F^{*n}; i.e. $R^*_{ijkh} = 0$ iff $R_{ijkh} = 0$. Under these conditions the equation (4.25) *reduces to*

$$R^*_{\alpha\beta\gamma\delta} = e^{2\sigma} p R_{\alpha\beta\gamma\delta}.$$

Thus, in view of the ***Theorem* 4.6**, we have the theorem:

***Theorem* 4.7.** *Let σ be homothetic, $b_i(x)$ be Cartan-parallel and F^n be a locally Minkowskian space. Then the hyperplane F^{*n-1} of the first kind is a locally Minkowskian space iff the hyperplane F^{n-1} of the first kind is a locally Minkowskian space.*

Acknowledgement. The authors are thankful to the Editor and referee for their valuable suggestions to bring the paper in its present form.

References

[1] S. H. Abed, Conformal $\beta-$changes in Finsler spaces, *Proc. Math. Phys. Soc., Egypt*, **86** (2008), 79-89.

[2] S. H. Abed, Cartan connections associated with a $\beta-$conformal change in Finsler geometry, *Tensor, N.S.*, **70** (2008), 146-158.

[3] M. Hashiguchi, On conformal transformations of Finsler metrics, *J. Math. Kyoto Univ.*, **16** (1976), 25-50.

[4] H. Izumi, Conformal transformations of Finsler spaces I, *Tensor, N.S.*, **31** (1977), 33-41.

[5] H. Izumi, Conformal transformations of Finsler spaces II, *Tensor, N.S.*, **33** (1980), 337-369.

[6] M. Kitayama, *Geometry of transformations of Finsler metrics*, Hokkaido University of Education, Kushiro Campus, Japan, 2000.

[7] M. Matsumoto, The induced and intrinsic connections of a hypersurface and Finslerian projective geometry, *J. Math. Kyoto Univ.*, **25** (1985), 107-144.

[8] M. Matsumoto, *Foundations of Finsler geometry and special Finsler spaces*, Kaiseisha Press, Saikawa, Otsu, Japan, 1986.

[9] S. K. Narasimhamurthy, G. N. Latha and C. S. Bagewadi, Geometric properties of Finsler hypersurface given by conformal $\beta-$change, *Analeb Universitatii de Vest din Timisoara. Seria Mathematica-Informatica*, **49** (2011), 77-87.

[10] B. N. Prasad and Bindu Kumari, The $\beta-$change of Finsler metric and imbedding classes of their tangent spaces, *Tensor, N.S.*, **74** (2013), 48-59.

[11] H. S. Shukla and Neelam Mishra, Conformal $\beta-$change of Finsler metric, *J. Int. Acad. Phys. Sci.*, **21(1)** (2017), 19-30.

[12] H. S. Shukla and Neelam Mishra, Some intrinsic properties of conformal $\beta-$change, *Int. J. Math. Combin.*, **2** (2018), 13-23.

[13] H. S. Shukla and Neelam Mishra, Homothetic $\beta-$change of Finsler metric, *J. Rajasthan Acad. Phys. Sci.*, **17(3&4)** (2018), 173-180.

[14] N. L. Youssef, S. H. Abed and S. G. Elgendi, Generalized $\beta-$conformal change of Finsler metrics, *Int. J. Geom. Meth. Mod. Phys.*, **7(4)** (2010). ArXiv No.: math. DG/0906.5369.

[15] N. L. Youssef, S. H. Abed and S. G. Elgendi, Generalized $\beta-$conformal change and special Finsler spaces, *Int. J. Geom. Meth. Mod. Phys.*, **8(6)** (2010). ArXiv No.: math. DG/1004.5478v3.

Jñānābha, Vol. 50(1) (2020), 57-64
(Dedicated to Honor Professor H.M. Srivastava on His 80th Birth Anniversary Celebrations)

ADJOINTNESS FOR SHEFFER POLYNOMIALS

By

Paolo Emilio Ricci
International Telematic University UniNettuno
Corso Vittorio Emanuele II, 39, 00186, Roma (Italia)
Email:paoloemilioricci@gmail.com

(*Received : March* 09, 2020 *; Revised: March* 27, 2020)
DOI: https://doi.org/10.58250/jnanabha.2020.50107

Abstract

In recent papers, new sets of Sheffer and Brenke polynomials based on higher order Bell numbers, and several integer sequences related to them have been studied. In this article new sets of Sheffer polynomials are introduced defining a sort of adjointness property. As an application, we show the adjoint set of Actuarial polynomials and derive their main characteristics.

2010 Mathematics Subject Classifications: 33C99, 34L99, 11B73
Keywords and phrases: Sheffer polynomials, Generating functions, Monomiality principle, Actuarial polynomials, Combinatorial analysis.

1 Introduction

In recent articles [24, 8], new sets of Sheffer and Brenke polynomials [7] based on higher order Bell numbers, have been studied. Furthermore, several integer sequences associated with the considered polynomials sets both of exponential and logarithmic type have been introduced.

In this article adjoint sets of Sheffer polynomials are considered and a particular case is analyzed.

2 Sheffer polynomials

The Sheffer polynomials $\{s_n(x)\}$ are introduced [26] by means of the exponential generating function [28] of the type:

$$A(t)\exp(xH(t)) = \sum_{n=0}^{\infty} s_n(x)\frac{t^n}{n!}\,, \tag{2.1}$$

where

$$A(t) = \sum_{n=0}^{\infty} a_n\frac{t^n}{n!}\,, \qquad (a_0 \neq 0), \tag{2.2}$$

$$H(t) = \sum_{n=0}^{\infty} h_n\frac{t^n}{n!}\,, \qquad (h_0 = 0).$$

According to a different characterization (see [25, p. 18]), the same polynomial sequence can be defined by means of the pair $(g(t), f(t))$, where $g(t)$ is an invertible series and $f(t)$ is a delta series:

$$g(t) = \sum_{n=0}^{\infty} g_n\frac{t^n}{n!}\,, \qquad (g_0 \neq 0), \tag{2.3}$$

$$f(t)=\sum_{n=0}^{\infty} f_n \frac{t^n}{n!}\,, \qquad (f_0=0, f_1\neq 0).$$

Denoting by $f^{-1}(t)$ the compositional inverse of $f(t)$, i.e. such that

$$f\left(f^{-1}(t)\right)=f^{-1}\left(f(t)\right)=t,$$

the exponential generating function of the sequence $\{s_n(x)\}$ is given by

$$\frac{1}{g[f^{-1}(t)]}\exp\left(xf^{-1}(t)\right)=\sum_{n=0}^{\infty} s_n(x)\frac{t^n}{n!}\,, \tag{2.4}$$

so that

$$A(t)=\frac{1}{g[f^{-1}(t)]}, \qquad H(t)=f^{-1}(t). \tag{2.5}$$

When $g(t)\equiv 1$, the Sheffer sequence corresponding to the pair $(1, f(t))$ is called the associated Sheffer sequence $\{\sigma_n(x)\}$ for $f(t)$, and its exponential generating function is given by

$$\exp\left(xf^{-1}(t)\right)=\sum_{n=0}^{\infty} \sigma_n(x)\frac{t^n}{n!}\,. \tag{2.6}$$

A list of known Sheffer polynomial sequences and their associated ones can be found in [5, 6].

3 Adjointness for Sheffer polynomial sequences

According to the above considerations, Sheffer polynomials are characterized both by the ordered couples $(A(t), H(t))$, or by $(g(t), f(t))$.

***Definition* 3.1.** *Adjoint Sheffer polynomials are defined by interchanging the ordered couple* $(A(t), H(t))$ *with* $(g(t), f(t))$*, when writing the generating function.*

Here and in the following the tilde *"∼", above the symbol of a polynomial set stands for the adjective* "adjoint"*.*

Examples of adjoint polynomial sets are listed here.

Adjoint-Hermite polynomials

$$A(t)=\exp(t^2/4), \qquad H(t)=t/2, \tag{3.1}$$

$$G(t,x)=\exp\left[\frac{t(t+2x)}{4}\right]=\sum_{n=0}^{\infty}\tilde{H}e_n(x)\frac{t^n}{n!}.$$

Adjoint-generalized Hermite polynomials

$$A(t)=\exp\left[(t/\nu)^m\right], \qquad H(t)=t/\nu, \tag{3.2}$$

$$G(t,x)=\exp\left[\left(\frac{t}{\nu}\right)^m+\frac{xt}{\nu}\right]=\sum_{n=0}^{\infty}\tilde{H}_n^{(m)}(x)\frac{t^n}{n!}.$$

Adjoint modified Pidduck polynomials

$$A(t)=\frac{2}{e^t+1}, \qquad H(t)=\frac{e^t-1}{e^t+1}, \tag{3.3}$$

$$G(t,x)=\frac{2}{e^t+1}\exp\left[x\left(\frac{e^t-1}{e^t+1}\right)\right]=\sum_{n=0}^{\infty}\tilde{\mathcal{P}}_n(x)\frac{t^n}{n!}.$$

Adjoint Actuarial polynomials

$$A(t)=(1-t)^{-\beta}, \qquad H(t)=\log(1-t), \tag{3.4}$$

$$G(t,x) = (1-t)^{x-\beta} = \sum_{n=0}^{\infty} \tilde{\alpha}_n^{(\beta)}(x)\frac{t^n}{n!}.$$

Adjoint Poisson-Charlier polynomials

(3.5) $$A(t) = \exp\left(a(e^t-1)\right), \qquad H(t) = a\left(e^t - 1\right),$$

$$G(t,x) = \exp\left(a(1+x)(e^t-1)\right) = \sum_{n=0}^{\infty} \tilde{c}_n(x;a)\frac{t^n}{n!}.$$

Adjoint Peters polynomials

(3.6) $$A(t) = \left(1+e^{\lambda t}\right)^{\mu}, \qquad H(t) = e^t - 1,$$

$$G(t,x) = \left(1+e^{\lambda t}\right)^{\mu} \exp\left(x(e^t-1)\right) = \sum_{n=0}^{\infty} \tilde{s}_n(x;\lambda,\mu)\frac{t^n}{n!}.$$

Adjoint Bernoulli polynomials of the second kind

(3.7) $$A(t) = \frac{t}{e^t-1}, \qquad H(t) = e^t - 1,$$

$$G(t,x) = \frac{t}{e^t-1}\exp\left(x\left(e^t-1\right)\right) = \sum_{n=0}^{\infty} \tilde{b}_n(x)\frac{t^n}{n!}.$$

Adjoint Related polynomials

(3.8) $$A(t) = \frac{1+e^t}{2}, \qquad H(t) = e^t - 1,$$

$$G(t,x) = \frac{1+e^t}{2}\exp\left[x\left(e^t-1\right)\right] = \sum_{n=0}^{\infty} \tilde{r}_n(x)\frac{t^n}{n!}.$$

Adjoint Hahn polynomials

(3.9) $$A(t) = \sec t, \qquad H(t) = \tan t,$$

$$G(t,x) = \sec t \exp(x \tan t) = \sum_{n=0}^{\infty} \tilde{R}_n(x)\frac{t^n}{n!}.$$

***Remark* 3.1.** *In the above list we have not considered the Laguerre polynomials $L_n(x)$ and their generalized form $L_n^{\alpha}(x)$, since they are self-adjoint, in the sense that they coincide respectively with $\tilde{L}_n(x)$ and $\tilde{L}_n^{\alpha}(x)$.*

Furthermore, recalling the Hermite polynomials $H_n(x)$ and their generalized form $H_n^m(x)$, introduced by M. Lahiri [19] (*see also the article by R.C.S. Chandel* [11])*, it is worth to note that their respective Adjoint form $\tilde{H}_n(x)$ and $\tilde{H}_n^{(m)}(x)$ reduce to a particular case of the Appell-Kampé de Fériet polynomials in two variables* [1]*, so that they will not be considered in the following.*

***Remark* 3.2.** *It is also worth to note that generalized Laguerre-type polynomial families have been considered by G. Dattoli* [13]*, and further extensions have been recently obtained by using umbral methods* [2, 20].

4 Adjoint Actuarial polynomials

Actuarial polynomials have been considered in the book by R.P. Boas and R.C. Buck [6]. They were previously introduced by J.F. Steffensen [29] and also sudied by L. Toscano [31].

Here we consider the adjoint Actuarial polynomials, defined through their generating function, i.e. by putting

(4.1) $$A(t) = (1-t)^{-\beta}, \qquad H(t) = \log(1-t),$$

$$G(t,x) = \exp\left[(x-\beta)\log(1-t)\right] = \sum_{k=0}^{\infty} \tilde{\alpha}_k^{(\beta)}(x)\frac{t^k}{k!}.$$

4.1 Recurrence relation

***Theorem* 4.1.** *For any $k \geq 0$, the polynomials $\tilde{\alpha}_k^{(\beta)}(x)$ satisfy the following recurrence relation:*

$$\tilde{\alpha}_{k+1}^{(\beta)}(x) = \textstyle\sum_{h=0}^{k} \frac{k!}{(k-h)!}(\beta - x)\tilde{\alpha}_{k-h}^{(\beta)}(x). \tag{4.2}$$

Proof. Differentiating $G(t, x)$ with respect to t, we have

$$\frac{\partial G(t, x)}{\partial t} = \left(\frac{x - \beta}{t - 1}\right) \exp\left[(\beta - x)\log(1 - t)\right], \tag{4.3}$$

$$\frac{\partial G(t, x)}{\partial t} = G(t, x)(\beta - x)\frac{1}{1 - t} = (\beta - x)\sum_{k=0}^{\infty} \tilde{\alpha}_k^{(\beta)}(x)\frac{t^k}{k!}\sum_{k=0}^{\infty} k!\frac{t^k}{k!} = \tag{4.4}$$

$$= \sum_{k=1}^{\infty} \tilde{\alpha}_k^{(\beta)}(x)\frac{t^{k-1}}{(k-1)!}.$$

i.e.

$$(\beta - x)\sum_{k=0}^{\infty}\left[\sum_{h=0}^{k}\binom{k}{h}\tilde{\alpha}_{k-h}^{(\beta)}(x)h!\right]\frac{t^k}{k!} = \sum_{k=0}^{\infty}\tilde{\alpha}_{k+1}^{(\beta)}(x)\frac{t^k}{k!}.$$

so that the recurrence relation (4.2) follows.

4.2 Generating function's PDE

***Theorem* 4.2.** *The generating function* (4.1) *satisfies the homogeneous linear PDE*

$$(x - \beta)\frac{\partial G}{\partial x} + (1 - t)\log(1 - t)\frac{\partial G}{\partial t} = 0. \tag{4.5}$$

Proof. Differentiating $G(t, x)$ with respect to x, we have

$$\frac{\partial G(t, x)}{\partial x} = \log(1 - t)\exp\left[(x - \beta)\log(1 - t)\right]. \tag{4.6}$$

From equation (4.6) we find

$$\exp\left[(x - \beta)\log(1 - t)\right] = \frac{1}{\log(1 - t)}\frac{\partial G}{\partial x}.$$

Eliminating the exponential function in equation (4.3), by using the above equation, we find

$$\frac{\partial G(t, x)}{\partial t} = \left(\frac{x - \beta}{t - 1}\right)\frac{1}{\log(1 - t)}\frac{\partial G}{\partial x},$$

so that our result is proved.

4.3 Shift operators

We recall that a polynomial set $\{p_n(x)\}$ is called quasi-monomial if and only if there exist two operators $\hat{P}$ and $\hat{M}$ such that

$$\hat{P}(p_n(x)) = np_{n-1}(x), \qquad \hat{M}(p_n(x)) = p_{n+1}(x), \qquad (n = 1, 2, \ldots). \tag{4.7}$$

$\hat{P}$ is called the *derivative* operator and $\hat{M}$ the *multiplication* operator, as they act in the same way of classical operators on monomials.

This definition traces back to a paper by J.F. Steffensen [30], recently improved by G. Dattoli [12] and widely used in several applications [14, 17].

Y. Ben Cheikh [3] proved that every polynomial set is quasi-monomial under the action of suitable derivative and multiplication operators (see also the article by G. Dattoli et al. [16]). In particular, in the same article (***Corollary* 3.2**), the following result is proved

***Theorem* 4.3.** *Let $(p_n(x))$ denote a Boas-Buck polynomial set, i.e. a set defined by the generating function*

$$A(t)\psi(xH(t)) = \sum_{n=0}^{\infty} p_n(x)\frac{t^n}{n!}\ , \tag{4.8}$$

where

$$A(t) = \sum_{n=0}^{\infty} \tilde{a}_n t^n\ , \qquad (\tilde{a}_0 \neq 0), \tag{4.9}$$

$$\psi(t) = \sum_{n=0}^{\infty} \tilde{\gamma}_n t^n\ , (\tilde{\gamma}_n \neq 0 \quad \text{for all } n),$$

with $\psi(t)$ not a polynomial, and lastly

$$H(t) = \sum_{n=0}^{\infty} \tilde{h}_n t^{n+1}\ , \qquad (\tilde{h}_0 \neq 0). \tag{4.10}$$

Let $\sigma \in \Lambda^{(-)}$ the lowering operator defined by

$$\sigma(1) = 0, \qquad \sigma(x^n) = \frac{\tilde{\gamma}_{n-1}}{\tilde{\gamma}_n}\, x^{n-1}, \qquad (n = 1, 2, \dots). \tag{4.11}$$

Put

$$\sigma^{-1}(x^n) = \frac{\tilde{\gamma}_{n+1}}{\tilde{\gamma}_n}\, x^{n+1} \quad (n = 0, 1, 2, \dots). \tag{4.12}$$

Denoting, as before, by $f(t)$ the compositional inverse of $H(t)$, the Boas-Buck polynomial set $\{p_n(x)\}$ is quasi-monomial under the action of the operators

$$\hat{P} = f(\sigma), \qquad \hat{M} = \frac{A'[f(\sigma)]}{A[f(\sigma)]} + xD_x H'[f(\sigma)]\sigma^{-1}, \tag{4.13}$$

where prime *denotes the ordinary derivatives with respect to t.*

Note that in our case we are dealing with a Sheffer polynomial set, so that since we have $\psi(t) = e^t$, the operator σ defined by equation (4.11) simply reduces to the derivative operator D_x. Furthermore, we have:

$$f(t) = H^{-1}(t) = 1 - e^t,$$

$$A(t) = (1-t)^{-\beta}, \qquad H(t) = \log(1-t),$$

$$\frac{A'(t)}{A(t)} = \beta(1-t)^{-1}, \qquad H'(t) = \frac{1}{t-1},$$

and consequently

$$f(\sigma) = 1 - e^{D_x},$$

$$\frac{A'[f(\sigma)]}{A[f(\sigma)]} = \beta[1 - f(\sigma)]^{-1} = \beta e^{-D_x},$$

$$H'[f(\sigma)] = -e^{-D_x}.$$

Comparing the last three equations with equation (4.13), the following result follows:

***Theorem* 4.4.** *The adjoint Actuarial polynomial set $\{\tilde{\alpha}_k^{(\beta)}(x)\}$ is quasi-monomial under the action of the operators*

$$\hat{P} = 1 - e^{D_x} = -\sum_{k=0}^{\infty} \frac{D_x^{k+1}}{(k+1)!}, \tag{4.14}$$

$$\hat{M} = (\beta - x)e^{-D_x} = (\beta - x)\sum_{k=0}^{\infty} \frac{(-1)^k D_x^k}{k!}.$$

There is no problem about the convergence of the above series, since they reduce to finite sums when applied to polynomials.

4.4 Differential equation

According to the results of monomiality principle [12, 17], the quasi-monomial polynomials $\{p_n(x)\}$ satisfy the differential equation

$$\hat{M}\hat{P}p_n(x) = np_n(x). \tag{4.15}$$

In the present case, recalling equations (4.14), we have

***Theorem* 4.5.** *The adjoint Actuarial polynomials* $\{\tilde{\alpha}_k^{(\beta)}(x)\}$ *satisfy the differential equation*

$$(\beta - x)\sum_{k=0}^{n-1}\frac{(-1)^{k+1}}{(k+1)!}D_x^{k+1}\tilde{\alpha}_n^{(\beta)}(x) = n\tilde{\alpha}_n^{(\beta)}(x). \tag{4.16}$$

Proof. Equation (4.15), by using equations (4.14), becomes

$$\begin{aligned}\hat{M}\hat{P}\tilde{\alpha}_n^{(\beta)}(x) &= (\beta - x)e^{-D_x}\left[1 - e^{D_x}\right]\tilde{\alpha}_n^{(\beta)}(x) = \\ &= (\beta - x)\left[e^{-D_x} - 1\right]\tilde{\alpha}_n^{(\beta)}(x) = n\tilde{\alpha}_n^{(\beta)}(x),\end{aligned}$$

i.e.

$$(\beta - x)\sum_{k=0}^{\infty}\frac{(-1)^{k+1}}{(k+1)!}D_x^{k+1}\tilde{\alpha}_n^{(\beta)}(x) = n\tilde{\alpha}_n^{(\beta)}(x),$$

and furthermore, for any fixed n, the last series expansion reduces to a finite sum, with upper limit $n-1$, when it is applied to a polynomial of degree n, because the last not vanishing term (for $k = n-1$) contains the derivative of order n.

4.5 First few values

Here we show the first few values for the adjoint Actuarial polynomials, defined by the generating function (4.1)

$$\begin{aligned}
\tilde{\alpha}_0^{(\beta)}(x) &= 1,\\
\tilde{\alpha}_1^{(\beta)}(x) &= (\beta - x),\\
\tilde{\alpha}_2^{(\beta)}(x) &= (\beta - x)^2 + (\beta - x),\\
\tilde{\alpha}_3^{(\beta)}(x) &= (\beta - x)^3 + 3(\beta - x)^2 + 2(\beta - x),\\
\tilde{\alpha}_4^{(\beta)}(x) &= (\beta - x)^4 + 6(\beta - x)^3 + 11(\beta - x)^2 + 6(\beta - x),\\
\tilde{\alpha}_5^{(\beta)}(x) &= (\beta - x)^5 + 10(\beta - x)^4 + 35(\beta - x)^3 + 50(\beta - x)^2 + 24(\beta - x),\\
\tilde{\alpha}_6^{(\beta)}(x) &= (\beta - x)^6 + 15(\beta - x)^5 + 85(\beta - x)^4 + 225(\beta - x)^3 + 274(\beta - x)^2 +\\
&\quad +120(\beta - x),\\
\tilde{\alpha}_7^{(\beta)}(x) &= (\beta - x)^7 + 21(\beta - x)^6 + 175(\beta - x)^5 + 735(\beta - x)^4 + 1624(\beta - x)^3 +\\
&\quad +1764(\beta - x)^2 + 720(\beta - x),\\
\tilde{\alpha}_8^{(\beta)}(x) &= (\beta - x)^8 + 28(\beta - x)^7 + 322(\beta - x)^6 + 1960(\beta - x)^5 + 6769(\beta - x)^4 +\\
&\quad +13132(\beta - x)^3 + 13068(\beta - x)^2 + 5040(\beta - x).
\end{aligned}$$

Note that the coefficients of the considered adjoint Actuarial polynomials appear in the Encyclopedia of Integer Sequences [27], under A094638: *Triangle read by rows*: $T(n,k) = |s(n, n+1-k)|$, *where* $s(n,k)$ *are the signed Stirling numbers of the first kind* ($1 \le k \le n$; in other words, the unsigned Stirling numbers of the first kind in reverse order).

In several articles [9, 10, 21, 22, 23], further adjoint sets of Sheffer polynomials have been examined.

Acknowledgements. I want to express my deep thanks to Professor Dr. Youssef Ben Cheikh and Professor G. Dattoli for useful advice and information to improve this paper.

References

[1] P. Appell and J. Kampé de Fériet, *Fonctions Hypergéométriques et Hypersphériques. Polynômes d'Hermite,* Gauthier-Villars, Paris, 1926.

[2] M. Artioli, G. Dattoli, S. Licciardi and M. R. Pidatella, Hermite and Laguerre Functions: a Unifying Point of View, (2019); 10.13140/RG.2.2.25384.83201.

[3] Y. Ben Cheikh, Some results on quasi-monomiality, *Appl. Math. Comput.*, **141** (2003), 63–76.

[4] P. Blasiack, G. Dattoli, H. Horzela and K. Penson, Representations of monomiality principle with Sheffer-type polynomials and boson normal ordering, *Phys. Lett. A*, **352**, (2006), 7–12.

[5] R. P. Boas and R. C. Buck, Polynomials defined by generating relations, *Amer. Math. Monthly*, **63** (1958), 626–632.

[6] R. P. Boas and R. C. Buck, *Polynomial Expansions of Analytic Functions*, Springer-Verlag, Berlin, Gottingen, Heidelberg, New York, 1958.

[7] W. C. Brenke, On generating functions of polynomial systems, *Amer. Math. Monthly*, **52** (1945), 297–301.

[8] G. Bretti, P. Natalini and P. E. Ricci, A new set of Sheffer-Bell polynomials and logarithmic numbers, *Georgian Math. J.*, 2019, https://doi.org/10.1515/gmj-2019-2007

[9] G. Bretti, P. Natalini and P. E. Ricci, New sets of Euler-type polynomials, *J. Anal. Number Theory*, **6(2)**, (2018), 51-55; available on line at: http://dx.doi. org/10.18576/jant/060203

[10] G. Bretti, P. Natalini and P. E. Ricci, Adjoint Peters and Pidduk polynomials, *Int. J. Adv. Math.,* (2018), (to appear).

[11] R.C.S. Chandel, Generalized Hermite Polynomials, *Jñānābha,*, Section A **(2)** (1972), 19–27.

[12] G. Dattoli, Hermite-Bessel and Laguerre-Bessel functions: A by-product of the monomiality principle, in *Advanced Special Functions and Applications* (Proceedings of the Melfi School on Advanced Topics in Mathematics and Physics; Melfi, 9-12 May, 1999) (D. Cocolicchio, G. Dattoli and H.M. Srivastava, Editors), Aracne Editrice, Rome, 2000, pp. 147–164.

[13] G. dattoli, Generalized polynomials, operational identities and their applications, *J. Comput. Appl. Math.,* **118**, (2000), 111–123.

[14] G. Dattoli, P.E. Ricci and H.M. Srivastava, Editors), Advanced Special Functions and Related Topics in Probability and in Differential Equations, (Proceedings of the Melfi School on Advanced Topics in Mathematics and Physics; Melfi, June 24-29, 2001), in: *Applied Mathematics and Computation*, **141(1)**, (2003), 1–230.

[15] G. Dattoli, M. Migliorati and H. M. Srivastava, Sheffer polynomials, monomiality principle, algebraic method and the theory of classical polynomials, *Math Computer Modelling,* **45**, (2007), 1033–1041.

[16] G. Dattoli, D. Levi and P. Winternitz, Heisenberg algebra, umbral calculus and orthogonal polynomials, *J. Math. Phys.,* **49**, N. 053509, (2008); arXiv:0712.2957.

[17] G. Dattoli, B. Germano, M.R. Martinelli and P.E. Ricci, Monomiality and partial differential equations, *Math. Comput. Modelling*, **50** (2009), 1332–1337.

[18] G. Dattoli, B. Germano, M. R. Martinelli and P. E. Ricci, Monomiality and partial differential equations, *Math. Comput. Modelling,* **50** (2009), 1332–1337.

[19] M. Lahiri, On a generalisation of Hermite polynomials, *Proc. Amer. Math. Soc.*, **27** (1971), 117–121.

[20] S. Licciardi, Umbral Calculus, a Different Mathematical Language, Mathematics and Computer Sciences, Dept. Math. Comput. Sci., XXIX cycle, University of Catania, 2018; arXiv:1803.03108[math.CA].

[21] P. Natalini, G. Bretti and P.E. Ricci, Adjoint Hermite and Bernoulli polynomials, Bull. Allahabad Math. Soc., (Dharma Prakash Gupta Memorial Volume), **33(2)**, (2018), 251–264.

[22] P. Natalini, G. Bretti and P.E. Ricci, Adjoint Poisson-Charlier and Related polynomials, *Int. J. Appl. Math. Stat.,* (2018), (to appear).

[23] P. E. Ricci, G. Bretti and P. Natalini, New sets of Hahn-type polynomials, *Bull. Allahabad Math.* Soc., **34**, Part 1, (2019), 135–146.

[24] P. E. Ricci, P. Natalini and G. Bretti, Sheffer and Brenke polynomials associated with generalized Bell numbers, *Jñānābha,*, **47(2)**, (2017), 337–352.

[25] S. M. Roman, *The Umbral Calculus*, Academic Press, New York, 1984.

[26] I. M. Sheffer, Some properties of polynomials sets of zero type, *Duke Math. J.*, **5** (1939), 590–622.

[27] N. J. A. Sloane, *The On-Line Encyclopedia of Integer Sequences*. Published electronically at http://oeis.org, 2016.

[28] H. M. Srivastava and H. L. Manocha, *A Treatise on Generating Functions*, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane and Toronto, 1984.

[29] J. F. Steffensen, On a class of polynomials and their application to actuarial problems, *Skand. Aktuarietidskrift*, **11** (1928), 75–97.

[30] J. F. Steffensen, The poweroid, an extension of the mathematical notion of power, *Acta Math.*, **73** (1941), 333–366.

[31] L. Toscano, Una classe di polinomi della matematica attuariale, *Riv. Mat. Univ. Parma*, **1** (1950), 459-470.

Jñānābha, Vol. 50(1) (2020), 65-71
(Dedicated to Honor Professor H.M. Srivastava on His 80th Birth Anniversary Celebrations)

A GENERALIZED SUBCLASS OF ALPHA CONVEX BI-UNIVALENT FUNCTIONS OF COMPLEX ORDER

By
Gurmeet Singh[1], Gagandeep Singh[2] and Gurcharanjit Singh[3]
[1] Department of Mathematics, GSSDOS Khalsa College, Patiala-147001, Punjab, India
Email:meetgur111@gmail.com
[2] Department of Mathematics, Khalsa College, Amritsar-143001, Punjab, India
Email:kamboj.gagandeep@yahoo.in
[3] Department of Mathematics, Punjabi University, Patiala-147002, Punjab, India
Department of Mathematics, GNDU College, Chungh-143001, Punjab, India
Email:dhillongs82@yahoo.com
(*Received : January* 22, 2020 *; Revised: May* 02, 2020)
DOI: https://doi.org/10.58250/jnanabha.2020.50108

Abstract

In this present investigation a subclass of alpha convex bi-univalent functions of complex order in the open unit disc $U = \{z : |z| < 1\}$, defined by Salagean operator and quasi-subordination is discussed. The estimates on the initial coefficients $|a_2|$ and $|a_3|$ for the functions in this subclass are studied. The results obtained in this paper would generalise those already proved by various authors.
2010 Mathematics Subject Classifications: 30C45.
Keywords and phrases: Bi-univalent functions, Salagean operator, Quasi-subordination, Coefficient estimate, Univalent functions.

1 Introduction and Preliminaries

Let A be the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic in the open unit disc $U = \{z : |z| < 1\}$. By S, we denote the class of functions $f(z) \in A$ and univalent in U.

Let us denote by B, the class of bounded or Schwarz functions $w(z)$ satisfying $w(0) = 0$ and $|w(z)| \leq 1$ which are analytic in the open unit disc U and of the form

$$w(z) = \sum_{n=1}^{\infty} c_n z^n, z \in U.$$

A function $f \in S$ is said to be starlike if it satisfies the inequality

$$Re\left(\frac{zf'(z)}{f(z)}\right) > 0 (z \in U).$$

The class of starlike functions is denoted by S^*.
A function $f \in S$ is said to be convex if it satisfies the inequality

$$Re\left(\frac{(zf'(z))'}{f'(z)}\right) > 0 (z \in U).$$

The class of convex functions is denoted by K.

A function $f \in S$ is said to be α-convex if it satisfies the inequality

$$Re\left((1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\frac{(zf'(z))'}{f'(z)}\right) > 0 (0 \leq \alpha \leq 1, z \in U).$$

The class of α-convex functions is denoted by $M(\alpha)$ and was introduced by Mocanu [8]. In particular $M(0) \equiv S^*$ and $M(1) \equiv K$.

For $f \in A$, Salagean [14] introduced the following operator:

$$D^0 f(z) = f(z), D^1 f(z) = zf'(z),$$

and in general,

$$D^n f(z) = D(D^{n-1} f(z)), n \in N$$

or equivalent to

$$D^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k, n \in N_0 = N \cup (0).$$

The inverse functions of the functions in the class S may not be defined on the entire unit disc U although the functions in the class S are invertible. However using Koebe-one quarter theorem [4] it is obvious that the image of U under every function $f \in S$ contains a disc of radius $\frac{1}{4}$. Hence every univalent function f has an inverse f^{-1}, defined by

$$f^{-1}(f(z)) = z (z \in U)$$

and

$$f(f^{-1}(w)) = w\left(| w |< r_0(f) : r_0(f) \geq \frac{1}{4}\right),$$

where

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \dots . \tag{1.2}$$

A function $f \in A$ is said to be bi-univalent in U if both f and f^{-1} are univalent in U.

By Σ, we denote the class of bi-univalent functions in U defined by (1.1).

Consider two functions f and g analytic in U. We say that f is subordinate to g (symbolically $f \prec g$) if there exists a bounded function $u(z) \in B$ for which $f(z) = g(u(z))$. This result is known as principle of subordination.

Robertson [13] introduced the concept of quasi-subordination in 1970. If f and ϕ are analytic functions, then we say that f is quasi-subordinate to ϕ (symbolically $f \prec_q \phi$) if there exists analytic functions k and ω with $|k(z)| \leq 1$, $\omega(0) = 0$ and $|\omega(z)| < 1$ such that

$$\frac{f(z)}{k(z)} \prec \phi(z),$$

or it is equivalent to

$$f(z) = k(z)\phi(\omega(z)). \tag{1.3}$$

In particular for $k(z) = 1$, $f(z) = \phi(\omega(z))$, so that $f(z) \prec \phi(z)$ in U. It is obvious to see that the quasi-subordination is a generalization of the usual subordination. The work on quasi-subordination is quite extensive which finds interesting dimensions in some recent investigations [1,5,7,12].

Lewin [6] discussed the class Σ of bi-univalent functions and obtained the bound for the second coefficient. Brannan and Taha [2] investigated certain subclasses of bi-univalent functions, similar to the familiar subclasses of univalent functions consisting of strongly starlike, starlike and convex functions. They introduced bi-starlike functions and bi-convex functions and obtained estimates on the initial coefficients. Also the subclasses of bi-univalent functions defined by Salagean operator were studied by various authors [3,9,11,15].

The earlier work on bi-univalent functions defined by quasi-subordination and Salagean operator motivate us to define the following subclass:

Also we assume that $\phi(z)$ is analytic in U with $\phi(0) = 1$ and let

$$\phi(z) = 1 + B_1 z + B_2 z^2 + ...(B_1 \in R^+) \tag{1.4}$$

and

$$k(z) = A_0 + A_1 z + A_2 z^2 + ...(|k(z)| \leq 1, z \in U). \tag{1.5}$$

To avoid repetition, throughout the paper we assume that $0 \leq \alpha \leq 1$ and $z \in U$.

***Definition* 1.1.** *A function $f \in \Sigma$ given by* (1.1) *is said to be in the class $M_\Sigma(n, \alpha, \gamma, \phi)$ if it satisfy the following conditions:*

$$\frac{1}{\gamma}\left[(1-\alpha)\frac{z(D^{n-1}f(z))'}{D^{n-1}f(z)} + \alpha\frac{z(D^n f(z))'}{D^n f(z)} - 1\right] \prec_q (\phi(z) - 1) \tag{1.6}$$

and

$$\frac{1}{\gamma}\left[(1-\alpha)\frac{w(D^{n-1}g(w))'}{D^{n-1}g(w)} + \alpha\frac{w(D^n g(w))'}{D^n g(w)} - 1\right] \prec_q (\phi(w) - 1), \tag{1.7}$$

where $g = f^{-1}$ and $z, w \in U$.

The folowing observations are obvious:

(i) $M_\Sigma(n, \alpha, 1, \phi) \equiv M_\Sigma(n, \alpha, \phi)$.

(ii) $M_\Sigma(1, \alpha, 1, \phi) \equiv M_\Sigma(\alpha, \phi)$.

(iii) $M_\Sigma(1, 0, 1, \phi) \equiv S^*_\Sigma(\phi)$*, the class of bi-starlike functions defined with quasi subordination.*

(iv) $M_\Sigma(1, 1, 1, \phi) \equiv K_\Sigma(\phi)$*, the class of bi-conves functions defined with quasi subordination.*

For deriving our main results, we need the following lemma:

***Lemma* 1.1.** [10] *If $p \in P$ be family of all functions p analytic in U for which* $Re[p(z)] > 0$ *and have the form $p(z) = 1 + p_1 z + p_2 z^2 + ...$ for $z \in U$, then $|p_n| \leq 2$ for each n.*

2 Coefficient bounds for the function class $M_\Sigma(n, \alpha, \gamma, \phi)$

***Theorem* 2.1.** *If $f \in M_\Sigma(n, \alpha, \gamma, \phi)$, then*

$$|a_2| \leq min.\left[\frac{|A_0\gamma|B_1}{(n+\alpha+1)}, \sqrt{\frac{|A_0\gamma|(B_1 + |B_2 - B_1|)}{(n+\alpha+1)}}\right] \tag{2.1}$$

and

$$|a_3| \leq min.\left[\frac{|A_0\gamma|(B_1 + |B_2 - B_1|)}{(n+\alpha+1)} + \frac{|A_1\gamma|B_1 + |A_0\gamma|B_1}{(n+2)(n+2\alpha+1)}\right. \tag{2.2}$$

$$\left.\frac{|\gamma|}{(n+2)(n+2\alpha+1)}\left[|\gamma|\left[\frac{(n+1)^2 + \alpha(2n+3)}{(n+\alpha+1)^2}\right]B_1^2|A_0|^2 + (B_1 + |B_2 - B_1|)|A_0| + |A_1|B_1\right]\right].$$

Proof. As $f \in M_\Sigma(n,\alpha,\gamma,\phi)$, so by ***Definition* 1.1**, using the concept of quasi-subordination, there exists Schwarz functions $r(z)$ and $s(z)$ and analytic function $k(z)$ such that

$$\frac{1}{\gamma}\left[(1-\alpha)\frac{z(D^{n-1}f(z))'}{D^{n-1}f(z)}+\alpha\frac{z(D^nf(z))'}{D^nf(z)}-1\right]=k(z)(\phi(r(z))-1) \tag{2.3}$$

and

$$\frac{1}{\gamma}\left[(1-\alpha)\frac{w(D^{n-1}g(w))'}{D^{n-1}g(w)}+\alpha\frac{w(D^ng(w))'}{D^ng(w)}-1\right]=k(w)(\phi(s(w))-1), \tag{2.4}$$

where $r(z)=1+r_1z+r_2z^2+...$ and $s(w)=1+s_1w+s_2w^2+....$

Define the functions $p(z)$ and $q(z)$ by

$$r(z)=\frac{p(z)-1}{p(z)+1}=\frac{1}{2}\left[c_1z+(c_2-\frac{c_1^2}{2})z^2+...\right] \tag{2.5}$$

and

$$s(z)=\frac{q(z)-1}{q(z)+1}=\frac{1}{2}\left[d_1z+(d_2-\frac{d_1^2}{2})z^2+...\right]. \tag{2.6}$$

Using (2.5) and (2.6) in (2.3) and (2.4) respectively, it yields

$$\frac{1}{\gamma}\left[(1-\alpha)\frac{z(D^{n-1}f(z))'}{D^{n-1}f(z)}+\alpha\frac{z(D^nf(z))'}{D^nf(z)}-1\right]=k(z)\left[\phi\left(\frac{p(z)-1}{p(z)+1}\right)-1\right] \tag{2.7}$$

and

$$\frac{1}{\gamma}\left[(1-\alpha)\frac{w(D^{n-1}g(w))'}{D^{n-1}g(w)}+\alpha\frac{w(D^ng(w))'}{D^ng(w)}-1\right]=k(w)\left[\phi\left(\frac{q(w)-1}{q(w)+1}\right)-1\right]. \tag{2.8}$$

But

$$\frac{1}{\gamma}\left[(1-\alpha)\frac{z(D^{n-1}f(z))'}{D^{n-1}f(z)}+\alpha\frac{z(D^nf(z))'}{D^nf(z)}-1\right]$$

$$=\frac{1}{\gamma}\left[(n+\alpha+1)a_2z+[(n+2)(n+2\alpha+1)a_3-((n+1)^2+\alpha(2n+3))a_2^2]z^2+...\right] \tag{2.9}$$

and

$$\frac{1}{\gamma}\left[(1-\alpha)\frac{w(D^{n-1}g(w))'}{D^{n-1}g(w)}+\alpha\frac{w(D^ng(w))'}{D^ng(w)}-1\right] \tag{2.10}$$

$$=\frac{1}{\gamma}\left[-(n+\alpha+1)a_2w+[(n+2)(n+2\alpha+1)(2a_2^2-a_3)-((n+1)^2+\alpha(2n+3))a_2^2]w^2+...\right].$$

Again using (1.4) and (1.5) in (2.5) and (2.6) respectively, we get

$$k(z)\left[\phi\left(\frac{p(z)-1}{p(z)+1}\right)-1\right]=\frac{1}{2}A_0B_1c_1z+\left[\frac{1}{2}A_1B_1c_1+\frac{1}{2}A_0B_1\left(c_2-\frac{c_1^2}{2}\right)+\frac{A_0B_2c_1^2}{4}\right]z^2+... \tag{2.11}$$

and

$$k(w)\left[\phi\left(\frac{q(w)-1}{q(w)+1}\right)-1\right]=\frac{1}{2}A_0B_1d_1w+\left[\frac{1}{2}A_1B_1d_1+\frac{1}{2}A_0B_1\left(d_2-\frac{d_1^2}{2}\right)+\frac{A_0B_2d_1^2}{4}\right]w^2+... \tag{2.12}$$

Using (2.9) and (2.11) in (2.7) and equating the coefficients of z and z^2, we get

$$\frac{(n+\alpha+1)}{\gamma}a_2=\frac{1}{2}A_0B_1c_1 \tag{2.13}$$

and

$$(2.14)\quad \frac{(n+2)(n+2\alpha+1)a_3-((n+1)^2+\alpha(2n+3))a_2^2}{\gamma}=\frac{1}{2}A_1B_1c_1+\frac{1}{2}A_0B_1\left(c_2-\frac{c_1^2}{2}\right)+\frac{A_0B_2}{4}c_1^2.$$

Again using (2.10) and (2.12) in (2.8) and equating the coefficients of w and w^2, we get

$$(2.15)\quad -\frac{(n+\alpha+1)}{\gamma}a_2=\frac{1}{2}A_0B_1d_1$$

and

$$(2.16)\quad \frac{(n+2)(n+2\alpha+1)(2a_2^2-a_3)-((n+1)^2+\alpha(2n+3))a_2^2}{\gamma}$$
$$=\frac{1}{2}A_1B_1d_1+\frac{1}{2}A_0B_1\left(d_2-\frac{d_1^2}{2}\right)+\frac{A_0B_2}{4}d_1^2.$$

From (2.13) and (2.15), it is clear that

$$(2.17)\quad c_1=-d_1$$

and

$$(2.18)\quad a_2=\frac{A_0B_1c_1\gamma}{2(n+\alpha+1)}=-\frac{A_0B_1d_1\gamma}{2(n+\alpha+1)}.$$

Therefore on applying triangle inequality and using ***Lemma* 1.1**, (2.18) yields

$$(2.19)\quad |a_2|\leq\frac{|A_0\gamma|B_1}{(n+\alpha+1)}.$$

Adding (2.14) and (2.16), it yields

$$(2.20)\quad \frac{2[(n+2)(n+2\alpha+1)-(n+1)^2-\alpha(2n+3)]}{\gamma}a_2^2=\frac{1}{2}A_0B_1(c_2+d_2)+\frac{A_0(B_2-B_1)}{4}(c_1^2+d_1^2).$$

Using ***Lemma* 1.1** and on applying triangle inequality in (2.20), we obtain

$$(2.21)\quad |a_2|^2\leq\frac{|A_0\gamma|(B_1+|B_2-B_1|)}{(n+\alpha+1)}.$$

So, the result (2.1) can be easily obtained from (2.19) and (2.21).

Now subtracting (2.16) from (2.14), we obtain

$$(2.22)\quad a_3=a_2^2+\frac{A_1B_1(c_1-d_1)+A_0B_1(c_2-d_2)}{4(n+2)(n+2\alpha+1)}\gamma.$$

Applying triangle inequality and using ***Lemma* 1.1** and (2.21) in (2.22), it yields

$$(2.23)\quad |a_3|\leq\frac{|A_0|(B_1+|B_2-B_1|)}{(n+\alpha+1)}+\frac{|A_1\gamma|B_1+|A_0\gamma|B_1}{(n+2)(n+2\alpha+1)}.$$

From (2.13) and (2.14), we have

(2.24)

$$|a_3|\leq\frac{|\gamma|}{(n+2)(n+2\alpha+1)}\left[|\gamma|\left[\frac{(n+1)^2+\alpha(2n+3)}{(n+\alpha+1)^2}\right]B_1^2|A_0|^2+(B_1+|B_2-B_1|)|A_0|+|A_1|B_1\right].$$

Again from (2.15) and (2.17), it gives

(2.25)

$$|a_3|\leq\frac{|\gamma|}{(n+2)(n+2\alpha+1)}\left[|\gamma|\left[\frac{n^2+2n\alpha+5\alpha+3}{(n+\alpha+1)^2}\right]B_1^2|A_0|^2+(B_1+|B_2-B_1|)|A_0|+|A_1|B_1\right].$$

Since R.H.S. of (2.25) is greater than that of (2.24), so result (2.2) is obvious.

For $\gamma=1$, ***Theorem* 2.1** gives the following result:

***Corollary* 2.1.** *If $M_\Sigma(n,\alpha,\phi)$, then*

$$|a_2| \leq min.\left[\frac{|A_0|B_1}{(n+\alpha+1)}, \sqrt{\frac{|A_0|(B_1+|B_2-B_1|)}{(n+\alpha+1)}}\right]$$

and

$$|a_3| \leq min.\left[\frac{|A_0|(B_1+|B_2-B_1|)}{(n+\alpha+1)} + \frac{|A_1|B_1+|A_0|B_1}{(n+2)(n+2\alpha+1)},\right.$$
$$\left.\frac{1}{(n+2)(n+2\alpha+1)}\left[\left[\frac{(n+1)^2+\alpha(2n+3)}{(n+\alpha+1)^2}\right]B_1^2|A_0|^2+(B_1+|B_2-B_1|)|A_0|+|A_1|B_1\right]\right].$$

For $\gamma = 1$ and $n = 1$, the following result is obvious from ***Theorem* 2.1**:

***Corollary* 2.2.** *If $f(z) \in M_\Sigma(\alpha,\phi)$, then*

$$|a_2| \leq min.\left[\frac{|A_0|B_1}{(2+\alpha)}, \sqrt{\frac{|A_0|(B_1+|B_2-B_1|)}{(2+\alpha)}}\right]$$

and

$$|a_3| \leq min.\left[\frac{|A_0|(B_1+|B_2-B_1|)}{(2+\alpha)} + \frac{|A_1|B_1+|A_0|B_1}{6(1+\alpha)},\right.$$
$$\left.\frac{1}{6(1+\alpha)}\left[\left[\frac{4+5\alpha}{(2+\alpha)^2}\right]B_1^2|A_0|^2+(B_1+|B_2-B_1|)|A_0|+|A_1|B_1\right]\right].$$

For $\gamma = 1, \alpha = 0$ and $n = 1$, ***Theorem* 2.1** gives the following result:

***Corollary* 2.3.** *If $f(z) \in S^*_\Sigma(\phi)$, then*

$$|a_2| \leq min.\left[\frac{|A_0|B_1}{2}, \sqrt{\frac{|A_0|(B_1+|B_2-B_1|)}{2}}\right]$$

and

$$|a_3| \leq min.\left[\frac{|A_0|(B_1+|B_2-B_1|)}{2} + \frac{|A_1|B_1+|A_0|B_1}{6},\right.$$
$$\left.\frac{1}{6}\left[B_1^2|A_0|^2+(B_1+|B_2-B_1|)|A_0|+|A_1|B_1\right]\right].$$

For $\gamma = 1, \alpha = 1$ and $n = 1$, the following result is obvious from *Theorem* 2.1:

***Corollary* 2.4.** *If $f(z) \in K_\Sigma(\phi)$, then*

$$|a_2| \leq min.\left[\frac{|A_0|B_1}{3}, \sqrt{\frac{|A_0|(B_1+|B_2-B_1|)}{3}}\right]$$

and

$$|a_3| \leq min.\left[\frac{|A_0|(B_1+|B_2-B_1|)}{3} + \frac{|A_1|B_1+|A_0|B_1}{12},\right.$$
$$\left.\frac{1}{12}\left[B_1^2|A_0|^2+(B_1+|B_2-B_1|)|A_0|+|A_1|B_1\right]\right].$$

3. Conclusion

This paper is concerened with a very generalized subclass of alpha convex bi-univalent functions of complex order in the open unit disc. The class is associated with Salagean operator and is defined by means of quasi-subordination. We have studied the estimates of the initial coefficients $|a_2|$ and $|a_3|$ for the functions in this class. By giving the particular values to the various paprameters like α, γ, n and q, the results already proved by earlier researchers can be easily obtained. So this paper will work as a milestone to the future researchers in this field.

Acknowledgement. The authors are very much grateful to the Editor and referee for their suggestions to bring the paper in its present form.

References

[1] O. Altintas and S. Owa, Mazorizations and quasi-subordinations for certain analytic functions, *Proc. Japan Acad. Ser.A.* **68**(7) (1992),181-185.

[2] D. A. Brannan and T. S. Taha, On some classes of bi-univalent functions, in: S. M. Mazhar, A. Hamoni, N. S. Faour (Eds.), Mathematical Analysis and its Applications, Kuwait; February 18-21, 1985, in: *KFAS Proceedings Series*, **3**, Pergamon Press, Elsevier Science Limited, Oxford, 1988, pp. 53-60. See also *Studia Univ. Babes-Bolyai Math.*, **31(2)**, (1986), 70-77.

[3] Murat Caglar and Erhan Deniz, Initial Coefficients for a subclass of bi-univalent functions defined by Salagean differential operator, *Commun. Fac. Sci. Univ. Auk. Ser. Al Math. Stat.* **66(1)**(2017), 85-91.

[4] P. L. Duren,*Univalent functions, Springer-Verlag, New York*, 1983.

[5] S. Y. Lee, Quasi-subordinate functions and coefficient conjectures, *J. Korean Math. Soc.* **12(1)**(1975), 43-50.

[6] M. Lewin, On a coefficient problem for bi-univalent functions, *Proc. Amer. Math. Soc.*, **18**(1967),63-68.

[7] N. Magesh, V. K. Balaji and J. Yamini, Certain subclasses of bi-starlike and bi-convex functions based on quasi-subordination, *Abstract and applied analysis*, Art. ID 3102960, 6 pages, 2016.

[8] P.T. Mocanu, Une propriete de convexite generalisee dans la theorie de la representation conforme, *Mathematica (Cluj)*, **11(34)**(1969), 127-133.

[9] G. Murugusundaramoorthy, C. Selvaraj and O. S. Babu, Coefficient estimates for Pascu-type subclasses of bi-univalent functions based on subordination, *Int. J. of Non-linear Science*, **19(1)**(2015), 47-52.

[10] Ch. Pommerenke, *Univalent functions, Vandenhoeck and Ruprecht, Gottingen*, 1975.

[11] C. Ramachandran and D. Kavitha, Coefficient estimates for a subclass of bi-univalent functions defined by Salagean operator using quasi subordination, *Applied Mathematical Sciences*, **11(35)**(2017), 1725-1732.

[12] F. Y. Ren, S. Owa and S. Fukui, Some inequalities on quasi-subordinate functions, *Bull. Austral. Math. Soc.*,**43(2)**(1991), 317-324.

[13] M. S. Robertson, Quasi-subordination and coefficient conjecture, *Bull. Amer. Math. Soc.*, **76**(1970), 1-9.

[14] G. S. Salagean, Subclasses of Univalent Functions, *Chapter in Lecture Notes in Mathematics, Springer-Verlag*, Vol. **1013**, 1983, 362-372.

[15] Bilal Seker and Veysi Mehmetoglu, Coefficient bounds for new subclasses of bi-univalent functions, *New Trends in Mathematical Sciences*, **4(3)**(2016), 197-203.

Jñānābha, Vol. 50(1) (2020), 72-78
(Dedicated to Honor Professor H.M. Srivastava on His 80th Birth Anniversary Celebrations)

BOUNDS FOR THE MAXIMUM MODULUS OF POLYNOMIAL NOT VANISHING IN A DISK

By

Roshan Lal
Department of Mathematics
V.S.K.C. Government Postgraduate College, Dakpathar
Dehradun-248125, Uttarakhand, India
Email:rlkeshtwal@gmail.com

(*Received : March* 23, 2020 *; Revised: April* 30, 2020)
DOI: https://doi.org/10.58250/jnanabha.2020.50109

Abstract

Let $p(z)$ be a polynomial of degree n. We have several results for the bounds of maximum modulus of polynomial in terms of coefficients of polynomial and radius of the disk having no zeros in it. In this paper we have proved some results for the bounds of maximum modulus of polynomial not vanishing in a disk of greater or smaller than unity. Our results improve the earlier proved results.

2010 Mathematics Subject Classifications: 26A15, 26A10
Keywords and phrases: Polynomials, zeros, bounds, inequalities, maximum modulus of polynomials.

1 Introduction and Statement of Results

Let $p(z)$ be a polynomial of degree n. Let us define $m = \min\limits_{|z|=k} |P(z)|$ and $M(p, r) = \max\limits_{|z|=r} |P(z)|$. Concerning the estimate for the maximum modulus of a polynomial on the circle $|z| = R$, $R > 0$, in terms of its degree and the maximum modulus on the unit circle, we know that for every $R \geq 1$,

$$\max_{|z|=R} |p(z)| \leq R^n \max_{|z|=1} |p(z)|. \tag{1.1}$$

The result is best possible for the polynomial having all its zeros at origin.

Inequality (1.1) is a simple deduction from the maximum modulus principle (for reference see [8] or [11]).

For the polynomial of degree n and the case $r \leq 1$, we have the following result due to Varga [13] who attributed it to Zerrantonello.

$$\max_{|z|=r} |p(z)| \geq r^n \max_{|z|=1} |p(z)|. \tag{1.2}$$

Again the result is best possible for the polynomial having all its zeros at origin.

For the class of polynomials having no zeros in $|z| < 1$, the inequalities (1.1) and (1.2) are sharpened by Ankeny and Rivlin [1] and Rivlin [12], by proving following inequality (1.3) and inequality (1.4) respectively

$$\max_{|z|=R\geq 1} |p(z)| \leq \frac{R^n + 1}{2} \max_{|z|=1} |p(z)|, \tag{1.3}$$

$$\max_{|z|=r} |p(z)| \geq \left(\frac{1+r}{2}\right)^n \max_{|z|=1} |p(z)|. \tag{1.4}$$

Aziz and Dawood [3] improved inequality (1.3) under the same hypothesis as

$$\max_{|z|=R\geq 1}|p(z)| \leq \left(\frac{R^n+1}{2}\right)\max_{|z|=1}|p(z)| - \left(\frac{R^n-1}{2}\right)\min_{|z|=1}|p(z)|. \tag{1.5}$$

There are several results concerning the refinement and generalizations of above mentioned inequalities (see [5], [7] and [14]).

For the case $0 < \rho \leq 1$, when polynomial does not vanish in $|z| < k$, $k \geq 1$ we have the following inequality due to Aziz [2].

$$\max_{|z|=\rho}|p(z)| \geq \left(\frac{\rho+k}{1+k}\right)^n \max_{|z|=1}|p(z)|. \tag{1.6}$$

The result is sharp and equality in (1.6) is attained for $p(z) = c(ze^{i\beta}+k)^n$, $c(\neq 0) \in C$ and $\beta \in R$.

Inequality (1.6) was improved by Govil, Qazi and Rahman [6] by introducing coefficients of polynomial under consideration in it as following

***Theorem* 1.1.** *Let $p(z) = \sum\limits_{j=0}^{n} a_j z^j$ does not vanish in $|z| < k$, $k \geq 1$ and let $\lambda = \lambda(k) = \frac{ka_1}{na_0}$. Then for $0 < \rho \leq 1$,*

$$\max_{|z|=\rho}|p(z)| \geq \left(\frac{k^2+2|\lambda|k\rho+\rho^2}{k^2+2|\lambda|k+1}\right)^{\frac{n}{2}} \max_{|z|=1}|p(z)|. \tag{1.7}$$

In the case when n is even equality in (1.7) is attained for

$$p(z) = c(z^2 e^{i2\beta} + 2kze^{i\beta}\cos\alpha + k^2)^{\frac{n}{2}}, \quad c(\neq 0) \in C \text{ and } \alpha, \beta \in R.$$

The following result is also due to Govil, Qazi and Rahman [6] and is complement to ***Theorem* 1.1**.

***Theorem* 1.2.** *Let $p(z) = \sum\limits_{j=0}^{n} a_j z^j$ does not vanish in $|z| < k$, $k \in (0,1)$ and let $\lambda = \lambda(k) = \frac{ka_1}{na_0}$. Then for $0 \leq \rho \leq k^2$*

$$\max_{|z|=\rho}|p(z)| \geq \left(\frac{k^2+2|\lambda|k\rho+\rho^2}{k^2+2|\lambda|k+1}\right)^{\frac{n}{2}} \max_{|z|=1}|p(z)|. \tag{1.8}$$

In the case when n is even equality in (1.8) is attained for

$$p(z) = c(z^2 e^{i2\beta} + 2kze^{i\beta}\cos\alpha + k^2)^{\frac{n}{2}}, \quad c(\neq 0) \in C \text{ and } \alpha, \beta \in R.$$

Recently, Mir et al. [7] proved the following interesting result and generalized a result due to Govil and Nwaeze [5] and many other results improving the ***Theorem*** of T. J. Rivlin [12].

***Theorem* 1.3.** *Let $p(z) = a_0 + \sum\limits_{j=\mu}^{n} a_j z^j$, $1 \leq \mu < n$ be a polynomial of degree n that does not vanish in $|z| < k$, $k \geq 1$.Then for $0 < r < R \leq 1$,*

$$M(p,r) \geq \frac{(1+r^\mu)^{\frac{n}{\mu}}}{(1+r^\mu)^{\frac{n}{\mu}} + (R^\mu+r^\mu)^{\frac{n}{\mu}} - (k^\mu+r^\mu)^{\frac{n}{\mu}}}\left\{M(p,R) + m\ln\left(\frac{(R^\mu+k^\mu)}{(r^\mu+k^\mu)}\right)^{\frac{n}{\mu}}\right\}, \tag{1.9}$$

where $m = \min\limits_{|z|=k}|p(z)|$ and $M(p,r) = \max\limits_{|z|=r}|p(z)|$ etc.

2 Main Theorems

In this paper, firstly we prove the following result for the class of polynomials not vanishing in a prescribed disk, which improves upon the bound obtained by ***Theorem* 1.1**.

***Theorem* 2.1.** *Let* $p(z) = \sum_{j=0}^{n} a_j z^j \neq 0$ *for* $|z| < k$, $k \geq 1$ *and let* $\lambda = \lambda(k) = \frac{ka_1}{na_0}$. *Then for* $0 < \rho \leq 1$, *we have*

$$\max_{|z|=\rho} |p(z)| \geq \left(\frac{\rho^2 + 2|\lambda|k\rho + k^2}{1 + 2|\lambda|k + k^2}\right)^{\frac{n}{2}} \max_{|z|=1} |p(z)| + \frac{1}{k^n}\left\{\left(\frac{\rho^2 + 2|\lambda|k\rho + k^2}{1 + 2|\lambda|k + k^2}\right)^{\frac{n}{2}} - \rho^n\right\} \min_{|z|=k} |p(z)|. \tag{2.1}$$

In the case where n is even, equality in (1.9) *is attained for*

$$p(z) = c(z^2 e^{i2\beta} + 2kze^{i\beta}\cos\alpha + k^2)^{\frac{n}{2}}, \quad c(\neq 0) \in C \text{ and } \alpha, \beta \in R.$$

The above inequality (2.1) always gives better bounds than inequality (1.7) except in the case when $\min_{|z|=k} |p(z)| = 0$.

Next we prove the following result, which is complement to ***Theorem* 2.1**, for the class of polynomials not vanishing in a disk of radius less than (or equal) unity and also improves upon ***Theorem* 1.2**.

***Theorem* 2.2.** *Let* $p(z) = \sum_{j=0}^{n} a_j z^j$ *does not vanish in* $|z| < k$, $k \in (0, 1)$ *and let* $\lambda = \lambda(k) = \frac{ka_1}{na_0}$. *Then for* $0 \leq \rho \leq k^2$

$$\max_{|z|=\rho} |p(z)| \geq \left(\frac{k^2 + 2|\lambda|k\rho + \rho^2}{k^2 + 2|\lambda|k + 1}\right)^{\frac{n}{2}} \max_{|z|=1} |p(z)| + \frac{1}{k^n}\left\{\left(\frac{k^2 + 2|\lambda|k\rho + \rho^2}{k^2 + 2|\lambda|k + 1}\right)^{\frac{n}{2}} - \rho^n\right\} \min_{|z|=k} |p(z)|. \tag{2.2}$$

In the case when n is even equality in (2.2) *is attained for*

$$p(z) = c(z^2 e^{i2\beta} + 2kze^{i\beta}\cos\alpha + k^2)^{\frac{n}{2}}, \quad c(\neq 0) \in C \text{ and } \alpha, \beta \in R.$$

Finally, we prove the following interesting result, which improves upon ***Theorem* 1.3** by Mir et al. [7] and hence also generalizes and improves upon all those results which are claimed to be improved by ***Theorem* 1.3** as well.

***Theorem* 2.3.** *Let* $p(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$, $1 \leq \mu < n$ *be a polynomial of degree n that does not vanish in* $|z| < k$, $k \geq 1$. *Then for* $0 < r < R \leq 1$,

$$M(p, r) \geq \frac{(1 + r^\mu)^{\frac{n}{\mu}}}{(1 + r^\mu)^{\frac{n}{\mu}} + (R^\mu + k^\mu)^{\frac{n}{\mu}} - (r^\mu + k^\mu)^{\frac{n}{\mu}}} \left[M(p, R) + \left[\frac{n}{k^n} I_n - \frac{r^n}{k^n}\left\{\left(\frac{(R^\mu + k^\mu)}{(r^\mu + k^\mu)}\right)^{\frac{n}{\mu}} - 1\right\} + \ln\left(\frac{(R^\mu + k^\mu)}{(r^\mu + k^\mu)}\right)^{\frac{n}{\mu}}\right] \min_{|z|=k} |p(z)|\right]. \tag{2.3}$$

Here, the integral I_n *is defined as*

$$I_n = \int_r^R \frac{t^{n+\mu-1}}{t^\mu + k^\mu} dt. \tag{2.4}$$

Here, the integrand being a rational algebraic function can be evaluated by reduction formulae for a given value of n. For example, $I_0 = \frac{1}{2}\ln\left(\frac{R}{k}\right)$ *and* $I_1 = (R - r) - k\ln\left(\frac{R+k}{r+k}\right)$.

***Remark* 2.1.** *The integrand in* (2.4) *is increasing function of t, so the least approximate value of* I_n *can be taken as* $\left(\dfrac{n(R - r)r^{n+\mu-1}}{k^n(r^\mu + k^\mu)}\right)$.

3 Lemmas

For the proof of the ***Theorems***, we need the following lemmas.

***Lemma* 3.1.** *Let $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \le \mu \le n$ is a polynomial of degree n having no zeros in $|z| < k$, $k \ge 1$. Then*

$$\max_{|z|=1} |p'(z)| \le \frac{n}{1+k^\mu} \left\{ \max_{|z|=1} |p(z)| - \min_{|z|=1} |p(z)| \right\}. \tag{3.1}$$

The result is sharp and equality holds for the polynomial $p(z) = (z^\mu + k^\mu)^{\frac{n}{\mu}}$.

The above ***Lemma* 3.1** is due to Pukhta [9].

The next ***lemma*** is due to Bidkham and Dewan [4].

***Lemma* 3.2.** *Let $p(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$, $1 \le \mu \le n$ is a polynomial of degree n, having no zeros in $|z| < k$, $k \ge 1$. Then for $0 < r < R \le 1$,*

$$\max_{|z|=r} |p(z)| \ge \left(\frac{k^\mu + r^\mu}{k^\mu + R^\mu} \right)^{\frac{n}{\mu}} \max_{|z|=R} |p(z)|. \tag{3.2}$$

We improve the above ***Lemma* 3.2** as follows.

***Lemma* 3.3.** *Let $p(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$ is a polynomial of degree n, having no zeros in $|z| < k$, $k \ge 1$. Then for $0 < r < R \le 1$,*

$$\max_{|z|=r} |p(z)| \ge \left(\frac{k^\mu + r^\mu}{k^\mu + R^\mu} \right)^{\frac{n}{\mu}} \max_{|z|=R} |p(z)| + \left\{ \frac{R^n}{k^n} \left(\frac{k^\mu + r^\mu}{k^\mu + R^\mu} \right)^{\frac{n}{\mu}} - \frac{r^n}{k^n} \right\} \min_{|z|=k} |p(z)|. \tag{3.3}$$

Proof. **[Proof of Lemma 3.3]** Since $p(z)$ does not vanish in $|z| < k$, $k \ge 1$ and $|p(z)| \ge m = \min_{|z|=k} |p(z)|$, therefore by Rouche's theorem, the polynomial $F(z) = p(z) + \lambda \frac{z^n}{k^n} m$, $|\lambda| < 1$, also does not vanish in $|z| < k$, $k \ge 1$. Therefore on applying inequality (3.2) to the polynomial $F(z) = p(z) + \lambda \frac{z^n}{k^n} m$, we have

$$\max_{|z|=r} \left| p(z) + \lambda \frac{z^n}{k^n} m \right| \ge \left(\frac{k^\mu + r^\mu}{k^\mu + R^\mu} \right)^{\frac{n}{\mu}} \max_{|z|=R} \left| p(z) + \lambda \frac{z^n}{k^n} m \right|,$$

or

$$\max_{|z|=r} |p(z)| + |\lambda| \frac{r^n}{k^n} m \ge \left(\frac{k^\mu + r^\mu}{k^\mu + R^\mu} \right)^{\frac{n}{\mu}} \left(\max_{|z|=R} \left| p(z) + \lambda \frac{z^n}{k^n} m \right| \right). \tag{3.4}$$

Now suitably choosing the argument of λ such that R.H.S. of inequality (3.4) becomes

$$\max_{|z|=R} \left| p(z) + \lambda \frac{z^n}{k^n} m \right| = \max_{|z|=R} |p(z)| + |\lambda| \frac{R^n}{k^n} m. \tag{3.5}$$

Now combining inequalities (3.4) and (3.5), we get

$$\max_{|z|=r} |p(z)| + |\lambda| \frac{r^n}{k^n} m \ge \left(\frac{k^\mu + r^\mu}{k^\mu + R^\mu} \right)^{\frac{n}{\mu}} \left(\max_{|z|=R} |p(z)| + |\lambda| \frac{R^n}{k^n} m \right).$$

Or equivalently,

$$\max_{|z|=r} |p(z)| \ge \left(\frac{k^\mu + r^\mu}{k^\mu + R^\mu} \right)^{\frac{n}{\mu}} \max_{|z|=R} |p(z)| + |\lambda| \left\{ \frac{R^n}{r^n} \left(\frac{k^\mu + r^\mu}{k^\mu + R^\mu} \right)^{\frac{n}{\mu}} - \frac{r^n}{k^n} \right\} \min_{|z|=k} |p(z)|.$$

Finally letting $|\lambda| \to 1$, we get the desired result.

4 Proof of the Main Theorems

Proof. **[Proof of *Theorem* 2.1]** Since $p(z)$ does not vanish in $|z| < k$, $k \geq 1$ and $|p(z)| \geq m = \min\limits_{|z|=k} |p(z)|$, therefore by Rouche's theorem, the polynomial $F(z) = p(z) + \mu \frac{z^n}{k^n} m$, $|\mu| < 1$, also does not vanish in $|z| < k$, $k \geq 1$. Therefore on applying inequality (1.7) to the polynomial $F(z) = p(z) + \mu \frac{z^n}{k^n} m$, we have

$$\max_{|z|=\rho} |F(z)| \geq \left(\frac{k^2 + 2|\lambda|k\rho + \rho^2}{k^2 + 2|\lambda|k + 1} \right)^{\frac{n}{2}} \max_{|z|=1} |F(z)|$$

or

$$\max_{|z|=\rho} \left| p(z) + \mu m \frac{z^n}{k^n} \right| \geq \left(\frac{k^2 + 2|\lambda|k\rho + \rho^2}{k^2 + 2|\lambda|k + 1} \right)^{\frac{n}{2}} \max_{|z|=1} \left| p(z) + \mu m \frac{z^n}{k^n} \right|$$

or

$$\max_{|z|=\rho} |p(z)| + |\mu|\, m \frac{\rho^n}{k^n} \geq \left(\frac{k^2 + 2|\lambda|k\rho + \rho^2}{k^2 + 2|\lambda|k + 1} \right)^{\frac{n}{2}} \max_{|z|=1} \left| p(z) + \mu m \frac{z^n}{k^n} \right| . \tag{4.1}$$

Now suitably choosing argument of μ on R.H.S. of (4.1), we have

$$\max_{|z|=1} \left| p(z) + \mu m \frac{z^n}{k^n} \right| = \max_{|z|=1} |p(z)| + |\mu| \frac{m}{k^n} . \tag{4.2}$$

Combining (4.1) and (4.2) we get

$$\max_{|z|=\rho} |p(z)| + |\mu|\, m \frac{\rho^n}{k^n} \geq \left(\frac{k^2 + 2|\lambda|k\rho + \rho^2}{k^2 + 2|\lambda|k + 1} \right)^{\frac{n}{2}} \left\{ \max_{|z|=1} |p(z)| + |\mu| \frac{m}{k^n} \right\}$$

or

$$\max_{|z|=\rho} |p(z)| \geq \left(\frac{\rho^2 + 2|\lambda|k\rho + k^2}{1 + 2|\lambda|k + k^2} \right)^{\frac{n}{2}} \max_{|z|=1} |p(z)| + \frac{|\mu|}{k^n} \left\{ \left(\frac{\rho^2 + 2|\lambda|k\rho + k^2}{1 + 2|\lambda|k + k^2} \right)^{\frac{n}{2}} - \rho^n \right\} \min_{|z|=k} |p(z)| .$$

Finally, on letting $|\mu| \to 1$ the proof of ***Theorem*** **2.1** is completed.

Proof. **[Proof of *Theorem* 2.2]** Proof of ***Theorem*** **2.2** follows on the same lines as that of proof of ***Theorem*** **2.1**. Here we use inequality (1.8) instead of (1.7). Hence we omit the details.

Proof. **[Proof of *Theorem* 2.3]** Let $0 < t \leq k$. Since $p(z)$ does not vanish in $|z| < k$, $k \geq 1$, the polynomial $F(z) = p(tz)$ does not vanish in $|z| < \frac{k}{t}$, $\frac{k}{t} \geq 1$, therefore applying ***Lemma*** **3.1** to $F(z)$, we have

$$\max_{|z|=1} |F'(z)| \leq \frac{n}{1 + \left(\frac{k}{t}\right)^{\mu}} \left\{ \max_{|z|=1} |F(z)| - \min_{|z|=\frac{k}{t}} |F(z)| \right\},$$

which is equivalent to

$$\max_{|z|=t} |p'(z)| \leq \frac{n\, t^{\mu-1}}{t^{\mu} + k^{\mu}} \left\{ \max_{|z|=t} |p(z)| - \min_{|z|=k} |p(z)| \right\}. \tag{4.3}$$

We have, now for $0 \leq \theta < 2\pi$ and $0 < r < R \leq 1$,

$$\left| p(Re^{i\theta}) - p(re^{i\theta}) \right| \leq \int_r^R \left| p'(te^{i\theta}) \right| dt.$$

$$\left| p\left(Re^{i\theta}\right) - p\left(re^{i\theta}\right) \right| \leq \int_r^R \frac{n\, t^{\mu-1}}{t^{\mu} + k^{\mu}} \left\{ \max_{|z|=t} |p(z)| - \min_{|z|=k} |p(z)| \right\} dt \quad \text{by using (4.3)} \tag{4.4}$$

Now applying inequality (3.3) of ***Lemma*** **3.3** to inequality (4.4), we get

$$\left| p\left(Re^{i\theta}\right) - p\left(re^{i\theta}\right) \right|$$

$$\leq \int_r^R \frac{nt^{\mu-1}}{t^\mu+k^\mu}\left[\left\{\left(\frac{t^\mu+k^\mu}{r^\mu+k^\mu}\right)^{\frac{n}{\mu}}\max_{|z|=r}|p(z)|-\left(\frac{t^n}{k^n}-\frac{r^n}{k^n}\left(\frac{t^\mu+k^\mu}{r^\mu+k^\mu}\right)^{\frac{n}{\mu}}\right)\min_{|z|=k}|p(z)|\right\}-\min_{|z|=k}|p(z)|\right]dt$$

$$=\int_r^R \frac{n\,t^{\mu-1}}{t^\mu+k^\mu}\left(\frac{t^\mu+k^\mu}{r^\mu+k^\mu}\right)^{\frac{n}{\mu}}\max_{|z|=r}|p(z)|dt-n\int_r^R \frac{t^{\mu-1}}{t^\mu+k^\mu}\left[\frac{t^n}{k^n}-\frac{r^n}{k^n}\left(\frac{t^\mu+k^\mu}{r^\mu+k^\mu}\right)^{\frac{n}{\mu}}+1\right]\min_{|z|=k}|p(z)|dt.$$

The above expression is equivalent to

$$\begin{aligned}
&\left|p\left(Re^{i\theta}\right)-p\left(re^{i\theta}\right)\right| \\
&\leq \int_r^R \frac{nt^{\mu-1}}{(r^\mu+k^\mu)^{\frac{n}{\mu}}}(t^\mu+k^\mu)^{\frac{n}{\mu}-1}\max_{|z|=r}|p(z)|dt \\
&\quad -n\min_{|z|=k}|p(z)|\int_r^R\left[\frac{t^{n+\mu-1}}{(t^\mu+k^\mu)k^n}-\frac{r^n}{k^n}\frac{t^{\mu-1}(t^\mu+k^\mu)^{\frac{n}{\mu}-1}}{(r^\mu+k^\mu)^{\frac{n}{\mu}}}+\frac{t^{\mu-1}}{(t^\mu+k^\mu)}\right]dt \\
&\leq \frac{n}{(r^\mu+k^\mu)^{\frac{n}{\mu}}}M(p,r)\int_r^R t^{\mu-1}(t^\mu+k^\mu)^{\frac{n}{\mu}-1}dt-n\min_{|z|=k}|p(z)| \\
&\quad \int_r^R\left[\frac{t^{n+\mu-1}}{(t^\mu+k^\mu)k^n}-\frac{r^n}{k^n}\frac{t^{\mu-1}(t^\mu+k^\mu)^{\frac{n}{\mu}-1}}{(r^\mu+k^\mu)^{\frac{n}{\mu}}}+\frac{t^{\mu-1}}{(t^\mu+k^\mu)}\right]dt, \\
&\leq \frac{n}{(r^\mu+1)^{\frac{n}{\mu}}}M(p,r)\int_r^R t^{\mu-1}(t^\mu+k^\mu)^{\frac{n}{\mu}-1}dt-\min_{|z|=k}|p(z)| \\
&\quad \left[\frac{n}{k^n}I_n-\frac{nr^n}{k^n(r^\mu+k^\mu)^{\frac{n}{\mu}}}\int_r^R t^{\mu-1}(t^\mu+k^\mu)^{\frac{n}{\mu}-1}+\int_r^R\frac{t^{\mu-1}}{(t^\mu+k^\mu)}dt\right] \\
&=\frac{(R^\mu+k^\mu)^{\frac{n}{\mu}}-(r^\mu+k^\mu)^{\frac{n}{\mu}}}{(1+r^\mu)^{\frac{n}{\mu}}}M(p,r) \\
&\quad -\left[\frac{n}{k^n}I_n-\frac{r^n}{k^n}\frac{(R^\mu+k^\mu)^{\frac{n}{\mu}}-(r^\mu+k^\mu)^{\frac{n}{\mu}}}{(k^\mu+r^\mu)^{\frac{n}{\mu}}}+\frac{n}{\mu}\,\ln\left(\frac{(R^\mu+k^\mu)}{(r^\mu+k^\mu)}\right)\right]\min_{|z|=k}|p(z)| \\
&=\frac{(R^\mu+k^\mu)^{\frac{n}{\mu}}-(r^\mu+k^\mu)^{\frac{n}{\mu}}}{(1+r^\mu)^{\frac{n}{\mu}}}M(p,r) \\
&\quad -\left[\frac{n}{k^n}I_n-\frac{r^n}{k^n}\left\{\left(\frac{R^\mu+k^\mu}{r^\mu+k^\mu}\right)^{\frac{n}{\mu}}-1\right\}+\ln\left(\frac{(R^\mu+k^\mu)}{(r^\mu+k^\mu)}\right)^{\frac{n}{\mu}}\right]\min_{|z|=k}|p(z)|
\end{aligned}$$

where the integral I_n is as defined in (2.4).

Thus we have shown that for $0 \leq \theta < 2\pi$ and $0 < r < R \leq 1$,

$$\begin{aligned}
\left|p\left(Re^{i\theta}\right)-p\left(re^{i\theta}\right)\right| &\leq \frac{(R^\mu+k^\mu)^{\frac{n}{\mu}}-(r^\mu+k^\mu)^{\frac{n}{\mu}}}{(1+r^\mu)^{\frac{n}{\mu}}}M(p,r) \\
&\quad -\left[\frac{n}{k^n}\,I_n-\frac{r^n}{k^n}\left\{\left(\frac{R^\mu+k^\mu}{r^\mu+k^\mu}\right)^{\frac{n}{\mu}}-1\right\}+\ln\left(\frac{(R^\mu+k^\mu)}{(r^\mu+k^\mu)}\right)^{\frac{n}{\mu}}\right]\min_{|z|=k}|p(z)|
\end{aligned}$$

Therefore, finally, we have the equivalent result

$$\begin{aligned}
M(p,R) &\leq M(p,r)+\frac{(R^\mu+k^\mu)^{\frac{n}{\mu}}-(r^\mu+k^\mu)^{\frac{n}{\mu}}}{(1+r^\mu)^{\frac{n}{\mu}}}M(p,r) \\
&\quad -\left[\frac{n}{k^n}I_n-\frac{r^n}{k^n}\left\{\left(\frac{(R^\mu+k^\mu)}{(r^\mu+k^\mu)}\right)^{\frac{n}{\mu}}-1\right\}+\ln\left(\frac{(R^\mu+k^\mu)}{(r^\mu+k^\mu)}\right)^{\frac{n}{\mu}}\right]\min_{|z|=k}|p(z)|
\end{aligned}$$

or

$$M(p,R) \le \frac{(1+r^{\mu})^{\frac{n}{\mu}} + (R^{\mu}+k^{\mu})^{\frac{n}{\mu}} - (r^{\mu}+k^{\mu})^{\frac{n}{\mu}}}{(1+r^{\mu})^{\frac{n}{\mu}}} M(p,r)$$
$$- \left[\frac{n}{k^n} I_n - \frac{r^n}{k^n} \left\{ \left(\frac{(R^{\mu}+k^{\mu})}{(r^{\mu}+k^{\mu})} \right)^{\frac{n}{\mu}} - 1 \right\} + \ln \left(\frac{(R^{\mu}+k^{\mu})}{(r^{\mu}+k^{\mu})} \right)^{\frac{n}{\mu}} \right] \min_{|z|=k} |p(z)|$$

or equivalently

$$M(p,r) \ge \frac{(1+r^{\mu})^{\frac{n}{\mu}}}{(1+r^{\mu})^{\frac{n}{\mu}} + (R^{\mu}+k^{\mu})^{\frac{n}{\mu}} - (r^{\mu}+k^{\mu})^{\frac{n}{\mu}}}$$
$$\left[M(p,R) + \left[\frac{n}{k^n} I_n - \frac{r^n}{k^n} \left\{ \left(\frac{(R^{\mu}+k^{\mu})}{(r^{\mu}+k^{\mu})} \right)^{\frac{n}{\mu}} - 1 \right\} + \ln \left(\frac{(R^{\mu}+k^{\mu})}{(r^{\mu}+k^{\mu})} \right)^{\frac{n}{\mu}} \right] \min_{|z|=k} |p(z)| \right].$$

Thus the desired result is proved.

Acknowledgment

The author is thankful to the anonymous referee and Editor for their valuable suggestions.

References

[1] N. C. Ankeny and T. J. Rivlin, On a Theorem of S. Bernstein, *Pacific J. Maths.*, **5** (1955), 249-252.

[2] A. Aziz, Growth of polynomials whose zeros are within or outside a circle, *Bull. Austral. Math. Soc.*, **35** (1987), 247-250.

[3] A. Aziz and Q. M. Dawood, Inequalities for a polynomial and its derivative, *J. Approx. Theory*, **54** (1988), 306-313.

[4] M. Bidkham and K. K. Dewan, Inequalities for a polynomial and its derivative, *J. Math. Anal. Appl.*, **166** (1992), 319-324.

[5] N. K. Govil and E. R. Nwaeze, Some sharpening and generalizations of a result of T. J. Rivlin, *Anal. Theory Appl.*, **33** (2017), 219-228.

[6] N. K. Govil M. A. Qazi and Q. I. Rahman, Inequalities describing the growth of polynomials not vanishing in a disk of prescribed radius, *Math. Inequal. Appl.*, **6** (3) (2003), 453-467.

[7] A. Mir, A. Wani and I. Hussain, A note on a theorem of T. J. Rivlin, *Anal. Theory Appl.*, Vol. **34**(4), (2018), 293-296

[8] G. Pólya and G. Szegö, *Problems and Theorems in Analysis*, Vol. 1, Springler-Verlag, Berlin, 1972.

[9] M. S. Pukhta, *Extremal Problems for Polynomials and on Location of Zeros of Polynomials, Ph.D.* Thesis submitted to the Jamia Millia Islamia, 1995.

[10] M. A. Qazi, On the maximum modulus of polynomials, *Proc. Amer. Math. Soc.*, **115** (1992), 337-343.

[11] M. Riesz, Uber enin Satz der Herrn Serge Bernstein, *Acta Math.*, **40** (1918), 337-347.

[12] T. J. Rivlin, On the maximum modulus of polynomials, *Amer. Math. Monthly*, **67** (1960), 251-253.

[13] R. S. Varga, A comparison of the successive over relaxation method and semi-iterative methods using Chebychev polynomials, *J. Soc. Indust. Appl. Math.*, **5** (1957), 44.

[14] A. Zireh, E. Khojastehnejhad and R. S. Musawi, Some results concerning growth of polynomials, *Anal. Theory Appl.*, **29** (2013), 37-46.

Jñānābha, Vol. 50(1) (2020), 79-88
(Dedicated to Honor Professor H.M. Srivastava on His 80th Birth Anniversary Celebrations)

KAMAL TRANSFORM OF STRONG BOEHMIANS

By
A. M. Mahajan
Department of Mathematics, Walchand College of Arts and Science
Solapur-413006,Maharashtra,India.
Email:ammahajan19@gmail.com
M. S. Chaudhary
Department of Mathematics, Shivaji University,
Vidya Nagar, Kolhapur - 416004, Maharashtra, India
Email:m_s_chaudhary@rediffmail.com
(*Received : January* 24, 2020 *; Revised: April* 22, 2020)
DOI: https://doi.org/10.58250/jnanabha.2020.50110

Abstract

The concept of Boehmian was motivated by the so called regular operators introduced by T.K.Boehme. The construction of Boehmians is similar to the construction of field of quotients. Several integral transforms have been extended to various class of Boehmians. We study here Kamal transform and extend it to Strong Boehmian space. This Kamal tranform is 1-1 and continuous in the space of Boehmians. Inverse Kamal transform is also defined.
2010 Mathematics Subject Classifications: 54C40, 14E20, 46E25, 20C20.
Keywords and phrases: Kamal Transform, Convolution Theorem, Boehmians, Distribution.

1 Introduction

Boehmians have an algebraic character of Mikusinski operators and do not have any restriction on the support. Here we discuss the Kamal transform defined by [11] on certain space of Strong Boehmians. Definition and some properties of Kamal transforms are given. The Kamal transform was introduced by Abdelilah Kamal [11] and many properties are discussed in [10, 13]. Some application related to population growth and decay of Kamal transform are given in[3]. Khandelwal [13] discussed Kamal transform and Kamal decomposition method for solving system of non linear PDE. Also Alomari and Kilicman[7] studied generalized Hartley-Hilbert and Fourier-Hilbert transform and extended them to a class of Boehmians. Al-omari [6] studied the distributional Elzaki transform and gave the extension to Boehmian space. The application of Natural transform and Boehmians [5] is also studied by Al-omari. Sudhansh Aggarwal [2, 3] gave application of Kamal transform for solving voltera integral equation, population growth & decay problems. S.K.Q Al-omari [6] gave application and the relation between Boehmians and Elzaki transform. E.R. Dill & P. Mikusinski [9] defined the concept of Strong Boehmians & its applications. The concept of Mikusinski operators was defined by T.K. Boehme[8]. R. Roopkumar & E.R. Negrin [17] discussed the unified extension of Stieltjes and Poission transform to Boehmians.

The Kamal transform of *f(t)* is defined by [11]

(1.1) $$K[f(t)] = F(v) = \int_0^\infty f(t)e^{-t/v}\,\mathrm{d}t \qquad J_1 \le v \le J_2$$

over the set of functions

(1.2) $$\mathscr{A} = \{f(t) : \exists M,\ J_1, J_2 > 0 \quad |f(t)| < Me^{|t|/J_j} \quad \text{if} \quad t \in (-1)^j \times [0,\infty)\}.$$

We denote the usual convolution of f and g by

(1.3) $$(f * g)(x) = \int_{R^+} f(x-t)\, g(t)\, \mathrm{d}t.$$

The Kamal transform of the convolution product is given by

(1.4) $$K(f * g) = K(f) \cdot K(g).$$

General properties of Kamal transforms:

1. If $\alpha, \beta \in \mathbb{R}$ and $K[f] = F(v)$ and $K[g] = G(v)$ then Kamal transform is linear.

$$\begin{aligned} K[\alpha f + \beta g] &= \int_0^\infty (\alpha f + \beta g)\,(t)\, e^{-t/v}\, \mathrm{d}t \\ &= \alpha \int_0^\infty f(t)\, e^{-t/v}\, \mathrm{d}t + \beta \int_0^\infty g(t)\, e^{-t/v}\, \mathrm{d}t \\ &= \alpha K[f] + \beta K[g] \\ &= \alpha F(v) + \beta G(v). \end{aligned}$$

2. If $f(t) = e^{at}, \sin at, \cos at$ then corresponding $K[f(t)]$ is given by $\dfrac{v}{1-av}, \dfrac{av^2}{1+a^2v^2}, \dfrac{v}{1+a^2v^2}$
For more properties see [11].

2 Strong Boehmians

We study Strong Boehmians [4, 9] and General Boehmians [5]-[8]. Let I_+ is the set of positive real numbers and $\mathscr{F}$ denote the Schwartz space of test functions ϕ with compact supports over I_+ and $\eta(\Omega)$ be the space of all infinitely differentiable functions over Ω where $\Omega = [1,\infty) \times I_+$. The dual of $\eta(\Omega)$ is $\eta'(\Omega)$ consists of distributions of compact supports. Let $f \in \eta(\Omega)$ and $\phi \in \mathscr{F}$ the convolution of f and ϕ is given by

(2.1) $$(f\#\phi)(x) = \int_{I_+} f(\alpha, t)\, \phi(x-t)\mathrm{d}t,$$

where $\alpha \in [1,\infty)$.

Let $\mu(I_+)$ be the subset of $\mathscr{F}$ of the test functions such that

(2.2) $$\int_{I_+} \phi(x)\, \mathrm{d}x = 1.$$

The pair (f,ϕ) or (f/ϕ) of functions such that $f \in \eta(\Omega)$, $\phi \in \mu(I_+)$ is said to be quotient of function denoted by (f,ϕ) or (f/ϕ) if and only if

(2.3) $$\{f(\alpha,x)\}\#\{\beta\phi\,(\beta x)\} = \{f(\beta,x)\}\#\{\alpha\phi\,(\alpha x)\},$$

for all $\alpha, \beta \in [1,\infty)$
or we define

$$f(\alpha,x)\,\#\,d_\beta\,\phi(x) = f(\beta,x)\,\#\,d_\alpha\,\phi(x),$$

where

$$d_\beta\,\phi(x) = \beta\,\phi(\beta x),$$
$$d_\alpha\,\phi(x) = \alpha\,\phi(\alpha x).$$

We use both the definitions whenever we required. Two quotients (f, ϕ) and (g, ψ) are said to be equivalent that is $(f, \phi) \sim (g, \psi)$ if and only if

$$f(\alpha, x)\#\beta\psi\,(\beta x) = g\,(\beta, x)\,\#\,\alpha\,\phi(\alpha x), \tag{2.4}$$

$\alpha,\beta \in [1,\infty)$.

Let the set be denoted by

$$\mathscr{B} = \{(f, \phi) | \forall f \in \eta(\Omega)\,, \phi \in \mu(I_+)\}. \tag{2.5}$$

Then the equivalence class $[(f, \phi)]$ containing (f, ϕ) is called Strong Boehmian. The space of all such Boehmians is denoted by $\mathscr{L}(\eta, \mu, \#)$ is called as space of Strong Boehmians. Following conclusions are given in [9]

1. Let ϕ, $\psi \in \mu(I_+)$ then $\phi\,\#\,\psi \in \mu(I_+)$,
2. Let f $\in \eta(\Omega)$ and $\phi \in \mu(I_+)$ then f # $\phi \in \eta(\Omega)$,
3. Let (f,ϕ) $\in \mathscr{B}$ and $\psi \in \mu(I_+)$ then

$$(f\#\psi, \phi\#\psi) \in \mathscr{B} \quad \text{and} \quad (f, \phi) \sim (f\#\psi, \phi\#\psi). \tag{2.6}$$

4. If $\phi \in \mu(I_+)$ then for $\alpha \geq 1$ $\quad \alpha\phi(\alpha\text{ x}) \in \mu(I_+)$,
5. Let $(f, \phi) \in \mathscr{B}, z > 0$ and $h(\alpha, x) = f(\alpha + z, x)$ and $\psi = z\phi(zx)$ then

$$(g, \psi) \in \mathscr{B} \quad \text{and} \quad (g, \psi) \sim (f, \phi). \tag{2.7}$$

Further the operation of addition and scalar multiplication in $\mathscr{L}(\eta, \mu, \#)$ are defined in the usual notation as,

$$\frac{f}{\phi} + \frac{g}{\psi} = \frac{f\#\psi + g\#\phi}{\phi\#\psi},\ \lambda \cdot \frac{f}{\phi} = \frac{\lambda f}{\phi}, \quad \frac{f}{\phi}\#\psi = \frac{f\#\psi}{\phi}. \tag{2.8}$$

The above operations are well defined in $\mathscr{L}$ and hence $\mathscr{L}$ is a vector space.

Let

$$D^P = (\frac{\partial}{\partial x_1})^{P_1}(\frac{\partial}{\partial x_2})^{P_2}(\frac{\partial}{\partial x_3})^{P_3}\cdots(\frac{\partial}{\partial x_N})^{P_N} \tag{2.9}$$

where $P = (p_1, p_2, \cdots, p_N)$ and $p_1, p_2, \cdots, p_N$ are nonnegative integers for $\frac{f}{\phi} \in \mathscr{L}(\eta, \mu, \#)$ define $D^p(\frac{f}{\phi}) = \frac{D^p f}{\phi}$, where D^p is well defined operation on $\mathscr{L}$. A sequence of Strong Boehmians $\{y_n\}$ is said to converge to a Strong Boehmian y if $y = \frac{f}{\phi}$ and $y_n = \frac{f_n}{\phi}$ for some $f, f_n \in \eta$ & $\phi \in \mu(I_+)$, $n \in \mathbb{N}$ and $f_n \to f$ uniformly on compact subset of Ω as n $\to\infty$.

3 General Construction of Boehmians

Mikusinski introduced a new class of generalised function space called Boehmian space, which is suitable for extending integral transforms. The construction of Boehmian space and its convergence is given in [15] The construction of Boehmians consists of following elements:

1. A set Γ,
2. Commutative semi group(S,$\otimes$),

3. An operation $\bigstar$: $\Gamma \times S \rightarrow \Gamma$ such that for each $x \in \Gamma$ and $\phi_1, \phi_2 \in \triangle \subset S$ $x\bigstar(\phi_1 \otimes \phi_2) = (x \bigstar \phi_1) \otimes (x \bigstar \phi_2)$,
4. (a) A collection $\triangle \subset S$ such that if $x, y \in \Gamma$, $\phi_n \in \triangle$ $x \bigstar \phi_n = y\bigstar \phi_n$ $\forall n \Rightarrow x = y$,

   (b) If $\phi_n \in \triangle$ and $\psi_n \in \triangle$ then $\phi_n \otimes \psi_n \in \triangle$, $\triangle$ is a set of all delta sequences.

Consider

$$\mathscr{B} = \{(x_n, \phi_n) : x_n \in \Gamma, \phi_n \in \triangle, x_n \bigstar \phi_m = x_m \bigstar \phi_n \quad \forall m, n \in \mathbb{N}\}. \tag{3.1}$$

If $(x_n, \phi_n), (y_n, \psi_n) \in \mathscr{B}$ $\quad x_n\bigstar\psi_m = y_m\bigstar\phi_n \quad \forall m, n \in N$ we say that $(x_n, \phi_n) \sim (y_n, \psi_n)$.The relation $\sim$ is an equivalence relation in $\mathscr{B}$.The space of equivalence classes in $\mathscr{B}$ is denoted by $\mathscr{L}_{\mathscr{B}}(\Gamma, S, \triangle)$. Elements of $\mathscr{L}_{\mathscr{B}}(\Gamma, S, \triangle)$ are called General Boehmians.We define a mapping which is a canonical mapping between Γ and $\mathscr{L}_{\mathscr{B}}$ as $x \rightarrow x \bigstar \phi_n/\phi_n$.
In $\mathscr{L}_{\mathscr{B}}(\Gamma, S, \triangle)$ there are two type convergences

1. A sequence q_n in $\mathscr{L}_{\mathscr{B}}(\Gamma, \rho, \triangle)$ is said to be δ convergent to q in $\mathscr{L}_{\mathscr{B}}(\Gamma, S, \triangle)$ denoted by $q_n \xrightarrow{\delta} q$ if there exist a delta sequence δ_n such that ($q_n \bigstar \delta_n$), ($q \bigstar \delta_n$) $\in \Gamma$ and for all k,n $\in \mathbb{N}$ ($q_n \bigstar \delta_k$)$\rightarrow$ ($q \bigstar \delta_k$) as n$\rightarrow \infty$ in Γ,
2. A sequence (q_n) in $\mathscr{L}_{\mathscr{B}}(\Gamma, S, \triangle)$ is said to be $\triangle$ convergent to q in $\mathscr{L}_{\mathscr{B}}(\Gamma, S, \triangle)$ denoted by $q_n \xrightarrow{\Delta} q$ if there exist $(\delta_n) \in \triangle$ such that $(q_n - q)\bigstar\ \delta_n \in \Gamma \quad \forall n \in N$ and
$(q_n - q)\bigstar\ \delta_n \rightarrow 0$ as $n \rightarrow \infty$ in Γ.

Following lemma is an equivalent statement for δ- convergence given by [17]

***Lemma* 3.1.** *$q_n \xrightarrow{\delta} q$ (as n$\rightarrow \infty$) in $\mathscr{L}_{\mathscr{B}}(\Gamma, S, \triangle)$ if and only if there exist $f_{n,k}, f_k \in \Gamma$ and $\delta_k \in \triangle$ such that*

$$q_n = [f_{n,k}/\delta_k] \qquad q = [f_n/\delta_k]$$

and for each $k \in N$ $f_{n,k} \rightarrow f_k$ as $n \rightarrow \infty$ on Γ.

4 Kamal Transform of Strong Boehmians

***Theorem* 4.1** (Convolution theorem)**.** *Let $f \in \eta(\Omega)$ and $\phi \in \mu(I_+)$ then*

$$\begin{aligned} K(f \# \beta\, \psi(\beta x))(\xi) &= K(f(x)) \cdot \beta K(\psi(\beta x)) \\ &= \hat{f}(\xi) \cdot \beta\hat{\psi}(\beta\xi). \end{aligned} \tag{4.1}$$

If onwards we define

$$f(\alpha, x) \# \beta\, \psi\, (\beta x) = f(\alpha, x)\#\, d_\beta\, \psi(x),$$

where,

$$d_\beta\, \psi(x) = \beta\, \psi(\beta x)$$

Then,

$$K(f \# d_\beta\ \psi)\, (\xi) = \hat{f}(\alpha, \xi) \cdot d_\beta\, \hat{\psi}(\xi),$$

where $\hat{f}$ and $\hat{\psi}$ are Kamal transforms of f and ψ.

Proof. By using definition of Kamal transform, Fubini's theorem,

$$K(f(\alpha,t)\# d_\alpha \psi(x))(\xi) = \int_0^\infty f(\alpha,t)\,\mathrm{d}t \int_0^\infty d_\alpha \psi(x-t)e^{-x/\xi}\,\mathrm{d}x \tag{4.2}$$

$$= \int_0^\infty f(\alpha,t)\,\mathrm{d}t \int_0^\infty \alpha\ \psi(\alpha x-\alpha t)\, e^{-x/\xi}\,\mathrm{d}x.$$

$$Put \qquad \alpha x-\alpha t = z \quad \text{i.e} \quad x = \frac{z}{\alpha}+t \quad \text{and} \quad dx = \frac{dz}{\alpha} \text{ to get} \tag{4.3}$$

$$K(f(\alpha,t)\# d_\alpha\psi(x))(\xi) = \int_0^\infty f(\alpha,t)\,\mathrm{d}t \int_0^\infty \alpha\,\psi(z)\ e^{-(\frac{z}{\alpha}+t)/\xi}\ \mathrm{d}z$$

$$= \int_0^\infty f(\alpha,t)\,\mathrm{d}t \int_0^\infty \alpha\,\psi(z)\ e^{-\frac{z}{\alpha\xi}}\ \cdot e^{-\frac{t}{\xi}}\ \mathrm{d}z$$

$$= \int_0^\infty e^{-\frac{t}{\xi}}\ f(\alpha,t)\,\mathrm{d}t \int_0^\infty \alpha\,\psi(z)\ e^{-\frac{z}{\alpha\xi}}\ \mathrm{d}z$$

$$= \hat{f}(\alpha,\xi)\cdot d_\alpha\ \hat{\psi}(\xi),$$

which completes the proof of the Theorem.

Now we define the images of Kamal transform of Strong Boehmians.

***Definition* 4.1.** *Let $\triangle_1(I_+)$ or $\triangle_1$ be a set of delta sequences such that $\psi_n \in \mu(I_+)$ and suppose $\psi_n \subset (0,\gamma_n)$ $\gamma_n > 0$, $\gamma_n \to 0$ as $n \to \infty$.*
Let $m(I_+)$ be the set of images of Kamal transforms of all $\mu(I_+)$ elements and $\triangle_2(I_+)$ be the set of Kamal transform of all delta sequences from $\triangle_1$ for $f \in \eta(\Omega)$ and $\hat{\psi} \in m(I_+)$ we define the operation $\circledast$ as

$$f(\alpha,\xi)\circledast\hat{\psi}(\xi) = f(\alpha,\xi)\ d_a\,\hat{\psi}(\xi) \quad as \in [1,\infty),$$

from which we see that

$$f\circledast\hat{\psi}\ \in \eta(\Omega) \quad as f \in \eta(\Omega) \quad and \quad d_a\cdot\hat{\psi} \in m(I_+). \tag{4.4}$$

***Lemma* 4.1.** 1. *If $\hat{\phi}_n, \hat{\psi}_n \in \triangle_2(I_+)$ then $\hat{\psi}_n \circledast \hat{\phi}_n \in \triangle_2(I_+)$ $\quad \forall n \in N$*
2. Let $f, g \in \eta(\Omega)$ and $\hat{\psi}_n \in \triangle_2(I_+)$ such that

$$f(c,\xi)\circledast\hat{\psi}_n(\xi) = g(d,\xi)\circledast\hat{\psi}_n(\xi) \tag{4.5}$$

$$then\ f(c,\xi) = g(d,\xi) \quad \forall c,d \in [1,\infty).$$

Proof. 1. $\hat{\phi}_n, \hat{\psi}_n \in \triangle_2(I_+)$ we find the sequences $\phi_n, \psi_n \in \triangle_1(I_+)$.
Since $\phi_n \# \psi_n\ \in \triangle_1(I_+)$, we get

$$K(\phi_n \# \psi_n) = \hat{\phi}_n(\xi)\cdot\hat{\psi}_n(\xi) = \hat{\phi}_n \circledast \hat{\psi}_n \in \triangle_2(I_+) \quad \forall n \in N \quad [by(4.4)]. \tag{4.6}$$

2. Let $\hat{\psi}_n \in \triangle_2(I_+)$ where $\hat{\psi}_n$ is delta sequence $d_a\hat{\psi}_n \to 1$ hence $\hat{\psi}_n \to 1$ as $n \to \infty$

$$f(c,\xi)\circledast\hat{\psi}_n(\xi) = f(c,\xi)\,d_a\hat{\psi}_n(\xi) \ \to f(c,\xi) \quad \text{as} \quad n\to\infty, \tag{4.7}$$

$$g(d,\xi)\circledast\hat{\psi}_n(\xi) = g(d,\xi)\,d_a\hat{\psi}_n(\xi) \ \to g(d,\xi) \quad \text{as} \quad n\to\infty, \tag{4.8}$$

from (4.7) and (4.8) $f(c,\xi) = g(d,\xi)$ $\quad \forall\xi \in I_+ \quad c,d \in [1,\infty)$, which completes the proof.

***Lemma* 4.2.** *The mapping $\eta(\Omega) \circledast m(I_+) \to m(I_+)$ defined by*

$$f(\alpha,\xi) \circledast \hat{\phi}_n(\xi) = f(\alpha,\xi)\ d_a\hat{\phi}_n(\xi) \tag{4.9}$$

satisfies the following properties:

1 $\hat{\phi}_n \circledast \hat{\psi}_n = \hat{\psi}_n \circledast \hat{\phi}_n$ *for every* $(\hat{\phi}_n), (\hat{\psi}_n) \in \triangle_2(I_+)$ *then*

$$(f+g) \circledast \hat{\phi}_n = f \circledast \hat{\phi}_n + g \circledast \hat{\phi}_n,$$

2 *If* $f \in \eta(\Omega)$, $\quad (\hat{\phi}_n), (\hat{\psi}_n) \in \triangle_2(I_+)$ *then*

$$(f \circledast \hat{\phi}_n) \circledast \hat{\psi}_n = f \circledast (\hat{\phi}_n \circledast \hat{\psi}_n),$$

3 *If* $f \in \eta(\Omega)$, $\quad (\hat{\phi}_n), (\hat{\psi}_n) \in \triangle_2(I_+)$ *then*

$$(f \circledast \hat{\phi}_n) \circledast \hat{\psi}_n = f \circledast (\hat{\phi}_n \circledast \hat{\psi}_n).$$

***Theorem* 4.2.** *The following are true*

1 *If* $f_n \to f$ *in* $\eta(\Omega)$ *and* $\hat{\psi} \in m(I_+)$ *then* $f_n \circledast \hat{\psi} \to f \circledast \hat{\psi}$ *as* $n \to \infty$,
2 *If* $f_n \to f$ *in* $\eta(\Omega)$ *and* $\hat{\psi}_n \in \triangle_2(I_+)$ *then* $f_n \circledast \hat{\psi}_n \to f$ *as* $n \to \infty$.

Proof. 1. $\hat{\psi} \in m(I_+)$ $\quad f_n,\ f \in \eta(\Omega)$ then

$$|D^k_\xi(f_n(\alpha,\xi) \circledast \psi(\hat{\xi}) - f(\alpha,\xi) \circledast \psi(\hat{\xi}))| = |D^k_\xi \cdot d_a\, \hat{\psi}(\xi)\,(f_n - f)(\alpha,\xi)| \to 0. \tag{4.10}$$

As $n \to \infty$ in $\eta(\Omega)$, therefore $f_n \circledast \hat{\psi} \to f \circledast \hat{\psi}$.

2. $\hat{\phi}_m\ \triangle_2(I_+)$ then $d_a\, \hat{\phi}_n(\xi) \to 1$ as $n \to \infty$ implies

$$|D^k_\xi(f_n \circledast \hat{\psi}_n(\xi) - f(\alpha,\xi)| \to |D^k_\xi(f_n(\alpha,\xi) - f(\alpha,\xi)| \to 0 \text{ as } n \to \infty. \tag{4.11}$$

Hence $f_n \circledast \hat{\psi}_n \to f$, which completes the proof.

The General Boehmian space $\mathscr{L}_{\mathscr{B}}(\eta,\ m,\ \triangle_2,\ \circledast)$ or $\mathscr{L}_{\mathscr{B}}$ is constructed. We give some properties of sum, scalar multiplication, differentiation as

$$[\frac{f_n}{\hat{\phi}_n}] + [\frac{g_n}{\hat{\psi}_n}] = [\frac{f_n \circledast \hat{\psi}_n + g_n \circledast \hat{\phi}_n}{\hat{\phi}_n \circledast \hat{\psi}_n}], \alpha[\frac{f_n}{\hat{\phi}_n}] = [\frac{\alpha f_n}{\hat{\phi}_n}]. \tag{4.12}$$

$$[\frac{f_n}{\hat{\phi}_n}] \circledast [\frac{g_n}{\hat{\psi}_n}] = [\frac{f_n \circledast g_n}{\hat{\phi}_n \circledast \hat{\psi}_n}], D^\alpha[\frac{f_n}{\hat{\phi}_n}] = [\frac{D^\alpha f_n}{\hat{\phi}_n}]. \tag{4.13}$$

Now we are concerned with the Strong Boehmians which are described by the set $(\eta, \#)$ and the subset $(\mu, \#)$ with the family $\triangle_1$ of delta sequences such a space is denoted by $\mathscr{L}(\eta,\ (\mu,\#),\ \triangle_1,\ \#)$ or simply by $\mathscr{L}$.This space preserve the operation of addition, scalar multiplication,differentiation and the convolution.

***Definition* 4.2.** *Let $f \in \eta(\Omega)$ and $\phi \in \mu(I_+)$ we define the Kamal transform of the Strong Boehmians $[f_n/\phi_n]$ in $\mathscr{L}$ by*

$$\tilde{\gamma}[\frac{f_n}{\phi_n}] = [\frac{\hat{f}_n}{d_a\hat{\phi}_n}] \in \mathscr{L}_{\mathscr{B}} \quad \textit{where } \mathscr{L}_{\mathscr{B}} \textit{ is General Boehmians.} \tag{4.14}$$

***Theorem* 4.3.** *The Kamal transform $\tilde{\gamma} : \mathscr{L} \to \mathscr{L}_{\mathscr{B}}$ is well defined.*

Proof. Let $[\frac{f_n}{\phi_n}], [\frac{g_n}{\psi_n}] \in \mathscr{L}$ are such that $[\frac{f_n}{\phi_n}] = [\frac{g_n}{\psi_n}]$. Then

$$f_n(\alpha, x) \# d_\beta \psi_n(x) = g_n(\beta, x) \# d_\alpha \phi_n(x). \tag{4.15}$$

Apply the convolution theorem on both sides of (4.15)

$$\frac{\hat{g}_n}{d_\beta \hat{\psi}_n} = \frac{\hat{f}_n}{d_\alpha \hat{\phi}_n}$$

Hence

$$\tilde{\gamma}\,[\frac{f_n}{\phi_n}] = \tilde{\gamma}\,[\frac{g_n}{\psi_n}], \tag{4.16}$$

which completes the proof.

***Theorem* 4.4.** *(ψ_n), $(\phi_n) \in \triangle_1(I_+)$ and $f, g \in \eta(\Omega)$ then mapping $\tilde{\gamma} : \mathscr{L} \to \mathscr{L}_{\mathscr{B}}$ is one-one.*

Proof. Now by (4.16)

$$\tilde{\gamma}\,[f_n/\phi_n] = \tilde{\gamma}\,[g_n/\psi_n] \text{ in } \mathscr{L}_{\mathscr{B}}.$$

Therefore

$$\begin{aligned} \hat{f}_n(\alpha, \xi) \circledast \hat{\psi}_n(\xi) &= \hat{g}_n(\beta, \xi) \circledast \hat{\phi}_n(\xi), \\ \hat{f}_n(\alpha, \xi)\, d_\beta \hat{\psi}_n(\xi) &= \hat{g}_n(\beta, \xi)\, d_\alpha \hat{\phi}_n(\xi), \\ K(f_n(\alpha, x) \# d_\beta \psi_n(x)) &= K(g_n(\beta, x) \# d_\alpha \phi_n(x)). \end{aligned}$$

Since Kamal transform is one-one.

$$\begin{aligned} f_n(\alpha, x) \# d_\beta \psi_n(x) &= g_n(\beta, x) \# d_\alpha \phi_n(x) \\ \Rightarrow \quad \frac{f_n(\alpha, x)}{\phi_n(x)} &\sim \frac{g_n(\beta, x)}{\psi_n(x)} \\ \Rightarrow \quad [\frac{f_n}{\phi_n}] &= [\frac{g_n}{\psi_n}], \end{aligned}$$

which completes the proof.

***Theorem* 4.5.** *$\tilde{\gamma} : \mathscr{L} \to \mathscr{L}_{\mathscr{B}}$ is continuous with respect to μ convergence.*

Proof. Let $y_n \to y \in \mathscr{L}$ by using the convergence concept in μ in $\mathscr{L}$[[9], [Theorem (2.6)]], we have ϕ for all y_n such that $y_n = [\frac{f_n}{\phi}]$ $\quad y = [\frac{f}{\phi}]$ and $f_n \to f$ as $n \to \infty$

Hence $\hat{f}_n \to \hat{f}$ as $n \to \infty$

$$\Rightarrow \frac{\hat{f}_n}{d_\alpha \hat{\phi}} \to \frac{\hat{f}}{d_\alpha \hat{\phi}} \quad \text{as } n \to \infty$$

Therefore $\tilde{\gamma}\, y_n \to \tilde{\gamma}\, y$ as $n \to \infty$ in $\mathscr{L}_{\mathscr{B}}$.

***Definition* 4.3.** *Let $z = [\hat{f}_n / d_\alpha \hat{\phi}_n] \in \mathscr{L}_{\mathscr{B}}$ then we can define $\tilde{\gamma}^{-1}$ of $\tilde{\gamma}$ by*

$$\tilde{\gamma}^{-1} z = [\frac{f_n}{\phi_n}] \in \mathscr{L}.$$

We can prove that $\tilde{\gamma}^{-1}$ is well defined,linear,continuous w.r.t. δ convergence.

***Theorem* 4.6.** *The mapping $\tilde{\gamma}^{-1} : \mathscr{L}_{\mathscr{B}} \to \mathscr{L}$ is well defined.*

Proof. Let

$$[\hat{f}_n/d_a\hat{\phi}_n] = [\hat{h_m}/d_b\hat{\psi}_n] \qquad a, b \in [1, \infty).$$

Then

$$\hat{f}_n(a,\xi) \circledast \hat{\psi_m}(\xi) = \hat{h_m}(b,\xi) \circledast \hat{\phi}_n(\xi).$$

By(4.4)

$$\hat{f}_n(a,\xi)\, d_b\, \hat{\psi_m}(\xi) = \hat{h_m}(b,\xi)\, d_a\, \hat{\phi}_n(\xi), \tag{4.17}$$

therefore by *Theorem* (4.1)

$$K(f_n(a,x) \,\#\, d_b\, \psi_m(x)) = K(h_m(b,x) \,\#\, d_a\, \phi_n(x)).$$

Hence

$$f_n(a,x) \,\#\, d_b\, \psi_m(x) = h_m(b,x) \,\#\, d_a\, \phi_n(x), \tag{4.18}$$

which completes the proof.

***Theorem* 4.7.** *The mapping $\tilde{\gamma}^{-1} : \mathscr{L}_{\mathscr{B}} \to \mathscr{L}$ is linear*

Proof. Let $[\hat{f}_n/d_a\hat{\phi}_n]$, $[\hat{h}_n/d_b\hat{\psi}_n] \in \mathscr{L}_{\mathscr{B}}$ & $c \in [1, \infty)$.
Then by (4.4)

$$\begin{aligned}
\tilde{\gamma}^{-1}\{[\hat{f}_n/d_a\hat{\phi}_n] + [\hat{h}_n/d_b\hat{\psi}_n]\} &= \tilde{\gamma}^{-1}\{\frac{\hat{f}_n(a,\xi)\, d_b\hat{\psi}_n(\xi) + \hat{h}_n(b,\xi)\, d_a\hat{\phi}_n(\xi)}{d_a\hat{\phi}_n(\xi) \,\#\, d_b\hat{\psi}_n(\xi)}\} \\
&= [\frac{f_n(a,x) \,\#\, d_b\psi_n(x) + h_n(b,x) \,\#\, d_a\phi_n(x)}{d_a\phi_n(x) \,\#\, d_b\psi_n(x)}] \\
&= [\frac{f_n}{\phi_n} + \frac{h_n}{\psi_n}] \\
&= \tilde{\gamma}^{-1}[\hat{f}_n/d_a\hat{\phi}_n] + \tilde{\gamma}^{-1}[\hat{h}_n/d_b\hat{\psi}_n].
\end{aligned} \tag{4.19}$$

We can also prove that

$$\tilde{\gamma}^{-1}[c\hat{\ }f_n/d_a\hat{\phi}_n] = c\,\tilde{\gamma}^{-1}[\hat{f}_n/d_a\hat{\phi}_n],$$

which completes the proof.

***Theorem* 4.8.** *The mapping $\tilde{\gamma}^{-1} : \mathscr{L}_{\mathscr{B}} \to \mathscr{L}$ is continuous w.r.t. δ convergence.*

Proof. Let $x_n \to x$ in $\mathscr{L}_{\mathscr{B}}$
Therefore,

$$x_n = [\hat{h}_{n,k}/d_a\hat{\phi}_k] \qquad x = [\hat{h}_k/d_a\hat{\phi}_k]$$

and

$$\hat{h}_{n,k} \to \hat{h}_k \qquad \text{as} \quad n \to \infty.$$

Applying inverse Kamal transform

$$\begin{aligned}
h_{n,k} &\to h_k \qquad \text{as} \quad n \to \infty \\
\Rightarrow h_n/\phi_n &\to f/\phi_n \qquad \text{as} \quad n \to \infty,
\end{aligned}$$

which completes the proof.

5 Conclusion

In this paper we defined the Strong Boehmians for Kamal transform and defined a mapping from Strong Boehmians to General Boehmians. Also we defined the convolution and inverse transform from General Boehmian to Strong Boehmians. An attempt is made to define Strong Boehmian with some references.

6 Acknowledgment

The authors would like to express their gratitude to the referee and Editor for their kind suggestions and corrections that helped to improve the original manuscript.

References

[1] Shudanshu Aggarwal, Kamal transform of Bessel's functions, *International Journal of Research and Innovation in Applied Science*, **3(7)** (2018), 1-4.

[2] Sudhanshu Aggarwal, R.Chauhan and Nidhi Sharma, A new Application of Kamal transform for solving linear Volterra integral equation, *International Journal of Research and Innovation in Applied Science*, **7(4)** (2018), 138-140.

[3] Sudhanshu Aggarwal, Application of Kamal transform for solving population growth and decay problems, *Global Journal of Engineering Science and Researches*, **5(9)** (2018), 254-260.

[4] S. K. Q. Al-omari, On the Sumudu Transform and its Extension to a class Boehmians, *Hindawi Publishing corporation ISRN Mathematical Analysis*, **4** (2014), 1-6.

[5] S. K. Q. Al-omari, on the Application of Natural transforms, *International Journal of Pure and Applied Mathematics*, **85(4)** (2013), 729-744.

[6] S. K. Q. Al-omari, Boehmians and Elzaki Transform, *Gen.Math notes*, **9(1)** (2012), 7-15.

[7] S. K. Q. Al-omari and A Kilicman, On the generalized Hartley Hilbert and Fourier Hilbert transforms, *Advances in Difference Equation*, **1** (2012), 1-11.

[8] T. K. Boehme, The support of Mikusinski operators, *Trans.Amer.Math.Soc.*, **176** (1973), 319-334.

[9] Ellen R. Dill and Piotr Mikusinski, Strong Boehmians, *Proc. of American Mathematical society*, **119** (1993), 885-888.

[10] Nidal E. Hasaan Taha, R. I. Naruddeen, A. Kamal and H. Sedeeg, Dualities between Kamal and Mahgoub integral transform and some famous integral transform, *British Journal of Applied science and technology*, **20(3)** (2017), 1-8.

[11] Abdelilah Kamal and H.Sedeeg, The New Integral Transform "Kamal Transform", *Advances in Theoretical and Applied Mathematics*, **11** (2016), 451-458.

[12] V. Karunakaran and N. V. Kalpakam, Hilbert transform for Boehmians, *Integral Transforms and Special Functions*, **9(1)** (2000), 19-36.

[13] Rachana Khandelwal , P.Choudhary and Y.Khandelwal, Solution of fractional ordinary differentiation equation by Kamal transform, *International Journal of Statistics and Applied Mathematics*, **3(2)** (2018), 279-284.

[14] Tarig M. Elzaki, The new integral transform " Elzaki Transform", *Global Journal of pure and Applied Mathematics*, **7(1)** (2011), 57-64.

[15] P. Mikusinski, Convergence of Boehmians, *Japan.J.Math.*, **9** (1983), 159-179.

[16] R. S. Pathak, *Integral Transform of Generalized Functions and their Application*, Gordon and Breach Science Publishers, Australia, 1997.

[17] R. Roop Kumar and E. R.Negrin, A unified extension of stieltjes and Poisson transforms to Boehmians, *Integral tranforms and special Functions*, **22(3)** (2011), 195-206.

[18] A. H. Zemanian, Generalized integral Transformation, *Interscience publishers,New York*, 1968.

Jñānābha, Vol. 50(1) (2020), 89-101
(Dedicated to Honor Professor H.M. Srivastava on His 80th Birth Anniversary Celebrations)

DEGREE-BASED TOPOLOGICAL INDICES OF DOX-LOADED PEG-PASP POLYMER

By
M. C. Shanmukha[1], N. S. Basavarajappa[2], K. N. Anil Kumar[3] and A. Usha[4]
[1]Department of Mathematics,
Jain Institute of Technology, Davanagere-577003, Karnataka, India
and affiliated to Visvesvaraya Technological University, Belagavi, Karnataka, India
Email:mcshanmukha@gmail.com
[2,3]Department of Mathematics,
Bapuji Institute of Engineering and Technology, Davanagere-577004, Karnataka, India
and affiliated to Visvesvaraya Technological University, Belagavi, Karnataka, India
Email:dr.nsbasavaraj@gmail.com, kn.anil5340@gmail.com
[4]Department of Mathematics,
Alliance College of Engineering and Design, Alliance University,
Anekal-Chandapura Road, Bangalore-562106, Karnataka, India
Email:usha.arcot@alliance.edu.in
(*Received : February* 13, 2020 *; Revised: May* 02, 2020)
DOI: https://doi.org/10.58250/jnanabha.2020.50111

Abstract

The drugs which are effective for malignant disease are anticancer drugs. They are also called antineo plastics. Anti cancer drugs are classified into several classes which include alkalyting agents, harmones and anti metabolites. Studies reveal the fact that, there will be an intrinsic relationship with the properties of alkanes and also drugs(e.g., Boiling Point-BP and Melting Point-MP) with its chemical structure. In this paper, various topological indices are defined on the drug to assist the researchers for better understanding of physical properties and chemical reactions. Here the topological indices are defined and computed for an anticancer drug .
2010 Mathematics Subject Classifications: 05C05, 05C12, 05C35.
Keywords and phrases: Atom Bond Connectivity *(ABC)* indices, Symmetric division index(*SDD*), $F-$indices, Multiplicative $F-$indices, $F-$polynomial indices, Anticancer drug.

1 Introduction and Terminologies

One of the main disease that lead to death in the world is Cancer. The proportion of death increases as increase of deaths caused by breast, stomach, lungs and colon cancers. When cells divide uncontrollably and invasively it causes cancer. It also invades the surrounding tissues there by causing damage to it. Cell division is a normal process in a human body. When cells get harmful or grow old, new cells take their place and old once die. This ordered process broke down as cancer grows. When cells became more harm, older or damage cells survive when they should die and new cells survive when those were not in need. All these extra cells divide without stopping and causes tumour. Tumours are malignant when they roll out into surrounding tissues. Unlike malignant tumours do not invade the neighbouring tissues. This dangerous disease can be cured by several treatments like surgery, radiotherapy, chemotherapy, harmone therapy, targeted therapy and more. Surgery of cancer includes some terms and conditions specified for the disease. Further this consist of not only taking away the tumour but also organism involvement. In various cases, the simple excision with extended resection beyond the tumour margin won't resolve the issue of recurrence and metastasis. Surgery is the first treatment for oncology, even though we have various limitations and constraints named adenopathies, metastases, etc. This solution will be very effective one, easy to perform and economical. Contradictions will get generate for some cases like multi centric cancer(leukosis), cancer with special locations, mesenteric lymph node tumour, facial, spinal, pelvic osteosarcomas, cancer characterized by important local extensions, tonsillar epidermoid carcinoma, bilateral thyroid adenocarcinoma, disseminated pancreatic insulinomas.

The therapeutic method will consist of satisfied defined plans: In the first-stage, for each patient this should be separated and particularized. Then it should consist of parallel treatment of metastases, para-neoplastic syndrome and its side effects also. The general condition of the patients are essentially evaluated and continuously maintained over the total duration of treatment being needed. The option to make use few anticancer drugs relies on many of the considerations like the type and location of the cancer, its gravity state, surgery or radiation therapy needed or not, and also considers the side effects of the drugs. Maximum of the drugs will be given intravenously where as few are taken orally and some others are given within the spinal cord.

This is an attempt to the application of graph theory in anticancer drugs. In this work, the drug is taken and using the degree based calculations few topological indices are determined. In this work the drug considered is dox-loaded micelle consisting of PEG-PAsp copolymer. First polymeric micelle developed was filled with anti cancer drug doxorubicin(DOX). Here DOX was covalently conjugated to side chains of the poly(aspartate)(PAsp)segmeny by an amide bond between the carboxylic group of the glycosidyl residue in DOX [17, 18].

The molecular structure topological index is described as a non-empirical numerical measure, which represents the molecular structures and their branching pattern. Also at this end, topological indices will map each molecule structure to a real number and used as a descriptor of the molecule under testing. Various important indices applicable in chemical-engineering (e.g., QSPR/QSAR study) for establishing the relationship within the molecular structures and the physico-chemical properties [4].

Usually, chemical compounds are modelled as a graph[16, 20] considering the atoms as the vertices and the links connecting them as the edges. In a similar fashion, the anti cancer drug under this study is considered as chemical compound and the said topological indices are determined. Applications of graph theory are QSAR, QSPR and QSTR where chemists or pharmacists are welcome to use this data for further research study.

All the graphs used in this work are simple graphs, i.e. cycle free and undirected [4].

Consider *G=(V(G), E(G))* will be a molecular graph, where *V(G)* and *E(G)* are vertex and edge respectively, which maps to atoms set and chemical bond set.

***Definition* 1.1.** *For a graph G, the First $F-$ index and Second $F-$index* [3, 14] *are defined respectively as*

$$F_1(G) = \sum_{e=uv\in E(G)} \left[(d_G(u))^2 + (d_G(v))^2\right].$$

$$F_2(G) = \sum_{e=uv\in E(G)} \left[(d_G(u))^2 \times (d_G(v))^2\right].$$

***Definition* 1.2.** *The multiplicative first $F-$index is defined as foolows* [1, 5]

$$F_1II(G) = \prod_{e=uv\in E(G)} \left[(d_G(u))^2 + (d_G(v))^2\right].$$

***Definition* 1.3.** *For a graph G, the second multiplicative F-index* [14] *can be defined as*

$$F_2II(G) = \prod_{e=uv\in E(G)} \left[(d_G(u))^2 \times (d_G(v))^2\right].$$

***Definition* 1.4.** *The multiplicative First and Second hyper F-index* [10] *for a graph G are*

$$HF_1II(G) = \prod_{e=uv\in E(G)} \left[(d_G(u))^2 + (d_G(v))^2\right]^2.$$

$$HF_2II(G) = \prod_{e=uv\in E(G)} \left[(d_G(u))^2 \times (d_G(v))^2\right]^2.$$

***Definition* 1.5.** *For a graph G, both Multiplicative Sum Connectivity and Multiplicative Product Connectivity F-indices* [14] *are*

$$SFII(G) = \prod_{e=uv\in E(G)} \frac{1}{\sqrt{(d_G(u))^2 + (d_G(v))^2}}.$$

$$PFII(G) = \prod_{e=uv\in E(G)} \frac{1}{\sqrt{(d_G(u))^2 \times (d_G(v))^2}}.$$

***Definition* 1.6.** *Again to the graph G, general multiplicative First and Second F-indices are stated* [12, 14] *as*

$$F_1^kII(G) = \prod_{e=uv\in E(G)} \left[(d_G(u))^2 + (d_G(v))^2\right]^k.$$

$$F_2^kII(G) = \prod_{e=uv\in E(G)} \left[(d_G(u))^2 \times (d_G(v))^2\right]^k.$$

***Definition* 1.7.** *The multiplicative atom bond connectivity F-index* [8, 14] *for a graph G is*

$$ABCFII(G) = \prod_{e=uv\in E(G)} \sqrt{\frac{(d_G(u))^2 + (d_G(v))^2 - 2}{(d_G(u))^2 \times (d_G(v))^2}}.$$

***Definition* 1.8.** *For a graph G, Multiplicative Geometric Arithmetic F-index* [14] *can be defined as*

$$GAFII(G) = \prod_{e=uv\in E(G)} \frac{2\sqrt{(d_G(u))^2 \times (d_G(v))^2}}{(d_G(u))^2 + (d_G(v))^2}.$$

***Definition* 1.9.** *Ghobadi et al.. defined the First F- polynomial* [5] *of a graph as*

$$F_1(G,x) = \sum_{e=uv\in E(G)} x^{(d_G(u))^2 + (d_G(v))^2}.$$

***Definition* 1.10.** *The Second F-polynomial, the First and Second hyper F- polynomial* [5, 15] *of a graph are stated as*

$$F_2(G,x) = \sum_{e=uv\in E(G)} x^{(d_G(u))^2\times(d_G(v))^2}.$$

$$HF_1(G,x) = \sum_{e=uv\in E(G)} x^{\left[(d_G(u))^2+(d_G(v))^2\right]^2}.$$

$$HF_2(G,x) = \sum_{e=uv\in E(G)} x^{\left[(d_G(u))^2\times(d_G(v))^2\right]^2}.$$

***Definition* 1.11.** *Consider a molecular graph $G = (V, E)$, $d_G(u)$ is the vertex degree of u and $d_G(v)$ is the vertex degree of v then first index ABC of G* [6] *can be stated as*

$$ABC(G) = \sum_{e=uv\in E(G)} \left[\sqrt{\frac{(d_G(u) + d_G(v) - 2)}{(d_G(u)) \times (d_G(v))}}\right].$$

***Definition* 1.12.** *In chemical graph theory field, there are some new degree based graph types, which plays an important role. These topological indices are required for finding total- surface-area and heat-formation of various chemical compounds. These graphs types are as follow Symmetric division* [7, 19],

$$SSD(G) = \sum_{e=uv\in E(G)} \left[\frac{A}{B} + \frac{B}{A}\right].$$

$$where,\ \ A = min\,[d_G(u), d_G(v)] \ \ and \ \ B = max\,[d_G(u), d_G(v)]\,.$$

In Section 2 we highlight main results obtained, in detail proofs and the calculations of topological indices of molecular graphs family.

2 Results and Discussions

Dox-loaded micelle consisting of $PEG - PAsp$ block polymer and copolymers with chemically conjugated Dox $SP[n]$ is considered in this study, as shown in ***Figure* 1**. The integer number n is step of growth in these form of polymers. Here ***Figure* 2** represents $SP[1]$, ***Figure* 3** represents $SP[2]$ and ***Figure* 4** represents $SP[3]$. And also for $n = 1, 2$ *and* 3 (consider the ***Figures* 2**, **3** and **4** respectively) are determined.

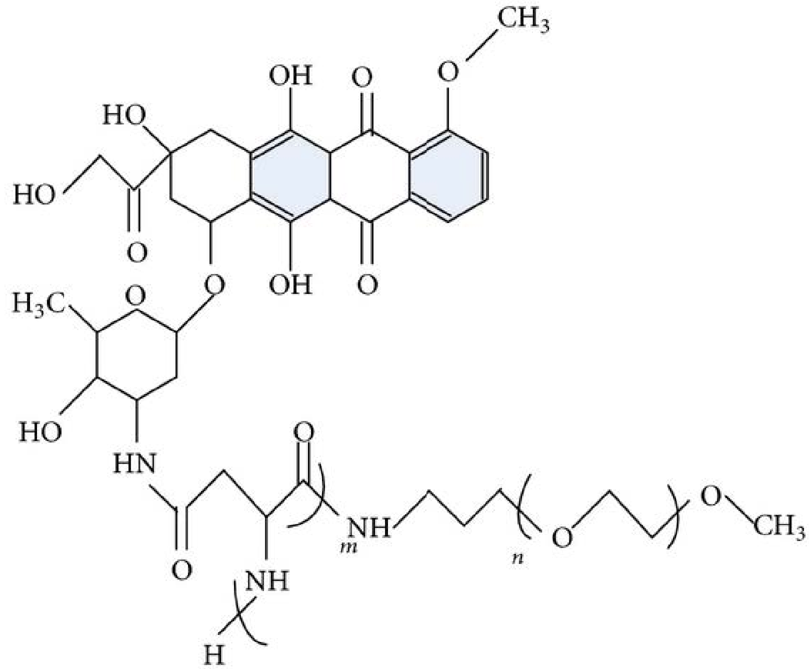


***Figure* 2.1:** Dox-loaded micelle consisting PEG-PAsp block copolymer with chemically conjugated Dox SP[n]

.

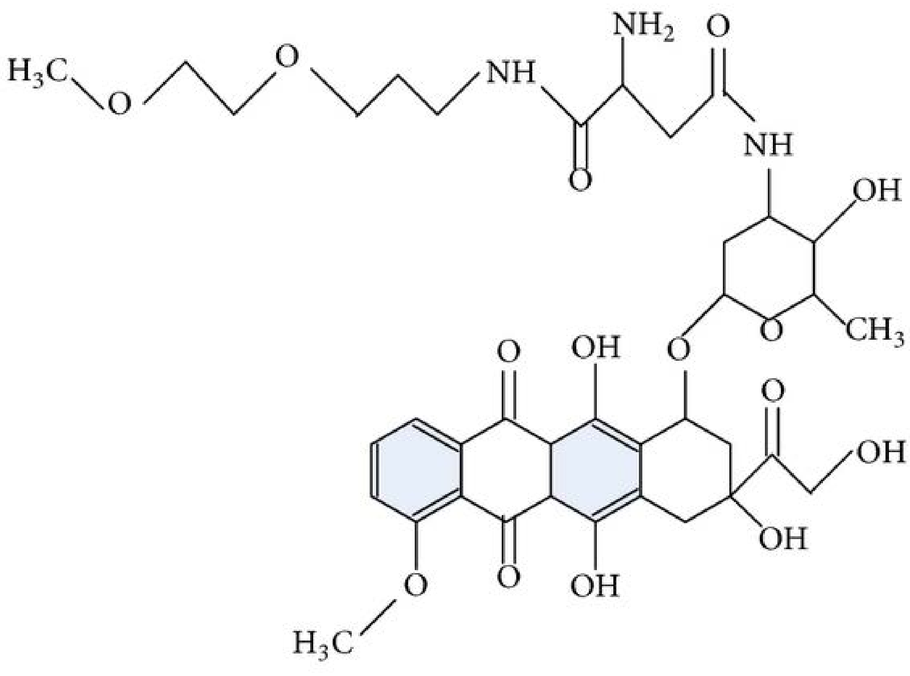


***Figure* 2.2:** The molecular structure of SP[1]

.

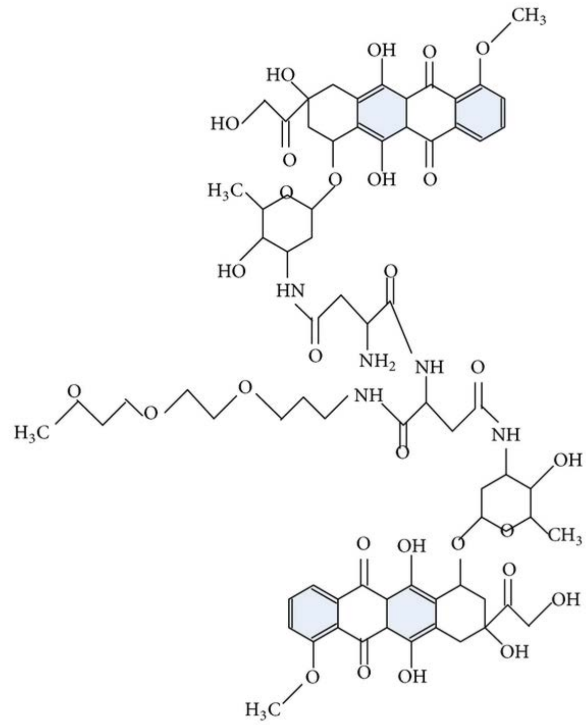


Figure **2.3:** The molecular structure of SP[2]

.

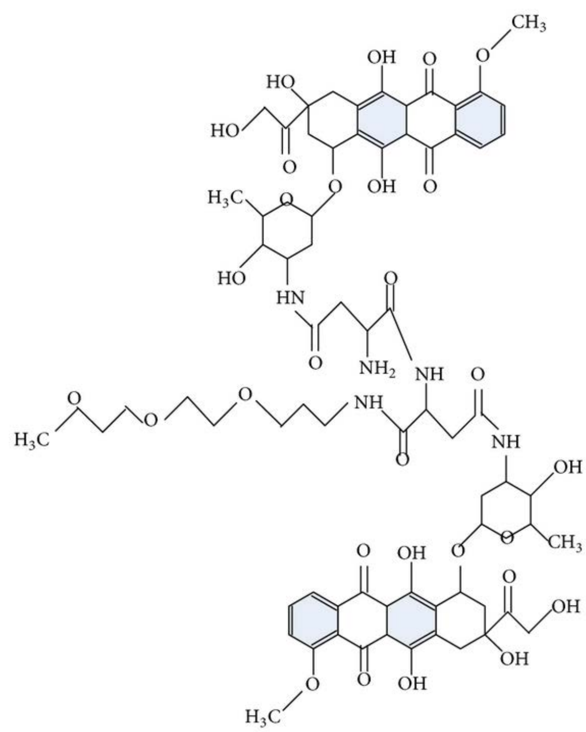


Figure **2.4:** The molecular structure of SP[3]

.

***Theorem* 2.1.** *The Dox-loaded micelle consisting PEG-PAsp block copolymer with chemically conjugated Dox $SP[n]$, One has*

$$SP[n] = F_1^K II(G) = 5^{k(18n+2)} \times 2^{k(44n+13)} \times 9^{k(16n)} \times 13^{k(18n-1)} \times 17^{k(n)}.$$

Proof. Let the lowest and highest degree of $SP[n]$ respectively are δ and Δ. Suppose the edge set $E(SP[n])$ can categorised as various divisions:

$$\begin{aligned}
&(i)E_3(orE_2^*) : d_G(u) = 1 \ \ and \ \ d_G(v) = 2;\\
&(ii)E_3^* : d_G(u) = 1 \ \ and \ \ d_G(v) = 3;\\
&(iii)E_5 \cap E_4^* : d_G(u) = 1 \ \ and \ \ d_G(v) = 4;\\
&(iv)E_4 \cap E_4^* : d_G(u) = 2 \ \ and \ \ d_G(v) = 2;\\
&(v)E_6^* : d_G(u) = 2 \ \ and \ \ d_G(v) = 3;\\
&(vi)E_8^* : d_G(u) = 2 \ \ and \ \ d_G(v) = 4;\\
&(vii)E_9^* : d_G(u) = d_G(v) = 3;\\
&(viii)E_7(orE_{12}^*) : d_G(u) = 3 \ \ and \ \ d_G(v) = 4.
\end{aligned}$$

Again Calculating in terms, we observe that $|V(SP[n])| = 49n + 6$ and $|E(SP[n])| = 54n + 5$. In specific, we define

$$\begin{aligned}
&|E_3| = |E_2^*| = 2n + 1,\ |E_3^*| = 9n + 1,\ |E_5 \cap E_4^*| = |E_7| = |E_{12}^*| = n,\\
&|E_4 \cap E_4^*| = 5n + 4, |E_6^*| = 18n - 1,\ |E_8^*| = 2n \ \ and \ \ |E_9^*| = 16n.
\end{aligned}$$

$\therefore$ The general multiplicative first F- index of a graph $SP[n]$ is

$$\begin{aligned}
F_1^k II(G) &= \prod_{e=uv\in E(G)} \left[(d_G(u))^2 + (d_G(v))^2\right]^k.\\
&= \prod_{uv\in E_3} \left[(1)^2 + (2)^2\right]^k \times \prod_{uv\in E_3^*} \left[(1)^2 + (3)^2\right]^k \times \prod_{uv\in E_5\cap E_4^*} \left[(1)^2 + (4)^2\right]^k\\
&\quad \times \prod_{uv\in E_4\cap E_4^*} \left[(2)^2 + (2)^2\right]^k \times \prod_{uv\in E_6^*} \left[(2)^2 + (3)^2\right]^k \times \prod_{uv\in E_8^*} \left[(2)^2 + (4)^2\right]^k\\
&\quad \times \prod_{uv\in E_9^*} \left[(3)^2 + (3)^2\right]^k \times \prod_{uv\in E_7} \left[(3)^2 + (4)^2\right]^k.\\
&= 5^{k(18n+2)} \times 2^{k(44n+13)} \times 9^{k(16n)} \times 13^{k(18n-1)} \times 17^{k(n)}.
\end{aligned}$$

We get the below results by using ***Theorem* 2.1**.

***Corollary* 2.1.** *The multiplicative first F- index of a graph $SP[n]$ is*

$$F_1 II(G) = 5^{(18n+2)} \times 2^{(44n+13)} \times 9^{(16n)} \times 13^{(18n-1)} \times 17^{(n)}.$$

Proof. Put $k = 1$ in ***Theorem* 2.1**, we gain the required result.

***Corollary* 2.2.** *The multiplicative first hyper F- index of a graph $SP[n]$ is*

$$HF_1 II(G) = 5^{(36n+4)} \times 2^{(88n+26)} \times 9^{(32n)} \times 13^{(36n-2)} \times 17^{(2n)}.$$

Proof. Put $k = 2$ in ***Theorem* 2.1**, we gain the required result.

***Corollary* 2.3.** *The multiplicative sum connectivity F- index of a graph $SP[n]$ will be*

$$SFII(G) = \left(\frac{1}{\sqrt{5}}\right)^{18n+2} \times \left(\frac{1}{\sqrt{2}}\right)^{44n+13} \times \left(\frac{1}{\sqrt{9}}\right)^{16n} \times \left(\frac{1}{\sqrt{13}}\right)^{18n-1} \times \left(\frac{1}{\sqrt{17}}\right)^{n}.$$

Proof. Put $k = \frac{-1}{2}$ in ***Theorem* 2.1**, we gain the desired result. We now determine the general multiplicative second F- index of SP[n].

***Theorem* 2.2.** *The general multiplicative second F- index of a graph $SP[n]$ is*

$$F_2^K II(G) = 4^{k(20n+4)} \times 9^{k(30n)}.$$

Proof.

$$\begin{aligned}
F_2^k II(G) &= \prod_{e=uv\in E(G)} \left[(d_G(u))^2 \cdot (d_G(v))^2\right]^k. \\
&= \prod_{uv\in E_3} \left[(1)^2(2)^2\right]^k \times \prod_{uv\in E_3^*} \left[(1)^2(3)^2\right]^k \times \prod_{uv\in E_5\cap 4^*} \left[(1)^2(4)^2\right]^k \\
&\quad \times \prod_{uv\in E_4\cap 4^*} \left[(2)^2(2)^2\right]^k \times \prod_{uv\in E_6^*} \left[(2)^2(3)^2\right]^k \times \prod_{uv\in E_8^*} \left[(2)^2(4)^2\right]^k \\
&\quad \times \prod_{uv\in E_9^*} \left[(3)^2(3)^2\right]^k \times \prod_{uv\in E_7} \left[(3)^2(4)^2\right]^k. \\
&= 4^{k(20n+4)} \times 9^{k(30n)}.
\end{aligned}$$

The following results are obtained by using ***Theorem* 2.5**.

***Corollary* 2.4.** *The multiplicative second F- index of a graph $SP[n]$ is*

$$F_2 II(G) = 4^{(20n+4)} \times 9^{(30n)}.$$

Proof. Put $k = 1$ in ***Theorem* 2.5**, we gain the required results.

***Corollary* 2.5.** *The multiplicative second hyper F- index of a graph $SP[n]$ will be*

$$HF_2 II(G) = 4^{(40n+8)} \times 9^{(60n)}.$$

Proof. Put $k = 2$ in ***Theorem* 2.5**, we acquire the required result.

***Corollary* 2.6.** *The multiplicative product connectivity F- index of a graph $SP[n]$ will be*

$$PFII(G) = \left(\frac{1}{4}\right)^{10n+2} \times \left(\frac{1}{9}\right)^{15n}.$$

Proof. Put $k = \frac{-1}{2}$ in ***Theorem* 2.5**, we obtained the required result. In the following theorems, we deduce the multiplicative atom bond connectivity F-index and multiplicative geometric-arithmetic F-index of $SP[n]$.

***Theorem* 2.3.** *The multiplicative atom bond connectivity F- index of $SP[n]$ will be*

$$\begin{aligned}
ABCFII(G) &= \left(\sqrt{\frac{3}{4}}\right)^{2n+1} \times \left(\sqrt{\frac{8}{9}}\right)^{9n+1} \times \left(\sqrt{\frac{15}{16}}\right)^{n} \times \left(\sqrt{\frac{6}{16}}\right)^{5n+4} \times \left(\sqrt{\frac{11}{36}}\right)^{18n-1} \\
&\quad \times \left(\sqrt{\frac{18}{64}}\right)^{2n} \times \left(\sqrt{\frac{16}{81}}\right)^{16n} \times \left(\sqrt{\frac{23}{144}}\right)^{n}.
\end{aligned}$$

Proof.

$$ABCFII(G) = \prod_{e=uv\in E(G)} \left[\sqrt{\frac{(d_G(u))^2 + (d_G(v))^2 - 2}{(d_G(u))^2 \cdot (d_G(v))^2}} \right].$$

$$= \left(\sqrt{\frac{(1)^2 + (2)^2 - 2}{(1)^2(2)^2}} \right)^{2n+1} \times \left(\sqrt{\frac{(1)^2 + (3)^2 - 2}{(1)^2(3)^2}} \right)^{9n+1} \times \left(\sqrt{\frac{(1)^2 + (4)^2 - 2}{(1)^2(4)^2}} \right)^{n}$$

$$\times \left(\sqrt{\frac{(2)^2 + (2)^2 - 2}{(2)^2(2)^2}} \right)^{5n+4} \times \left(\sqrt{\frac{(2)^2 + (3)^2 - 2}{(2)^2(3)^2}} \right)^{18n-1} \times \left(\sqrt{\frac{(2)^2 + (4)^2 - 2}{(2)^2(4)^2}} \right)^{2n}$$

$$\times \left(\sqrt{\frac{(3)^2 + (3)^2 - 2}{(3)^2(3)^2}} \right)^{16n} \times \left(\sqrt{\frac{(3)^2 + (4)^2 - 2}{(3)^2(4)^2}} \right)^{n}.$$

$$= \left(\sqrt{\frac{3}{4}} \right)^{2n+1} \times \left(\sqrt{\frac{8}{9}} \right)^{9n+1} \times \left(\sqrt{\frac{15}{16}} \right)^{n} \times \left(\sqrt{\frac{6}{16}} \right)^{5n+4} \times \left(\sqrt{\frac{11}{36}} \right)^{18n-1} \times \left(\sqrt{\frac{18}{64}} \right)^{2n}$$

$$\times \left(\sqrt{\frac{16}{81}} \right)^{16n} \times \left(\sqrt{\frac{23}{144}} \right)^{n}.$$

Theorem 2.4. *The multiplicative geometric-airtmetic F- index of $SP[n]$ is*

$$= \left(\frac{24}{25} \right)^{7n} \times \left(\frac{12}{13} \right)^{33n-1} \times \left(\frac{4}{5} \right)^{4n} \times \left(\frac{3}{5} \right)^{16n} \times \left(\frac{8}{17} \right)^{4n}.$$

Proof.

$$GAFII(G) = \prod_{e=uv\in E(G)} \frac{2\sqrt{(d_G(u))^2 \cdot (d_G(v))^2}}{(d_G(u))^2 + (d_G(v))^2}.$$

$$= \left(\frac{2\sqrt{(1)^2(2)^2}}{(1)^2 + (2)^2} \right)^{2n+1} \times \left(\frac{2\sqrt{(1)^2(3)^2}}{(1)^2 + (3)^2} \right)^{9n+1} \times \left(\frac{2\sqrt{(1)^2(4)^2}}{(1)^2 + (4)^2} \right)^{n}$$

$$\times \left(\frac{2\sqrt{(2)^2(2)^2}}{(2)^2 + (2)^2} \right)^{5n+4} \times \left(\frac{2\sqrt{(2)^2(3)^2}}{(2)^2 + (3)^2} \right)^{18n-1} \times \left(\frac{2\sqrt{(2)^2(4)^2}}{(2)^2 + (4)^2} \right)^{2n}$$

$$\times \left(\frac{2\sqrt{(3)^2(3)^2}}{(3)^2 + (3)^2} \right)^{16n} \times \left(\frac{2\sqrt{(3)^2(4)^2}}{(3)^2 + (4)^2} \right)^{n}.$$

$$= \left(\frac{24}{25} \right)^{7n} \times \left(\frac{12}{13} \right)^{33n-1} \times \left(\frac{4}{5} \right)^{4n} \times \left(\frac{3}{5} \right)^{16n} \times \left(\frac{8}{17} \right)^{4n}.$$

Theorem 2.5. *The $SP[n]$ for the first F- polynomial of a graph is*

$$= (2n+1)x^5 + (9n+1)x^{10} + (n)x^{17} + (5n+4)x^8 + (18n-1)x^{13} + (2n)x^{20}$$
$$+ (16n)x^{18} + (n)x^{25}.$$

Proof.

$$F_1(G, x) = \sum_{e=uv\in E(G)} x^{(d_G(u))^2 + (d_G(v))^2}.$$

$$= (2n+1)\left[x^{(1+4)}\right] + (9n+1)\left[x^{(1+9)}\right] + (n)\left[x^{(1+16)}\right] + (5n+4)\left[x^{(4+4)}\right]$$

$$+ (18n-1)\left[x^{(4+9)}\right] + (2n)\left[x^{(4+16)}\right] + (16n)\left[x^{(9+9)}\right] + (n)\left[x^{(9+16)}\right].$$

$$= (2n+1)x^{5} + (9n+1)x^{10} + (n)x^{17} + (5n+4)x^{8} + (18n-1)x^{13} + (2n)x^{20} + (16n)x^{18} + (n)x^{25}.$$

Theorem 2.6. *The $SP[n]$ for the second F- polynomial of a graph is*

$$(2n+1)x^{4} + (9n+1)x^{9} + (6n+4)x^{16} + (18n-1)x^{36} + (2n)x^{64} + (16n)x^{81} + (n)x^{144}.$$

Proof.

$$F_2(G,x) = \sum_{e=uv\in E(G)} x^{(d_G(u))^2\cdot(d_G(v))^2}.$$

$$= (2n+1)\left[x^{(1)(4)}\right] + (9n+1)\left[x^{(1)(9)}\right] + (n)\left[x^{(1)(16)}\right] + (5n+4)\left[x^{(4)(4)}\right]$$

$$+ (18n-1)\left[x^{(4)(9)}\right] + (2n)\left[x^{(4)(16)}\right] + (16n)\left[x^{(9)(9)}\right] + (n)\left[x^{(9)(16)}\right].$$

$$= (2n+1)x^{4} + (9n+1)x^{9} + (6n+4)x^{16} + (18n-1)x^{36} + (2n)x^{64} + (16n)x^{81} + (n)x^{144}.$$

Theorem 2.7. *The $SP[n]$ for the first hyper F- polynomial of a graph is*

$$(2n+1)x^{10} + (9n+1)x^{20} + n(x)^{34} + (5n+4)x^{16} + (18n-1)x^{26} + (2n)x^{40} + (16n)x^{36} + (n)x^{50}.$$

Proof.

$$HF_1(G,x) = \sum_{e=uv\in E(G)} x^{\left[(d_G(u))^2+(d_G(v))^2\right]^2}.$$

$$= (2n+1)\left[x^{(1+4)^2}\right] + (9n+1)\left[x^{(1+9)^2}\right] + (n)\left[x^{(1+16)^2}\right] + (5n+4)\left[x^{(4+4)^2}\right]$$

$$+ (18n-1)\left[x^{(4+9)^2}\right] + (2n)\left[x^{(4+16)^2}\right] + (16n)\left[x^{(9+9)^2}\right] + (n)\left[x^{(9+16)^2}\right].$$

$$= (2n+1)x^{10} + (9n+1)x^{20} + n(x)^{34} + (5n+4)x^{16} + (18n-1)x^{26} + (2n)x^{40} + (16n)x^{36} + (n)x^{50}.$$

Theorem 2.8. *The $SP[n]$ for the second hyper F- polynomial of a graph is*

$$(2n+1)x^{8} + (9n+1)x^{18} + (6n+4)x^{32} + (18n-1)x^{72} + (2n)x^{128} + (16n)x^{162} + (n)x^{288}.$$

Proof.

$$HF_2(G,x) = \sum_{e=uv\in E(G)} x^{\left[(d_G(u))^2\cdot(d_G(v))^2\right]^2}.$$

$$= (2n+1)\left[x^{(1)(4)}\right]^2 + (9n+1)\left[x^{(1)(9)}\right]^2 + (n)\left[x^{(1)(16)}\right]^2 + (5n+4)\left[x^{(4)(4)}\right]^2$$

$$+ (18n-1)\left[x^{(4)(9)}\right]^2 + (2n)\left[x^{(4)(16)}\right]^2 + (16n)\left[x^{(9)(9)}\right]^2 + (n)\left[x^{(9)(16)}\right]^2.$$

$$= (2n+1)x^{8} + (9n+1)x^{18} + (6n+4)x^{32} + (18n-1)x^{72} + (2n)x^{128} + (16n)x^{162} + (n)x^{288}.$$

***Theorem* 2.9.** *For a graph G, The ABC index of $SP[n]$ will be*

$$ABC(G) = \frac{194n}{5} + \frac{18}{5}.$$

Proof. By the definition, the ABC index of a graph SP[n] is

$$ABC(G) = \sum_{e=uv\in E(G)} \left[\sqrt{\frac{(d_G(u) + d_G(v) - 2)}{(d_G(u))(d_G(v))}} \right].$$

$$= \sum_{uv\in E_3} \left[\sqrt{\frac{(1+2-2)}{(1)(2)}} \right] + \sum_{uv\in E_3^*} \left[\sqrt{\frac{(1+3-2)}{(1)(3)}} \right] + \sum_{uv\in E_5\cap E_4^*} \left[\sqrt{\frac{(1+4-2)}{(1)(4)}} \right]$$
$$+ \sum_{uv\in E_4\cap E_4^*} \left[\sqrt{\frac{(2+2-2)}{(2)(2)}} \right] + \sum_{uv\in E_6^*} \left[\sqrt{\frac{(2+3-2)}{(2)(3)}} \right] + \sum_{uv\in E_8^*} \left[\sqrt{\frac{(2+4-2)}{(2)(4)}} \right]$$
$$+ \sum_{uv\in E_9^*} \left[\sqrt{\frac{(3+3-2)}{(3)(3)}} \right] + \sum_{uv\in E_7} \left[\sqrt{\frac{(3+4-2)}{(3)(4)}} \right].$$

$$= (25n+4)\left(\sqrt{\frac{1}{2}}\right) + (9n+1)\left(\sqrt{\frac{2}{3}}\right) + n\left(\sqrt{\frac{3}{4}}\right) + 2n\left(\sqrt{\frac{5}{8}}\right) + 16n\left(\sqrt{\frac{4}{9}}\right)$$
$$+ n\left(\sqrt{\frac{5}{12}}\right).$$

$$ABC(G) = \frac{194n}{5} + \frac{18}{5}.$$

***Theorem* 2.10.** *The Symmetric division index of a graph $SP[n]$ is*

$$SDD(G) = \frac{1273n}{10} + \frac{117}{10}.$$

Proof.

$$SSD(G) = \sum_{e=uv\in E(G)} \left[\frac{A(d_G(u), d_G(v))}{B(d_G(u), d_G(v))} + \frac{B(d_G(u), d_G(v))}{A(d_G(u), d_G(v))} \right].$$
$$= \sum_{uv\in E_3} \left[\frac{A(1,2)}{B(1,2)} + \frac{B(1,2)}{A(1,2)} \right] + \sum_{uv\in E_3^*} \left[\frac{A(1,3)}{B(1,3)} + \frac{B(1,3)}{A(1,3)} \right]$$
$$+ \sum_{uv\in E_5\cap E_4^*} \left[\frac{A(1,4)}{B(1,4)} + \frac{B(1,4)}{A(1,4)} \right] + \sum_{uv\in E_4\cap E_4^*} \left[\frac{A(2,2)}{B(2,2)} + \frac{B(2,2)}{A(2,2)} \right]$$
$$+ \sum_{uv\in E_6^*} \left[\frac{A(2,3)}{B(2,3)} + \frac{B(2,3)}{A(2,3)} \right] + \sum_{uv\in E_8^*} \left[\frac{A(2,4)}{B(2,4)} + \frac{B(2,4)}{A(2,4)} \right]$$
$$+ \sum_{uv\in E_9^*} \left[\frac{A(3,3)}{B(3,3)} + \frac{B(3,3)}{A(3,3)} \right] + \sum_{uv\in E_7} \left[\frac{A(3,4)}{B(3,4)} + \frac{B(3,4)}{A(3,4)} \right].$$

$$
\begin{aligned}
&= (2n+1)\left(\frac{5}{2}\right) + (9n+1)\left(\frac{10}{3}\right) + n\left(\frac{17}{4}\right) + (5n+4)(2) + (18n-1)\left(\frac{13}{6}\right) + 2n\left(\frac{20}{8}\right) \\
&\quad + 16n(2) + n\left(\frac{25}{12}\right). \\
&= \frac{1273n}{10} + \frac{117}{10}.
\end{aligned}
$$

3 Conclusion

In this work, various topological indices are obtained with the values inspired by Dox-loaded micelle consisting of PEG-PAsp block copolymer for an anti cancer drug. By means of which the exact expressions are denoted for several important indices. These formulae help in correlating chemical structure of polymers with the physical properties. The outcome obtained in this work demonstrate the optimistic applications in chemical and pharmaceutical engineering field.

Acknowledgement

The authors would like to express their grateful thanks to the referee and Editor for their kind suggestions and corrections that helped to improve the original manuscript .

References

[1] M. Bhanumathi and K. Easu Julia Rani, On multiplicative harmonic index, multi-plicative ISI index and multiplicative F-index of TUC4C8[m; n] and TUC4[m; n] nanotubes, *International J. on Recent Trends in Life Science and Mathematics,* **4(9)**(2017), 1-88.

[2] M. Eliasi, A. Iranmanesh and I. Gutman, Multiplicative versions of first Zagreb index, *MATCH Commun. Math. Comput. Chem,* **68**(2012), 217- 230.

[3] B. Furtula and I. Gutman, A forgotten topological index, *J. Math. Chem.,* **53**(2015), 1184-1190.

[4] W. Gao, W. F. Wang and M. R. Farahani, Topological indices study of molecular structure in anticancer drugs, *J.Chem.,* **10**(2016), 1-8, DOI:10.1155/2016/3216327.

[5] S. Ghobadi and M. Ghorbaninejad, On F-polynomial, Multiple and hyper F-index of sum molecular graphs, *Bulletin of Mathematical Sciences and Applications,* **20**(2018), 36-43.

[6] M. Ghorbani and M. A. Hosseinzateh, Computing ABC_4 index of Nanostar dendrimers, *Optoelectron.Adv.Mater-Rapid Commun.,* **4(9)**(2010), 419-1422.

[7] M. Javaid, A. Raheem, M. Abbas and J. Cao, M-polynomial method for topological indices of 3-Layered probabilistic neural networks, *TWMS J.APP.Eng.Math.,* **9**(2019), 864-875.

[8] V.R. Kulli, Edge version of multiplicative atom bond connectivity index of certain nanotubes and nanotorus, *International Journal of Mathematics And its Applications,* **6(1-E)**(2018), 977-982.

[9] V. R. Kulli, Edge version of F-index, General sum connectivity index of certain nan-otubes, *Annals of Pure and Applied Mathematics,* **14(3)**(2017), 449-455.

[10] V. R. Kulli, Multiplicative hyper-Zagreb indices and coindices of graphs, *International Journal of Pure Algebra,* **67**(2016), 342-347.

[11] V. R. Kulli, Branden Stone and Bing Wei, Generalized multiplicative indices of poly-cyclic aromatic hydrocarbons and benzenoid systems, *Zeitschrift fur Naturforschung,* **72(6)**(2017), 573-576.

[12] V. R. Kulli, Multiplicative connectivity indices of certain nano-tubes, *Annals of Pure and Applied Mathematics,* **12(2)**(2016), 169-176.

[13] V. R. Kulli, Multiplicative Connectivity Indices of Nano-structures, *Journal of Ultra Scientist of Physical Sciences,* **29(1)**(2017), 1-10.

[14] V. R. Kulli, On multiplicative F -indices and multiplicative connectivity F -indices of chemical networks, *International Journal of current research in Science and Technology,* **5(2)**(2019), 1-10.

[15] V. R. Kulli, F -indices of chemical networks, *International Journal of Mathematical Archieve,* **10(3)**(2019), 21-30.

[16] V. Lokesha and P. S. Ranjini, Eccentric Connectivity index, Hyper and reverse-Wiener indices of the subdivision Graph, *General Mathematics Notes,* **2(2)**(2011), 34-46.

[17] N. Nishiyama, K. Kataoka, Polymeric micelle drug carrier systems: PEG-PAsp(Dox) and second generation of micellar drugs, in Polymer Drugs in the Clinical Stage. *Advances in Experimental Medicine and Biology,* **519**(2003), 155-177.

[18] K. Osada, R. J. Christie and K. Kataoka, Polymeric micelles from poly(ethyleneglycol)-poly(amino acid) block copolymer, *Journal of the Royal Society Interface,* **6(3)**(2019), 325-339.

[19] H.L. Parashivamurthy, M. R. Rajesh Kanna and R. Jagdeesh, *Toplological Indices of Nicotine,* **6(3)**(2019), 20-28.

[20] P. S. Ranjini, V. Lokesha and M. A. Rajan, On Wiener Polynomial of the Subdivision Graphs, *Chineese Journal of Engineering Mathematics,* **3**(2011), 411-418.

Jñānābha, Vol. 50(1) (2020), 102-114
(Dedicated to Honor Professor H.M. Srivastava on His 80th Birth Anniversary Celebrations)

ON SOFT AND FUZZY SOFT RELATIONS WITH THEIR APPLICATIONS

By
Seema Singh
Department of Statistics, M. D. University, Rohtak-124001, Haryana, India
Email:seemamotsara2@gmail.com
D. S. Hooda
Honorary Professor of Mathematics, G. J. University of Science and Technology, Hisar- 125001, Haryana, India
Corresponding Author D.S. Hooda
Email:ds_hooda@rediffmail.com
S. C. Malik
Department of Statistics, M. D. University, Rohtak-124001, Haryana, India
Email:sc_malik@rediffmail.com
(*Received : March* 08*,* 2020 *; Revised: May* 05*,* 2020)
DOI: https://doi.org/10.58250/jnanabha.2020.50112

Abstract

In our day to day life we face many problems which are abstract or vague in nature. These problems cannot be solved only by using simple mathematical tools. To deal with such kind of problems a new technique popularly known as Fuzzy Set Theory was discovered. Fuzzy set is the generalization of crisp set and is used in almost every field of life including Medical Sciences, Business, Administration, Social Science and Operation Research. Later on, a new concept of parameterization of power set of the universal set was introduced. Consequently, Fuzzy Soft Set Theory was defined by embedding Fuzzy Set and Soft Set. In the present communication fuzzy binary relation is described and its applications are studied. The concepts of soft and fuzzy soft relations are also defined with their applications in decision making problems.
2010 Mathematics Subject Classifications: 03E72, 03E75, 62C86, 11M50.
Keywords and phrases: Fuzzy Set, Soft Set, Fuzzy Soft Set, Soft Relation and Fuzzy Soft Relation, Decision making.

1 Introduction

In real life there exist many problems which contain vague or linguistic data and these data cannot be analysed merely by mathematical tools. However, other concepts, like Game Theory, Fuzzy Set, Soft Set, Fuzzy Soft Set and Rough Set, etc. have been developed to deal with such types of imprecise data. Fuzzy Set theory is very popular now these days and so several studies have been carried out in this area during past few years.

L.A. Zadeh [10] defined the concept of Fuzzy Set to handle the imprecise data. Fuzzy Set theory is the extension of crisp set. In addition to this, Rough Set theory was developed by Computer Scientist Pawlak [8] and that was known as Pawlak Rough Set Theory. These theories have been successfully applied to the fields of Decision Making, Dimension Reduction, Data Mining, etc.

Besides the Fuzzy Set Theory, the Soft Set Theory is another mathematical tool to deal with vague data. Soft Set is the parameterization of power set of fuzzy subsets of universal set.

Molodstov [4] was the first to introduce soft set theory for modelling uncertainties. He not only defined fundamental concepts of soft set theory, but also showed how this theory had removed parameterization insufficiency which was existing in the case of game theory, probability theory, rough set and fuzzy set theories. Many models are unique in case of soft set theory. Soft set has its wide applications in the fields of medical and environmental sciences, economic and business management.

Extending soft set theory a new approach known as Fuzzy Soft Set was introduced by Maji et al. [6]. He proposed Fuzzy Soft Set theory by embedding the concept of fuzzy set and soft set. Later on this concept was generalized by Majumdar and Samanta [7] in many ways and frequently used in decision making problems. Recently, Hooda and Kumari [4] have applied this theory in dimension reduction and medical diagnosis.

Aktas and Cagman [1] compared soft sets with the related concepts of fuzzy sets and rough sets. Yang et al. [10] worked on different operations for fuzzy soft sets. Zou and Xiao [11] introduced the soft set and fuzzy soft set into incomplete environment.

Decision making problems are the centre of attraction in every field. Optimum decision must be taken on the basis of uncertain or vague data. Fuzzy set theory and soft set theory are the best technique to handle such situations. Also, there may be the situation where two or more soft sets or fuzzy soft sets are given and we have to form a relation between them. On the basis of that relation decision is taken.

The concepts of soft relation and fuzzy soft relation are applied in forming relations between two attributes, like the relation of weight with height. A person is considered to be fit if his/her weight is within the range prescribed corresponding to his/her height. Otherwise he/she is considered as overweight or underweight according to the situation. The binary relation of height and weight is one of the important studies in health. The study of fuzzy relation was extended to soft and fuzzy soft relations which have found interesting and useful applications in decision making problems and medical diagnosis.

In the present paper basic concepts and definitions are described in Section 2, Section 3 fuzzy binary relation and its applications are studied. Soft and fuzzy soft relations with their applications are discussed in Sections 4 and 5 respectively. The conclusion and future scope are given in Section 6 with the references in the end.

2 Basic Concepts and Definitions

In this section we define some basic concepts and definitions which are used later on development of the paper.

2.1 Fuzzy Set

***Definition* 2.1.** *Let us consider X as a set of universe, then a fuzzy subset "A" of X is defined as a set of ordered pair given by*

$$A = \{< x, \mu_A(x) > / x \epsilon X\} \tag{2.1}$$

where, $\mu_A(x)$ *is called membership function from X to* $[0, 1]$ *with the following properties:*

$$\mu_A(x) = \begin{cases} 0, & \textit{if } x \notin A \textit{ there is no ambiguity} \\ 1, & x \epsilon A \textit{ there is no ambiguity} \\ 0.5 & \textit{whether } x \epsilon A \textit{ or } x \notin A, \textit{ there is maximum ambiguity} . \end{cases} \tag{2.2}$$

***Example* 2.1.** *A possible membership function defined for the set of real number close to* 9 *is*

$$\mu_A(x) = \frac{1}{1 + (x - 9)^2} \quad ; (x, \mu_A(x)) \epsilon R^2. \tag{2.3}$$

Here, number 7 *is assigned a membership value* 0.2*,* 11 *is assigned a membership value* 0.2*,* 21 *is assigned a membership value* 0.0068 *and for* 9 *membership value is* 1*. Thus the set A given by* $A = \{(7, 0.2), (11, 0.2), (21, 0.0068), (9, 1)\}$ *is a fuzzy set.*

2.2 Soft Set

***Definition* 2.2.** *Soft set is the extension of fuzzy set theory proposed by Molodstov* [5] *to handle uncertainty of the non-probabilistic approach.*

Let's consider 'U' as the universal set and T be the parametric set, then the pair (π, T) is defined as soft set over U iff π is a function of T into power set of U i.e. $\pi : T \to P(U)$.

***Example* 2.2.** *Assuming $S = \{s_1, s_2, s_3, s_4, s_5\}$ be the set of scooters under study and let 'T' be the parametric set given as $T = \{t_1 = expensive, t_2 = beautiful, t_3 = cheap, t_4 = in\ good\ repair, t_5 = latest\}$. Then the soft set is given by (π, T) describe the attractiveness of the scooters and given by*

$$(\pi, T) = \begin{cases} expensive = s_1, s_2, s_3, \\ beautiful = s_3, s_5, \\ cheap = s_3, s_4, s_5 \\ in\ good\ repair = s_1, s_2, s_3, s_4, s_5 \\ latest = s_1, s_3, s_4, s_5. \end{cases} \tag{2.4}$$

***Table* 2.1:** Tabular representation of (π, T)

X/E	t_1	t_2	t_3	t_4	t_5
s_1	1	0	0	1	1
s_2	1	0	0	1	0
s_3	0	1	1	1	1
s_4	0	0	1	1	1
s_5	1	1	1	1	1

2.3 Fuzzy Soft Set

***Definition* 2.3.** [6] *Fuzzy soft set was defined was by Maji et al. as a hybridization of soft set and fuzzy set. Let X be a universal set and E be the set of parameters and $A \subset E$. Let $F(x)$ be the set of all fuzzy subsets of X, then the pair (F, A) is called fuzzy soft subset of X, where F is a mapping from A to fuzzy set $F(x)$.*

***Example* 2.3.** *Let $X = \{c_1, c_2, c_3\}$ be set of* 3 *cars and $E = \{costly(e_1), getup(e_2), colour(e_3)\}$ be the set of parameters and let $A = \{e_1, e_2\} \epsilon E$. Then,*

$$(F, A) = \begin{cases} F(e_1) = \{c_1/0.6, c_2/0.4, c_3/0.4\} \\ F(e_2) = \{c_1/0.6, c_2/0.3, c_3/0.8\}, \end{cases} \tag{2.5}$$

(F, A) is a Fuzzy Soft Set over U to give the "attractiveness of cars".

3 Fuzzy Relations

A relation is a subset of $X \times Y$, where X and Y are crisp sets, where fuzzy relation is a fuzzy subset of $X \times Y$ i.e. a mapping from X to Y. There are many applications of fuzzy relations.

***Definition* 3.1.** [2] *A fuzzy relation is as a fuzzy set defined on Cartesian product of crisp set* $(X_1, X_2, , X_n)$ *with membership grade* $(x_1, x_2, , x_n)$*. The membership grade indicates the strength of the relation present between the elements of the tuples.*

A fuzzy relation can also be conveniently represented by n-dimensional membership array whose entries correspond to n-tuples in the universal set. These entries take values representing the membership grades of the corresponding n-tuples. In other words, an n- dimensional fuzzy relation R is a fuzzy set of Cartesian product $(X_1 \times X_2 \times \cdots \times X_n)$*, where* $(X_1, X_2, , X_n)$ *are domain.*

Definition* 3.2.** [2] ***Binary Fuzzy Relation
When fuzzy relation is taken over only two crisp sets i.e. between X and Y is known as binary fuzzy relation.

***Example* 3.1.** *Let* $X = (a, b, c)$ *and* $Y = (x, y)$*, then binary fuzzy relation '*R' *on* $X \times Y$ *is given in **Table* 3.1.**

***Table* 3.1**

R	x	y
a	0.6	1.0
b	0.3	0.5
c	0.4	0.2

Definition* 3.3. *Ternary Binary Relation

When fuzzy relation is taken over three crisp sets i.e. between X, Y *and* Z *is known as Ternary Binary Relation.*

***Example* 3.2.** *Let* $X = (x_1, x_2, x_3)$*,* $Y = (y_1, y_2)$ *and* $Z = (z_1, z_2)$*, then fuzzy relation* $X \times Y \times Z$ *is given as*

$$R = \frac{0.21}{< x_1, y_1, z_1 >} + \frac{0.38}{< x_2, y_2, z_1 >} + \frac{0.9}{< x_1, y_2, z_2 >}. \tag{3.1}$$

*The tabular representation of the fuzzy relation R is given in **Tables** **3.2** and **3.3**.*

***Table* 3.2**

R	y_1	y_2
x_1	0	0.9
x_2	0	0
x_3	0	0

***Table* 3.3**

R	y_1	y_2
x_1	0.21	0
x_2	0	0.38
x_3	0	0

Definition* 3.4. *Height of a Fuzzy Relation
It is a number denoted by $h(R)$*, where R is a fuzzy relation given as:*

$$h(R) = \max_{x \epsilon X, y \epsilon Y} R(x, y) \tag{3.2}$$

i.e. it is the largest membership grade attained be any pair (x, y) *in fuzzy relation R. If* $h(R) = 1$*, then it is a normal fuzzy relation*

Definition* 3.5. *Inverse of a Fuzzy Relation

Inverse of a fuzzy relation $R(x, y)$ is denoted by $R^{-1}(x, y)$, a fuzzy relation over $Y \times X$ is given as

$$R^{-1}(x, y) = R(x, y); x \epsilon X, y \epsilon Y. \tag{3.3}$$

Definition* 3.6. *Max-Min Composition of Two Fuzzy Relations

Let us consider R be a binary fuzzy relation on $X \times Y$ and S be a binary fuzzy relation over $Y \times Z$. Then, max-min composition of R followed by S is a binary fuzzy relation on $X \times Z$. It is denoted by $S \circ R$, given by

$$(S \circ R)(x, z) = \max[\min(R(x, y), R(y, z))], \tag{3.4}$$

where max is taken over all y in Y.

***Example* 3.3.** *Consider the fuzzy relation 'R' on $X \times Y$ and 'S' on $Y \times Z$. X, Y and Z are given as: $X = (a, b, c), Y = (d, e, f)$ and $Z = (\star, \#)$. Fuzzy relations 'R' and 'S' in matrix form are given in the following **Tables** 3.4 and 3.5 respectively:*

***Table* 3.4**

R	d	e	F
a	1.0	0.4	0.5
b	0.3	0.0	0.4
c	0.6	0.3	0.2

***Table* 3.5**

S	$*$	#
d	0.7	0.1
e	0.2	0.9
f	0.3	0.4

Then the composition $S \circ R$ is defined and that is also a fuzzy relation over $X \times Z$ described as follows:

$$\begin{aligned}
S \circ R(a, \star) &= \max[\min(1, 0.7), \min(0.4, 0.2), \min(0.5, 0.3)] \\
&= max[0.7, 0.2, 0.3] \\
&= 0.7 \\
S \circ R(a, \#) &= \max[\min(1, 0.1), \min(0.4, 0.9), \min(0.5, 0.4)] \\
&= \max[0.1, 0.4, 0.4] \\
&= 0.4 \\
S \circ R(b, \star) &= \max[\min(0.3, 0.7), \min(0.0, 0.2), \min(0.4, 0.3)] \\
&= \max[0.3, 0.0, 0.3] \\
&= 0.3 \\
S \circ R(b, \#) &= \max[\min(0.3, 0.1), \min(0.0, 0.9), \min(0.4, 0.4)] \\
&= \max[0.1, 0.0, 0.4] \\
&= 0.4 \\
S \circ R(c, \star) &= \max[\min(0.6, 0.7), \min(0.3, 0.2), \min(0.2, 0.3)] \\
&= max[0.6, 0.2, 0.2] \\
&= 0.6 \\
S \circ R(c, \#) &= \max[\min(0.6, 0.1), \min(0.3, 0.9), min(0.2, 0.4)] \\
&= \max[0.1, 0.3, 0.2] = 0.3.
\end{aligned}$$

Thus, $S \circ R$ in matrix form is given as

0.7	0.4
0.3	0.4
0.6	0.3

Definition* 3.7. *Max Product Composition *If R and S are the fuzzy relation over $X \times Y$ and $Y \times Z$ respectively, then max-product composition of R followed by S is given as*

$$(S \circ R)(x, z) = \max[R(x, y) * S(z, y)] \tag{3.5}$$

where '' is the ordinary product of real numbers and max is taken over all elements y in Y.*

***Definition* 3.8.** *Let we consider R and S two fuzzy relations over $X \times Y$ and $Y \times Z$ respectively given in following Tables*

***Table* 3.6:** Fuzzy Relation *R*

0.3	0.5	0.8
0.0	0.7	1.0
0.4	0.6	0.5

***Table* 3.7:** Fuzzy Relation *S*

0.9	0.5	0.7	0.7
0.3	0.2	0.0	0.9
1.0	0.0	0.5	0.5

*Then the max-product composition of $S \circ R$ is given in the **Table 3.8** given below:*

***Table* 3.8:** $S \circ R$

0.8	0.15	0.4	0.45
1.0	0.14	0.5	0.63
0.5	0.20	0.28	0.54

4 Soft Relation and its Application

In this section the concept of soft relation is introduced and its application in decision making is studied with an example.

***Definition* 4.1.** *Let us consider U and V as two initial universal sets and E be the parametric set and let (F, E) and (G, E) be two soft set over U and V respectively, then (H, E) is a soft relation between (F, E) and (G, E) over $U \times V$ if*

$$H : E \to 2^{U \times V},$$

where H is a mapping such that

$$H(e) = \begin{cases} (u_i, v_j); & \text{if } u_i \epsilon F(e) \text{ and } v_j \epsilon G(e) \ \forall \ e \epsilon E \\ \phi; & \text{otherwise} \end{cases}. \tag{4.1}$$

***Example* 4.1.** *Let $U = (h_1, h_2, h_3, h_4)$ be the set of four houses and let $V = (f_1, f_2, f_3)$, be the set of three farm houses. Also, let 'E' be the parametric set namely*

$E = (e_1(green\ surrounding), e_2(cheap), e_3(wooden))$. Then the soft sets (F, E) and (G, E) over U and V are given by:

$(F, E) = F(e_1) = (h_1, h_3), F(e_2) = (h_2, h_4), F(e_3) = (h_1, h_2)$ and

$(G,E) = G(e_1) = (f_1, f_3), G(e_2) = (f_1, f_2), G(e_3) = (f_2, f_3)$.
Then, the soft relation (H,E) between (F,E) and (G,E) is given as
$H(e_1) = (h_1, f_1), (h_1, f_3), (h_3, f_1), (h_3, f_3)$,
$H(e_2) = (h_2, f_1), (h_2, f_2), (h_4, f_1), (h_4, f_2)$ and
$H(e_3) = (h_1, f_2), (h_1, f_3), (h_2, f_2), (h_2, f_3)$.

*The tabular representation of soft sets (F, E) and (G, E) are given in **Tables 4.1** and **4.2**.*

***Table* 4.1:** Soft Set (F, E)

U	e_1	e_2	e_3
h_1	1	0	1
h_2	0	1	1
h_3	1	0	0
h_4	0	1	0

***Table* 4.2:** Soft Set (G, E)

V	e_1	e_2	e_3
f_1	1	1	0
f_2	0	1	1
f_3	1	0	1

The soft relation (H,E) is given in the following Table:

***Table* 4.3:** Soft Relation (H, E)

$U \times V$	e_1	e_2	e_3
(h_1, f_1)	1	0	0
(h_1, f_2)	0	0	1
(h_1, f_3)	1	0	1
(h_2, f_1)	0	1	0
(h_2, f_2)	0	1	1
(h_2, f_3)	0	0	1
(h_3, f_1)	1	0	0
(h_3, f_2)	0	0	0
(h_3, f_3)	1	0	0
(h_4, f_1)	0	1	0
(h_4, f_2)	0	1	0
(h_4, f_3)	0	0	0

4.1 Properties of Soft Relation

Let (H_1, E) and (H_2, E) be the two soft relations between (F, E) and (G, E) over $U \times V$, then the following results hold:

(a) Union of two soft relations is also a soft relation, i.e., $(H, E) = (H_1, E) \cup (H_2, E)$ such that $H(e) = H_1(e) \cup H_2(e) \;\; ; \;\; \forall\, e \,\epsilon E$.
(b) Intersection of soft relation is also a soft relation, i.e., $(K, E) = (H_1, E) \cap (H_2, E)$; such that $K(e) = H_1(e) \cap H_2(e) \;\; ; \;\; \forall e \,\epsilon\, E$.
(c) Complement of soft relation is also a soft relation. Let us consider $(F, E)^C$ and $(G, E)^C$ as the complement of soft set (F, E) and (G, E). Then soft relation between $(F, E)^C$ and $(G, E)^C$ is given by
$H^C = \rceil E \to 2^{U \times V}$, where H^C is a mapping such that
$$H^C(\rceil, e) = (u_i, v_j), \text{ where } u_i \,\epsilon\, F(\rceil e) \text{ and } v_j\; G(\rceil e), \;\forall\, e \,\epsilon E.$$
The symbol “$\rceil$” stands for “not in”.

(d) Composition of soft relation is also a soft relation. Let $(F, E), (G, E)$ and (H, E) are three soft sets over U, V and W respectively. Also let (K_1, E) and (K_2, E) are soft relation between $(F, E)\&(G, E)$ and $(G, E)\&(H, E)$ over $U \times V$ and $V \times W$ respectively. Then composition of (K_1, E) and (K_2, E) is also a soft relation over $U \times W$ given as $K : E \to 2^{U \times W}$: such that

$$K(e) = \{(u, v); \exists v \epsilon V, u \epsilon U \text{ and } w \epsilon W, \text{ also } (u, v) \epsilon (K_1, E) \text{ and } (v, w) \epsilon (K_2, E)\}$$

(e) A soft relation is reflexive iff $(u_i, u_i) \epsilon H(e)\ \forall e \epsilon E$ and$u_i \epsilon U$.

(f) A soft relation is symmetric iff $(u_i, u_j) \epsilon H(e)$ and $(u_j, u_j) \in H(e)\ \forall\ e \epsilon E$ and $u_i, u_j, u_k \epsilon U$

(g) A soft relation is transitive iff $(u_i, u_j) \epsilon H(e)$ and $(u_j, u_i) \in H(e)\ \forall\ e \epsilon E$ and $u_i, u_j \epsilon U$

(h) A soft relation is soft tolerance relation if it is reflexive and symmetric.

(i) A soft relation is soft equivalence relation if it is reflexive, symmetric and transitive.

4.2 Application of Soft Relation in Decision Making Problem

Here we discuss the application of soft relation in decision making problem. Suppose $B = b_1, b_2, b_3, b_4$ is the set of four boys and $G = g_1, g_2, g_3$ is the set of three girls who play badminton. From these players a pair of boy and girl is to be chosen for sponsorship. Also, let 'E' be the set of parameters to judge the capability of a badminton players. E is given as $=e_1$ (physical fitness), e_2 (average matches win), e_3 (judgement capability), e_4 (average matches played), e_5 (height).

Suppose Mr X is interested to sponsor a mixed pair of badminton player on the basis of his choice of parameters.

$A = e_1$ (physical fitness), e_2 (average matches win), e_3 (judgement capability).

4.2.1 Soft Set Formation

Let us consider soft sets (F, A) and (H, A) over B and G respectively, given as

$$(F, A) = F(e_1) = (b_1, b_2), F(e_2) = (b_1, b_2, b_4), F(e_3) = (b_1, b_3)$$

$$(H, A) = H(e_1) = (g_1, g_3), H(e_2) = (g_1, g_2), H(e_3) = (g_1, g_2, g_3)$$

Tabular representation of soft sets (F, A) and (H, A) are given in the following tables respectively.

***Table* 4.4:** Soft Set (F, A)

B	e_1	e_2	e_3
b_1	1	1	1
b_2	1	1	0
b_3	0	0	1
b_4	0	1	0

***Table* 4.5:** Soft Set (H, A)

B	e_1	e_2	e_3
g_1	1	1	1
g_2	0	1	1
g_3	1	0	1

4.2.2 Algorithm for Selection of Mixed Double Players using Soft Relation

(i) Firstly, input the soft set (F, A) and (H, A).

(ii) Secondly, construct the soft relation (I, A) table using soft sets (F, A) and (H, A) w.r.t the choice of the paramters of Mr X.

(iii) Then compute the choice value $r_{i,j}$ i.e $r_{i,j} = \sum_{e \epsilon E} (b_i, g_j)$ for the soft relation (I, A).

(iv) Find $m = \max_{1 \le i \le 4, 1 \le j \le 3} r_{i,j}$,

If two or more values of 'm' are same then Mr X can choose any one of them by his opinion.

As we already construct the soft set(F, A) and (H, A) in ***Tables* 4.4** and **4.5**. Now we construct the soft relation (I, A) over $B \times G$ and then compute the choice value$r_{i,j}$. By applying ***Algorithm* 4.2.2**, we get the following soft relation as given below:

Table **4.6:** Soft Relation over $B \times G$

$B \times G$	e_1	e_2	e_3	Choice value ($r_{i,j}$)
(b_1, g_1)	1	1	1	3
(b_1, g_2)	0	1	1	2
(b_1, g_3)	1	0	1	2
(b_2, g_1)	1	1	0	2
(b_2, g_2)	0	1	0	1
(b_2, g_3)	1	0	0	1
(b_3, g_1)	0	0	1	1
(b_3, g_2)	0	0	1	1
(b_3, g_3)	1	0	1	1
(b_4, g_1)	0	1	0	1
(b_4, g_2)	0	1	0	1
(b_4, g_3)	0	0	0	0

Here maximum value of $r_{ij} = 3$ given by r_{11} for pair (b_1, g_1). Hence, Mr X will sponsor the pair (b_1, g_1) for mixed double badminton game

5 Fuzzy Soft Relation

In this section we shall define fuzzy soft relation with its application in decision making problem by considering example.

***Definition* 5.1.** *Let U and V be two initial universal sets and E be the set of parameter. Also, let (F, E) and (G, E) be two fuzzy soft set over U and V respectively and $\phi(U \times V)$ be the set of all fuzzy subset of $U \times V$, then (H, E) is a fuzzy soft relation between (F, E) and (G, E) over $U \times V$ if $H : E \rightarrow \phi(U \times V)$, where H is a mapping such that*

$$H(e) = \{(u_i, v_j)/u_{ij};\ u_{ij} = \min(u_i, u_j)\ \forall e \in E, (u_i, u_i) \in F(e) \text{ and } (v_j, u_j) \in G(e)\}. \tag{5.1}$$

***Example* 5.1.** *Let U = {Paris, Berlin, Amsterdam} and V = {Rome, Madrid, Lisbon} are sets of cities and let 'E' be the set of given parameters, where E ={e_1 (far), e_2 (very far), e_3 (near), e_4 (crowded), e_5 (well managed)}. Let H be the fuzzy soft relation over U and V given by*

(H, E) = {$H(e_1)$ = *(Paris, Rome)*/0.60, *(Paris, Madrid)*/0.45,*(Paris, Lisbon)*/0.40, *(Berlin, Rome)*/0.55, *(Berlin, Madrid)*/0.65, *(Amsterdam, Lisbon)*/0.70, *(Amsterdam, Rome)*/0.75, *(Amsterdam, Madrid)*/0.50, *(Amsterdam, Lisbon)*/0.80}

Tabular Representation of Fuzzy Soft Relation over $U \times V$ is given in ***Table* 5.1**.

Table **5.1:** Fuzzy Soft Relation over $U \times V$

H (far)	Rome	Madrid	Lisbon
Paris	0.60	0.45	0.40
Berlin	0.55	0.65	0.70
Amsterdam	0.75	0.50	0.80

5.1 Properties of Fuzzy Soft Relation

Let (H_1, E) and (H_2, E) be the two fuzzy soft relations between (F, E) and (G, E) over $U \times V$, then

(a) Union of two fuzzy soft relations is also a fuzzy soft relation i.e.

$$(H, E) = (H_1, E) \cup (H_2, E), \text{ where } H(e) = H_1(e) \cup H_2(e) \text{ ; } \forall \ e \ \epsilon E.$$

(b) Intersection of fuzzy soft relation is also a fuzzy soft relation i.e

$$(K, E) = (H_1, E) \cap (H_2, E), \text{ where } K(e) = H_1(e) \cap H_2(e); \ \ \forall \ e \ \epsilon \ E.$$

(c) Complement of fuzzy soft relation is also a fuzzy soft relation.
Let us consider $(F, E)^(C)$ and (G, E) as the complement of fuzzy soft set (F, E) and (G, E), then fuzzy soft relation between $(F, E)^C$ and $(G, E)^C$ is given by

$H^C = \rceil E \to 2^{U \times V}$, where H^C is a mapping such that

$$H^C(\rceil e) = \frac{(u_i, v_j)}{u_{ij}}; \ u_{jj} = \min(\mu_i, \mu_j), \forall \ e \epsilon E, \ (u_i, \mu_i) \ \epsilon F(\rceil e) \text{ and } (v_j, \mu_j) \ \epsilon G(\rceil e).$$

The symbol "$\rceil$" stands for "not in"

(d) A fuzzy soft relation is reflexive iff $\mu_{H(e)} \ (h_i, h_i) = 1 \ \forall \ e \epsilon E$ and $h_i \epsilon U$.

(e) A fuzzy soft relation is symmetric iff $\mu_{H(e)}(h_i, h_j) = \mu_{H(e)}(h_j, h_i) \ \forall \ e \ \epsilon E$ and $h_i, h_j \ \epsilon \ U$.

(f) A fuzzy soft relation is transitive iff $\mu_{H(e)}(h_i, h_j) = \lambda_1$ and $\mu_{H(e)}(h_j, h_k) = \lambda_2 \to \mu_{H(e)}(h_i, h_k) = \lambda \ \forall e \ \epsilon \ E, \lambda \geq \min(\lambda_1, \lambda_2)$ and $h_i, h_j, h_k \ \epsilon \ U$.

(g) A fuzzy soft relation is fuzzy tolerance relation if it is reflexive and symmetric.

(h) A fuzzy soft relation is fuzzy equivalence relation if it is reflexive, symmetric and transitive.

5.2 Application of Fuzzy Soft Relation in Decision Making Problem

Let us consider $B = b_1, b_2, b_3, b_4$ the set of four boys and $G = g_1, g_2, g_3$ the set of three girls who play badminton. We are to choose a pair for double badminton game. Further, let 'E' be a set of parameters to judge the capability of a badminton players and is given as

=$\{e_1$ (physical fitness), e_2 (average matches win), e_3 (judgement capability),

e_4 (average matches played), e_5 (height)$\}$

Suppose Mr X is interested to sponsor a mixed pair of badminton player on the basis of his choice of parameters. Let A be a subset of E given as

=$\{e_1$ (physical fitness), e_2 (average matches win), e_3 (judgement capability)$\}$

5.2.1 Algorithm for Selection of Mixed Double Using Fuzzy Soft Relation

(i) Firstly, input the soft sets (F, A) and (H, A) w.r.t. choice of parameters of Mr X.

(ii) Secondly, covert the soft sets (F, A) and (H, A) into fuzzy soft sets using suitable technique.

(iii) Form the fuzzy soft relation (I, A) between fuzzy soft sets (F, A) and (H, A).

(iv) Then compute the comparison table for fuzzy soft relation (I, A).

(v) Then compute row-sum and column-sum of comparison table as

$$s_{ij} = \sum_{e \epsilon A} r_{ij} \ \text{ and } \ p_{ij} = \sum_{e \epsilon A} r_{ij}$$

(vi) Then find the score value $S_{ij} = s_{ij} - p_{ij}$.

(vii) Finally, maximum of S_{ij} will be the choice. In case two or more values of $S_i j$ are same, then Mr X can choose any one the pair according to his opinion

5.2.2 Illustration with Example

Let us consider soft sets (F, A) and (H, A) over B and G respectively, given as

$$(F, A) = \{F(e_1) = (b_1, b_2), F(e_2) = (b_1, b_2, b_4), F(e_3) = (b_1, b_3)\}$$
$$(H, A) = \{H(e_1) = (g_1, g_3), H(e_2) = (g_1, g_2), H(e_3) = (g_1, g_2, g_3)\}.$$

Tabular representation of soft sets (F, A) and (H, A) are given in ***Tables*** **5.2** and **5.3** respectively.

Table **5.2:** Soft Set (F, A)

B	e_1	e_2	e_3
b_1	1	1	1
b_2	1	1	0
b_3	0	0	1
b_4	0	1	0

Table **5.3:** Soft Set (H, A)

G	e_1	e_2	e_3
g_1	1	1	1
g_2	0	1	1
g_3	1	0	1

These soft sets are converted respectively to fuzzy soft set by applying ***Algorithm*** **5.2.1** given in ***Tables*** **5.4** and **5.5** below:

Table **5.4:** Fuzzy Soft Set (F, A)

B	e_1	e_2	e_3	
b_1	0.50	0.75	0.50	3/3
b_2	0.34	0.50	0	2/3
b_3	0	0	0.17	1/3
b_4	0	0.26	0	1/3
	2/4	3/4	2/4	

Table **5.5:** Fuzzy Soft Set (H, A)

G	e_1	e_2	e_3	
g_1	0.67	0.67	1	3/3
g_2	0	0.45	0.67	2/3
g_3	0.45	0	0.67	2/3
	2/3	2/3	3/3	

By applying the ***Algorithm*** **5.2.1** the fuzzy soft relation (I, A) is found out as given below in the ***Table*** **5.6**.

Table **5.6:** Fuzzy Soft Relation (I, A)

$B \times G$	e_1	e_2	e_3
(b_1, g_1)	0.50	0.67	0.50
(b_1, g_2)	0	0.45	0.50
(b_1, g_3)	0.45	0	0.50
(b_2, g_1)	0.34	0.50	0
(b_2, g_2)	0	0.45	0
(b_2, g_3)	0.34	0	0
(b_3, g_1)	0	0	0.17
(b_3, g_2)	0	0	0.17
(b_3, g_3)	0	0	0.17
(b_4, g_1)	0	0.26	0
(b_4, g_2)	0	0.26	0
(b_4, g_3)	0	0	0

Now the comparison table is given below in ***Table*** **5.7**

***Table* 5.7:** Comparison Table

$B \times G$	(b_1, g_1)	(b_1, g_2)	(b_1, g_3)	(b_2, g_1)	(b_2, g_2)	(b_2, g_3)	(b_3, g_1)	(b_3, g_2)	(b_3, g_3)	(b_4, g_1)	(b_4, g_2)	(b_4, g_3)
(b_1, g_1)	3	3	3	3	3	3	3	3	3	3	3	3
(b_1, g_2)	1	3	2	1	2	2	3	3	3	3	3	3
(b_1, g_3)	1	2	3	2	2	3	3	3	3	2	2	3
(b_2, g_1)	0	2	1	3	3	3	2	2	2	3	3	3
(b_2, g_2)	0	2	1	1	3	2	2	2	2	3	3	3
(b_2, g_3)	0	1	1	2	2	3	2	2	2	2	2	3
(b_3, g_1)	0	1	1	1	2	2	3	3	3	2	2	3
(b_3, g_2)	0	1	1	1	2	2	3	3	3	2	2	3
(b_3, g_3)	0	1	1	1	2	2	3	3	3	2	2	3
(b_4, g_1)	0	1	1	1	2	2	2	2	2	3	3	3
(b_4, g_2)	0	1	1	1	2	2	2	2	2	3	3	3
(b_4, g_3)	0	1	1	1	2	2	2	2	2	2	2	3

***Table* 5.8:** Comparison between Row-sum and Column-sum

Row-sum	Colum sum	Score Value
36	5	31
29	9	20
29	17	12
27	18	9
24	27	-3
22	28	-6
23	30	-7
23	30	-7
23	30	-7
22	30	-8
22	30	-8
20	36	-16

Here maximum score value $S_{ij} = 31$ corresponding to the pair (b_1, g_1). So Mr X would like to sponsor (b_1, g_1) for mixed double badminton game.

6 Conclusion and Future Scope

The concepts of soft set and fuzzy soft set are recently emerged important topics to deal with uncertainties and ambiguities present in our day to day life. The availability of the parameterization tools in these sets has further enhanced the flexibility of their applications. Thus, soft and fuzzy soft relations which are extensions to crisp and fuzzy relations have been introduced and their applications in decision making problems have been studied with examples.

In our view the theory of soft and fuzzy soft relations based on soft set and fuzzy soft set respectively, can be extended to interval valued soft and fuzzy soft sets. Also, the theory can be extended to intuitionistic soft and fuzzy soft sets, generating a new class of relations. Their application to decision making and medical diagnosis problems can be further considered and studied.

Compliance with Ethical Standards

Conflict of interest: On behalf of all the authors, the corresponding author declares that there is no conflict of interest. This article does not contain any studies with human participants or animals performed by any of the authors.

References

[1] H. Aktas and N. Cagman, Soft Sets and Soft Groups, *Information Sciences*, **177**(2007), 2726-2735.

[2] D. S. Hooda and V. Raich, Fuzzy Set Theory and Fuzzy Controller, *Narora Publishing House, New Delhi*, 2015

[3] D. S Hooda and Rakesh Bajaj, Useful Fuzzy Measures of information, Integrated Ambiguity and Directed Divergence, *International Journal of General System*, **39**(2010), 647-858.

[4] D. S. Hooda and R. Kumari, On Application of Fuzzy Soft Sets in Dimension Reduction and Medical Diagnosis, *Advance in Research*, **12**(2017), 1-9.

[5] D. Molodstov [1999], Soft Set Theory- First Result, *Computer and Mathematics with Application*, **37**(1999), 19-31.

[6] P. K. Maji, R. Biswas and A. R. Roy, Fuzzy Soft Sets, *Journal of Fuzzy Mathematics*, **9**(2001), 589-602.

[7] P. Majumdar and S. K. Samanta, Generalised Fuzzy Soft Sets, *Computer Mathematics Applications*, **59**(2010), 1425-1432.

[8] Z. Pawlak, Rough Sets, *International Journal of Information and Computer Sciences*, **11**(1982), 341-356.

[9] X. Yang, D. Yu, J. Yang and C. Wu, Generalization of Soft Set Theory: From Crisp to Fuzzy Case, *Advance in Soft Computing*, **40**(2007), 345-354.

[10] L. A. Zadeh, Fuzzy Sets, *Information and Control*, **8**(1965), 338-353.

[11] Y. Zou and Z. Xiao, Data Analysis Approaches of Soft Sets under Incomplete Information, *Knowledge- Based System*, **21**(2008), 941-945.

Jñānābha, Vol. 50(1) (2020), 115-133
(Dedicated to Honor Professor H.M. Srivastava on His 80th Birth Anniversary Celebrations)

CUBICAL POLYNOMIAL TIME FUNCTION DEMAND RATE AND PARETO TYPE PERISHABLE RATE BASED INVENTORY MODEL WITH PERMISSIBLE DELAY IN PAYMENTS UNDER PARTIAL BACKLOGGING

By
Ganesh Kumar[1], Sunita[2] and Ramesh Inaniyan[3]
[1,2,3]Departmentof Mathematics, University of Rajasthan, Jaipur-302004
Email:ganeshmandha1988@gmail.com, sunitarolan95@gmail.com and rameshinaniyan05@gmail.com

(*Received : March* 28, 2020 *; Revised: May* 17, 2020)
DOI: https://doi.org/10.58250/jnanabha.2020.50113

Abstract

In present paper an inventory model is produced for immediate transient things with cubical polynomial time function demand rate and pareto type perishable rate with permissible delay in payments. Deficiencies are permitted and partially backlogged for the next replenishment cycle. Holding cost is linear function of time. The fundamental motivation of this paper is to examine the retailer's ideal strategy that minimizes the retailer's yearly total cost per unit time under reasonable deferral in instalments inside the EOQ structure. Numerical results are shown for proposed model. Sensitivity investigation of the ideal arrangements regarding various parameters is examined.

2010 Mathematics Subject Classifications: 90B05, 90B10, 90B15
Keywords and phrases: Inventory, Instantaneous deterioration, Pareto perishable rate, Cubical polynomial time function demand rate, Permissible delay in payments.

1 Introduction

It is day to day challange that concerns the need and availability of products relevant to distributers, wholesalers and consumers. On large scale inventory problems can be devided into raw materials, process work and finished products. The basic EOQ model (1915) recognized three categories of costs, such as inventory prices, purchasing/set up cost and keeping/holding/carrying costs. Deterioration of commodities is unavoidable and a growing occurence of every day life. Deterioration plays significant role in the control of inventories. Instant and non-instant are two categories of deterioration. Stock actions under the permissible delay in payments is another crucial function in actual world scenarios. Partial backlogging is another factor which effect a slight decline in profit if there is waiting for shipment. Significant studies performed by a number of researchers in the field. Goyal [9] established an EOQ model that provided a reasonable pause in payments. Aggarwal and Jaggi [1] developed an EOQ model to achieve optimum order quantity of deteriorating products. Chen and Kang [5] have established an optimized inventory model with a pricing approach focussed on the permitted pause in payments. Prasad and Kansal [17] also established a dealer EOQ model where the manufacturer has an incremental interest rate within the allowable delay in payments. A lot size model for decaying items was provided by Shah [24, 25] in order to assess the optimum specific turnaround duration where the manufacturer only gave a specific extended payments for one duration. Optimum cost and lot size was evaluated by Teng et al.[30] under delay in payments. Shalini Singh [27] developed a procuring strategy of an inventory model for single item and multi supplier with allowing shortages.

An inventory model for deteriorating products with power dependent demand and linear deterioration was discussed by Sharma and Vijay [26]. Mohanty et al.[19] considered random review period and discounts for deteriorating items. An EOQ model for instantaneous items with cubic demand and constant deterioration rate was analyzed by Rangarajan and Karthikeyan [21]. Jaggi et al.[8] had taken ramp type demand to develop inventory model. Incremental holding cost were considered under partial backlogging by Singh et al. [28]. Jain et al.[14] produced an inventory model for a supplier. They assumed stock dependent demand rate for perishable items in inflationary environment. Ideal purchasing strategies carried out in [3, 18] with delay in payments. Chang and Dye [4] produced an EOQ model for deteriorating products with time dependent demand and marginally backlogged. When the ordered quantity is less than the prescribed quantity hypothetical theory developed in [11, 20]. Time dependent demand and perishable rate based EOQ model was designed by Sarkar [23] under permissible delay in payments. Strategies of inventory replenishment were discussed and Goyal's model modified by [6, 7]. Jamal et al. [15, 16] constructed an EOQ model for perishable goods and permissible delay in payments. Salameh et al.[22] discussed the continuous inventory analysis paradigm. Teng [29] revised Goyal's model by recognizing the gap between product price and product expenses. An EOQ model with time dependent perishable cost and constant demand rate was presented by Amutha and Chandrasekaran [2]. Teng et al.[31] extended the existing models by introducing some additional parameters. Geetha and Uthayakumar [32] provided an EOQ model for non-immediate depreciation and allowable delay in payments. Inventory models for decaying goods with delay in payments were established in [12, 13, 25].

In this paper, we develop an inventory model with cubical polynomial time function demand rate, perishable rate is taken pareto type, shortages are allowed and are partially backlogged with permissible delay in payments. Holding cost has been taken linear function of time. Replenishment rate is taken infinite and instantaneous. This paper is structured as follows: some notations and assumptions are mentioned in Section 2. Mathematical formulation and solution of model carried out in Section 3. Numerical investigation by assigning values of parameters is performed in Section 4. In Section 5, we present sensitivity analysis of the developed model by varying parameters. Results and observations are reported in Section 6. In Section 7, consequences of the paper are concluded.

2 Notations and Assumptions

We have used the following postulates to develop this inventory model:

1. Lead time is assigned to zero.
2. The replenishment rate is taken as infinite and instantaneous.
3. The finite planning horizon is reckoned.
4. The demand for the aspect is snatched as a cubical polynomial function of time.
5. Perishable rate is put up within the Pareto type.
6. The supplier didn't have provided the replacement or return strategy. Entities that have terminated will be demolished.
7. Shortages are tolerated and are partially backlogged. The backlogging rate is dangling on the customer's waiting duration for the subsequent replenishment, throughout the stock-out duration i.e. for the negative inventory, the backlogging rate is distinguished as $B(t) = \frac{1}{1+\delta(T-t)}$; $\delta > 0$ denotes the backlogging parameter and $t_1 \leq t \leq T$.

8. During the permissible delay deriod of payments, the retailer does not have to settle down the account with the supplier. The retailer deposites the generated sales revenue in an interest-bearing account. The supplier starts charging interest as soon as the deadline ends.

The following notations have been used in compiling this inventory model:

$D(t)$: The demand rate $D(t) = \begin{cases} a + bt + ct^2 + dt^3, & 0 \le t \le t_1 \\ D_0(constant), & t_1 \le t \le T \end{cases}$

where a=initial demand, b=initial rate of change of demand,
c=acceleration of demand,
d=rate of change of acceleration of demand,
$\theta(t)$: $\theta(t) = \frac{\theta_1\theta_2}{1+\theta_2 t}$ (Pareto type),
where $\theta_1 > 0$ and $0 < \theta_2 < 1$,
HC: Holding cost has taken as a linear function of time.$HC = \alpha + \beta t$, $\alpha > 0$, $\beta > 0$,
t_1: Time to exhaust stock within a replenishment cycle, $0 < t_1 < T$,
T: Length of a replenishment cycle,
M: Permissible delay period,
A: Fixed ordering cost per unit,
$p(t)$: The selling price at time t, $p(0) = p$,
C_b: Unit shortage cost of an item,
C_p: Unit purchasing cost of an item,
C_l: Unit lost sale cost of an item,
I_p: The interest charged per unit of money per year by the supplier, $0 < I_p < 1$,
I_e: The interest earned per unit of money per year, $0 < I_e < 1$.

3 Mathematical Formulation and Solution of the Model

The inventory system for instantaneous deteriorating items with shortages is portrayed in the following ***Figure* 3.1**

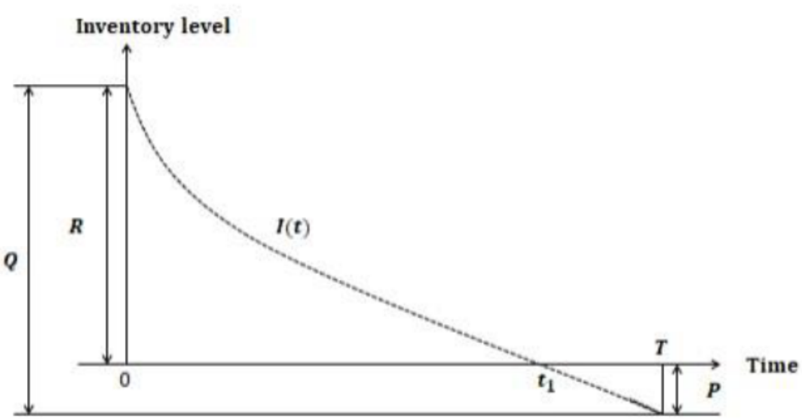


***Figure* 3.1:** Inventory System for Instantaneous Deteriorating Items with Shortages

The instantaneous inventory level $I(t)$ at any time 't' during the cycle time $[0, t_1]$ is represented by the following governing differential equation:

(3.1) $$\frac{dI(t)}{dt} + \theta(t)I(t) = -D(t),\ 0 \le t \le t_1$$

$$\implies \frac{dI(t)}{dt} + \left(\frac{\theta_1\theta_2}{1+\theta_2 t}\right) I(t) = -(a + bt + ct^2 + dt^3),\ 0 \le t \le t_1.$$

The solution of above equation (3.1) with boundary condition $I(t_1) = 0$ is

$$I(t) = \left[a(t_1 - t) + \frac{b}{2}(t_1^2 - t^2) + \frac{c}{3}(t_1^3 - t^3) + \frac{d}{4}(t_1^4 - t^4) + \frac{a\theta_1\theta_2}{2}(t_1^2 - 2tt_1 + t^2) + \frac{b\theta_1\theta_2}{6}(2t_1^3 - 3tt_1^2 + t^3) + \frac{c\theta_1\theta_2}{12}(3t_1^4 - 4tt_1^3 + t^4) + \frac{d\theta_1\theta_2}{20}(4t_1^5 - 5tt_1^4 + t^5)\right],\ 0 \le t \le t_1. \tag{3.2}$$

and by using boundary condition $I(0) = R$, we get the maximum positive inventory

$$R = \left[\frac{at_1}{2}(2 + \theta_1\theta_2 t_1) + \frac{bt_1^2}{6}(3 + 2\theta_1\theta_2 t_1) + \frac{ct_1^3}{12}(4 + 3\theta_1\theta_2 t_1) + \frac{dt_1^4}{20}(5 + 4\theta_1\theta_2 t_1)\right]. \tag{3.3}$$

Partial Backlogging Model

The instantaneous inventory level $I_1(t)$ at any time 't' during the shortage period $[t_1, T]$ is represented by the governing differential equation

$$\frac{dI_1(t)}{dt} = -\frac{D_0}{1 + \delta(T - t)},\ t_1 \le t \le T. \tag{3.4}$$

The solution of above equation (3.4) with boundary condition $I_1(t_1) = 0$ is

$$I_1(t) = \frac{D_0}{\delta} \log\left[\frac{1 + \delta(T - t)}{1 + \delta(T - t_1)}\right],\ t_1 \le t \le T. \tag{3.5}$$

Using boundary condition $-I_1(T) = P$, we get the negative inventory

$$-I_1(T) = P = \frac{D_0}{\delta} \log\{1 + \delta(T - t_1)\}. \tag{3.6}$$

Total inventory, $Q = R + P$

$$\implies Q = \left[\frac{at_1}{2}(2 + \theta_1\theta_2 t_1) + \frac{bt_1^2}{6}(3 + 2\theta_1\theta_2 t_1) + \frac{ct_1^3}{12}(4 + 3\theta_1\theta_2 t_1) + \frac{dt_1^4}{20}(5 + 4\theta_1\theta_2 t_1) + \frac{D_0}{\delta} \log\{1 + \delta(T - t_1)\}\right]. \tag{3.7}$$

The total average inventory cost (TC) per cycle consists of the following costs:

(i) Ordering cost per cycle:

$$OC = \frac{A}{T}. \tag{3.8}$$

(ii) Holding cost per cycle:

(3.9)

$$HC = \frac{1}{T}\int_0^{t_1} (\alpha + \beta t) I(t) dt$$
$$= \frac{\alpha}{T}\left\{\frac{at_1^2}{6}(3 + \theta_1\theta_2 t_1) + \frac{bt_1^3}{24}(8 + 3\theta_1\theta_2 t_1) + \frac{ct_1^4}{20}(5 + 2\theta_1\theta_2 t_1) + \frac{dt_1^5}{60}(12 + 5\theta_1\theta_2 t_1)\right\}$$

$$+\frac{\beta}{T}\left\{\frac{at_1^3}{24}(4+\theta_1\theta_2 t_1)+\frac{bt_1^4}{120}(15+4\theta_1\theta_2 t_1)+\frac{ct_1^5}{180}(18+5\theta_1\theta_2 t_1)+\frac{dt_1^6}{84}(7+2\theta_1\theta_2 t_1)\right\}.$$

(iii) Shortage cost per cycle:

$$SC = \frac{C_b}{T}\int_{t_1}^{T}[-I_1(t)]\,dt = \frac{C_b D_0}{\delta^2 T}\left[\delta(T-t_1)-\log\{1+\delta(T-t_1)\}\right]. \tag{3.10}$$

(iv) Perishable cost per cycle:

$$PC = \frac{C_p}{T}\left\{R-\int_0^{t_1} D(t)\,dt\right\} = \frac{C_p}{T}\left\{\frac{a}{2}(\theta_1\theta_2 t_1^2)+\frac{b}{3}(\theta_1\theta_2 t_1^3)+\frac{c}{4}(\theta_1\theta_2 t_1^4)+\frac{d}{5}(\theta_1\theta_2 t_1^5)\right\}. \tag{3.11}$$

(v) Cost due to lost sales per cycle:

$$CLS = \frac{C_l D_0}{T}\int_{t_1}^{T}\left[1-\frac{1}{1+\delta(T-t)}\right]dt = \frac{C_l D_0}{T}\left[T-t_1-\frac{1}{\delta}\log\{1+\delta(T-t_1)\}\right]. \tag{3.12}$$

(vi) Interest earned per cycle: $(t_1 < M)$

$$I_nE = \frac{pI_e}{T}\int_0^{M} tD(t)\,dt = \frac{pI_e}{T}\left[\frac{at_1^2}{2}+\frac{bt_1^3}{3}+\frac{ct_1^4}{4}+\frac{dt_1^5}{5}+\frac{D_0}{2}(M^2-t_1^2)\right]. \tag{3.13}$$

(vii) Interest payable per cycle: $(t_1 \geq M)$

$$\begin{aligned} I_nP_1 &= \frac{C_pI_p}{T}\int_M^{t_1} I(t)\,dt \\ &= \frac{C_pI_p}{T}\Big[\frac{a}{2}(t_1^2-2Mt_1+M^2)+\frac{b}{6}(2t_1^3-3Mt_1^2+M^3) \\ &\quad+\frac{c}{12}(3t_1^4-4Mt_1^3+M^4)+\frac{d}{20}(4t_1^5-5Mt_1^4+M^5) \\ &\quad+\frac{a\theta_1\theta_2}{6}(t_1^3-3Mt_1^2+3M^2t_1-M^3) \\ &\quad+\frac{b\theta_1\theta_2}{24}(3t_1^4-8Mt_1^3+6M^2t_1^2-M^4) \\ &\quad+\frac{c\theta_1\theta_2}{60}(6t_1^5-15Mt_1^4+10M^2t_1^3-M^5) \\ &\quad+\frac{d\theta_1\theta_2}{120}(10t_1^6-24Mt_1^5+15M^2t_1^4-M^6)\Big]. \end{aligned} \tag{3.14}$$

Total average inventory cost per cycle,

$$TC = OC + HC + PC + SC + CLS + I_nP_1 - I_nE. \tag{3.15}$$

$$TC = \frac{A}{T}+\frac{\alpha}{T}\left\{\frac{at_1^2}{6}(3+\theta_1\theta_2 t_1)+\frac{bt_1^3}{24}(8+3\theta_1\theta_2 t_1)+\frac{ct_1^4}{20}(5+2\theta_1\theta_2 t_1)+\frac{dt_1^5}{60}(12+5\theta_1\theta_2 t_1)\right\} \tag{3.16}$$

$$+\frac{\beta}{T}\left\{\frac{at_1^3}{24}(4+\theta_1\theta_2 t_1)+\frac{bt_1^4}{120}(15+4\theta_1\theta_2 t_1)+\frac{ct_1^5}{180}(18+5\theta_1\theta_2 t_1)+\frac{dt_1^6}{84}(7+2\theta_1\theta_2 t_1)\right\}$$
$$+\frac{C_p}{T}\left\{\frac{a}{2}(\theta_1\theta_2 t_1^2)+\frac{b}{3}(\theta_1\theta_2 t_1^3)+\frac{c}{4}(\theta_1\theta_2 t_1^4)+\frac{d}{5}(\theta_1\theta_2 t_1^5)\right\}$$
$$+\frac{C_b D_0}{\delta^2 T}\left[\delta(T-t_1)-\log\{1+\delta(T-t_1)\}\right]$$
$$+\frac{C_l D_0}{T}\left[T-t_1-\frac{1}{\delta}\log\{1+\delta(T-t_1)\}\right]$$
$$+\frac{C_p I_p}{T}\left[\frac{a}{2}(t_1^2-2Mt_1+M^2)+\frac{b}{6}(2t_1^3-3Mt_1^2+M^3)\right.$$
$$+\frac{c}{12}(3t_1^4-4Mt_1^3+M^4)+\frac{d}{20}(4t_1^5-5Mt_1^4+M^5)$$
$$+\frac{a\theta_1\theta_2}{6}(t_1^3-3Mt_1^2+3M^2t_1-M^3)$$
$$+\frac{b\theta_1\theta_2}{24}(3t_1^4-8Mt_1^3+6M^2t_1^2-M^4)$$
$$+\frac{c\theta_1\theta_2}{60}(6t_1^5-15Mt_1^4+10M^2t_1^3-M^5)$$
$$\left.+\frac{d\theta_1\theta_2}{120}(10t_1^6-24Mt_1^5+15M^2t_1^4-M^6)\right]$$
$$-\frac{pI_e}{T}\left[\frac{at_1^2}{2}+\frac{bt_1^3}{3}+\frac{ct_1^4}{4}+\frac{dt_1^5}{5}+\frac{D_0}{2}(M^2-t_1^2)\right].$$

Our objective is to minimize the total average inventory cost per cycle. Firstly we consider the derivative of TC with respect to the decision variable t_1 i.e. $\frac{d(TC)}{dt_1}$. Setting the derivative equal to zero, we have, $\frac{d(TC)}{dt_1}=0$.

Secondly we consider the second order derivative of TC with respect to the decision variable t_1. Provided that TC satisfies the following condition: $\frac{d^2(TC)}{dt_1^2}>0$.

By solving above non-linear equation by MATLAB software, the value of t_1^* can be obtained and then from Eqns. (3.7) and (3.16), the optimal values of Q^* and TC^* can be found out respectively. Assume suitable values for A, T, a, b, c, d, α, β, p, θ_1, θ_2, D_0, δ, M, C_b, C_l, C_p, I_e and I_p with appropriate units.

4 Numerical Example

Suppose that there is a product of pareto type decreasing function $\theta(t)=\left(\frac{\theta_1\theta_2}{1+\theta_2 t}\right)$, where $\theta_1>0$ and $0<\theta_2<1$. The parameters of the inventory system are $A=1000$, $T=2$, $a=10$, $b=12$, $c=15$, $d=20$, $\alpha=5$, $\beta=0.5$, $p=50$, $\theta_1=0.2$, $\theta_2=0.4$, $D_0=0.5$, $\delta=0.5$, $M=0.5$, $C_b=2$, $C_l=4$, $C_p=5$, $I_e=0.12$, $I_p=0.15$.

Under the above given parameters we obtain the optimum solutions $t_1^*=0.7087$, $TC^*=499.7216$ and $Q^*=14.0885$.

5 Sensitivity Analysis

On the basis of the data given in above example the sensitivity analysis is studied by changing the values of parameters a, b, c, d, θ_1, θ_2, δ, α, β and M by +50%, +25%, −25% and −50% and supervising the halting parameters at their original values

***Table* 5.1:** Sensitivity Analysis of Partial Backlogging Inventory Model

Changing parameter	Changing %	Optimal values		
		t_1^*	TC^*	Q^*
a	-50 %	0.7003	499.9223	10.1999
	-25 %	0.7049	499.8220	12.1445
	+25 %	0.7120	499.6210	16.0322
	+50 %	0.7148	499.5202	17.9756
b	-50 %	0.7019	499.8175	12.3190
	-25 %	0.7055	499.7696	13.2034
	+25 %	0.7115	499.6734	14.9743
	+50 %	0.7140	499.6251	15.8605
c	-50 %	0.7028	499.7771	12.9799
	-25 %	0.7059	499.7494	13.5335
	+25 %	0.7112	499.6937	14.6447
	+50 %	0.7135	499.6657	15.2018
d	-50 %	0.7032	499.7589	13.2595
	-25 %	0.7061	499.7403	13.6730
	+25 %	0.7111	499.7028	14.5057
	+50 %	0.7133	499.6839	14.9241
δ	-50 %	0.6960	499.5806	13.7148
	-25 %	0.7027	499.6524	13.9103
	+25 %	0.7147	499.7945	14.2725
	+50 %	0.7212	499.8772	14.4804
θ_1	-50 %	1.0409	498.3046	29.5772
	-25 %	0.8457	499.2445	19.3630
	+25 %	0.6091	500.0015	11.0248
	+50 %	0.5341	500.1836	9.0719
θ_2	-50 %	1.0409	498.3046	29.5772
	-25 %	0.8457	499.2445	19.3630
	+25 %	0.6091	500.0015	11.0248
	+50 %	0.5341	500.1836	9.0719
α	-50 %	2.0000	262.7969	183.1858
	-25 %	2.0000	422.3803	183.1858
	+25 %	0.2546	500.7201	3.6986
	+50 %	0.1526	500.8670	2.3513
β	-50 %	0.8028	499.4838	17.6810
	-25 %	0.7508	499.6180	15.6120
	+25 %	0.6737	499.8046	12.9142
	+50 %	0.6440	499.8731	11.9795
M	-50 %	0.4589	500.3020	7.2639
	-25 %	0.5895	500.0489	10.4045
	+25 %	0.8173	499.3249	18.2999
	+50 %	0.9167	498.8731	23.0253

6 Results and Observations

Graphical representations and effect of different parameters on t_1^*, TC^*, Q^* is as follows:

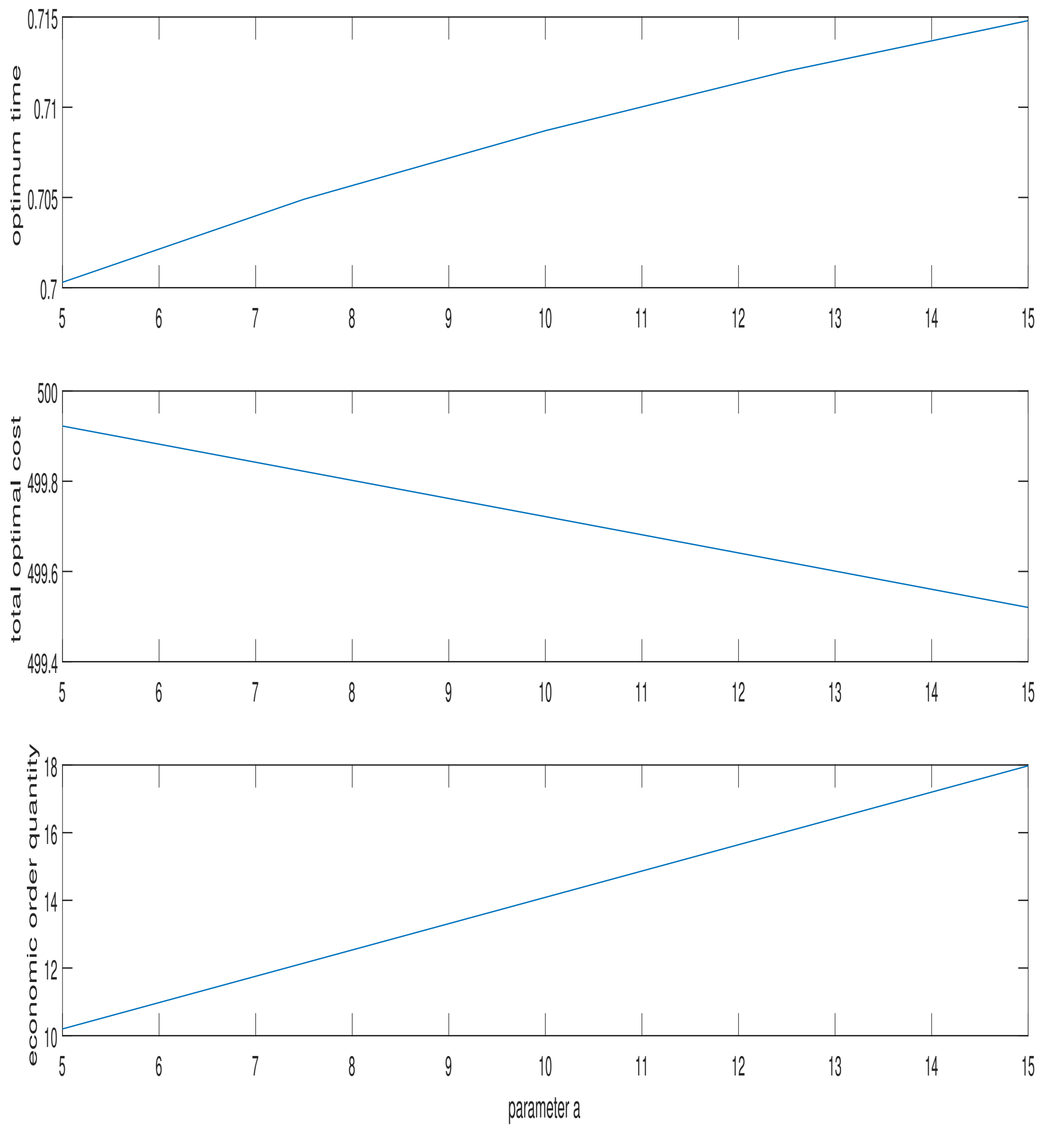


***Figure* 6.1:** graph of percentage change in optimum time, total optimal cost and economic order quantity with respect to parameter a

From above graph we observe that the values of t_1^* and Q^* increase linearly and TC^* decreases linearly with respect to the parameter a.

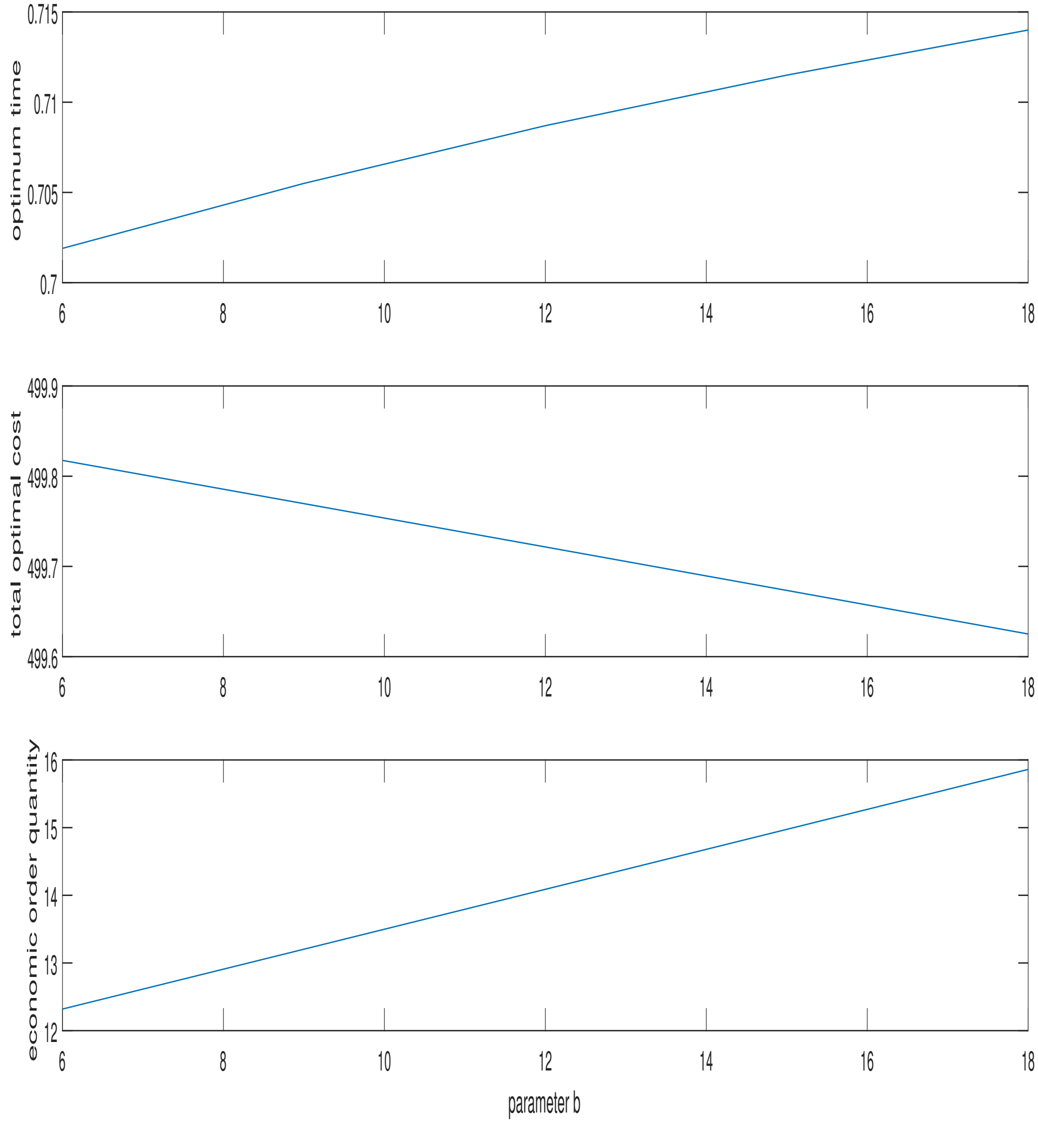


***Figure* 6.2:** graph of percentage change in optimum time, total optimal cost and economic order quantity with respect to parameter *b*

From above graph we observe that the values of t_1^* and Q^* increase linearly and TC^* decreases linearly with respect to the parameter *b*.

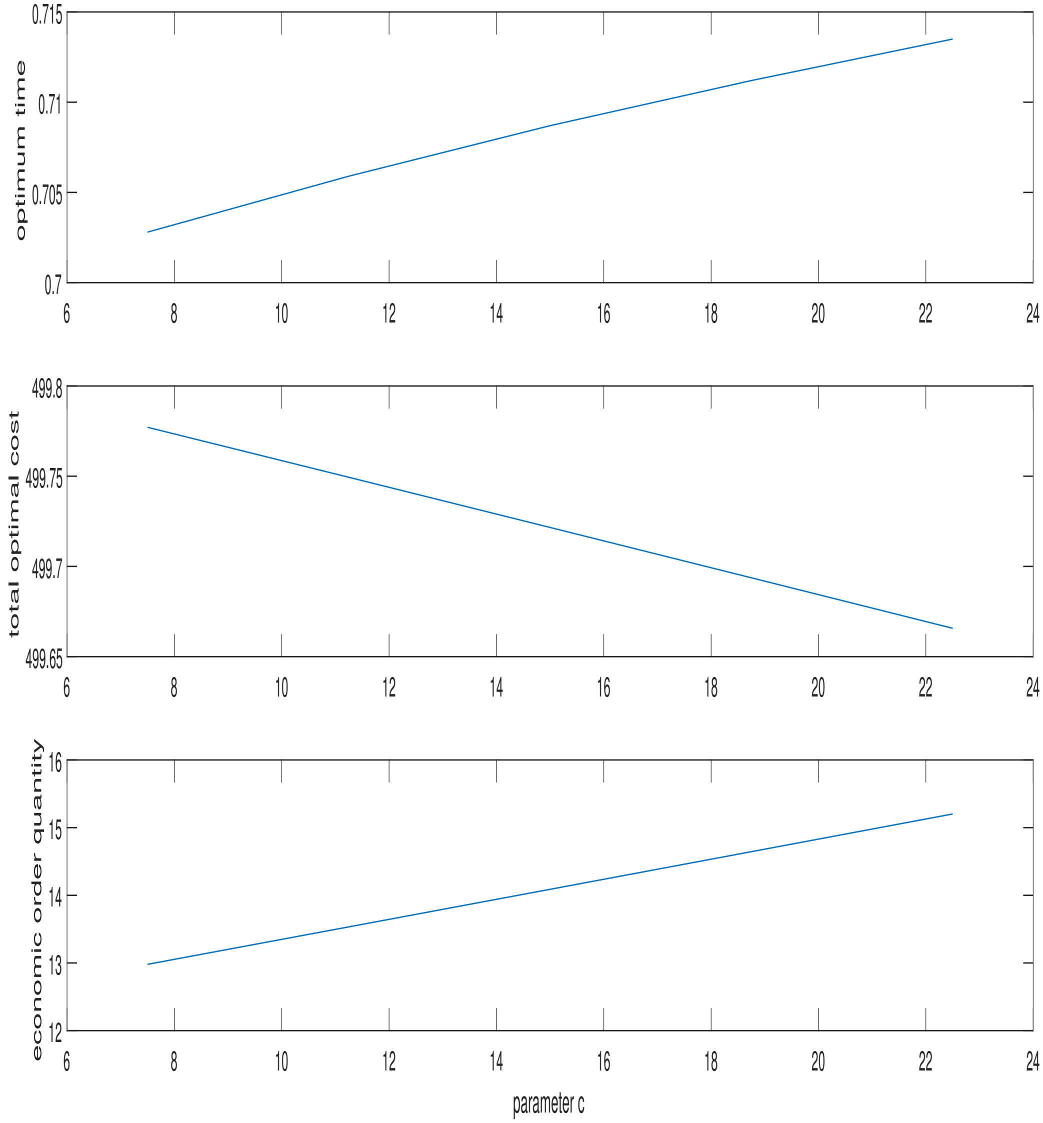


***Figure* 6.3:** graph of percentage change in optimum time, total optimal cost and economic order quantity with respect to parameter *c*

From above graph we observe that the values of t_1^* and Q^* increase linearly and TC^* decreases linearly with respect to the parameter c.

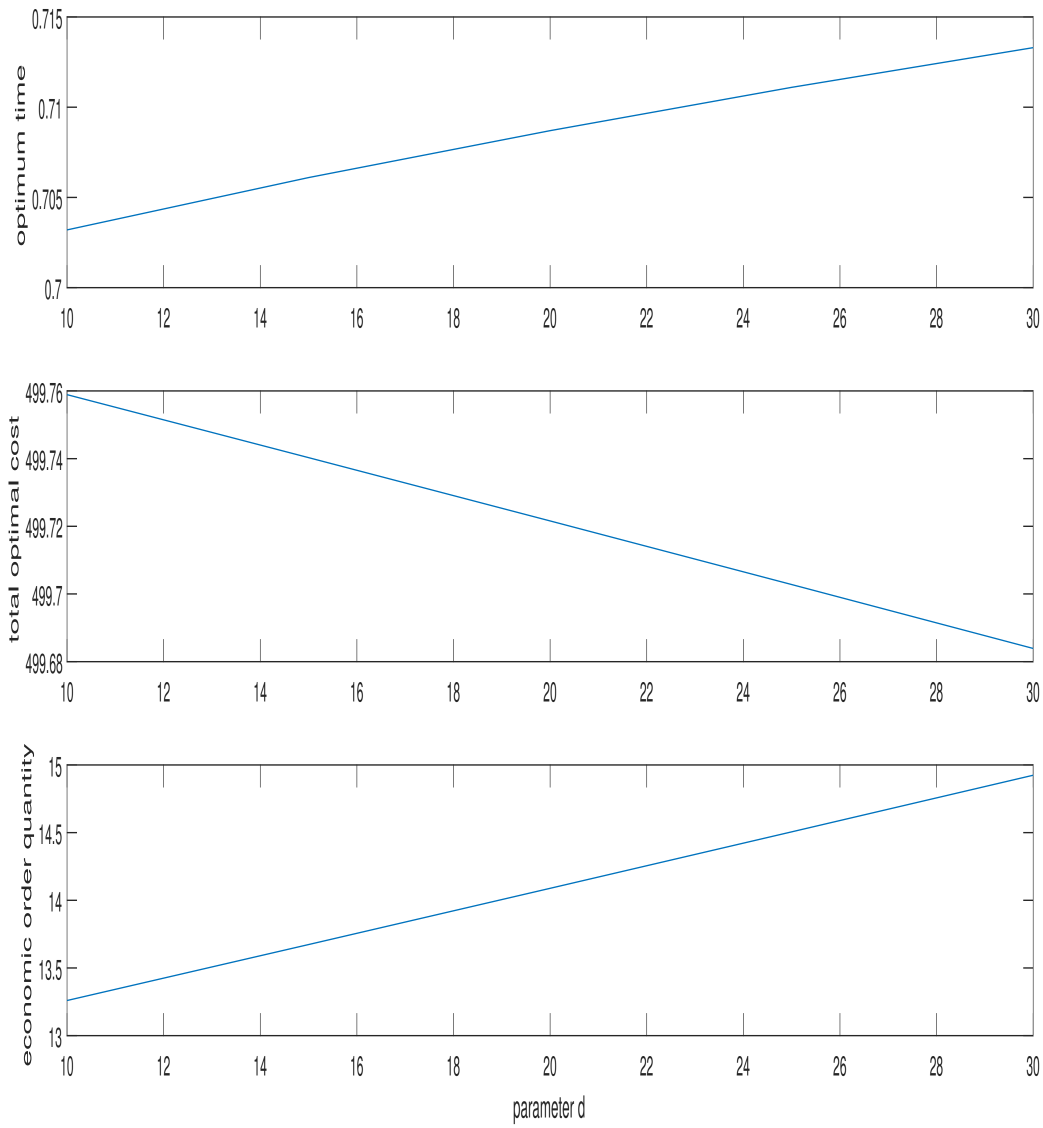


***Figure* 6.4:** graph of percentage change in optimum time, total optimal cost and economic order quantity with respect to parameter *d*

From above graph we observe that the values of t_1^* and Q^* increase linearly and TC^* decreases linearly with respect to the parameter d.

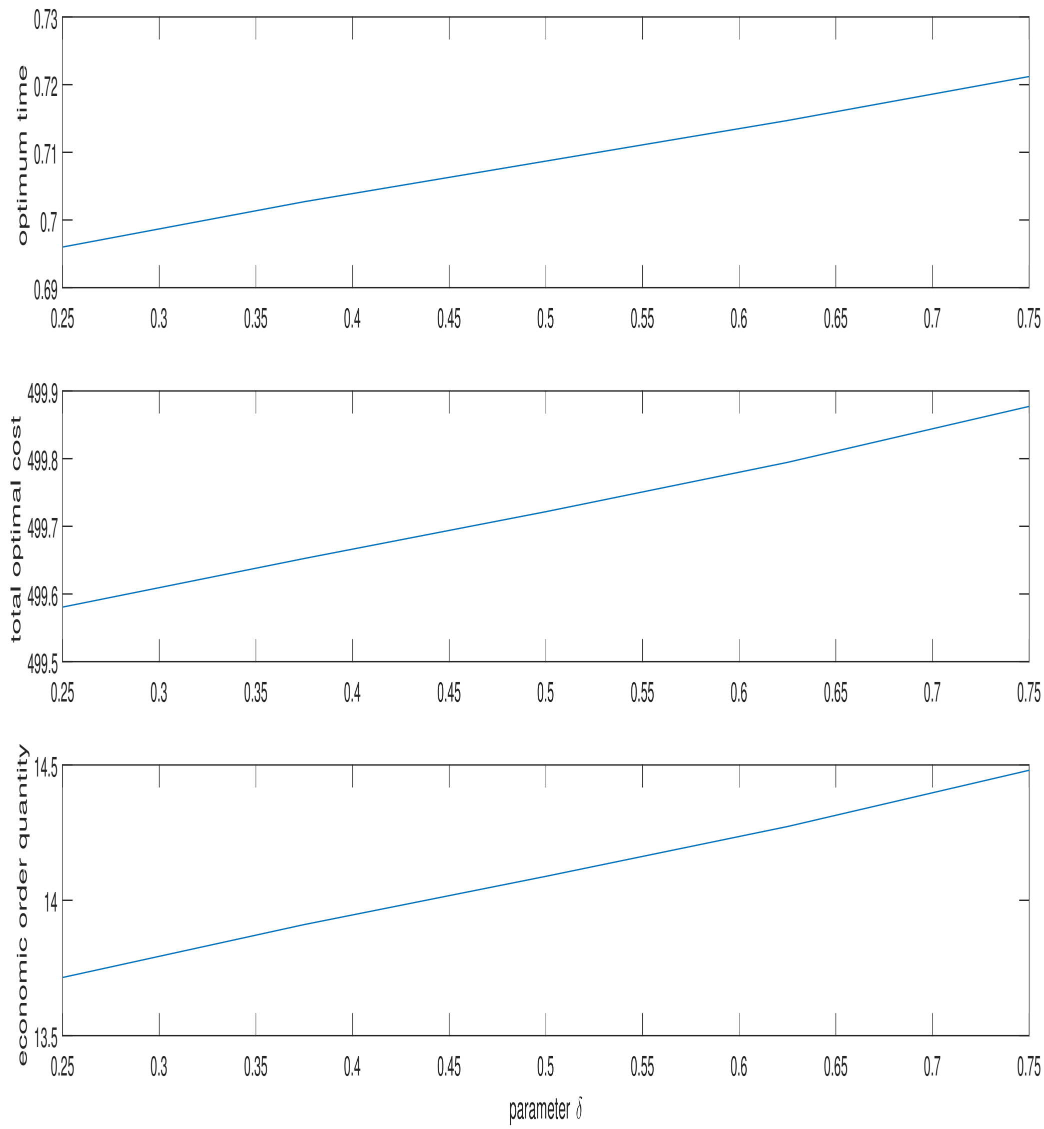


***Figure* 6.5:** graph of percentage change in optimum time, total optimal cost and economic order quantity with respect to parameter δ

From above graph we observe that the values of t_1^*, Q^* and TC^* increase linearly with respect to the parameter δ.

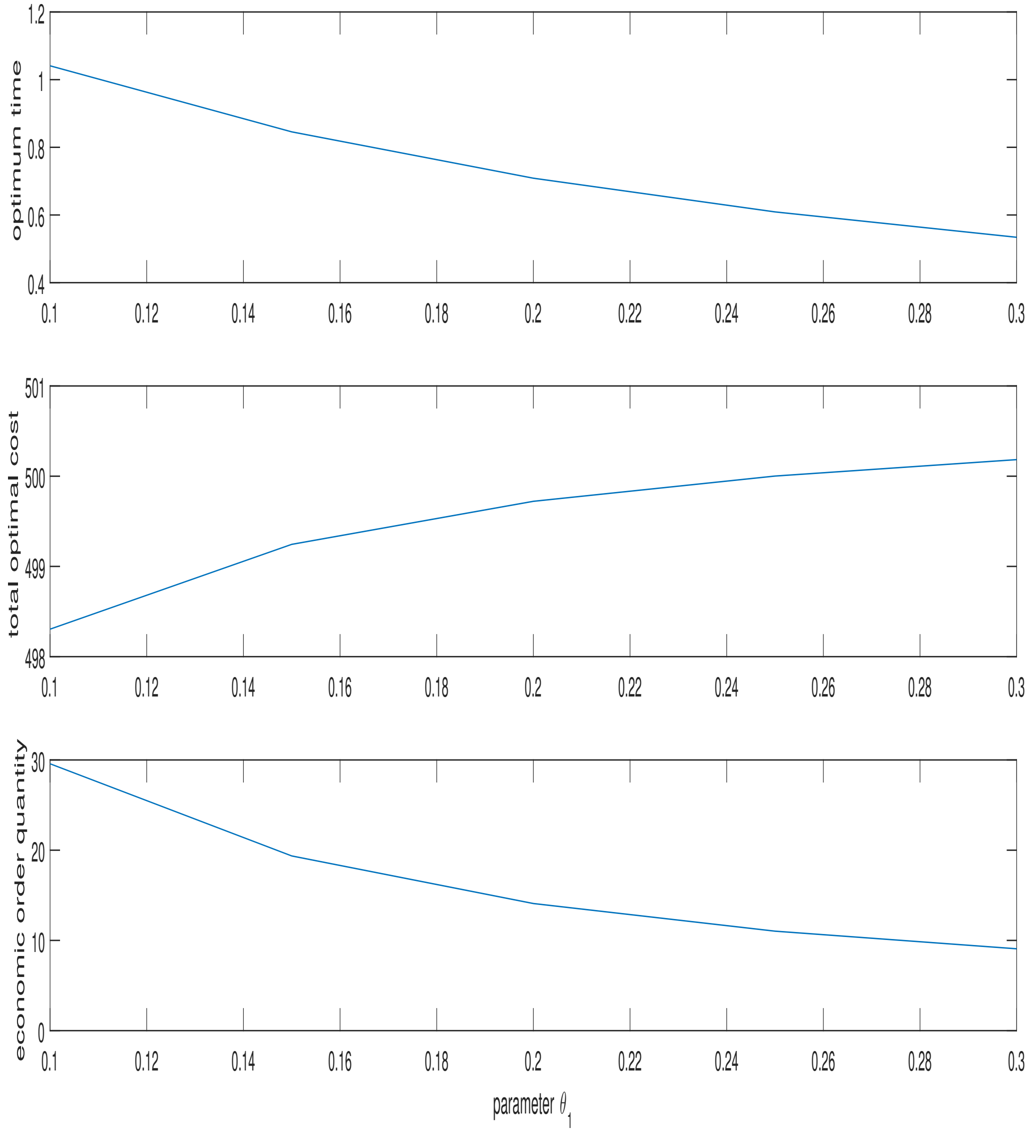


***Figure* 6.6:** graph of percentage change in optimum time, total optimal cost and economic order quantity with respect to parameter θ_1

From above graph we observe that on adjusting the percentage value of the parameter θ_1, t_1^* decreases approximately linear, TC^* increases initially and after a certain time it goes flat and Q^* decreases initially and after a certain time it goes flat.

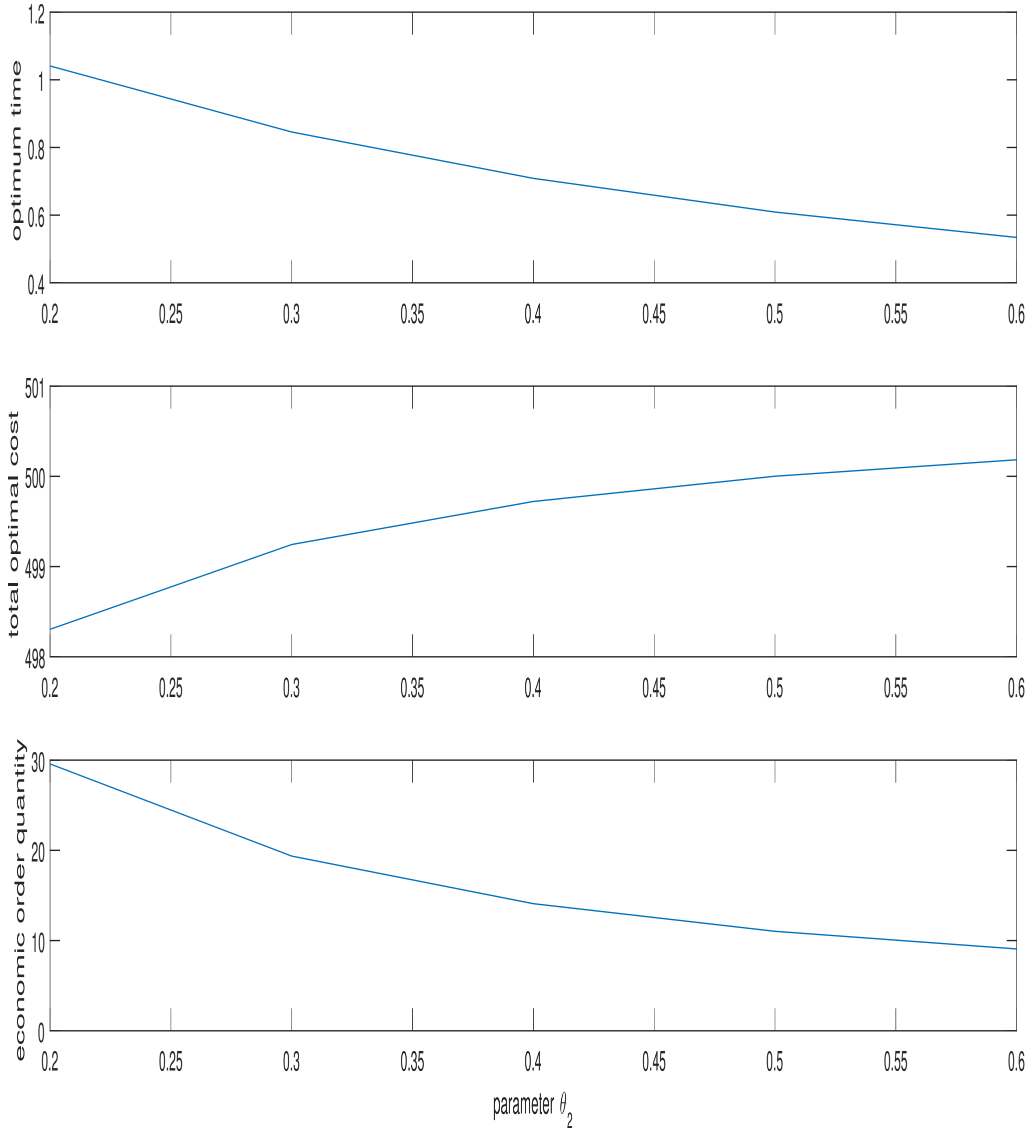


***Figure* 6.7:** graph of percentage change in optimum time, total optimal cost and economic order quantity with respect to parameter θ_2

From above graph we observe that on adjusting the percentage value of the parameter θ_2, t_1^* decreases approximately linear, TC^* increases initially and after a certain time it goes flat and Q^* decreases initially and after a certain time it goes flat.

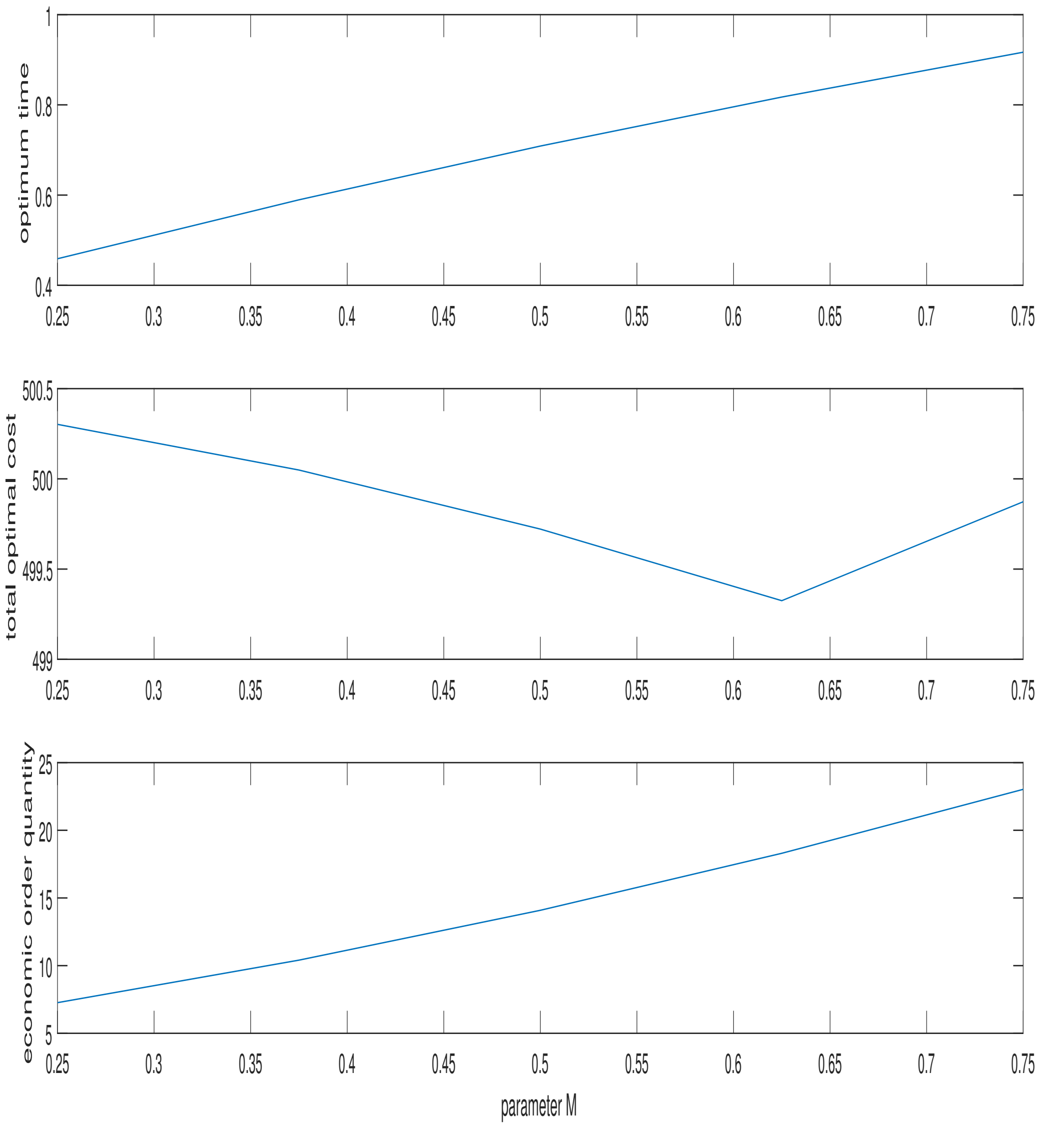


Figure* 6.8:** graph of percentage change in optimum time, total optimal cost and economic order quantity with respect to parameter ***M

From above graph we observe that the values of t_1^* and Q^* increase linearly with respect to the parameter M. TC^* decreases linearly and then suddenly it increases with respect to the parameter M.

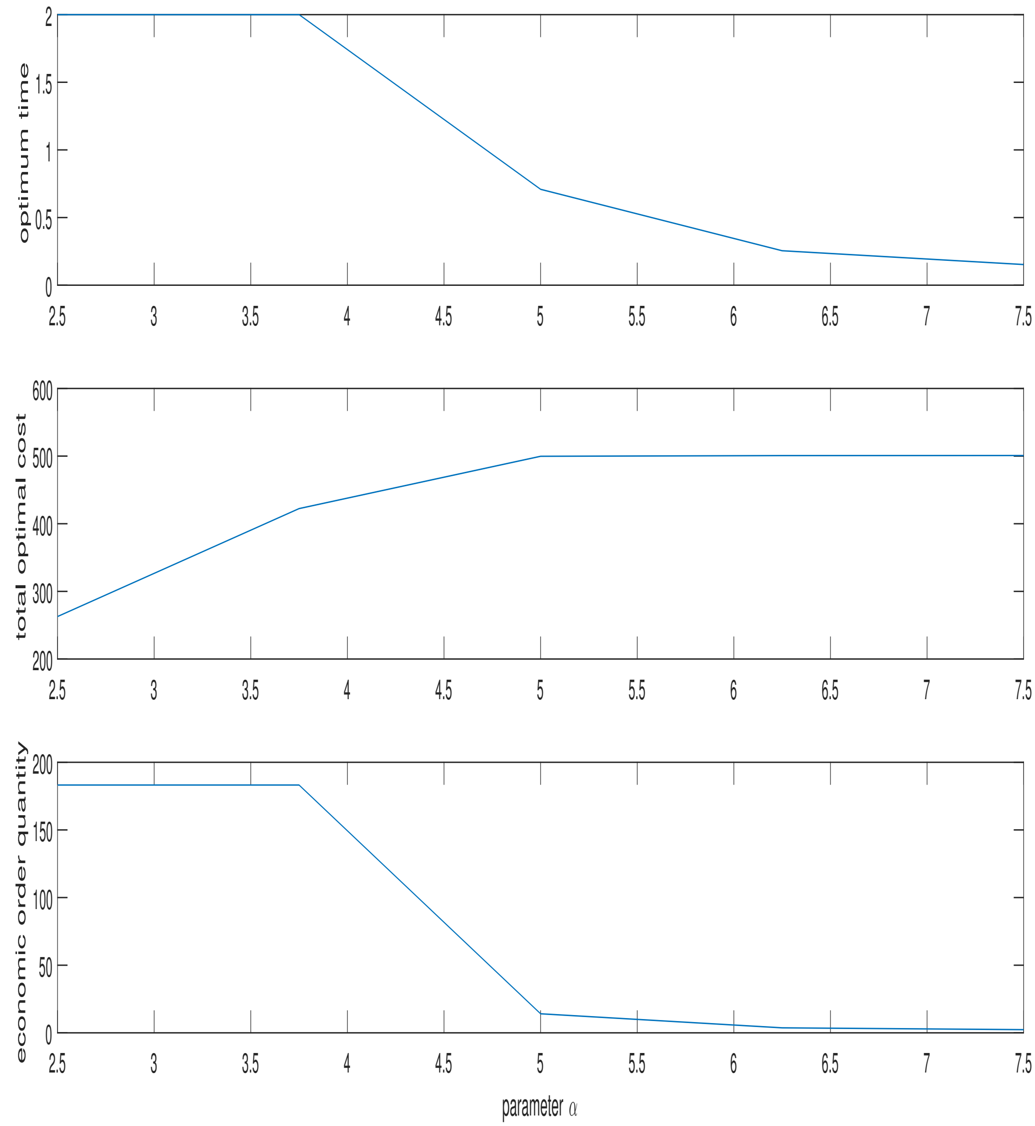


***Figure* 6.9:** graph of percentage change in optimum time, total optimal cost and economic order quantity with respect to parameter α

From above graph we observe that t_1^* decreases with non-constant slop and TC^* increases with non-constant slope with respect to the parameter α. Q^* with respect to α initially remains constant, then decreases linearly and finally goes flat.

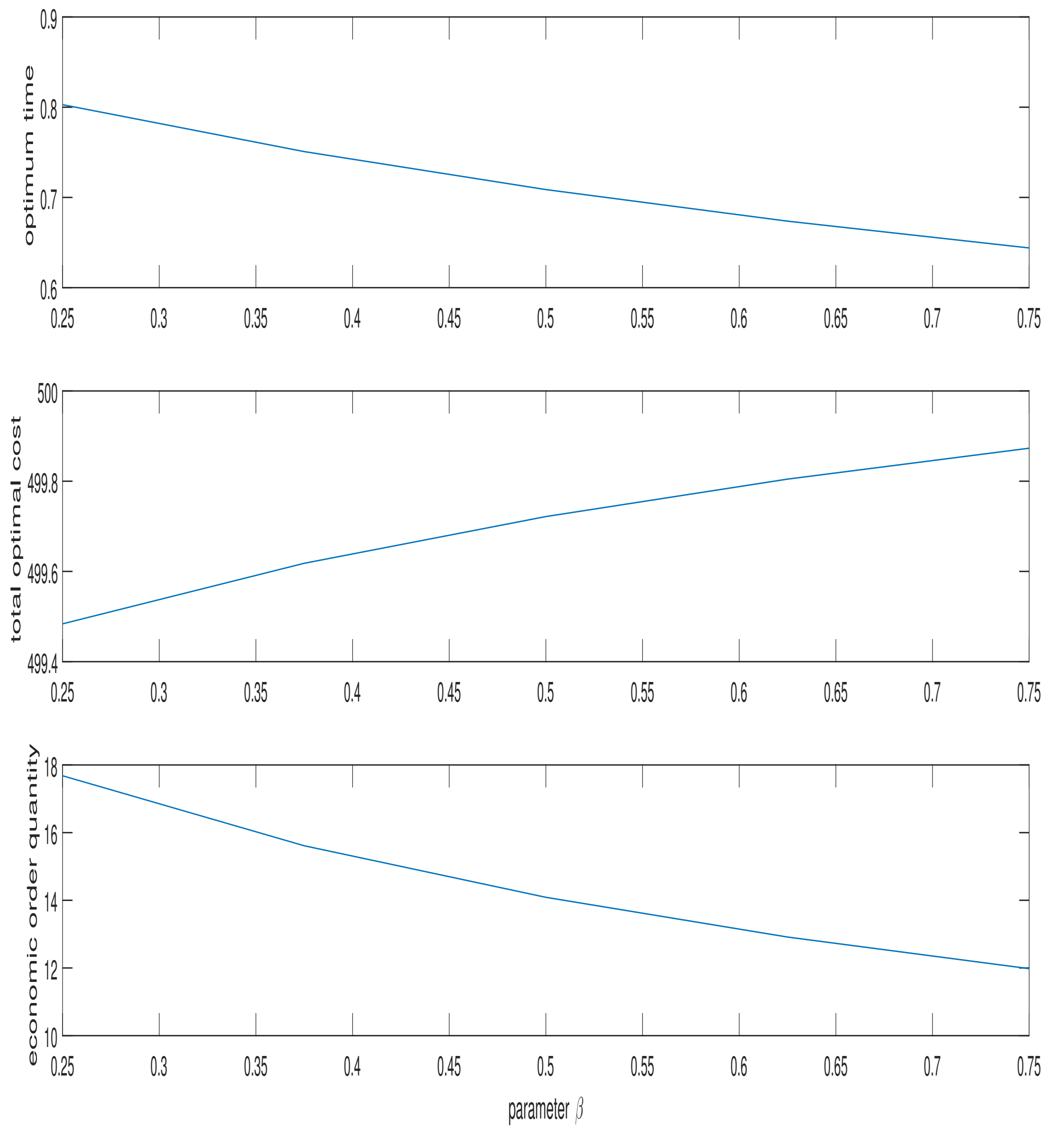


***Figure* 6.10:** graph of percentage change in optimum time, total optimal cost and economic order quantity with respect to parameter β

From above graph we observe that the values of t_1^* and Q^* increase linearly and TC^* decreases linearly with respect to the parameter β.

7 Conclusion

The suggested model offers an effective path for the management of a company enterprise where customer's demand rate is cubical polynomial time function. Pareto type perishable rate is taken. Delay in payment is allowed and shortages are taken partially backlogged. The model is solved analytically by minimizing the total inventory cost. Numerical example of the parameters is also presented to illustrate the model. By sensitivity analysis the decision maker can plan for the optimal value for total cost and for other related parameters. The proposed model can further be extended by taking more realistic assumptions such as probabilistic demand rate, finite replenishment rate etc.

Acknowledgement
Authors would like to thank the editor and anonymous reviewers for their valuable comments that improved the quality of this paper.

References

[1] S. P. Aggarwal and C. K. Jaggi, Ordering Policies of Deteriorating Items under Conditions of Permissible Delay in Payments, *J. Operat. Res.Soc.*, 46 (1995), 658-662.

[2] R. Amutha and E. Chandrasekaran, An Inventory Model for Constant Demand with Shortages under Permissible Delay in Payments, *IOSR Journal of Mathematics*, **6(5)** (2013), 28-33.

[3] B. Bhaula, J. K. Dash and M. R. Kumar, An Optimal Inventory Model for Perishable Items under Succesive Price Discounts with Permissible Delay in Payments, *Operational Research Society of India*, **56** (2019), 261-281.

[4] H. J. Chang and C. Y. Dye, An Inventorymodel for Deteriorating Items with Partial Backlogging and Permissible Delay in Payments, *International Journal of Systems Science*, **32(3)** (2001), 345-352.

[5] L. H. Chen and F. S. Kang, Integrated Inventory Models Considering Permissible Delay in Payments and Variant Pricing Strategy, *Applied Mathematical Modelling*, **34(1)** (2010), 36-46.

[6] K. J. Chung, The Inventory replenishment Policy for Deteriorating Items under Permissible Delay in Payments, *OPSEARCH*, **37** (2000), 267-281.

[7] K. J. Chung, Y. F. Huang and C. K. Huang, The Replenishment Decision for EOQ Inventory Model under Permissible Delay in Payments, *OPSEARCH*, **39(5-6)** (2002), 327-340.

[8] C. K. Jaggi, S. Pareek, S. K. Goel and Nidhi, An Inventory Model for Deteriorating Items with Ramp Type Demand under Fuzzy Environment, *International Journal of Logistics Systems and Management*, **22(4)** (2015), 436-463.

[9] S. K. Goyal, Economic Order Quantity under Conditions of Permissible Delay in Payments, *J. Operat. Res.Soc.*, **36** (1985), 335-338.

[10] Y. F. Huang, Optimal Retailers Ordering Policies in the EOQ Model under Trade Credit Financing, *Journal of the Operational Research Society*, **54** (2003), 1011-1015.

[11] Y. F. Huang, Retailers Inventory Policy under Supplier's Partial Trade Credit Policy, *Journal of the Operations Research Society of Japan*, **48(3)** (2005), 173-182.

[12] Y. F. Huang and K. J. Chung, Optimal Replenishment and Payment Policies in the EOQ Model under Cash Discount and Trade Credit, *Asia-Pacific Journal of Operational Research*, **20** (2003), 177-190.

[13] H. Hwang and S. W. Shinn, Retailers Pricing and Lot Sizing Policy for Exponentially Deteriorating Product under Condition of Permissible Delay in Payments, *Computer & Operations Research*, **24** (1997), 539-547.

[14] M. Jain, G. C. Sharma and V. Rani, Cost Analysis for a Supplier in an Inflationary Environment with Stock Dependent Demand Rate for Perishable Items, *Advances in Decision Sciences*, ID457276 (2014), 1-10.

[15] A. M. Jamal, B. R. Sarker and S. Wang, An Ordering Policy for Deteriorating Items with Allowable Shortage and Permissible Delay in Payments, *Journal of the Operational Research Society*, **48** (1997), 826-833.

[16] A. M. Jamal,B. R. Sarker and S. Wang, Optimal Payment Time for a Retailer under Permitted Delay of Payment by the Wholesaler, *International Journal of Production Economics*, **66** (2000), 59-66.

[17] A. Kansal and D. Prasad, An Inventory Model for Deteriorating Items with Linearly Time Varying Demand Rate and Shortages under Inflation, *International Journal of Applied Mathematics and Applications*, **2(2)** (2010), 207-216.

[18] H.C. Liao, C. H. Tsai and C. T. Su, An Inventory Model for Deteriorating Items under Inflation When a Delay in Payment is Permissible, *International Journal of Production Economics*, **63** (2001), 207-214.

[19] D. J. Mohanty, R. S. Kumar and A. Goswami, An Improved Inventory Model with Random Review Period and Temporary Price Discount for Deteriorating Items, *International Journal Syst. Assur. Eng. Manag.*, **7(1)** (2016), 62-72.

[20] M. Pal and S. K. Ghosh, An Inventory Model with Shortage and Quantity Dependent Permissible Delay in Payment, *Aust. Soc. Operat. Res. Bull.*, **25(3)** (2006), 1-12.

[21] K. Rangarajan and K. Karthikeyan, Analysis of an EOQ Inventory Model for Instantaneous Deteriorating Items with Various Time Dependent Demand Rates, *International Journal of Pure and Applied Mathematics*, **106(7)** (2016), 77-86.

[22] M. K. Salameh, N .E. Abboud, A. N. El-Kassar and R. E. Ghattas, Continuous Review Inventory Model with Delay in Payments, *International Journal of Production Economics*, **85(1)** (2003), 91-95.

[23] B. Sarkar, An EOQ Model with Delay in Payments and Time Varying Deterioration Rate, *Mathematical and Computer Modelling*, **55(3-4)** (2012), 367-377.

[24] N. H. Shah, A Probabilistic Order Level System When Delay in Payments is Permissible, *Journal of the Korean Operations Research and Management Science*, **18(2)** (1993), 175-183.

[25] N. H. Shah and Y. K. Shah, A Discrete-in-Time Probabilistic Inventory Model for Deteriorating Items under Conditions of Permissible Delay in Payments, *Inter. J.Syst. Sci.*, **29** (1998), 121-126.

[26] S. C. Sharma and V. Vijay, Replenishment Policy for Deteriorating Inventory System with Power Demand and Backlogging, *Jñānābha*, **47(1)** (2017), 7-16.

[27] S. Singh, Inventory Model for Single Item-Multi Supplier with Shortages, *Jñānābha*, **48(2)** (2018), 71-80.

[28] S. Singh, S. Sharma and S. R. Singh, Inventory Model for Deteriorating Items with Incremental HOlding Cost under Partial Backlogging, *International Journal of Mathematics in Operations Research*, **15(1)** (2019), 110-126.

[29] J. T. Teng, On the Economic Order Quantity Under Conditions of Permissible Delay in Payments, *Journal of the Operational Research Society*, 53 (2002), 915-918.

[30] J.T. Teng, C.T. Chang and S.K. Goyal, Optimal Pricing and Ordering Policy under Permissible Delay in Payments, *International Journal of Production Economics*, **97(2)** (2005), 121-129.

[31] J. T. Teng, H-L. Yang and L-Y Ouyang, On an EOQ Model for Deteriorating Items with Time-Varying Demand and Partial Backlogging, *Journal of the Operational Research Society*, **54** (2003), 432-436.

[32] R. Uthayakumar and K.V. Geetha, Replenishment Policy for Single Item Inventory Model with Money Inflation, *OPSEARCH*, **46(3)** (2009), 345-357.

Jñānābha, Vol. 50(1) (2020), 134-143
(Dedicated to Honor Professor H.M. Srivastava on His 80th Birth Anniversary Celebrations)

SUBCLASS OF MEROMORPHIC FUNCTIONS WITH POSITIVE COEFFICIENTS INVOLVING POLYLOGARITHM FUNCTION

By
Santosh M. Popade
Department of Mathematics, Sant Tukaram College of Arts & Science,
Parbhani - 431 401, Maharastra, India.
smpopade2007@gmail.com
Rajkumar N. Ingle
Department of Mathematics, Bahirji Smarak Mahavidyalay,
Bashmathnagar - 431 512, Hingoli Dist., Maharastra, India.
Email:ingleraju11@gmail.com
P. Thirupathi Reddy
Department of Mathematics, Kakatiya University,
Warangal- 506 009, Telangana, India.
Email:reddypt2@gmail.com

(*Received : March* 19, 2019 ; *Revised: April* 13, 2020)
DOI: https://doi.org/10.58250/jnanabha.2020.50114

Abstract

In this paper, we introduce and study a new subclass of meromorphic functions with positive coefficients involving the polylogaritham function and obtain coefficient estimates, growth and distortion theorem, radius of convexity, integral transforms, convex linear combinations and convolution properties for the class $\sigma_{c,p}(\alpha, \beta, \lambda)$.

2010 Mathematics Subject Classifications: 30C45 .
Keywords and phrases: Meromorphic, Polylogarithm, Coefficient estimates.

1 Introduction

Historically,the classical polylogarithm function was invented in 1696, by Leibnitz and Bernoulli, as mentioned in [3]. For $|z| < 1$ and c a natural number with $c \geq 2$, the polylogarithm function (which is also known as Jonquiere's function) is defined by the absolutely convergent series:

$$Li_c(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^c}. \tag{1.1}$$

Later on, many mathematicians studied the polylogarithm function such as Euler, Spence, Abel, Lobachevsky, Rogers, Ramanujan and many others [6], where they discovered many functional identities by using polylogarithm function. However, the work employing polylogarithm has been stopped many decades later. During the past four decades, the work using polylogarithm has again been intensified vividly due to its importance in many fields of mathematics, such as complex analysis, algebra, geometry, topology, and mathematical physics (quantum field theory) [5, 7, 9]. In [10], Ponnusamy and Sabapathy discussed the geometric mapping properties of the generalized polylogarithm. Recently, Al-Shaqsi and Darus [1] generalized Ruscheweyh and Salagean operators, using polylogarithm functions on class A of analytic functions in the open unit disk $U = \{z : |z| < 1\}$. By making use of the generalized operator they introduced certain new subclasses of A and investigated many related polylogarithm function to define a multiplier transformation on the class A in U [2]. To the best of our knowledge, no research work has discussed the polylogarithm function conjunction with meromorphic functions. Thus, in this present paper, we redefine the polylogarithm function to be on meromorphic type. Let $\sum$ denote the class of functions of the form

$$f(z) = \frac{1}{z} + \sum_{m=0}^{\infty} a_m z^m, \tag{1.2}$$

which are analytic in the punctured open unit disk

$$U^* := \{z : z \in C,\ 0 < |z| < 1\} = U \setminus \{0\}. \tag{1.3}$$

A function f in $\sum$ is said to be meromorphically starlike of order δ if and only if

$$\Re\Big\{ -\frac{zf'(z)}{f(z)}\Big\} > \delta;\ (z \in U^*), \tag{1.4}$$

for some δ $(0 \le \delta < 1)$. We denote by $\sum(\delta)$ the class of all meromorphically starlike order δ. Furthermore, a function f in $\sum$ is said to be meromorphically convex of order δ if and only if

$$\Re\Big\{ -\Big(1 + \frac{zf''(z)}{f'(z)}\Big)\Big\} > \delta;\ (z \in U^*), \tag{1.5}$$

for some δ, $(0 \le \delta < 1)$. We denote by $\sum_k(\delta)$ the class of all meromorphically convex order δ. For functions $f \in \sum$ given by (1.2) and $g \in \sum$

$$g(z) = \frac{1}{z} + \sum_{m=0}^{\infty} b_m z^m, \tag{1.6}$$

we define the Hadamard product (or convolution) of f and g by

$$(f * g)(z) = \frac{1}{z} + \sum_{m=0}^{\infty} a_m b_m z^m. \tag{1.7}$$

Let $\sum_p$ be the class of functions of the form

$$f(z) = \frac{1}{z} + \sum_{m=0}^{\infty} a_m z^m;\ a_m \ge 0 \tag{1.8}$$

which are analytic and univalent in U^*. Liu and Srivastava [8] defined a function $h_p(\alpha_1, \cdots, \alpha_q; \beta_1, \cdots, \beta_s; z)$ by multiplying the well known generalized hypergeometric function ${}_qF_s$, with z^{-p} as follows:

$$h_p(\alpha_1, \cdots, \alpha_q; \beta_1, \cdots, \beta_s; z) = z^{-p}\ {}_qF_s(\alpha_1, \cdots, \alpha_q; \beta_1, \cdots, \beta_s; z), \tag{1.9}$$

where $\alpha_1, \cdots, \alpha_q; \beta_1, \cdots, \beta_s$ are complex parameters and $q \le s + 1$, $p \in N$. Analogous to Liu and Srivastava work [8] and corresponding to a function $\phi_c(z)$ given by

$$\phi_c(z) = z^{-2} Li_c(z) = \frac{1}{z} + \sum_{m=0}^{\infty} \frac{1}{(m+2)^c} z^m. \tag{1.10}$$

We consider a linear operator $\Omega_c f(z) : \sum \to \sum$ which is defined by the following Hadamard product (or Convolution):

$$\begin{aligned} \Omega_c f(z) &= \phi_c(z) * f(z) \\ &= \frac{1}{z} + \sum_{m=0}^{\infty} \frac{1}{(m+2)^c} a_m z^m. \end{aligned} \tag{1.11}$$

Next, we define the linear operator $\mathfrak{D}_c f(z) : \sum \to \sum$ as follows:

$$\begin{aligned} \mathfrak{D}_c f(z) &= \{\Omega_c f(z) - \frac{1}{2^c} a_0\} \\ &= \frac{1}{z} + \sum_{m=1}^{\infty} \frac{1}{(m+2)^c} a_m z^m. \end{aligned} \tag{1.12}$$

For function f in the class $\sum_p$, we define a linear operator $\mathfrak{D}^n_{c,\,\lambda} f(z)$ as follows

$$\begin{aligned} \mathfrak{D}^0_{c,\,\lambda} f(z) &= f(z), \\ \mathfrak{D}^1_{c,\,\lambda} f(z) &= (1-\lambda)\mathfrak{D}_c f(z) + \lambda \frac{(z^2 \mathfrak{D}_c f(z))'}{z}\ \lambda \geq 0, \\ &= (1+\lambda)\mathfrak{D}_c f(z) + \lambda z(\mathfrak{D}_c f(z)') = \mathfrak{D}_{c,\,\lambda} f(z), \\ \mathfrak{D}^2_{c,\,\lambda} f(z) &= \mathfrak{D}_{c,\,\lambda} f(z)(\mathfrak{D}^1_{c,\,\lambda} f(z)), \\ &\vdots \\ \mathfrak{D}^n_{c,\,\lambda} f(z) &= \mathfrak{D}_{c,\,\lambda} f(z)(\mathfrak{D}^{n-1}_{c,\,\lambda} f(z)), \\ &= \frac{1}{z} + \sum_{m=1}^{\infty} \frac{[1+\lambda(m+1)]^n}{(m+2)^c} a_m z^m \text{ for } n = 1, 2, \cdots . \end{aligned} \tag{1.13}$$

Now, by making use of operator $\mathfrak{D}^n_{c,\,\lambda} f(z)$, we define a new subclass of functions in $\sum_p$ as follows.

***Definition* 1.1.** *For* $-1 \leq \alpha < 1$, $\beta \geq 1$, *and* $\lambda \geq 0$ *we let* $\sigma_{c,p}(\alpha, \beta, \lambda)$ *be the subclass of* $\sum_p$ *consisting of functions of the form* (1.8) *and satisfying the analytic criterion*

$$-Re\left\{\frac{z(\mathfrak{D}^n_{c,\,\lambda} f(z))'}{\mathfrak{D}^n_{c,\,\lambda} f(z)} + \alpha\right\} > \beta \left|\frac{z(\mathfrak{D}^n_{c,\,\lambda} f(z))'}{\mathfrak{D}^n_{c,\,\lambda} f(z)} + 1\right|. \tag{1.14}$$

$\mathfrak{D}^n_{c,\,\lambda} f(z)$ *is given by* (1.13). *The main object of the paper is to study some usual properties of the geometric function theory such as coefficient bounds, growth and distortion properties, radius of convexity, convex linear combination and convolution properties, and integral operators for the class* $\sigma_{c,p}(\alpha, \beta, \lambda)$.

2 Coefficient inequality

***Theorem* 2.1.** *A function* f *of the form* (1.8) *is in* $\sigma_{c,p}(\alpha, \beta, \lambda)$ *if*

$$\sum_{m=1}^{\infty} \frac{[1+\lambda(m+1)]^n[(1+\beta)(m+1)+1-\alpha]}{(m+2)^c}|a_m| \leq 1-\alpha, \tag{2.1}$$

$-1 \leq \alpha < 1$, $\beta \geq 1$ *and* $\lambda \geq 0$.

Proof. It is sufficient to show that

$$\beta\left|\frac{z(\mathfrak{D}^n_{c,\,\lambda} f(z))'}{\mathfrak{D}^n_{c,\,\lambda} f(z)} + 1\right| + Re\left\{\frac{z(\mathfrak{D}^n_{c,\,\lambda} f(z))'}{\mathfrak{D}^n_{c,\,\lambda} f(z)} + 1\right\} \leq 1-\alpha.$$

We have

$$\beta\left|\frac{z(\mathfrak{D}^n_{c,\,\lambda} f(z))'}{\mathfrak{D}^n_{c,\,\lambda} f(z)} + 1\right| + Re\left\{\frac{z(\mathfrak{D}^n_{c,\,\lambda} f(z))'}{\mathfrak{D}^n_{c,\,\lambda} f(z)} + 1\right\}$$

$$\leq (1+\beta)\Big|\frac{z(\mathfrak{D}^n_{c,\,\lambda} f(z))'}{\mathfrak{D}_c f(z)}+1\Big|$$

$$\leq \frac{(1+\beta)\sum\limits_{m=1}^{\infty}\frac{[1+\lambda(m+1)]^n}{(m+2)^c}(m+1)|a_m||z|^m}{\frac{1}{|z|}-\sum\limits_{m=1}^{\infty}\frac{[1+\lambda(m+1)]^n}{(m+2)^c}|a_m||z|^m}.$$

Letting $z \to 1$ along the real axis, we obtain

$$\leq \frac{(1+\beta)\sum\limits_{m=1}^{\infty}\frac{[1+\lambda(m+1)]^n}{(m+2)^c}(m+1)|a_m|}{1-\sum\limits_{m=1}^{\infty}\frac{[1+\lambda(m+1)]^n}{(m+2)^c}|a_m|}.$$

The last expression is bounded by $(1-\alpha)$ if

$$\sum_{m=1}^{\infty}\frac{[1+\lambda(m+1)]^n[(1+\beta)(m+1)+1-\alpha]}{(m+2)^c}|a_m| \leq 1-\alpha.$$

Hence the theorem is proved.

***Corollary* 2.1.** *Let the function f defined by* (1.8) *be in the class $\sigma_{c,p}(\alpha,\beta,\lambda)$. Then*

$$a_m \leq \sum_{m=1}^{\infty}\frac{(m+2)^c(1-\alpha)}{[1+\lambda(m+1)]^n[(1+\beta)(m+1)+1-\alpha]},\ (m\geq 1). \tag{2.2}$$

Equality holds for the functions of the form

$$f_m(z) = \frac{1}{z}+\sum_{m=1}^{\infty}\frac{(m+2)^c(1-\alpha)}{[1+\lambda(m+1)]^n[(1+\beta)(m+1)+1-\alpha]}z^m. \tag{2.3}$$

3 Distortion Theorems

***Theorem* 3.1.** *Let the function f defined by* (1.8) *be in the class $\sigma_{c,p}(\alpha,\beta,\lambda)$. Then for $0<|z|=r<1$,*

$$\frac{1}{r}-\frac{3^c(1-\alpha)}{[1+2\lambda]^n(3+2\beta-\alpha)}r \leq |f(z)| \leq \frac{1}{r}+\frac{3^c(1-\alpha)}{[1+2\lambda]^n(3+2\beta-\alpha)}r \tag{3.1}$$

with equality for the function

$$f(z) = \frac{1}{z}+\frac{3^c(1-\alpha)}{[1+2\lambda]^n(3+2\beta-\alpha)}z. \tag{3.2}$$

Proof. Suppose f is in $\sigma_{c,p}(\alpha,\beta,\lambda)$. In view of ***Theorem* 2.1**, we have

$$\frac{[1+2\lambda]^n(3+2\beta-\alpha)}{3^c}\sum_{m=1}^{\infty}a_m \leq \sum_{m=1}^{\infty}\frac{[(1+\beta)(m+1)+1-\alpha][1+\lambda(m+1)]^n}{(m+2)^c} \leq (1-\alpha)$$

which evidently yields

$$\sum_{m=1}^{\infty}a_m \leq \frac{3^c(1-\alpha)}{[1+2\lambda]^n(3+2\beta-\alpha)}.$$

Consequently, we obtain

$$|f(z)| = \Big|\frac{1}{z}+\sum_{m=1}^{\infty}a_m z^m\Big|$$

$$\leq |\frac{1}{z}| + \sum_{m=1}^{\infty} a_m |z|^m \leq \frac{1}{r} + r\sum_{m=1}^{\infty} a_m$$
$$\leq \frac{1}{r} + \frac{3^c(1-\alpha)}{[1+2\lambda]^n(3+2\beta-\alpha)} r.$$

Also

$$|f(z)| = \left|\frac{1}{z} + \sum_{m=1}^{\infty} a_m z^m\right|$$
$$\geq \left|\frac{1}{z}\right| - \sum_{m=1}^{\infty} a_m |z|^m$$
$$\geq \frac{1}{r} - r\sum_{m=1}^{\infty} a_m$$
$$\geq \frac{1}{r} - \frac{3^c(1-\alpha)}{[1+2\lambda]^n(3+2\beta-\alpha)} r.$$

Hence the results (3.1) follow.

***Theorem* 3.2.** *Let the function f defined by* (1.8) *be in the class $\sigma_{c,p}(\alpha, \beta, \lambda)$. Then for $0 < |z| = r < 1$,*

$$\frac{1}{r^2} - \frac{3^c(1-\alpha)}{[1+2\lambda]^n(3+2\beta-\alpha)} \leq |f'(z)| \leq \frac{1}{r^2} + \frac{3^c(1-\alpha)}{[1+2\lambda]^n(3+2\beta-\alpha)}.$$

The result is sharp, the extremal function being of the form (2.3).

Proof. From Theorem 2.1, we have

$$\frac{[1+2\lambda]^n(3+2\beta-\alpha)}{3^c}\sum_{m=1}^{\infty} m a_m \leq \sum_{m=1}^{\infty} \frac{[(1+\beta)(m+1)+1-\alpha][1+\lambda(m+1)]^n}{(m+2)^c} \leq (1-\alpha)$$

which evidently yields

$$\sum_{m=1}^{\infty} m a_m \leq \frac{3^c(1-\alpha)}{[1+2\lambda]^n(3+2\beta-\alpha)}$$

Consequently, we obtain

$$|f(z)| = \frac{1}{r^2} + \sum_{m=1}^{\infty} m a_m r^{m-1}$$
$$\leq \frac{1}{r^2} + \sum_{m=1}^{\infty} m a_m$$
$$\leq \frac{1}{r^2} + \frac{3^c(1-\alpha)}{[1+2\lambda]^n(3+2\beta-\alpha)}.$$

Also

$$|f(z)| \geq \frac{1}{r^2} - \sum_{m=1}^{\infty} m a_m r^{m-1}$$
$$\geq \frac{1}{r^2} - \sum_{m=1}^{\infty} m a_m$$
$$\geq \frac{1}{r^2} - \frac{3^c(1-\alpha)}{[1+2\lambda]^n(3+2\beta-\alpha)}.$$

This completes the proof.

4 Class Preserving Integral Operators

In this section we consider the class preserving integral operators of the form (1.8).

***Theorem* 4.1.** *Let the function* f *be defined by* (1.8) *be in the class* $\sigma_{c,p}(\alpha,\beta,\lambda)$. *Then*

$$F(z) = \mu z^{-\mu-1}\int_0^z t^{\mu} f(t)dt = \frac{1}{z} + \sum_{m=1}^{\infty}\frac{\mu}{\mu+m+1}a_m z^m,\ \mu > 0 \tag{4.1}$$

belongs to the class $\sigma[\delta(\alpha,\beta,\lambda,m,\mu)]$, *where*

$$\delta(\alpha,\beta,\lambda,m,\mu) = \frac{[1+2\lambda]^n(3+2\beta-\alpha)(\mu+2) - 3^c\mu(1-\alpha)}{[1+2\lambda]^n(3+2\beta-\alpha)(\mu+2) + 3^c\mu(1-\alpha)}. \tag{4.2}$$

The result is sharp for $f(z) = \frac{1}{z} + \frac{3^c(1-\alpha)}{[1+2\lambda]^n(3+2\beta-\alpha)}z$.

Proof. Suppose $f(z) = \frac{1}{z} + \sum\limits_{m=1}^{\infty} a_m z^m$ is in $\sigma_{c,p}(\alpha,\beta,\lambda)$.

We have

$$F(z) = \mu z^{-\mu-1}\int_0^z t^{\mu} f(t)dt = \frac{1}{z} + \sum_{m=1}^{\infty}\frac{\mu}{\mu+m+1}a_m z^m,\ \mu > 0$$

It is sufficient to show that

$$\sum_{m=1}^{\infty}\frac{m+\delta}{1-\delta}\frac{\mu a_m}{m+\mu+1} \le 1. \tag{4.3}$$

Since $f(z)$ is in $\sigma_{c,p}(\alpha,\beta,\lambda)$, we have

$$\sum_{m=1}^{\infty}\frac{[(1+\beta)(m+1)+1-\alpha][1+\lambda(m+1)]^n}{(m+2)^c(1-\alpha)}|a_m| \le 1. \tag{4.4}$$

Thus (4.3) will be satisfied if

$$\frac{(m+\delta)\mu}{(1-\delta)(m+\mu+1)} \le \frac{[(1+\beta)(m+1)+1-\alpha][1+\lambda(m+1)]^n}{(m+2)^c(1-\alpha)},\quad \text{for each } m \tag{4.5}$$

or

$$\delta \le \frac{[1+\lambda(m+1)]^n[(1+\beta)(m+1)+(1-\alpha)](\mu+m+1) - m\mu(1-\alpha)(m+2)^c}{[1+\lambda(m+1)]^n[(1+\beta)(m+1)+(1-\alpha)](\mu+m+1) + m\mu(1-\alpha)(m+2)^c}$$

$$G(m) = \frac{[1+\lambda(m+1)]^n[(1+\beta)(m+1)+(1-\alpha)](\mu+m+1) - m\mu(1-\alpha)(m+2)^c}{[1+\lambda(m+1)]^n[(1+\beta)(m+1)+(1-\alpha)](\mu+m+1) + m\mu(1-\alpha)(m+2)^c}.$$

Then $G(m+1) - G(m) > 0$, for each m. Hence $G(m)$ is increasing function of m. Since

$$G(1) = \frac{[1+2\lambda]^n(3+2\beta-\alpha)(\mu+2) - 3^c\mu(1-\alpha)}{[1+2\lambda]^n(3+2\beta-\alpha)(\mu+2) + 3^c\mu(1-\alpha)}.$$

The result follows.

5 Convex Linear Combinations and Convolution Properties

***Theorem* 5.1.** *If the function f is in $\sigma_{c,p}(\alpha,\beta,\lambda)$ then $f(z)$ is meromorphically convex of order δ $(0 \le \delta < 1)$ in $|z| < r = r(\alpha,\beta,\lambda,\delta)$ where*

$$r(\alpha,\beta,\lambda,\delta) = \inf_{n\ge1}\Big\{\frac{[1+\lambda(m+1)]^n(1-\delta)[(1+\beta)(1+m)+1-\alpha]}{(m+2)^c(1-\alpha)m(m+2-\delta)}\Big\}^{\frac{1}{m+1}}$$

The result is sharp.

Proof. Let $f(z)$ is in $\sigma_{c,p}(\alpha,\beta,\lambda)$. Then by *Theorem* 2.1, we have

$$\sum_{m=1}^{\infty}\frac{[1+\lambda(m+1)]^n[(1+\beta)(m+1)+1-\alpha]}{(m+2)^c}|a_m| \le (1-\alpha). \tag{5.1}$$

It is sufficient to show that $\left|2+\frac{zf''(z)}{f'(z)}\right| \le 1-\delta$, for $|z| < r = r(\alpha,\beta,\lambda,\delta)$, where $r(\alpha,\beta,\lambda,\delta)$ is specified in the statement of the theorem. Then

$$\left|2+\frac{zf''(z)}{f'(z)}\right| = \left|\frac{\sum\limits_{m=1}^{\infty} m(m+1)a_m z^{m-1}}{\frac{-1}{z^2}+\sum\limits_{m=1}^{\infty} m a_m z^{m-1}}\right| \le \sum_{m=1}^{\infty}\frac{m(m+1)a_m|z|^{m+1}}{1-\sum\limits_{m=1}^{\infty} m a_m|z|^{m+1}}.$$

This will be bounded by $(1-\delta)$ if

$$\sum_{m=1}^{\infty}\frac{m(m+2-\delta)}{1-\delta}a_m|z|^{m+1} \le 1. \tag{5.2}$$

By (5.1), it follow that (5.2) is true if

$$\frac{m(m+2-\delta)}{1-\delta}|z|^{m+1} \le \frac{[1+\lambda(m+1)]^n[(1+\beta)(m+1)+1-\alpha]}{(m+2)^c(1-\alpha)},\ m \ge 1$$

or

$$|z| \le \Big\{\frac{[1+\lambda(m+1)]^n(1-\delta)[(1+\beta)(1+m)+1-\alpha]}{(m+2)^c(1-\alpha)m(m+2-\delta)}\Big\}^{\frac{1}{m+1}}. \tag{5.3}$$

Setting $|z| = r(\alpha,\beta,\lambda,\delta)$ in (5.3), the result follows.
The result is sharp for the function

$$f_m(z) = \frac{1}{z} + \frac{(m+2)^c(1-\alpha)}{[1+\lambda(m+1)]^n[(1+\beta)(m+1)+1-\alpha]}z^m,\ (m \ge 1).$$

***Theorem* 5.2.** *Let $f_0(z) = \frac{1}{z}$ and*

$$f_m(z) = \frac{1}{z} + \frac{(m+2)^c(1-\alpha)}{[1+\lambda(m+1)]^n[(1+\beta)(m+1)+1-\alpha]}z^m,\ (m \ge 1).$$

Then $f(z) = \frac{1}{z} + \sum\limits_{m=1}^{\infty} a_m z^m$ is in the class $\sigma_{c,p}(\alpha,\beta,\lambda)$ if and only if it can be expressed in the form $f(z) = \lambda_0 f_0(z) + \sum\limits_{m=1}^{\infty}\lambda_m f_m(z)$, where $\lambda_0 \ge 0, \lambda_m \ge 0$ $(m \ge 1)$ and $\lambda_0 + \sum\limits_{m=1}^{\infty}\lambda_m = 1$.

Proof. Let $f(z) = \lambda_0 f_0(z) + \sum\limits_{m=1}^{\infty}\lambda_m f_m(z)$ with $\lambda_0 \ge 0, \lambda_m \ge 0$ $(m \ge 1)$ and

$$\lambda_0 + \sum_{m=1}^{\infty}\lambda_m = 1.$$

Then

$$f(z) = \lambda_0 f_0(z) + \sum_{m=1}^{\infty} \lambda_m f_m(z)$$
$$= \frac{1}{z} + \sum_{m=1}^{\infty} \lambda_m \frac{(m+2)^c(1-\alpha)}{[1+\lambda(m+1)]^n[(1+\beta)(m+1)+1-\alpha]} z^m.$$

Since

$$\sum_{m=1}^{\infty} \frac{[1+\lambda(m+1)]^n[(1+\beta)(m+1)+1-\alpha]}{(m+2)^c(1-\alpha)} \lambda_m \frac{(m+2)^c(1-\alpha)}{[1+\lambda(m+1)]^n[(1+\beta)(m+1)+1-\alpha]}$$
$$= \sum_{m=1}^{\infty} \lambda_m = 1 - \lambda_0 \le 1.$$

By *Theorem* 2.1, f is in the class $\sigma_{c,p}(\alpha, \beta, \lambda)$.

Conversely suppose that the function f is in the class $\sigma_{c,p}(\alpha, \beta, \lambda)$, since

$$a_m \le \frac{(m+2)^c(1-\alpha)}{[1+\lambda(m+1)]^n[(1+\beta)(m+1)+1-\alpha]}, \ (m \ge 1)$$
$$\lambda_m = \frac{[1+\lambda(m+1)]^n[(1+\beta)(m+1)+1-\alpha]}{(m+2)^c(1-\alpha)} a_m,$$

and $\lambda_0 = 1 - \sum_{m=1}^{\infty} \lambda_m$, it follows that $f(z) = \lambda_0 f_0(z) + \sum_{m=1}^{\infty} \lambda_m f_m(z)$.

This completes the proof of the *Theorem*.

For the functions $f(z) = \frac{1}{z} + \sum_{m=1}^{\infty} a_m z^m$ and $g(z) = \frac{1}{z} + \sum_{m=1}^{\infty} b_m z^m$ belongs to $\sum_p$ we denote by $(f * g)(z)$ the convolution of $f(z)$ and $g(z)$ or

$$(f * g)(z) = \frac{1}{z} + \sum_{m=1}^{\infty} a_m b_m z^m.$$

***Theorem* 5.3.** *If the functions* $f(z) = \frac{1}{z} + \sum_{m=1}^{\infty} a_m z^m$ *and* $g(z) = \frac{1}{z} + \sum_{m=1}^{\infty} b_m z^m$ *are in the class* $\sigma_{c,p}(\alpha, \beta, \lambda)$, *then*

$$(f * g)(z) = \frac{1}{z} + \sum_{m=1}^{\infty} a_m b_m z^m$$

is in the class $\sigma_{c,p}(\alpha, \beta, \lambda)$.

Proof. Suppose $f(z)$ and $g(z)$ are in $\sigma_{c,p}(\alpha, \beta, \lambda)$. By ***Theorem* 2.1**, we have

$$\sum_{m=1}^{\infty} \frac{[1+\lambda(m+1)]^n[(1+\beta)(m+1)+1-\alpha]}{(m+2)^c(1-\alpha)} a_m \le 1$$
$$\sum_{m=1}^{\infty} \frac{[1+\lambda(m+1)]^n[(1+\beta)(m+1)+1-\alpha]}{(m+2)^c(1-\alpha)} b_m \le 1.$$

Since $f(z)$ and $g(z)$ are regular are in E, so is $(f * g)(z)$. Furthermore,

$$\sum_{m=1}^{\infty} \frac{[1+\lambda(m+1)]^n[(1+\beta)(m+1)+1-\alpha]}{(m+2)^c(1-\alpha)} a_m b_m$$

$$\leq \Big\{ \frac{[1+\lambda(m+1)]^n[(1+\beta)(m+1)+1-\alpha]}{(m+2)^c(1-\alpha)} \Big\}^2 a_m b_m$$
$$\leq \Big(\sum_{m=1}^{\infty} \frac{[1+\lambda(m+1)]^n[(1+\beta)(m+1)+1-\alpha]}{(m+2)^c(1-\alpha)} a_m \Big)$$
$$\Big(\sum_{m=1}^{\infty} \frac{[1+\lambda(m+1)]^n[(1+\beta)(m+1)+1-\alpha]}{(m+2)^c(1-\alpha)} b_m \Big)$$
$$\leq 1.$$

Hence by *Theorem* 2.1, $(f * g)(z)$ is in the class $\sigma_{c,p}(\alpha, \beta, \lambda)$.

6 Neighborhoods for the class $\sigma_{c,p}(\alpha, \beta, \lambda)$

Neighborhoods for the class $\sigma_{c,p}(\alpha, \beta, \lambda)$ which we define as follows:

***Definition* 6.1.** *A function* $f \in \sum_p$ *is said to in the class* $\sigma_{c,p}(\alpha, \beta, \lambda, \gamma)$ *if there exists a function* $g \in \sigma_{c,p}(\alpha, \beta, \lambda)$ *such that*

$$\Big| \frac{f(z)}{g(z)} - 1 \Big| < 1 - \gamma, \; z \in U, \; (0 \leq \gamma < 1). \tag{6.1}$$

Following the earlier works on neighborhoods of analytic functions by Goodman [4] *and Ruschweyh* [11], *we define the* δ*−neighborhhod of a function* $f \in \sum_p$ *by*

$$N_\delta(f) := \Big\{ g \in \sum_p : \; g(z) = \frac{1}{z} + \sum_{m=1}^{\infty} b_m z^m : \; \sum_{m=1}^{\infty} m|a_m - b_m| \leq \delta \Big\}. \tag{6.2}$$

***Theorem* 6.1.** *If* $g \in \sigma_{c,p}(\alpha, \beta, \lambda)$ *and*

$$\gamma = 1 - \frac{\delta(3+2\beta-\alpha)(1+2\lambda)}{(3+2\beta-\alpha)(2+2\beta) - 3^c(1-\alpha)}. \tag{6.3}$$

Then $N_\delta(g) \subset \sigma_{c,p}(\alpha, \beta, \lambda, \gamma)$.

Proof. Let $f \in N_\delta(g)$. Then we find from (6.2) that

$$\sum_{m=1}^{\infty} m|a_m - b_m| \leq \delta, \tag{6.4}$$

which implies the coefficient inequality

$$\sum_{m=1}^{\infty} |a_m - b_m| \leq \delta, \; (m \in N). \tag{6.5}$$

Since $g \in \sigma_{c,p}(\alpha, \beta, \lambda)$, we have $\sum_{m=1}^{\infty} b_m < \frac{3^c(1-\alpha)}{(1+2\lambda)^n(3+2\beta-\alpha)}$. So that

$$\Big| \frac{f(z)}{g(z)} - 1 \Big| \leq \frac{\sum_{m=1}^{\infty} |a_m - b_m|}{1 - \sum_{m=1}^{\infty} b_m} \leq \frac{\delta(1+2\lambda)(3+3\beta-\alpha)}{(1+2\lambda)(3+3\beta-\alpha) - 3^c(1-\alpha)} = 1 - \gamma$$

provided γ is given by (6.3). Hence, by definition $f \in \sigma_{c,p}(\alpha, \beta, \lambda, \gamma)$ for γ given by (6.3), which completes the proof.

***Remark* 6.1.**

(i.) *For* $\lambda = 0$ *in the results mentioned in all the sections above the class are the same as those of Venkateswarlu et al:* [13].

(ii.) *For* $\lambda = 0$ *and* $\beta = 1$ *in the results mentioned in all the sections above the class are the same as those of Thirupathi Reddy et al:* [12].

7 Conclusion

This research has introduced a new linear differential operator related to polylogarithm function and studied some properties were studied. Accordingly, some results related to closure theorems have also been considered, inviting future research for this field of study.

Acknowledgement

We are very much thankful to the Editor and referees for their valuable suggestions to improve the manuscript in its present form.

References

[1] K. Al-Shaqsi and M. Darus, An operator defined by convolution involving the polylogarithms functions, *J. of Math. and Stat.,* **4 (1)** (2008), 46-50.

[2] K. Al-Shaqsi and M. Darus, A multiplier transformation defined by convolution involving nth order polylogarithm functions, *Int. Math. Form,* 4 (2009), 1823-1837.

[3] C. I. Gerhardt and G. W. Leibnitz, *Mathematische Schriften III*, I, Georg Olms, New York, NY, USA, 1971.

[4] A. W. Goodman, Univalent functions and non-analytic curves, *Proc. Amer. Math. Soc.,* **8(3)** (1957), 598-601.

[5] A. B. Goncharov, Polylogarithms in arthmetic and geometry, *Proc. of the Int. Congress of Math., Zurich, Switzerland,* (1994), 374-387.

[6] L. Lewin, Polylogarithms and Associated Functions, *North-Holland, Oxford, UK,* 1981.

[7] L. Lewin, Structural properties of polylogarithms, *Math. Surveys and Monographs, Amer. Math. Soc.,* **73**, 1991.

[8] J. Liu and H. M. Srivastava, Classes of meromorphically multivalent functions associated with the generalized Hypergeometric function, *Math. and Computer Modeling,* **39(1)** (2004), 21-34.

[9] S. Oi, Gauss hypergeometric functions, multiple polylogarithms, and multiple zeta values, *Publ. of the Research Ins. for Math. Sci.,* **45 (4)** (2009), 981-1009.

[10] S. Ponnusamy and S. Sabapathy, Polylogarithms in the theory of univalent functions, *Results in Math.,* **30 (1-2)** (1996), 136-150.

[11] St. Ruscheweyh, Neighbourhoods of univalent functions, *Proc. Amer Math. Soc.,* **81** (1981), 521-527.

[12] P. Thirupathi Reddy, B. Venkateswarlu, Rajkumar N. Ingle and S. Sreelakshmi, On a class of meromorphic functions with positive coeffcients involving polylogarithm functions, *Libertas Math. (new series),* **39 (2)** (2019), 45–57.

[13] B. Venkateswarlu, P. Thirupathi Reddy and N. Rani, On a certain subclass of meromorphic functions with positive coefficients involving polylogarithm functions, *Commun. Optim. Theory,* **2020** (2020), Article ID 6.

Jñānābha, Vol. 50(1) (2020), 144-157
(Dedicated to Honor Professor H.M. Srivastava on His 80th Birth Anniversary Celebrations)

N-POLICY FOR $M/E_k/1$ QUEUEING MODEL WITH SERVICE INTERRUPTION

By
Pankaj Sharma
Department of Mathematics, School of Science,
Noida International University, Gautam Budh Nagar, 203201, G. Noida, Uttar Pradesh, India
Email:sharma_ibspankaj@rediffmail.com
(*Received : May* 29, 2019 ; *Revised: May* 18, 2020)
DOI: https://doi.org/10.58250/jnanabha.2020.50115

Abstract

This study deals with an unreliable single service station Erlangian queueing model with k-phase service and l-phase repair under N-policy. The arriving customers follow Poisson process with arrival rates dependent upon the state of the service station, which may be idle, operating, broken down, and under setup or repair states. Due to N-policy the service station turns on only when at least $N(\geq 1)$ customers are accumulated in the system and turns off only when the system becomes empty. While providing service, the service station may breakdown according to Poisson process. An optimal operating N-policy is proposed to minimize the total expected cost. If the service station breakdowns, then it is sent for repair at the repair facility which renders repair after a set up time. After repairing, the service station works as good as before breakdown. Recursive technique and generating functions are employed for solution purpose. Explicit expressions for various performance indices are established. Cost analysis and sensitivity analysis have been done to explore the effects of different parameters.
2010 Mathematics Subject Classifications: 90B22, 60K25.
Keywords and phrases: N-Policy, Erlangian queue, Breakdown, Phase repair, Setup, Balking Recursive technique, Generating function, Cost analysis.

1 Introduction

There is an extensive literature on the Erlangian queueing model, which has been studied in various forms by numerous authors. Earlangeian queueing system represent a simple single server with negative exponential inter arrival and service time distribution. In this direction Conolly [4] studied the generalized state dependent Erlangian queue. Wang and Kuo [18] considered the profit analysis of the $M/E_k/1$ machine repair problem with a non-reliable server. The transient solution to $M/E_k/1$ queue was investigated by Griffiths et al. [5]. The transient phase probabilities are obtained in terms of a new generalisation of the modified Bessel function, and the mean waiting time in the queue is evaluated. Kim et al. [10] developed the erlang loss queueing system with batch arrivals operating in a random environment. Sharma [15] analyzed unreliable server $M^x/G/1$ queue with loss and delay, balking and second optional service. After receiving the essential service, the customers may opt for the optional service with some probability or may leave the system.

The congestion situation where the service does not start until some specified number of customers, say N are accumulated in the system during an idle period and once server starts serving, goes on serving till the system become empty is called N-policy. Medhi and Templeton [11] developed a Poisson input queue under N-policy restriction and a general start up time. Optimal NT policies for $M/G/1$ system with a startup and unreliable server was analysed by Ke [9]. Sharma [14] investigated a single unreliable server interdependent loss and delay queueing model with controllable arrival rate under N-policy. When there is no customer present in the system then the server goes on vacation and returns back in the system whenever the specified N ($\geq$ 1) or more customers are accumulated. Sharma [13] developed machine repair problem with spares, balking, reneging and n-policy for vacation. In this system, there are two repairmen, the first repairman is always available for providing service to the failed units while the second repairman goes on vacation when the failed units are less than a threshold value (say N).

One important fact that has been overlooked in most studies is that the server is subject to breakdown while serving a customer. This phenomenon is encountered in the manufacturing system, communication system, computer system and many others. Some queueing problems with breakdowns were studied by Avi-Itzhak and Naor [2], $M/G/1$ queue with breakdown was investigated by Ke [8]. Wang et al. [17] comprised the two randomized policy M/G/1 queues with second optional service, server breakdown and startup.

In the real life congestion situations, the customer may balk from the system due to some reason; one most common reason is long queue. Blackburn [3] considered optimal control of single server queue with balking and reneging. Multi server queue with balking and reneging was investigated by Abou and Hariri [1]. Controllable multi server queue with balking was studied by Jain and Sharma [6]. We incorporate an additional server which is added and removed at pre-specified threshold levels of queue size to control the balking behaviour of the customers. Finite capacity queueing system with queue dependent servers and discouragement was analysed by Jain and Sharma [7]. The service rates of the servers are different and the number of servers in the system changes depending on the queue length. The first server starts service only when N customers are accumulated in the queue and once he starts serving, continues to serve until the system becomes empty. Sharma [12] developed loss and delay multi server state dependent queue with discouragement, additional server and no-passing. The customers arrive according to Poisson process and depart from the system in the same chronological order in which they join the system, due to no-passing restriction.

The present investigation deals with N-policy for Earlangian service queueing model with unreliable service station, ℓ phase repair, setup and balking. Here, we consider the state dependent arrival rates. The steady state results for different states are obtained by using probability generating function and recursive techniques. The rest of the paper is organized as follows. In the next section, the model is described by stating requisite notations and assumptions. In Sections 3 and 4, we construct the steady state equations and obtain the distribution for the queue size, respectively. Some performance measures are given in Section 5. In Section 6, we discuss the optimal N-policy by constructing the cost function. Special cases are deduced in section 7. Sensitivity analysis is carried out in Section 8 by taking numerical illustrations. In the final Section 9, we conclude the investigation.

2 Model Description

Consider an Earlangian queueing system under N-policy restriction and unreliable single service station. The notations and basic assumptions governing the model are stated below.

1. The service station renders service under N-policy restriction, according to which the service station starts service only when there are N-customers are accumulated in the system and keeps providing service until the system becomes empty.
2. The state of the system is defined by (n, i, j); where $n(n = 0, 1, 2...$ denotes the number of customers present in the system, the customer in service is in phase $i(i = 0, 1, 2, \ldots, k)$, and

$j(j = 0, b, d, 1, 2 \ldots, l)$ represents the state of the service station, respectively. The state j of the service station at any time t is stared as follows:

$$j = \begin{cases} 0, & \text{for turn-off state of the service station} \\ b, & \text{for turn-on and busy state of the service station} \\ d, & \text{for brokendown and under setup state of the service station} \\ m, & \text{for } m^{th} \text{ phase repair state of the server where } m = 1, 2, 3, ..., l \end{cases}$$

3. The customers arrive at the service station according to Poisson fashion with rate λ. The customers may balk on finding service station busy, under setup state when broken down, and under phase repair, with balking function (i.e. the joining probabilities) of customers γ_j when $j(j = 0, b, d, 1, 2, \ldots, l)$ denotes the state of the service station.
4. The service time is k-phase Earlangian distributed with service rate μ.
5. The life time and setup time of service station follow negative exponential distribution with mean $\frac{1}{\alpha}$ and $\frac{1}{\delta}$, respectively.
6. It is assumed that the repair time of m^{th} phase is negative exponentially distributed with mean $1/\beta_m (m = 1, 2, 3, \ldots, l)$.
7. After repair the service station performs its duty with same efficiency as before breakdown.
8. The service discipline is first come first served.

3 The Mathematical Formulation

The steady state probabilities for mathematical formulation of the model are given as follows:

$P^0_{0,0}$ The probability that there is no customer present in the system and service station is in turned off state.

$P^0_{n,k}$ The probability that there are $n(1 \le n \le N - 1)$ customers present in the system but the service of customer is not initiated (i.e. $i = k$) as service station is in turned off state.

$P^b_{n,i}$ The probability that there are $n(\ge 1)$ customers present in the system, the customer is in $i^{th}(i = 1, 2, ..., k)$ phase of service and service station is in turned on busy in rendering service i.e. in operation.

$P^d_{n,i}$ The probability that there are $n(\ge 1)$ customers present in the system, the customer is in $i^{th}(i = 1, 2, ..., k)$ phase of service and the service station is in brokendown and under setup state.

$P^m_{n,i}$ The probability that there are $n(\ge 1)$ customers in the system, customer is in $i^{th}(i = 1, 2, ..., k)$ phase of service and the service station is receiving m^{th} phase repair $m = 1, 2, 3, ..., l$.

The steady states equations governing the model are obtained as follows:

$$\lambda P^0_{1,k} = \lambda P^0_{0,0}, \tag{3.1}$$

$$\lambda P^0_{n,k} = \lambda P^0_{n-1,k}; \;\; 2 \le n \le N - 1, \tag{3.2}$$

$$\lambda P^0_{0,0} = k\mu P^b_{1,1}, \tag{3.3}$$

$$(\lambda\gamma_0 + \alpha + k\mu)\, P^b_{1,i} = k\mu P^b_{1,i+1} + \beta_l P^l_{1,i}; \;\; 1 \le i \le k - 1, \tag{3.4}$$

$$(\lambda\gamma_0 + \alpha + k\mu)\, P^b_{1,k} = k\mu P^b_{2,1} + \beta_l P^l_{1,k}, \tag{3.5}$$

$$(\lambda\gamma_0 + \alpha + k\mu)\, P^b_{n,i} = \lambda\gamma_0 P^b_{n-1,i} + k\mu P^b_{n,i+1} + \beta_l P^l_{n,i}; \;\; 2 \le n \le N, \;\; 1 \le i \le k - 1, \tag{3.6}$$

$$(\lambda\gamma_0 + \alpha + k\mu)\, P^b_{n,k} = \lambda\gamma_0 P^b_{n-1,k} + k\mu P^b_{n+1,1} + \beta_l P^l_{n,k}; \;\; 2 \le n \le N - 1, \tag{3.7}$$

(3.8) $$(\lambda\gamma_0 + \alpha + k\mu)\, P^b_{N,k} = \lambda\gamma_0 P^b_{N-1,k} + k\mu P^b_{N+1,1} + \beta_l P^l_{N,k} + \lambda P^0_{N-1,k},$$

(3.9) $$(\lambda\gamma_0 + \alpha + k\mu)\, P^b_{n,i} = \lambda\gamma_0 P^b_{n-1,i} + k\mu P^b_{n,i+1} + \beta_l P^l_{n,i};\ n \geq N+1,\ 2 \leq i \leq k-1,$$

(3.10) $$(\lambda\gamma_0 + \alpha + k\mu)\, P^b_{n,k} = \lambda\gamma_0 P^b_{n-1,k} + k\mu P^b_{n+1,1} + \beta_l P^l_{n,k};\ n \geq N+1,$$

(3.11) $$(\lambda\gamma_1 + \delta)\, P^d_{1,i} = \alpha P^b_{1,i};\ 1 \leq i \leq k,$$

(3.12) $$(\lambda\gamma_1 + \delta)\, P^d_{n,i} = \lambda\gamma_1 P^d_{n-1,i} + \alpha P^b_{n,i};\ n \geq 2,\ 1 \leq i \leq k,$$

(3.13) $$(\lambda\gamma_2 + \beta_1)\, P^1_{1,i} = \delta P^d_{1,i};\ 1 \leq i \leq k,$$

(3.14) $$(\lambda\gamma_2 + \beta_1)\, P^1_{n,i} = \lambda\gamma_2 P^1_{n-1,i} + \delta P^d_{n,i};\ n \geq 2,\ 1 \leq i \leq k,$$

(3.15) $$\left(\lambda\gamma_{j+1} + \beta_j\right) P^j_{1,i} = \beta_{j-1} P^{j-1}_{1,i};\ 1 \leq i \leq k,\ \ j = 2, 3, ..., l,$$

(3.16) $$\left(\lambda\gamma_{j+1} + \beta_j\right) P^j_{n,i} = \lambda\gamma_{j+1} P^j_{n-1,i} + \beta_{j-1} P^{j-1}_{n,i};\ n \geq 2,\ 1 \leq i \leq k,\ \ j = 2, 3, ...l.$$

4 The Generating Function Method

By using the recursive technique it is not possible to obtain the explicit result for$P^0_{0,0}$, so we employ probability-generating function technique to obtain analytic solution in neat closed form. The partial probability generating functions are defined as

(4.1) $$X_i(z) = \sum_{n=1}^{\infty} P^b_{n,i} z^n;\ 1 \leq i \leq k,\ |z| < 1,$$

(4.2) $$Y^d_i(z) = \sum_{n=1}^{\infty} P^d_{n,i} z^n;\ 1 \leq i \leq k,\ |z| < 1,$$

(4.3) $$Y^j_i(z) = \sum_{n=1}^{\infty} P^j_{n,i} z^n;\ 1 \leq i \leq k,\ |z| < 1,\ \text{ and },\ j = 1, 2, ..., l,$$

(4.4) $$G_0(z) = P^0_{0,0} + \sum_{n=1}^{N-1} P^0_{n,0} z^n;\ |z| \leq 1,$$

(4.5) $$G_1(z) = \sum_{i=1}^{k} X_i(z);\ |z| \leq 1,$$

(4.6) $$G_2(z) = \sum_{i=1}^{k} Y^d_i(z);\ |z| \leq 1,$$

(4.7) $$G^j_3(z) = \sum_{i=1}^{k} Y^j_i(z);\ |z| \leq 1,\ \ j = 1, 2, ..., l.$$

Equations (3.1), (3.2) and (4.3) yield

(4.8) $$G_0(z) = \frac{\left(1 - z^N\right)}{(1 - z)} P^0_{0,0}.$$

On multiplying equation (3.4) by z, equation (3.6) by $z^n (2 \leq n \leq N)$ and equation (3.9) by $z^n (n \geq N+1)$ respectively and summing over all n, we obtain

(4.9) $$X_{i+1}(z) = (1 + a_0 - b_0 z) X_i(z) - r_l Y^l_i(z), 1 \leq i \leq k,$$

$$a_0 = \frac{\lambda\gamma_0 + \alpha}{k\mu}, b_j = \frac{\lambda\gamma_j}{k\mu}(j = 0, 1, 2, ..., l+1), r_j = \frac{\beta_j}{k\mu} \text{ for } (j = 1, 2, ..., l).$$

Again, multiplying equation (3.5) by z, equation (3.7) by $z^n(2 \leq n \leq N-1)$, equation (3.8) by z^n and equation (3.10) by $z^n(n \geq N+1)$, respectively and summing over all n and simplifying, we obtain

$$X_1(z) = z\,(1 + a_0 - b_0 z)\,X_k(z) - r_l z Y_k^l(z) - pz\left(z^N - 1\right)P_{0,0}^0, \tag{4.10}$$

where $p = \frac{\lambda}{k\mu}$.

Now, multiplying equation (3.11) by z, equation (3.12) by $z^n(n \geq 2)$, summing over all n, we have

$$Y_i^d(z) = \frac{(a_0 - b_0)}{(b_1 + c - b_1 z)}X_i(z);\ \ 1 \leq i \leq k. \text{ where } c = \frac{\delta}{k\mu}. \tag{4.11}$$

Similarly, multiplying equation (3.13) by z, equation (3.4) by $z^n(n \geq 2)$, summing over all n, we get

$$Y_i^1(z) = \frac{c}{(b_2 + r_1 - b_2 z)}Y_i^d(z);\ \ 1 \leq i \leq k. \tag{4.12}$$

In the similar manner, we obtain

$$Y_i^2(z) = \frac{r_1}{(b_3 + r_2 - b_3 z)}Y_i^1(z);\ \ 1 \leq i \leq k. \tag{4.13}$$

In general, multiplying equations (3.15) and (3.16) by the appropriate power of z and summing over all n, one can have

$$Y_i^j(z) = \frac{r_{j-1}}{\left(b_{j+1} + r_j - b_{j+1} z\right)}Y_i^{j-1}(z);\ \ 1 \leq i \leq k,\ \ j = 3, 4, ..., l. \tag{4.14}$$

By using equation, (4.9), (4.10) and (4.11) in (4.14), we have

$$Y_i^l(z) = \frac{c\,(a_0 - b_0)\prod_{k=1}^{l-1} r_k}{(b_1 + c - b_1 z)\displaystyle\prod_{k=1}^{l}(b_{k+1} + r_k - b_{k+1} z)}X_i(z);\ \ 1 \leq i \leq k. \tag{4.15}$$

Substituting the value from equation (4.15) in equation (4.9), we obtain

$$X_{i+1}(z) = w(z)X_i(z) \tag{4.16}$$

where

$$w(z) = (1 + a_0 - b_0 z) - \frac{c\,(a_0 - b_0)\displaystyle\prod_{k=1}^{l} r_k}{(b_1 + c - b_1 z)\displaystyle\prod_{k=1}^{l}(b_{k+1} + r_k - b_{k+1} z)}.$$

Here,

$$w(1) = 1, \tag{4.17}$$

(4.18)

$$w'(z) = -b_0 - \frac{c(a_0 - b_0)\displaystyle\prod_{k=1}^{l} r_k\left\{b_1\prod_{k=1}^{l}(b_{k+1} + r_k - b_{k+1} z) + (b_1 + c - b_1 z)\left(\sum_{m=1}^{l} b_m \prod_{\substack{k=1\\k\neq m}}^{l}(b_{k+1} + r_k - b_{k+1} z)\right)\right\}}{\left\{(b_1 + c - b_1 z)\displaystyle\prod_{k=1}^{l}(b_{k+1} + r_k - b_{k+1} z)\right\}^2},$$

$$w'(1) = \frac{(b_0 - a_0)\left\{b_1 \prod_{k=1}^{l}(r_k) + c\left(\sum_{m=1}^{l} b_m \prod_{\substack{k=1\\k\neq m}}^{l}(r_k)\right)\right\} - b_0 c \prod_{k=1}^{l}(r_k)}{c\prod_{k=1}^{l}(r_k)}, \tag{4.19}$$

(4.20)

$$w''(z) = \frac{\begin{array}{c} c\,(a_0 - b_0)\prod_{k=1}^{l} r_k \times \left[\left\{(b_1 + c - b_1 z)\prod_{k=1}^{l}(b_{k+1} + r_k - b_{k+1}z)\right\}^2 \left[b_1\left\{\sum_{m=1}^{l} b_m \prod_{\substack{k=1\\k\neq m}}^{l}(b_{k+1} + r_k - b_{k+1}z)\right\}\right.\right. \\ \left. +2b_1\left(\sum_{m=1}^{l} b_m \prod_{\substack{k=1\\k\neq m}}^{l}(b_{k+1} + r_k - b_{k+1}z)\right) + (b_1 + c - b_1 z)\sum_{p=1}^{l} b_p \left\{\sum_{m=1}^{l} b_m \prod_{\substack{k=1\\k\neq m}}^{l}(b_{k+1} + r_k - b_{k+1}z)\right\}\right] \\ +2\left\{(b_1 + c - b_1 z)\prod_{k=1}^{l}(b_{k+1} + r_k - b_{k+1}z)\right\}\times \\ \left.\left[b_1 \prod_{k=1}^{l}(b_{k+1} + r_k - b_{k+1}z) + (b_1 + c - b_1 z)\sum_{m=1}^{l} b_m \prod_{\substack{k=1\\k\neq m}}^{l}(b_{k+1} + r_k - b_{k+1}z)\right]^2\right] \end{array}}{\left\{(b_1 + c - b_1 z)\prod_{k=1}^{l}(b_{k+1} + r_k - b_{k+1}z)\right\}^4}$$

(4.21)

$$w''(z) = \frac{c\prod_{k=1}^{l} r_k \left\{2b_1\left(\sum_{m=1}^{l} b_m \prod_{\substack{k=1\\k\neq m}}^{l}(r_k)\right) + c\left(\sum_{p=1}^{l} b_p\left\{\sum_{m=1}^{l} b_m \prod_{\substack{k=1\\k\neq m}}^{l}(r_k)\right\}\right)\right\} + 2\left[b_1 \prod_{\substack{k=1\\k\neq m}}^{l}(r_k) + c\left\{\sum_{m=1}^{l} b_m \prod_{\substack{k=1\\k\neq m}}^{l}(r_k)\right\}\right]^2}{\left\{c\prod_{k=1}^{l} r_k\right\}^2}.$$

Equation (4.16) gives

$$X_i(z) = w^{i-1}(z)X_1(z);\ 1 \le i \le k. \tag{4.22}$$

On putting $i = k$ in equation (4.21), we have

$$X_k(z) = w^{k-1}(z)X_1(z). \tag{4.23}$$

Substituting the values from equations (4.23) and (4.15) in equation (4.10), and after simplification, we get

$$X_1(z) = \frac{zp\left(1 - z^N\right)}{(1 - zw^k(z))}P^0_{0,0}. \tag{4.24}$$

Using above equation in equation (4.22), we get

$$X_i(z) = \frac{zp\left(1 - z^N\right)w^{i-1}(z)}{(1 - zw^k(z))}P^0_{0,0}. \tag{4.25}$$

Now we have

$$G_1(z) = \frac{zp\left(1 - z^N\right)}{(1 - zw^k(z))}\left(\frac{1 - w^k(z)}{1 - w(z)}\right)P_{0,0}^0, \tag{4.26}$$

$$G_2(z) = \frac{(a_0 - b_0)}{(b_1 + c - b_1 z)}G_1(z), \tag{4.27}$$

$$G_3^j(z) = \frac{c\,(a_0 - b_0)\prod_{k=1}^{l-1} r_k}{(b_1 + c - b_1 z)\prod_{k=1}^{l}(b_{k+1} + r_k - b_{k+1}z)}G_1(z);\ \text{for } j = 1, 2, ..., l. \tag{4.28}$$

Evaluation of $P_{0,0}^0$

To obtain the value of $P_{0,0}^0$, we use the normalizing condition

$$G\,(1) = G_0\,(1) + G_1\,(1) + G_2\,(1) + \sum_{j=1}^{l} G_3^j\,(1) = 1. \tag{4.29}$$

Using equations (4.4), (4.25), (4.26) and (4.27) in equation (4.28) and applying the L-Hospital rule to get the limiting values when z→1, we get

$$G_0\,(1) = NP_{0,0,}^0 \tag{4.30}$$

$$G_1\,(1) = \frac{Npk}{(1 + kw'\,(1))}P_{0,0}^0, \tag{4.31}$$

$$G_2\,(1) = \frac{(a_0 - b_0)\,Npk}{c\,(1 + kw'\,(1))}P_{0,0}^0, \tag{4.32}$$

$$G_3^j\,(1) = \frac{(a_0 - b_0)\,Npk}{r_j\,(1 + kw'\,(1))}P_{0,0}^0; \quad j = 1, 2, 3...., l \tag{4.33}$$

and

$$P_{0,0}^0 = \frac{1}{N\left[1 + \frac{pk}{(1+kw'(1))} + \frac{(a_0-b_0)pk}{(1+kw'(1))}\left\{\frac{1}{c} + \sum_{j=1}^{l}\frac{1}{r_j}\right\}\right]}. \tag{4.34}$$

5 Performance Measures

In order to derive expressions for various performance measures, we use the generating functions. Let P_I, P_B, P_D and P_R^m denote the long run fraction of time for which service station is idle, busy, breakdown and under setup, and under j^{th} phase repair ($m = 1, 2, 3, \ldots, l$) states respectively. We compute the long run probabilities P_I, P_B, P_D, and $P_R^m(m = 1, 2, 3, \ldots, l)$, respectively in the following manner:

$$P_I = \lim_{z\to 1} G_0(z) = NP_{0,0}^0 = \frac{1}{\left[1 + \frac{pk}{(1+kw'(1))} + \frac{(a_0-b_0)pk}{(1+kw'(1))}\left\{\frac{1}{c} + \sum_{j=1}^{l}\frac{1}{r_j}\right\}\right]}, \tag{5.1}$$

$$P_B = \lim_{z\to 1} G_1(z) = \frac{pk}{(1 + kw'\,(1))\left[1 + \frac{pk}{(1+kw'(1))} + \frac{(a_0-b_0)pk}{(1+kw'(1))}\left\{\frac{1}{c} + \sum_{j=1}^{l}\frac{1}{r_j}\right\}\right]}, \tag{5.2}$$

$$P_D = \lim_{z\to 1} G_2(z) = \frac{(a_0 - b_0)\,pk}{c\,(1 + kw'\,(1))\left[1 + \frac{pk}{(1+kw'(1))} + \frac{(a_0-b_0)pk}{(1+kw'(1))}\left\{\frac{1}{c} + \sum_{j=1}^{l}\frac{1}{r_j}\right\}\right]}, \tag{5.3}$$

Similarly

(5.4)
$$P_R^m = \lim_{z\to 1} G_3^m(z) = \frac{(a_0-b_0)\,pk}{r_m\,(1+kw'(1))\left[1+\frac{pk}{(1+kw'(1))}+\frac{(a_0-b_0)pk}{(1+kw'(1))}\left\{\frac{1}{c}+\sum_{j=1}^{l}\frac{1}{r_j}\right\}\right]};(m=1,\,2,\,3,...,l)\,.$$

Expected number of customers in the system when the service station is idle, is obtained as

$$E[N_0] = G_0'(z)\,|_{z=1} = \frac{N\,(N-1)\,P_{0,0}^{0}}{2}. \tag{5.5}$$

Similarly, we can compute the expected number of customers in the system when the service station is busy, breakdown and under setup, and under m^{th} phase $(m = 1,\,2,\,3,...,l)$ repair state by the following formulae:

$$E[N_1] = G_1'(z)\,|_{z=1} = \frac{q[MR-LQ]P_{0,0}^{0}}{2[R]^2}, \tag{5.6}$$

$$E[N_2] = G_2'(z)|_{z=1} = \frac{\begin{array}{l} c'(1)d'(1)[2a'(1)b'(1)+a''(1)b'(1)+a'(1)b''(1)]\times \\ -a'(1)b'(1)[c''(1)d'(1)+c'(1)d''(1)]\end{array}}{2\{c'(1)d'(1)\}^2}, \tag{5.7}$$

$$E[N_{j+1}] = (G_3^j(z))'|_{z=1} = \frac{\begin{array}{l} 2c'(1)d'(1)e(1)[2a'(1)b'(1)+a''(1)b'(1)+a'(1)b''(1)]-\times \\ a'(1)b'(1)[c''(1)d'(1)e(1)+2c'(1)d''(1)e'(1)]\end{array}}{12\{c'(1)d'(1)e(1)\}^2} \tag{5.8}$$

$$(j = 2,\,3,....,l+1),$$

where $a(z) = 1-z^N, b(z) = 1-W^k(z), c(z) = 1-zW^k(z), d(z) = 1-W(z), e(z) = b_1+c-b_1z.$

Then the expected number of customers in the system is given by

$$E[N] = E[N_0]+E[N_1]+E[N_2]+\sum_{j=1}^{l}E[N_{j+2}]. \tag{5.9}$$

6 Optimal N-Policy

Let the expected length of the idle period, busy period, breakdown period under m^{th} phase repair $(m = 1,\,2,\,3,...,l)$ period and busy cycle be denoted by $E[I], E[B], E[D], E[R^m](m = 1,2,3,...,l)$ and $E[C]$ respectively. We have

$$E[C] = E[B]+E[I]+E[D]+\sum_{m=1}^{l}E[R^m]. \tag{6.1}$$

The length of the idle period is the sum of N exponential distributed random variables with mean $1/\lambda$. This implies that

$$E[I] = \frac{N}{\lambda}. \tag{6.2}$$

Also

$$P_I = \frac{E[I]}{E[C]}, P_B = \frac{E[B]}{E[C]}, P_D = \frac{E[D]}{E[C]} \text{ and } P_R^m = \frac{E[R^m]}{E[C]}\,(m = 1,\,2,\,3,...,l)\,. \tag{6.3}$$

Now, we define the following cost elements to determine the optimal value of control parameters N for an Erlangian queueing model with k-phase service and l-phase repair under N-policy restriction.

C_d holding cost per unit time per customer present in the system,
C_u start up cost per unit time for turning the service station on,
C_B shut down cost per unit time for turning the service station off,
C_I cost per unit time of the service station in idle state,
C_D cost per unit time of the service station in broken down state when repairman is under set up state,
C_R^m repair cost per unit time rendering m^{th} phaserepair $(m = 1, 2, 3, ..., l)$,

The expected total cost per unit time is formulated as:

$$E\,(TC) = C_h E[N_s] + (C_u + C_d)\,\frac{1}{E[C]} + C_I P_I + C_B P_B + C_D P_D + \sum_{m=1}^{l} C_R^{,m} P_R^m. \tag{6.4}$$

The optimal value (say N^*) of the decision variable N, could be determined by setting

$$\frac{d\,\{E(TC)\}}{dN} = 0. \tag{6.5}$$

In case when N^* is not an integer, then the best positive integer value N^* is achieved by rounding off the N^*.

7 Special Cases

Case I: When $\lambda = \lambda_0, \lambda\gamma_1 = \lambda_1, \lambda\gamma_2 = \lambda_2,\ l = 1,$ and $\delta = 0$, then we get the results for this model.

Case II: If $\lambda = \lambda\gamma_1 = \lambda\gamma_2 = \Lambda,\ l = 1,$ and $\delta = 0$, then we get results for N-policy $M/M/1$ queueing system with breakdown.

Case III: In case when $\gamma_1 = \gamma_2 = 1,\ l = 1,\ \delta = 0,\ \alpha = 0,$ and $\beta = 1$ then our model coincides with the model developed by Wang and Huang [16].

8 Sensitivity Analysis

In order to show the validity of analytical results, we perform extensive numerical experiment by using MATLAB. The effects of different parameters on the average queue length are shown in ***Figs.*** **8.1-8.8**. The numerical results of the expected total cost with the variation of different parameters are presented in ***Tables*** **8.1** and **8.2**.

The effect of arrival rate (λ), failure rate (α), service rate (μ), setup rate (σ), repair rate of first phase (β_i), repair rate of second phase (β_2), optimal threshold parameter N^* and number of phases of service (k) respectively, on $E(TC)$ are examined for different sets of cost elements which are given as follows:

Set 1:	C_U=10,	C_F =5,	C_I=5,	C_B=10,	C_D=2,	C_H=5,	C_{R1}=2	C_{R2}=1
Set 2:	C_U=20,	C_F =5,	C_I=5,	C_B=10,	C_D=2,	C_H=5,	C_{R1}=2	C_{R2}=1
Set 3:	C_U=10,	C_F =10,	C_I=5,	C_B=10,	C_D=2,	C_H=5,	C_{R1}=2	C_{R2}=1
Set 4:	C_U=10,	C_F =5,	C_I=10,	C_B=10,	C_D=2,	C_H=5,	C_{R1}=2	C_{R2}=1
Set 5:	C_U=10,	C_F =5,	C_I=5,	C_B=20,	C_D=2,	C_H=5,	C_{R1}=2	C_{R2}=1
Set 6:	C_U=10,	C_F =5,	C_I=5,	C_B=10,	C_D=4,	C_H=5,	C_{R1}=2	C_{R2}=1
Set 7:	C_U=10,	C_F =5,	C_I=5,	C_B=10,	C_D=2,	C_H=10,	C_{R1}=2	C_{R2}=1
Set 8:	C_U=10,	C_F =5,	C_I=5,	C_B=10,	C_D=2,	C_H=5,	C_{R1}=4	C_{R2}=1
Set 9:	C_U=10,	C_F =5,	C_I=5,	C_B=10,	C_D=2,	C_H=5,	C_{R1}=2	C_{R2}=2

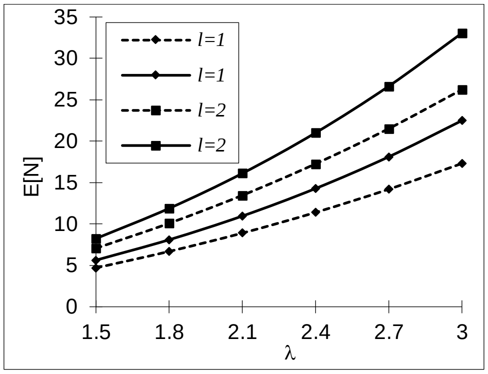


***Fig*. 8.1:** Expected queue length vs. λ

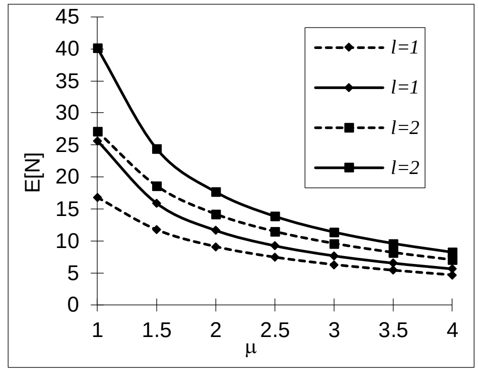


***Fig*. 8.2:** Expected queue length vs. μ

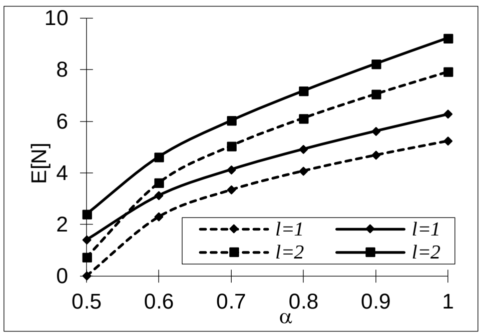


***Fig*. 8.3:** Expected queue length vs. α

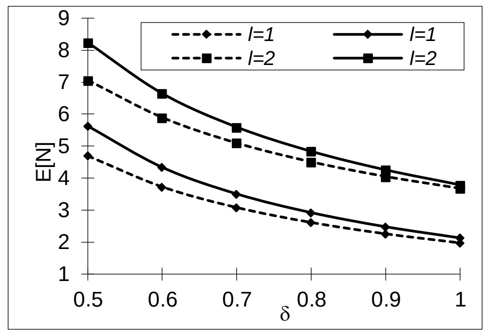


***Fig*. 8.4:** Expected queue length vs. δ

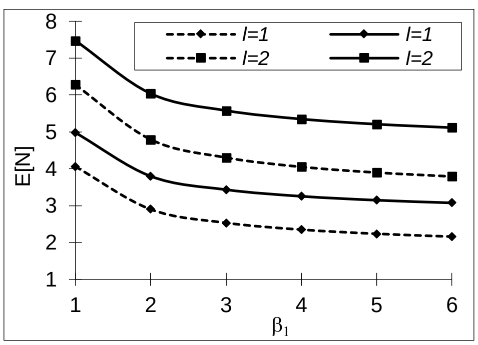


***Fig*. 8.5:** Expected queue length vs. β_1

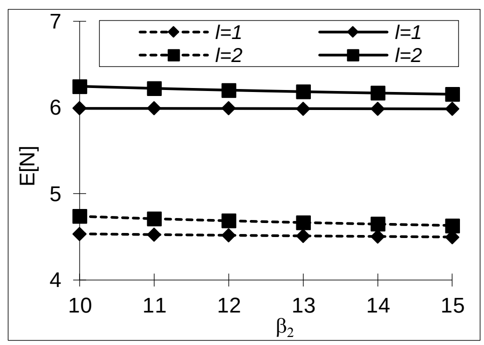


***Fig*. 8.6:** Expected queue length vs. β_2

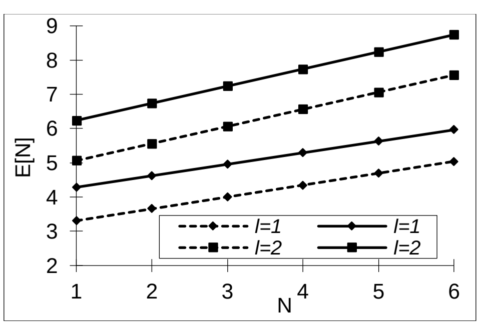


***Fig*. 8.7:** Expected queue length vs. N

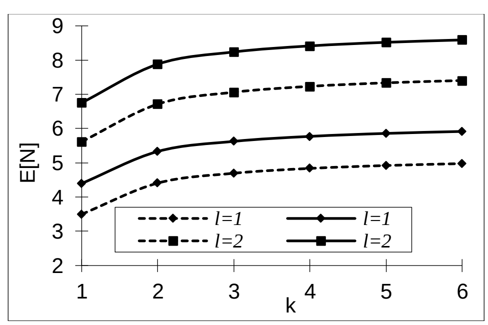


***Fig*. 8.8:** Expected queue length vs. k

From ***Table* 8.1** it is observed that by increasing λ, α, N and k, the expected total cost increases. But as we increase the parameters μ, β_1, β_2 and δ the expected total cost decreases as can be seen from ***Table* 8.2**.

***Table* 8.1:** Effect of parameters (λ, α, k, N) on the expected total cost for different sets of cost elements

λ	SET 1	SET 2	SET 3	SET 4	SET 5	SET 6	SET 7	SET 8	SET 9
1.00	6.69	6.31	6.50	5.76	9.19	7.59	10.69	7.25	6.89
1.20	21.69	20.68	21.18	19.58	24.69	22.77	42.02	22.37	21.94
1.40	30.24	28.39	29.32	26.94	33.74	31.50	60.72	31.03	30.53
1.60	38.08	35.21	36.64	33.59	42.08	39.52	78.28	38.98	38.41
1.80	46.14	42.05	44.10	40.47	50.64	47.76	96.57	47.15	46.51
2.00	54.71	49.22	51.96	47.85	59.71	56.51	116.17	55.83	55.12
α	SET 1	SET 2	SET 3	SET 4	SET 5	SET 6	SET 7	SET 8	SET 9
0.50	7.35	6.89	7.12	6.58	11.10	8.10	11.04	7.82	7.52
0.60	20.56	19.62	20.09	19.00	24.31	21.46	38.65	21.12	20.76
0.70	26.53	25.13	25.83	24.19	30.28	27.58	51.80	27.18	26.77
0.80	30.71	28.84	29.77	27.59	34.46	31.91	61.35	31.46	30.98
0.90	34.16	31.83	33.00	30.27	37.91	35.51	69.48	35.01	34.47
1.00	37.27	34.46	35.86	32.59	41.02	38.77	76.88	38.20	37.61
N	SET 1	SET 2	SET 3	SET 4	SET 5	SET 6	SET 7	SET 8	SET 9
1.00	10.15	1.53	4.31	6.25	13.90	11.50	35.46	10.99	10.46
2.00	21.41	15.57	18.49	17.52	25.16	22.76	49.22	22.25	21.72
3.00	26.83	22.94	24.88	22.94	30.58	28.18	57.14	27.67	27.14
4.00	30.79	27.87	29.33	26.90	34.54	32.14	63.60	31.63	31.10
5.00	34.16	31.83	33.00	30.27	37.91	35.51	69.48	35.01	34.47
6.00	37.25	35.30	36.28	33.36	41.00	38.60	75.06	38.09	37.56
k	SET 1	SET 2	SET 3	SET 4	SET 5	SET 6	SET 7	SET 8	SET 9
1.00	26.94	24.60	25.77	23.05	30.69	28.29	55.03	27.78	27.25
2.00	32.43	30.09	31.26	28.54	36.18	33.78	66.01	33.27	32.74
3.00	34.16	31.83	33.00	30.27	37.91	35.51	69.48	35.01	34.47
4.00	35.02	32.69	33.85	31.13	38.77	36.37	71.19	35.87	35.33
5.00	35.53	33.20	34.36	31.64	39.28	36.88	72.21	36.38	35.84
6.00	35.87	33.54	34.70	31.98	39.62	37.22	72.89	36.72	36.18

***Table* 8.2:** Effect of parameters ($\mu, \delta, \beta_i, \beta_2$) on the expected total cost for different sets of cost elements.

μ	SET 1	SET 2	SET 3	SET 4	SET 5	SET 6	SET 7	SET 8	SET 9
1.00	102.53	84.19	93.36	71.96	117.53	107.93	238.15	105.91	103.76
2.00	59.29	51.61	55.45	46.50	66.79	61.99	130.37	60.97	59.90
3.00	43.42	39.31	41.37	36.57	48.42	45.22	91.54	44.55	43.83
4.00	34.16	31.83	33.00	30.27	37.91	35.51	69.48	35.01	34.47
5.00	26.11	24.84	25.48	24.00	29.11	27.19	51.24	26.79	26.36
6.00	13.08	12.52	12.80	12.15	15.58	13.98	23.76	13.64	13.28
δ	SET 1	SET 2	SET 3	SET 4	SET 5	SET 6	SET 7	SET 8	SET 9
0.50	34.16	31.83	33.00	30.27	37.91	35.51	69.48	35.01	34.47
0.60	29.18	27.18	28.18	25.85	32.93	30.31	58.67	30.02	29.49
0.70	25.79	24.04	24.92	22.87	29.54	26.76	51.29	26.64	26.10
0.80	23.31	21.73	22.52	20.68	27.06	24.15	45.86	24.15	23.61
0.90	21.37	19.93	20.65	18.97	25.12	22.12	41.63	22.21	21.67
1.00	19.79	18.47	19.13	17.58	23.54	20.46	38.19	20.63	20.10
β_1	SET 1	SET 2	SET 3	SET 4	SET 5	SET 6	SET 7	SET 8	SET 9
1.00	30.87	28.79	29.83	27.40	34.62	32.22	62.25	31.54	31.18
2.00	24.71	23.14	23.92	22.09	28.46	26.06	48.67	25.05	25.02
3.00	22.68	21.27	21.98	20.34	26.43	24.03	44.19	22.91	22.99
4.00	21.64	20.31	20.98	19.43	25.39	22.99	41.89	21.81	21.94
5.00	20.99	19.72	20.35	18.87	24.74	22.34	40.47	21.13	21.30
6.00	20.55	19.31	19.93	18.48	24.30	21.90	39.50	20.66	20.85
β_2	SET 1	SET 2	SET 3	SET 4	SET 5	SET 6	SET 7	SET 8	SET 9
1.00	35.31	32.88	34.09	31.26	39.06	36.66	72.02	36.15	35.64
2.00	29.35	27.43	28.39	26.15	33.10	30.70	58.67	30.19	29.52
3.00	27.50	25.74	26.62	24.57	31.25	28.85	54.49	28.34	27.61
4.00	26.58	24.91	25.74	23.79	30.33	27.93	52.41	27.42	26.66
5.00	26.02	24.40	25.21	23.32	29.77	27.37	51.15	26.86	26.09
6.00	25.65	24.06	24.85	23.01	29.40	27.00	50.31	26.49	25.70

***Figs* 8.1-8.8** depict the effect of parameters $\lambda, \alpha, \mu, \delta, \beta_1, \beta_2, N$ and k respectively, on the average queue length. From all the graphs it is observed that the average queue length is higher for heterogeneous arrival rates in comparison of homogeneous arrival rates. Also, it is noticed that average queue length increases as we increase the number of phases of repair. From ***Figs.* 8.1, 8.2** we examine the effect of arrival rate (service rate) on the average queue length and observed that $E[N]$ increases (decreases) with the increase in λ and μ however the effect of μ on $E[N]$ is more prominent for lower values. Substantially the same effects with respect to failure rate (setup rate) have been seen in ***Figs.* 8.3**, **8.4**. In ***Figs.* 8.5** and **8.6**, the average queue length $E[N]$ is plotted against the parameters β_1 & β_2. As we expect, initially $E[N]$ decreases sharply by increasing β_1 but after some time it becomes almost constant; moreover the decreasing effect due to increment in β_2 is almost not negligible. ***Fig.* 8.7** illustrates the effect of threshold parameter N on the average queue length and we notice that $E[N]$ increases linearly as N increases. From ***Fig.* 8.8** it is seen that initially $E[N]$ increases sharply and then after slowly as we increase k.

From the tables and graphs, overall we conclude that the average queue length increases as

λ, α, N and k increase but decreases as μ, δ, β_1, and β_2 increase, which is in agreement with physical situations.

9 Concluding Remarks

In this paper we have analysed an Erlangian queueing model with phase service and phase repair under N-policy. We have employed the generating function approach for computing the steady state probability distribution for various performance measures. The cost analysis facilitated may be helpful to assist the decision makers in determining the optimal value of threshold parameter N, so as to minimize total expected cost per unit time. The incorporation of balking behavior of the customers makes our model more realistic to depict the day-to-day as well as industrial congestion situations.

Acknowledgement. We are very much thankful to the Editor and referee for their suggestions to bring the paper in its present form.

References

[1] M. O. Abou-El-Ata and A. M. Hariri, $M/M/c/N$ queue with balking and reneging, *Comput. Oper. Res.*, **19 (8)** (1992), 713-716.

[2] B. Avi-Itzhak and P. Naor, Some queueing problems with the service station subject to breakdown, *Oper. Res.*, **10** (1962), 303-320.

[3] J. D. Blackburn, Optimal control of single service station queue with balking and reneging, *Manage Sci.*, **19(8)** (1972), 297-379.

[4] B. Conally, The generalized state dependent Erlangian queue, the busy period Erlangian, *J. Appl. Prob.*, **2** (1974), 618-623.

[5] J. D Griffiths, G. M. Leonenko, and J.E. Williams, The transient solution to $M/E_k/1$ queue, *Oper. Res. Letters*, **34(3)**(2006), 349-354.

[6] M. Jain and P. Sharma, Controllable multi service station queue with balking, *Int. J. Engg.*, **18(3)** (2005), 263-271.

[7] M. Jain and P. Sharma, Finite capacity queueing system with queue dependent servers and discouragement, *Jnanabha*, **38** (2008), 1-12.

[8] J. C. Ke, The optimal control of an $M/G/1$ queueing system with service station vacations, startup and breakdowns, *Comput. Math. Appl.*, **44(4)**(2003), 567-579.

[9] J. C. Ke, Optimal NT policies for $M/G/1$ system with a startup and unreliable server, *Comput. Ind. Engg.*, **50(3)** (2006), 248-262.

[10] C. S. Kim, A. Dudin, V. Klimenok, and V. Khramova, Erlang loss queueing system with batch arrivals operating in a random environment, *Comput. Oper. Res.*, **36(3)** (2009), 674-697.

[11] J. Medhi and J. G. C. Templeton, A Poisson input queue under N-policy and a general start up time, *Comput. Oper. Res.*, **19** (1992), 35-41.

[12] P. Sharma, Loss and Delay Multi Server State Dependent queue with Discouragement, Additional Server and No-Passing, *Int. J. Res. Sci. Innv. (IJRSI)*, **5(8)** (2018), 33-48.

[13] P. Sharma, Machine Repair Problem With Spares, Balking, Reneging And N-Policy For Vacation, *J. Raj. Aca. Phy. Sci.*, **14(3&4)** (2015), 337-344.

[14] P. Sharma, Single Unreliable Server Interdependent Loss and Delay queueing Model with Controllable Arrival Rate Under N-Policy, *Int. J. Sci. Res. Pub. (IJSRP)*, **3(8)** (2013), 1-13.

[15] P. Sharma, Unreliable Server $M^x/G/1$ Queue with Loss-Delay, Balking and Second Optional Service, *Int. J. Engg. Iran*, **30(2)** (2017), 243-252.

[16] K. H. Wang and H. M. Huang, Optimal control of an $M/E_k/1$ queueing system with a removable service station, *J. Oper. Res. Soc.*, **46** (1995), 1014-1022.
[17] K. H. Wang, D. Y. Yang, and W. L. Pearn, Comparison of two randomized policy M/G/1 queues with second optional service, server breakdown and startup, *J. Comp. App. Math.*, **234 (3)** (2010), 812-824.
[18] K. H. Wang and M. Y. Kuo, Profit analysis of the $M/E_k/1$ machine repair problem with a non reliable service station, *Comput. Indus. Engg.*, **32(3)** (1997), 587-594.

Jñānābha, Vol. 50(1) (2020), 158-163
(Dedicated to Honor Professor H.M. Srivastava on His 80th Birth Anniversary Celebrations)

THE THEORETICAL OVERVIEW OF THE HARTLEY TRANSFORM AND THE GENERALIZED *R*-FUNCTION

By

[1]**Naseer Ahmad Malik**
Department of Mathematics,
Government Postgraduate College for Women
Gandhi Nagar Jammu-180004 Jammu and Kashmir, India.
email: drnaseerulhassan@gmail.com
[2]**Farooq Ahmad**
Department of Mathematics,
Government College for Women
Nawakadal-190001, Jammu and Kashmir, India.
email: sheikhfarooq85@gmail.com
[3]**D. K. Jain**
Department of Mathematics,
Madhav Institute of Technology and Science,
Gwalior-474005, Madhya Pradash India.
email: jain_dkj@yahoo.co.in
(*Received : April* 03, 2020 ; *Revised: June* 02, 2020)
DOI: https://doi.org/10.58250/jnanabha.2020.50116

Abstract

In this paper the R-functions have been mentioned in connection with integral operator named as Hartely transform. The Hartley transform is a mathematical transformation which is closely related to the better known Fourier transform. The properties that differentiate the Hartley Transform from its Fourier counterpart are that the forward and the inverse transforms are identical and also that the Hartley transform of a real signal is a real function of frequency. The Whitened Hartley spectrum, which stems from the Hartley transform, is a bounded function that encapsulates the phase content of a signal. The Whitened Hartley spectrum, unlike the Fourier phase spectrum, is a function that does not suffer from discontinuities or wrapping ambiguities. An overview on how the Whitened Hartley spectrum encapsulates the phase content of a signal more efficiently compared with its Fourier counterpart as well as the reason that phase unwrapping is not necessary for the Whitened Hartley spectrum, are provided in this study. Moreover, in this study, we deal with the function which is significant generalization of Fox's H-function which was introduced by Hartley and Lorenzo and later on modified by Jain et al.

2010 Mathematics Subject Classifications: 26A33, 33C05, 33C10, 33C20.
Keywords and phrases: Generalized fractional integral operators, H- Function, I-function and R-function.

1 Introduction

1.1 Fox's H-function

The H-function series introduced by Fox [4] will be represented and defined in the following manner

$$H_{p,q}^{m,n}\left\{\begin{array}{l}(a_1,A_1)(a_2,A_2)...(a_p,A_p)\\(b_1,B_1)(b_2,B_2)...(b_q,B_q)\end{array}|x\right\}=\frac{1}{2\pi i}\int_L\frac{\prod_{j=1}^{m}\Gamma(b_j-B_js)\prod_{j=1}^{n}\Gamma(1-a_j+A_js)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+B_js)\prod_{j=n+1}^{p}\Gamma(a_j-A_js)}x^sds,$$

where L is a suitable contour.

1.2 Hartley transform

The Hartley transform is an integral transformation that maps a real-valued temporal or spacial function into a real-valued frequency function via the kernel, $cas(vx) \equiv cos(vx) + sin(vx)$. This novel symmetrical formulation of the traditional Fourier transform, attributed to Ralph Vinton Lyon Hartley [5], leads to a parallelism that exists between the function of the original variable and that of its transform. Furthermore, the Hartley transform permits a function to be decomposed into two independent sets of sinusoidal components; these sets are represented in terms of positive and negative frequency components, respectively. This is in contrast to the complex exponential, $exp(jwx)$, used in classical Fourier analysis. For periodic power signals, various mathematical forms of the familiar Fourier series come to mind. For a periodic energy and power signals of either finite or infinite duration, the Fourier integral can be used. In either case, signal and systems analysis and design in the frequency domain using the Hartley transform may be deserving of increased awareness due necessarily to the existence of a fast algorithm that can substantially lessen the computational burden when compared to the classical complex-valued Fast Fourier Transform (FFT). Perhaps one of Hartley's most long-lasting contributions was a more symmetrical Fourier integral originally developed for steady-state and transient analysis of telephone transmission system problems. Although this transform remained in a quiescent state for over 40 years, the Hartley transform was rediscovered more than a decade ago by Wang [12],[14] and Bracewell [1],[3] who authored definitive treatises on the subject.

The Hartley transform of a function $f(x)$ can be expressed as either

$$H(v) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) cas(vx) dx, \tag{1.1}$$

$$H(f) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) cas(2\pi f x) dx. \tag{1.2}$$

Here the integral kernel, known as the *cosine* and *sine* or *cas* function, is defined as

$$cas(vx) = cos(vx) + sin(vx),$$
$$cas(vx) = \sqrt{2} sin(vx + \frac{\pi}{4}),$$
$$cas(vx) = \sqrt{2} cos(vx - \frac{\pi}{4}).$$

The Hartley transform has the convenient property of being its own inverse

$$f = \{H\{Hf\}\}.$$

1.3 Generalized Functions for the Fractional Calculus(R-function)

It is of significant usefulness to develop a generalized function which when fractionally differ integrated (by any order) returns itself. Such a function would greatly ease the analysis of fractional order differential equations. To end this process the following was proposed by Hartley and Lorenzo,[7]. The R-function is unique in that it contains all of the derivatives and integrals of the F-function. The R-function has the Eigen property that is it returns itself on q^{th} order differ-integration. Special cases of the R-function also include the exponential function, the sine, cosine, hyperbolic sine and hyperbolic cosine functions. The value of the R-function is clearly demonstrated in the dynamic thermocouple problem where it enables the analyst to directly inverse transform the Laplace domain solution, to obtain the time domain solution, and is defined as follows

(1.3) $$R_{q,v}[a,c,t] = \sum_{n=0}^{\infty} \frac{a^n(t-c)^{(n+1)q-1-v}}{\Gamma((n+1)q-v)}.$$

The more compact notation

$$R_{q,v}[a,t-c] = \sum_{n=0}^{\infty} \frac{a^n(t-c)^{(n+1)q-1-v}}{\Gamma((n+1)q-v)}.$$

When $c = 0$, we get

$$R_{q,v}[a,t] = \sum_{n=0}^{\infty} \frac{a^n(t)^{(n+1)q-1-v}}{\Gamma((n+1)q-v)}.$$

Put $v = q-1$, we get Mittag-Leffler function

$$R_{q,q-1}[a,t] = \sum_{n=0}^{\infty} \frac{a^n(t)^{(nq)}}{\Gamma(nq+1)} = E(at^q).$$

Taking $a = 1, v = q-\beta$

$$R_{q,q-\beta}[1,t] = \sum_{n=0}^{\infty} \frac{1^n(t)^{(n+1)q-1-q+\beta}}{\Gamma((n+1)q-q+\beta)}.$$

$$\Rightarrow R_{q,q-\beta}[1,t] = t^{\beta-1}E_{q,\beta}(t^q).$$

2 Main Result

In this section, the authors have derived the Hartley transform of R- functions as follows

***Theorem* 2.1.** *The Hartley transform H of R-functions*

$$H\{R_{q,v}[a,0,t]\} = \frac{1}{\Gamma((n+1)q-v)}[1+(-1)^n]cos(\frac{n\pi}{2})\Gamma(n+1).$$

Proof. The Hartley transform of R- functions in terms of Fox's H-function is given by

$$H\{R_{q,v}[a,c,t]\} = H\left\{\sum_{n=0}^{\infty} \frac{a^n(t-c)^{(n+1)q-1-v}}{\Gamma((n+1)q-v)}\right\},$$

or

$$H\{R_{q,v}[a,c,t]\} = \sum_{n=0}^{\infty}(a)^n H\left\{\frac{(t-c)^{(n+1)q-1-v}}{\Gamma((n+1)q-v)}\right\}.$$

Taking $c = 0$, we get

$$\begin{aligned}H\{R_{q,v}[a,0,t]\} &= \sum_{n=0}^{\infty}(a)^n H\left\{\frac{(t)^{(n+1)q-1-v}}{\Gamma((n+1)q-v)}\right\}, Re((n+1)q-v) > 0\\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{(t)^{(n+1)q-1-v}}{\Gamma((n+1)q-v)} cas(vt)dt,\\ &= \frac{1}{\Gamma((n+1)q-v)}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(t)^{(n+1)q-1-v}\{cos(vt)+sin(vt)\}dt,\\ &= \frac{1}{\Gamma((n+1)q-v)}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(t)^{(n+1)q-1-v}\sqrt{2}sin(vt+\frac{\pi}{4})dt.\end{aligned}$$

This gives

$$H\{R_{q,v}[a,0,t]\} = \frac{1}{\Gamma((n+1)q-v)}[1+(-1)^n]cos(\frac{n\pi}{2})\Gamma(n+1).$$

This proves the ***Theorem* 2.1**.

Special Case: Putting $c = 0$ and $v = q-1$, we get Mittag-Leffler function as special case of the above result, the result follows as

***Theorem* 2.2.** *The Hartley transform of Fox-Wright function in terms Fox's H-function*

$$H\left\{p\Psi q\left[\begin{array}{c}(a_1,A_1)(a_2,A_2)(a_3,A_3)...(a_p,A_p)\\(b_1,B_1)(b_2,B_2)(b_3,B_3)...(b_q,B_q)\end{array}|z\right]\right\}$$
$$= \frac{1}{s}H^{1,p}_{1,q}\left[\begin{array}{c}(1-a_1,-A_1)(1-a_2,-A_2)(1-a_3,-A_3)...(1-a_p,-A_p)\\(1-b_1,-B_1)(1-b_2,-B_2)(1-b_3,-B_3)...(1-b_q,-B_q)\end{array}|S\right].$$

Proof. The Hartley transform of Fox-Wright function in terms Fox's H-function is given by

$$H\left\{p\Psi q\left[\begin{array}{c}(a_1,A_1)(a_2,A_2)...(a_p,A_p)\\(b_1,B_1)(b_2,B_2)...(b_q,B_q)\end{array}|z\right]\right\}.$$

This implies

$$H\left\{p\Psi q\left[\begin{array}{c}(a_1,A_1)(a_2,A_2)...(a_p,A_p)\\(b_1,B_1)(b_2,B_2)...(b_q,B_q)\end{array}|z\right]\right\}$$
$$=\left\{\sum_{n=0}^{\infty}\frac{\Gamma(a_1+nA_1)\Gamma(a_2+nA_2)...\Gamma(a_p+nA_p)}{\Gamma(b_1+nB_1)\Gamma(b_2+nB_2)...\Gamma(b_q+nB_q)}\right\}H\left\{\frac{z^n}{n!}\right\}$$
$$=\left\{\sum_{n=0}^{\infty}\frac{\Gamma(a_1+nA_1)\Gamma(a_2+nA_2)...\Gamma(a_p+nA_p)}{\Gamma(b_1+nB_1)\Gamma(b_2+nB_2)...\Gamma(b_q+nB_q)}\right\}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{z^n}{n!}cas(vz)dz$$
$$=\left\{\sum_{n=0}^{\infty}\frac{\Gamma(a_1+nA_1)\Gamma(a_2+nA_2)...\Gamma(a_p+nA_p)}{\Gamma(b_1+nB_1)\Gamma(b_2+nB_2)...\Gamma(b_q+nB_q)}\right\}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{z^n}{n!}\{cos(vz)+sin(vz)\}dz$$
$$=\left\{\sum_{n=0}^{\infty}\frac{\Gamma(a_1+nA_1)\Gamma(a_2+nA_2)...\Gamma(a_p+nA_p)}{\Gamma(b_1+nB_1)\Gamma(b_2+nB_2)...\Gamma(b_q+nB_q)}\right\}\frac{1}{\Gamma(n+1)}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}z^n\sqrt{2}sin(vz+\frac{\pi}{4})dz$$
$$=\left\{\sum_{n=0}^{\infty}\frac{\Gamma(a_1+nA_1)\Gamma(a_2+nA_2)...\Gamma(a_p+nA_p)}{\Gamma(b_1+nB_1)\Gamma(b_2+nB_2)...\Gamma(b_q+nB_q)}\right\}\frac{-1}{\Gamma(n+1)}[1+(-1)^n]cos(\frac{n\pi}{2})\Gamma(n+1)$$
$$H\left\{p\Psi q\left[\begin{array}{c}(a_1,A_1)(a_2,A_2)...(a_p,A_p)\\(b_1,B_1)(b_2,B_2)...(b_q,B_q)\end{array}|z\right]\right\}$$
$$=\frac{1}{s}\left\{\sum_{n=0}^{\infty}\frac{\Gamma(1-(1-a_1)+nA_1)\Gamma(1-(1-a_2)+nA_2)...\Gamma(1-(1-a_p)+nA_p)}{\Gamma(1-(1-b_1)+nB_1)\Gamma(1-(1-b_2)+nB_2)...\Gamma(1-(1-b_q)+nB_q)}\right\}s^{-n}$$
$$=\frac{1}{s}H^{1,p}_{1,q}\left[\begin{array}{c}(1-a_1,-A_1)(1-a_2,-A_2)(1-a_3,-A_3)...(1-a_p,-A_p)\\(1-b_1,-B_1)(1-b_2,-B_2)(1-b_3,-B_3)...(1-b_q,-B_q)\end{array}|S\right].$$

This proves the ***Theorem* 2.2**.

3 Application of the Hartley Transform via the Fast Hartley Transform

The discretized versions of the continuous Fourier and Hartley transform integrals may be put in an amenable form for digital computation. Consider the discrete Fourier transform (*DFT*) and inverse *DFT* (*IDFT*) of a periodic function of period *NT* seconds.

- The *DHT* requires only half the memory storage for real data arrays vs. complex data arrays.
- For a sequence of length N, the *DHT* performs $O(N \log 2N)$ real operations vs. the *DFT* $O(N \log 2N)$ complex operations.
- The *DHT* performs fewer operations that may lead to fewer truncation and rounding errors from computer finite word length.
- The *DHT* is its own inverse (i.e., it has a self-inverse) For reasons of computational advantage either occurring through waveform symmetry or simply the use of real quantities, the Hartley transform is recommended as a serious alternative to the Fourier transform for frequency-domain analysis. The salient disadvantage of the Hartley approach is that Fourier amplitude and phase information is not readily interpreted. This is not a difficulty in many applications because this information is typically used as an intermediate stage toward a final goal. Due to the cited advantages above, it is clear that the Hartley transform has much to offer when engineering applications warrant digital filtering of real-valued signals. In particular, the *FHT* should be used when either the computation time is to be minimized; for example, in real-time signal processing. The minimization of computing time includes many other issues, such as memory allocation, real vs. complex variables, computing platforms, and so forth. However, when one is interested in computing the Hartley transform or the convolution or correlation integral, the Hartley transform is the method of choice. In general, most engineering applications based on the *FFT* can be reformulated in terms of the all-real *FHT* in order to realize a computational advantage. This is due primarily to the vast amounts of research within the past decade on *FHT* algorithm development as evidenced in [16]. A voluminous number of applications exist for the Hartley transform H some of which are listed below
- Fast convolution, correlation, interpolation and extrapolation, finite-impulse response and multidimensional filter design.

4 Conclusion.

In this paper, an overview of the Hartley transform is presented, the relationship between the Hartley transform and the Fourier transform is provided and the Hartley transform properties are analyzed. More importantly, the Whitened Hartley spectrum is defined, its properties for phase spectral estimation are highlighted, its short time analysis is provided and its advantages compared with the Fourier phase spectrum are underlined. The properties of the Whitened Hartley spectrum are also demonstrated via an example involving time-delay measurement. Summarizing, the Whitened Hartley spectrum is proposed as an alternative to the Fourier phase spectrum for applications related to phase spectral processing. Specifically, the Whitened Hartley spectrum, unlike its Fourier counterpart, does not convey extrinsic discontinuities since it is not using the inverse tangent function, whereas the discontinuities of the signal in the phase spectrum which are caused because of intrinsic characteristics of the signal can be compensated. Finally, it is important to mention that the phase spectrum which is developed via the Whitened Hartley spectrum does not only have important advantages compared with the Fourier phase spectrum but it is also very straightforward in terms of its implementation and processing.

Acknowledgement. We are very much thankful to the editor and referee for their valuable suggestions to revise the paper in the present form.

References

[1] R. N. Bracewell, Discrete Hartley transform, *J. Opt. Soc. Amer.*, **73** (1983), 1832-1835.
[2] R. N. Bracewell, The fast Hartley transform, *Proc. IEEE*, **72** (1984), 1010-1018.
[3] R. N. Bracewell, The Hartley Transform, *Oxford University Press, New York*, 1986.
[4] C. Fox, The G and H function as symmetric Fourier kernals , *Trans. Am. Math. Soc.*, **98** (1968).
[5] R. V. L. Hartley, A more symmetrical Fourier analysis applied to transmission problems, *Proc. Of the I.R.E.*, **30** (1942), 144-150.
[6] R. V. L. Hartley, Transmission of information, *Bell Sys. Tech. J.*, **7** (1928), 535-563.
[7] T. T. Hartley, and C. F. Lorenzo, A solution to the fundamental linear fractional order differential equation, *NASATP*, 1998.
[8] D. K. Jain , R. Jain. and Farooq Ahmad, Some Transformation Formula for Basic Analogue of I-function; Asian Journal of Mathematics and Statistic, *Publication*, **5** (2012), 158-162.
[9] G. L. N. Rao, The generalized Laplace transform of generalized function, *Ranchi Univ.,Math.J.*, **5** (1974).
[10] I. N. Sneddon, Fourier Transform, *Mc Graw-Hill, New York*, 1951.
[11] R. K. Saxena, Some theorems on generalized Laplace transform, *Int., Proc. Nat. Inst. Sci. India Part A*, **26** (1960), 400-413.
[12] Z. Wang, Harmonic analysis with a real frequency function-I. Aperiodic case, *Appl. Math. And Comput.*, **9** (1981), 53-73.
[13] Z. Wang, Harmonic analysis with a real frequency function-II. Periodic and bounded case, *Appl. Math. and Comput.*, **9** (1981), 153-163.
[14] Z. Wang, Harmonic analysis with a real frequency function-III. Data sequence, *Appl. Math. And Comput.*, **9** (1981), 245-255.
[15] Z. Wang, Fast algorithms for the discrete W transform and for the discrete Fourier transform, IEEE Trans. Acoust., Speech, Signal Process., *ASSP*, **32** (1984), 803-816.
[16] A. H. Zemanian, Generalized integral transform, *Pure Appl.Math. Inter Science* Publ. John Wiley and Sons, New York., **18**, 1968.

Jñānābha, Vol. 50(1) (2020), 164-178
(Dedicated to Honor Professor H.M. Srivastava on His 80th Birth Anniversary Celebrations)

GENERALIZED FRACTIONAL CALCULUS OF *I*-FUNCTION OF TWO VARIABLES

By
Dheerandra Shanker Sachan[1] and Shailesh Jaloree[2]
[1]St.Mary's Postgraduate College, Vidisha-464001, Madhya Pradesh, India.
[2]Samrat Ashok Technological Institute, Vidisha-464001, Madhya Pradesh, India.
Email:sachan.dheerandra17@gmail.com,shailesh_jaloree@rediffmail.com
(*Received : March* 01, 2020 *; Revised: June* 05, 2020)
DOI: https://doi.org/10.58250/jnanabha.2020.50117

Abstract

This paper is devoted to study and develop the generalized fractional calculus of arbitrary order for the *I*-function of two variables which is based on generalized fractional integration and differentiation operators of arbitrary complex order involving Appell hypergeometric function F_3 as a kernel due to Saigo and Maeda. On account of general nature of the Saigo-Maeda operators, a large number of results involving Saigo and Riemann-Liouville operetors are found as corollaries. Some special cases also have been considered.
2010 Mathematics Subject Classifications: 26A33, 33C60, 33C70.
Keywords and phrases: Generalized fractional calculus operators, Appell function, Fractional calculus, *I*-function of two variables, Mellin-Barnes type integrals.

1 Introduction

In last some decades, considerable amount of research work in fractional calculus is published due to its applicability in the various fields of science and engineering such as dynamical system in control theory, electrical circuits, viscoelasticity, electrochemistry, fluid mechanics, mathematical biology, image processing, astrophysics and quantum mechanics.There is no doubt that fractional calculus has become an important mathematical tool to solve diverse problems of mathematics, science and engineering. The fractional calculus operators involving various special functions have been successfully applied to frame relevant system in various fields of science and engineering. see [2], [3], [19], [20]. Therefore number of authors have investigated different unifications and extentions of various fractional calculus operators. For more detail about fractional calculus operators, reader may refer to the research monograph by Miller and Ross [16], Samko *et al*.[22], and Kiryakova [9].

The image formulas for special functions of one and more variables under various fractional calculus operators have been obtained by number of authors such as Gupta *et al*.[7] obtained the image formulas of the product of two H functions using Saigo operators, Agarwal [1] studied and developed the generalized fractional integration of the product of $\overline{H}$-function and a general class of polynomials in Saigo operators, Kumar [10] established some new unified integral and differential formulas associated with $\overline{H}$-function applying Saigo and Maeda operator. For more information, we may also refer to Chandel [4]; Chandel and Vishwakarma [5]; Chandel and Gupta [6]; Kumar, Purohit and Choi [12]; Kumar [13]; Kumar, Chandel and Srivastava [14]; Kumar, Pathan and Kumari [15]; Mathai, Saxena and Houbold [17]; Pathan, Kumar, Srivastava and Chandel [28]; Srivastava, Saxena and Ram [28]; Srivastava, Chandel and vishwakarma [29]. In order to stimulate more interest in the subject, we have established some image formulas concerning to *I*-function of two variables.

In 1995, Goyal and Agrawal [8] introduced I-function of two variables by means of Mellin-Barnes type integrals in the following manner

$$(1.1)\ I^{m_1,n_1:m_2,n_2;m_3,n_3}_{p,q:p_i^{(1)},q_i^{(1)};p_i^{(2)},q_i^{(2)}:r}\left[\begin{array}{c|c} z_1 & [(e_p:E_p,E_p^{'})]:[(a_j,\alpha_j)_{1,n_2}],[(a_{ji},\alpha_{ji})_{n_2+1,p_i^{(1)}}];[(c_j,\gamma_j)_{1,n_3}],[(c_{ji},\gamma_{ji})_{n_3+1,p_i^{(2)}}] \\ z_2 & [(f_q:F_q,F_q^{'})]:[(b_j,\beta_j)_{1,m_2}],[(b_{ji},\beta_{ji})_{m_2+1,q_i^{(1)}}];[(d_j,\delta_j)_{1,m_3}],[(d_{ji},\delta_{ji})_{m_3+1,q_i^{(2)}}]\end{array}\right]$$

$$= \frac{1}{(2\pi\omega)^2}\int_{L_1}\int_{L_2}\phi_1(\xi)\,\phi_2(\eta)\,\psi(\xi,\eta)\,z_1^{\xi}\,z_2^{\eta}\,d\xi\,d\eta,$$

where $\omega = \sqrt{-1}$ and $\phi_1(\xi)$, $\phi_2(\eta)$, $\psi(\xi,\eta)$ are given by

$$(1.2)\qquad \phi_1(\xi) = \frac{\prod_{j=1}^{m_2}\Gamma(b_j-\beta_j\xi)\prod_{j=1}^{n_2}\Gamma(1-a_j+\alpha_j\xi)}{\sum_{i=1}^{r}\left[\prod_{j=m_2+1}^{q_i^{(1)}}\Gamma(1-b_{ji}+\beta_{ji}\xi)\prod_{j=n_2+1}^{p_i^{(1)}}\Gamma(a_{ji}-\alpha_{ji}\xi)\right]},$$

$$(1.3)\qquad \phi_2(\eta) = \frac{\prod_{j=1}^{m_3}\Gamma(d_j-\delta_j\eta)\prod_{j=1}^{n_3}\Gamma(1-c_j+\gamma_j\eta)}{\sum_{i=1}^{r}\left[\prod_{j=m_3+1}^{q_i^{(2)}}\Gamma(1-d_{ji}+\delta_{ji}\eta)\prod_{j=n_3+1}^{p_i^{(2)}}\Gamma(c_{ji}-\gamma_{ji}\eta)\right]},$$

$$(1.4)\qquad \psi(\xi,\eta) = \frac{\prod_{j=1}^{m_1}\Gamma(f_j-F_j\xi-F_j^{'}\eta)\prod_{j=1}^{n_1}\Gamma(1-e_j+E_j\xi+E_j^{'}\eta)}{\prod_{j=m_1+1}^{q}\Gamma(1-f_j+F_j\xi+F_j^{'}\eta)\prod_{j=n_1+1}^{p}\Gamma(e_j-E_j\xi-E_j^{'}\eta)},$$

and an empty product is interpreted as unity. z_1, z_2 are two non zero complex variables, L_1, L_2 are two Mellin-Barnes type contour integrals and $m_1,n_1;\ m_2,n_2;\ m_3,n_3,\ p,q;\ p_i^{(1)}$, $q_i^{(1)};\ p_i^{(2)},q_i^{(2)}$ are non-negative integers satisfying the conditions $0\le n_1\le p$, $0\le n_2\le p_i^{(1)}$, $0\le n_3\le p_i^{(2)}$, $0\le m_1\le q$, $0\le m_2\le q_i^{(1)}$, $0\le m_3\le q_i^{(2)}$ for all $i=1,2,3,\cdots,r$ where r is also a positive integer. $\alpha_j(j=1,\cdots,n_2),\beta_j(j=1,\cdots,m_2),\gamma_j(j=1,\cdots,n_3),\delta_j(j=1,\cdots,m_3),\alpha_{ji}(j=n_2+1,\cdots,p_i^{(1)}),\beta_{ji}(j=m_2+1,\cdots,q_i^{(1)}),\gamma_{ji}(j=n_3+1,\cdots,p_i^{(2)}),\delta_{ji}(j=m_3+1,\cdots,q_i^{(2)})$ are assumed to be positive quantities for standardization purposes. $E_j,E_j^{'},F_j,F_j^{'}$ are also positive. $a_j(j=1,\cdots,n_2),b_j(j=1,\cdots,m_2),c_j(j=1,\cdots,n_3),d_j(j=1,\cdots,m_3),a_{ji}(j=n_2+1,\cdots,p_i^{(1)}),b_{ji}(j=m_2+1,\cdots,q_i^{(1)}),c_{ji}(j=n_3+1,\cdots,p_i^{(2)}),d_{ji}(j=m_3+1,\cdots,q_i^{(2)})$ are complex for all $i=1,2,3,\cdots,r$.

The contour L_1 lies in the complex ξ-plane and runs from $-\omega\infty$ to $+\omega\infty$ with loops, if necessary, to ensure that the poles of $\Gamma(b_j-\beta_j\xi)(j=1,\cdots,m_2),\Gamma(f_j-F_j\xi-F_j^{'}\eta)(j=1,\cdots,m_1)$

lies to the right and the poles of $\Gamma(1-a_j+\alpha_j\xi)(j=1,\cdots,n_2), \Gamma(1-e_j+E_j\xi+E_j^{'}\eta)(j=1,\cdots,n_1)$ to the left of the contour L_1.The contour L_2 lies in the complex η plane and runs from $-\omega\infty$ to $+\omega\infty$ with loops, if necessary, to ensure that the poles of $\Gamma(d_j-\delta_j\eta)(j=1,\cdots,m_3), \Gamma(f_j-F_j\xi-F_j^{'}\eta)(j=1,\cdots,m_1)$ lies to the right and the poles of $\Gamma(1-c_j+\gamma_j\xi)(j=1,\cdots,n_3), \Gamma(1-e_j+E_j\xi+E_j^{'}\eta)(j=1,\cdots,n_1)$ to the left of the contour L_2. All the poles are simple poles.

Convergence conditions are as follows:

$$|arg\, z_1| < \frac{A_i\pi}{2},\ |arg\, z_2| < \frac{B_i\pi}{2}, \tag{1.5}$$

where

$$A_i = \sum_{j=1}^{n_1} E_j - \sum_{j=n_1+1}^{p} E_j + \sum_{j=1}^{m_1} F_j - \sum_{j=m_1+1}^{q} F_j + \sum_{j=1}^{m_2} \beta_j - \sum_{j=m_2+1}^{q_i^{(1)}} \beta_{ji} + \sum_{j=1}^{n_2} \alpha_j - \sum_{j=n_2+1}^{p_i^{(1)}} \alpha_{ji} > 0, \tag{1.6}$$

and

$$B_i = \sum_{j=1}^{n_1} E_j^{'} - \sum_{j=n_1+1}^{p} E_j^{'} + \sum_{j=1}^{m_1} F_j^{'} - \sum_{j=m_1+1}^{q} F_j^{'} + \sum_{j=1}^{m_3} \delta_j - \sum_{j=m_3+1}^{q_i^{(2)}} \delta_{ji} + \sum_{j=1}^{n_3} \gamma_j - \sum_{j=n_3+1}^{p_i^{(2)}} \gamma_{ji} > 0, \tag{1.7}$$

for $i = 1, ..., r$.

By considering the behaviour of the Gamma functions involved in $I[z_1, z_2]$ defined by (1.1), it can be shown that $I[z_1, z_2]$ is of certain algebraic order of z_1, z_2 for large values of z_1, z_2, if the validity conditions (1.5)-(1.7) are satisfied.

Also, for small values of z_1 and z_2

$$I\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = o\left(|z_1|^{\lambda_j}|z_2|^{\mu_k}\right) \text{ for all } j = 1,2,\cdots,m_2;\ k = 1,2,\cdots,m_3,$$

where $\lambda_j = min\, Re\left(\frac{b_j}{\beta_j}\right),\ \mu_k = min\, Re\left(\frac{d_k}{\delta_k}\right)$ provided $A_i > 0,\ B_i > 0$,

Further, we observe, for large values of z_1 and z_2, that

$$I\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = o\left(|z_1|^{\lambda_j^{'}}|z_2|^{\mu_k^{'}}\right) \text{ for all } j = 1,2,\cdots,n_2;\ k = 1,2,\cdots,n_3,$$

where $\lambda_j^{'} = max\, Re\left(\frac{a_j-1}{\alpha_j}\right),\ \mu_k^{'} = max\, Re\left(\frac{c_k-1}{\gamma_k}\right)$ provided $A_i > 0,\ B_i > 0$,

For the sake of brevity throughout the paper we shall use following notations:
$P = m_2, n_2; m_3, n_3,$
$Q = p_i^{(1)}, q_i^{(1)}; p_i^{(2)}, q_i^{(2)} : r,$
$U = [(a_j, \alpha_j)_{1,n_2}], [(a_{ji}, \alpha_{ji})_{n_2+1,p_i^{(1)}}]; [(c_j, \gamma_j)_{1,n_3}], [(c_{ji}, \gamma_{ji})_{n_3+1,p_i^{(2)}}],$
$V = [(b_j, \beta_j)_{1,m_2}], [(b_{ji}, \beta_{ji})_{m_2+1,q_i^{(1)}}]; [(d_j, \delta_j)_{1,m_3}], [(d_{ji}, \delta_{ji})_{m_3+1,q_i^{(2)}}],$

If $\alpha, \alpha', \beta, \beta', \gamma \in C$ and $x > 0$, then the generalized fractional calculus operators containing Appell hypergeometric function F_3 given by Saigo and Maeda [23] are defined in the following manner:

$$\left(I_{0_+}^{\alpha,\alpha',\beta,\beta',\gamma} f\right)(x) = \frac{x^{-\alpha}}{\Gamma(\gamma)} \int_0^x t^{-\alpha'}(x-t)^{\gamma-1} F_3\left(\alpha, \alpha', \beta, \beta'; \gamma; 1-\frac{t}{x}, 1-\frac{x}{t}\right) f(t)\, dt,\ \ Re(\gamma) > 0, \tag{1.8}$$

(1.9) $$= \left(\frac{d}{dx}\right)^k \left(I_{0_+}^{\alpha,\alpha',\beta+k,\beta',\gamma+k} f\right)(x),\ \ Re(\gamma) \le 0; k = [-Re(\gamma)+1], \left(I_{-}^{\alpha,\alpha',\beta,\beta',\gamma} f\right)(x)$$

(1.10) $$= \frac{x^{-\alpha'}}{\Gamma(\gamma)} \int_x^{\infty} t^{-\alpha}(t-x)^{\gamma-1} F_3\left(\alpha,\alpha',\beta,\beta';\gamma;1-\frac{x}{t},1-\frac{t}{x}\right) f(t)\,dt,\ \ Re(\gamma) > 0,$$

(1.11) $$= (-1)^k \left(\frac{d}{dx}\right)^k \left(I_{-}^{\alpha,\alpha',\beta,\beta'+k,\gamma+k} f\right)(x),\ \ Re(\gamma) \le 0; k = [-Re(\gamma)+1],$$

(1.12)
$$\left(D_{0_+}^{\alpha,\alpha',\beta,\beta',\gamma} f\right)(x) = \left(I_{0_+}^{-\alpha',-\alpha,-\beta',-\beta,-\gamma} f\right)(x),\ \ Re(\gamma) > 0,$$

(1.13) $$= \left(\frac{d}{dx}\right)^k \left(I_{0_+}^{-\alpha',-\alpha,-\beta'+k,-\beta,-\gamma+k} f\right)(x),\ \ Re(\gamma) > 0; k = [Re(\gamma)+1],$$

(1.14)
$$\left(D_{-}^{\alpha,\alpha',\beta,\beta',\gamma} f\right)(x) = \left(I_{-}^{-\alpha',-\alpha,-\beta',-\beta,-\gamma} f\right)(x),\ \ Re(\gamma) > 0,$$

(1.15) $$= (-1)^k \left(\frac{d}{dx}\right)^k \left(I_{-}^{-\alpha',-\alpha,-\beta',-\beta+k,-\gamma+k} f\right)(x),\ \ Re(\gamma) > 0; k = [Re(\gamma)+1],$$

These generalized fractional calculus operators reduces to Saigo's [24] fractional calculus operators by means of the following relationship:

(1.16) $$\left(I_{0_+}^{\alpha,0,\beta,\beta',\gamma} f\right)(x) = \left(I_{0_+}^{\gamma,\alpha-\gamma,-\beta} f\right)(x),\ \ (\gamma \in C),$$

(1.17) $$\left(I_{-}^{\alpha,0,\beta,\beta',\gamma} f\right)(x) = \left(I_{-}^{\gamma,\alpha-\gamma,-\beta} f\right)(x),\ \ (\gamma \in C),$$

(1.18) $$\left(D_{0_+}^{0,\alpha',\beta,\beta',\gamma} f\right)(x) = \left(D_{0_+}^{\gamma,\alpha'-\gamma,\beta'-\gamma} f\right)(x),\ \ Re(\gamma) > 0,$$

(1.19) $$\left(D_{-}^{0,\alpha',\beta,\beta',\gamma} f\right)(x) = \left(D_{-}^{\gamma,\alpha'-\gamma,\beta'-\gamma} f\right)(x),\ \ Re(\gamma) > 0.$$

Our main findings in the next section are based on the following composition formula due to Saigo-Maeda [23].

***Lemma* 1.1.** *If* $\alpha,\alpha',\beta,\beta',\gamma \in C, Re(\gamma) > 0$ *and* $Re(\rho) > max[0, Re(\alpha+\alpha'+\beta-\gamma), Re(\alpha'-\beta')]$ *then there hold the formula*

(1.20) $$\left(I_{0_+}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1}\right)(x) = x^{\rho-\alpha-\alpha'+\gamma-1} \frac{\Gamma(\rho)\Gamma(\rho+\gamma-\alpha-\alpha'-\beta)\Gamma(\rho+\beta'-\alpha')}{\Gamma(\rho+\gamma-\alpha-\alpha')\Gamma(\rho+\gamma-\alpha'-\beta)\Gamma(\rho+\beta')},$$

***Lemma* 1.2.** *If* $\alpha,\alpha',\beta,\beta',\gamma \in C, Re(\gamma) > 0$ *and* $Re(\rho) < 1 + min[Re(-\beta), Re(\alpha+\alpha'-\gamma), Re(\alpha+\beta'-\gamma)]$ *then there hold the formula*

(1.21) $$\left(I_{-}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1}\right)(x) = x^{\rho-\alpha-\alpha'+\gamma-1} \frac{\Gamma(1+\alpha+\alpha'-\gamma-\rho)\Gamma(1+\alpha+\beta'-\gamma-\rho)\Gamma(1-\beta-\rho)}{\Gamma(1-\rho)\Gamma(1+\alpha+\alpha'+\beta'-\gamma-\rho)\Gamma(1+\alpha-\beta-\rho)},$$

2 Main Results

In this section we have established fractional calculus formulas associated to I-function of two variables with the help of Saigo-Maeda generalized fractional calculus operators. Further by specializing the parameters, we have found some corollaries concerning to Saigo fractional calculus operators and Riemann-Liouville fractional calculus operators.The results are presented in the form of theorems stated below.

***Theorem* 2.1.** *Let $\alpha, \alpha', \beta, \beta', \gamma, \rho \in C$, $z_1, z_2 \in C$, $Re(\gamma) > 0$, $\mu, \nu \in R_+$. Further let the constants $m_1, n_1, p, q \in N_0$, $a_j, b_j, a_{ji}, b_{ji} \in C$, $\alpha_j, \beta_j, \alpha_{ji}, \beta_{ji} \in R_+(i = 1, \cdots, p_i^{(1)}; \ j = 1, \cdots, q_i^{(1)})$, $c_j, d_j, c_{ji}, d_{ji} \in C$, $\gamma_j, \delta_j, \gamma_{ji}, \delta_{ji} \in R_+(i = 1, \cdots, p_i^{(2)}; \ j = 1, \cdots, q_i^{(2)})$, $|arg\, z_1| < \frac{A_i \pi}{2}$, $|arg\, z_2| < \frac{B_i \pi}{2}$, $A_i > 0$, $B_i > 0$ and satisfy the condition*

$$Re(\rho) + \mu \min_{1 \le j \le m_2} Re\left(\frac{b_j}{\beta_j}\right) + \nu \min_{1 \le j \le m_3} Re\left(\frac{d_j}{\delta_j}\right) > max\left[0, Re(\alpha + \alpha' + \beta - \gamma), Re(\alpha' - \beta')\right].$$

Then the fractional integration $I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}$ of the I-function of two variables exists and the following relation holds:

$$\left\{ I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1} I_{p,q:Q}^{m_1,n_1:P} \left[\begin{array}{c|c} z_1 t^{\mu} & [(e_p : E_p, E_p^{'})] : U \\ z_2 t^{\nu} & [(f_q : F_q, F_q^{'})] : V \end{array} \right] \right\}(x) \tag{2.1}$$

$$= x^{\rho+\gamma-\alpha-\alpha'-1} I_{p+3,q+3:Q}^{m_1,n_1+3:P} \left[\begin{array}{c|c} z_1 x^{\mu} & X_1, [(e_p : E_p, E_p^{'})] : U \\ z_2 x^{\nu} & [(f_q : F_q, F_q^{'})], X_2 : V \end{array} \right],$$

where

$$X_1 = [(1-\rho : \mu, \nu)], [(1-\rho+\alpha+\alpha^{'}+\beta-\gamma : \mu, \nu)], [(1-\rho+\alpha^{'}-\beta^{'} : \mu, \nu)],$$
$$X_2 = [(1-\rho+\alpha+\alpha^{'}-\gamma : \mu, \nu)], [(1-\rho+\alpha^{'}+\beta-\gamma : \mu, \nu)], [(1-\rho-\beta^{'} : \mu, \nu)].$$

Proof. In order to prove (2.1), we first express I-function of two variables occurring on the left hand side of (2.1) in terms of Mellin-Barnes contour integral with the help of equation (1.1) and interchanging the order of integration, which is justified under the conditions stated with the ***Theorem***, we obtain (say I_1):

$$I_1 = \frac{1}{(2\pi\omega)^2} \int_{L_1} \int_{L_2} \phi_1(\xi)\, \phi_2(\eta)\, \psi(\xi, \eta)\, z_1^{\xi}\, z_2^{\eta}\, (I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}\, t^{\rho+\mu\xi+\nu\eta-1})(x)\, d\xi\, d\eta, \tag{2.2}$$

Now by applying ***Lemma* 1.1**, we arrive at

$$I_1 = x^{\rho+\gamma-\alpha-\alpha'-1} \frac{1}{(2\pi\omega)^2} \int_{L_1} \int_{L_2} \phi_1(\xi)\, \phi_2(\eta)\, \psi(\xi, \eta)\, (z_1 x^{\mu})^{\xi}\, (z_2 x^{\nu})^{\eta}$$
$$\times \frac{\Gamma(\rho+\mu\xi+\nu\eta)\Gamma(\rho+\mu\xi+\nu\eta+\gamma-\alpha-\alpha'-\beta)\Gamma(\rho+\mu\xi+\nu\eta+\beta'-\alpha')}{\Gamma(\rho+\mu\xi+\nu\eta+\gamma-\alpha-\alpha')\Gamma(\rho+\mu\xi+\nu\eta+\gamma-\alpha'-\beta)\Gamma(\rho+\mu\xi+\nu\eta+\beta')}\, d\xi\, d\eta.$$

By re-interpreting the Mellin-Barnes contour integral in terms of I-function of two variables defined by (1.1) , we obtain the right hand side of (2.1) after little simplifications. This completes proof of ***Theorem* 2.1**.
In view of the relation (1.16), we get the following corollary concerning left-sided Saigo fractional integral operator [24].

***Corollary* 2.1.** *Let $\alpha, \beta, \gamma, \rho \in C$, $z_1, z_2 \in C$, $Re(\alpha) > 0$, $\mu, \nu \in R_+$. Further let the constants $m_1, n_1, p, q \in N_0$, $a_j, b_j, a_{ji}, b_{ji} \in C$, $\alpha_j, \beta_j, \alpha_{ji}, \beta_{ji} \in R_+(i = 1, \cdots, p_i^{(1)}; \ j = 1, \cdots, q_i^{(1)})$, $c_j, d_j, c_{ji}, d_{ji} \in C$, $\gamma_j, \delta_j, \gamma_{ji}, \delta_{ji} \in R_+(i = 1, \cdots, p_i^{(2)}; \ j = 1, \cdots, q_i^{(2)})$, $|arg\, z_1| < \frac{A_i \pi}{2}$, $|arg\, z_2| < \frac{B_i \pi}{2}$, $A_i > 0$, $B_i > 0$ and satisfy the condition*

$$Re(\rho) + \mu \min_{1 \le j \le m_2} Re\left(\frac{b_j}{\beta_j}\right) + \nu \min_{1 \le j \le m_3} Re\left(\frac{d_j}{\delta_j}\right) > max\left[0, Re(\beta - \gamma)\right].$$

Then the fractional integration $I_{0+}^{\alpha,\beta,\gamma}$ of the I-function of two variables exists and the following relation holds:

$$\left\{I_{0+}^{\alpha,\beta,\gamma} t^{\rho-1} I_{p,q:Q}^{m_1,n_1:P}\left[\begin{array}{c|c} z_1 t^{\mu} & [(e_p : E_p, E_p^{'})] : U \\ z_2 t^{\nu} & [(f_q : F_q, F_q^{'})] : V \end{array}\right]\right\}(x) \tag{2.3}$$

$$= x^{\rho-\beta-1} I_{p+2,q+2:Q}^{m_1,n_1+2:P}\left[\begin{array}{c|c} z_1 x^{\mu} & [(1-\rho:\mu,\nu)],[(1-\rho-\gamma+\beta:\mu,\nu)],[(e_p : E_p, E_p^{'})] : U \\ z_2 x^{\nu} & [(f_q : F_q, F_q^{'})],[(1-\rho+\beta:\mu,\nu)],[(1-\rho-\alpha-\gamma:\mu,\nu)] : V \end{array}\right].$$

Now if we set $\beta = -\alpha$ in (2.3), we obtain the following result concerning left-sided Riemann-Liouville fractional integral operator [24].

Corollary 2.2. *Let $\alpha,\rho \in C$, $z_1, z_2 \in C$, $Re(\alpha) > 0$, $\mu,\nu \in R_+$. Further let the constants $m_1, n_1, p, q \in N_0$, $a_j, b_j, a_{ji}, b_{ji} \in C$, $\alpha_j,\beta_j,\alpha_{ji},\beta_{ji} \in R_+(i = 1,\cdots,p_i^{(1)};\ j = 1,\cdots,q_i^{(1)})$, $c_j, d_j, c_{ji}, d_{ji} \in C$, $\gamma_j,\delta_j,\gamma_{ji},\delta_{ji} \in R_+(i = 1,\cdots,p_i^{(2)};\ j = 1,\cdots,q_i^{(2)})$, $|arg\, z_1| < \frac{A_i\pi}{2}$, $|arg\, z_2| < \frac{B_i\pi}{2}$, $A_i > 0$, $B_i > 0$ and satisfy the condition*

$$Re(\rho) + \mu \min_{1\le j\le m_2} Re\left(\frac{b_j}{\beta_j}\right) + \nu \min_{1\le j\le m_3} Re\left(\frac{d_j}{\delta_j}\right) > 0.$$

Then the fractional integration I_{0+}^{α} of the I-function of two variables exists and the following relation holds:

$$\left\{I_{0+}^{\alpha} t^{\rho-1} I_{p,q:Q}^{m_1,n_1:P}\left[\begin{array}{c|c} z_1 t^{\mu} & [(e_p : E_p, E_p^{'})] : U \\ z_2 t^{\nu} & [(f_q : F_q, F_q^{'})] : V \end{array}\right]\right\}(x) \tag{2.4}$$

$$= x^{\rho-\alpha-1} I_{p+1,q+1:Q}^{m_1,n_1+1:P}\left[\begin{array}{c|c} z_1 x^{\mu} & [(1-\rho:\mu,\nu)],[(e_p : E_p, E_p^{'})] : U \\ z_2 x^{\nu} & [(f_q : F_q, F_q^{'})],[(1-\rho-\alpha:\mu,\nu)] : V \end{array}\right].$$

Theorem 2.2. *Let $\alpha,\alpha',\beta,\beta',\gamma,\rho \in C$, $z_1, z_2 \in C$, $Re(\gamma) > 0$, $\mu,\nu \in R_+$. Further let the constants $m_1, n_1, p, q \in N_0$, $a_j, b_j, a_{ji}, b_{ji} \in C$, $\alpha_j,\beta_j,\alpha_{ji},\beta_{ji} \in R_+(i = 1,\cdots,p_i^{(1)};\ j = 1,\cdots,q_i^{(1)})$, $c_j, d_j, c_{ji}, d_{ji} \in C$, $\gamma_j,\delta_j,\gamma_{ji},\delta_{ji} \in R_+(i = 1,\cdots,p_i^{(2)};\ j = 1,\cdots,q_i^{(2)})$, $|arg\, z_1| < \frac{A_i\pi}{2}$, $|arg\, z_2| < \frac{B_i\pi}{2}$, $A_i > 0$, $B_i > 0$ and satisfy the condition*

$$Re(\rho) + \mu \max_{1\le j\le n_2}\left[\frac{Re(a_j)-1}{\alpha_j}\right] + \nu \max_{1\le j\le n_3}\left[\frac{Re(c_j)-1}{\gamma_j}\right] < 1 + min\left[Re(-\beta), Re(\alpha+\alpha'-\gamma), Re(\alpha+\beta'-\gamma)\right]$$

Then the fractional integration $I_{-}^{\alpha,\alpha',\beta,\beta',\gamma}$ of the I-function of two variables exists and the following relation holds:

$$\left\{I_{-}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1} I_{p,q:Q}^{m_1,n_1:P}\left[\begin{array}{c|c} z_1 t^{\mu} & [(e_p : E_p, E_p^{'})] : U \\ z_2 t^{\nu} & [(f_q : F_q, F_q^{'})] : V \end{array}\right]\right\}(x) \tag{2.5}$$

$$= x^{\rho-\alpha-\alpha^{'}+\gamma-1} I_{p+3,q+3:Q}^{m_1+3,n_1:P}\left[\begin{array}{c|c} z_1 x^{\mu} & [(e_p : E_p, E_p^{'})], X_3 : U \\ z_2 x^{\nu} & X_4, [(f_q : F_q, F_q^{'})] : V \end{array}\right],$$

where

$$X_3 = [(1-\rho:\mu,\nu)],[(1-\rho+\alpha+\alpha^{'}+\beta'-\gamma:\mu,\nu)],[(1-\rho+\alpha-\beta:\mu,\nu)],$$
$$X_4 = [(1-\rho+\alpha+\alpha^{'}-\gamma:\mu,\nu)],[(1-\rho+\alpha+\beta'-\gamma:\mu,\nu)],[(1-\rho-\beta:\mu,\nu)].$$

Proof. In order to prove (2.5), we first express I-function of two variables occurring on the left hand side of (2.5) in terms of Mellin-Barnes contour integral with the help of equation (1.1) and interchanging the order of integration, which is justified under the conditions stated with the ***Theorem***, we obtain (say I_2):

$$I_2 = \frac{1}{(2\pi\omega)^2}\int_{L_1}\int_{L_2} \phi_1(\xi)\,\phi_2(\eta)\,\psi(\xi,\eta)\,z_1^{\xi}\,z_2^{\eta}\,(I_-^{\alpha,\alpha',\beta,\beta',\gamma}\,t^{\rho+\mu\xi+\nu\eta-1})(x)\,d\xi\,d\eta, \tag{2.6}$$

Now by applying ***Lemma*** 1.2, we arrive at

$$I_2 = x^{\rho-\alpha-\alpha'+\gamma-1}\frac{1}{(2\pi\omega)^2}\int_{L_1}\int_{L_2} \phi_1(\xi)\,\phi_2(\eta)\,\psi(\xi,\eta)\,(z_1x^{\mu})^{\xi}\,(z_2x^{\nu})^{\eta}$$
$$\times\frac{\Gamma(1-\rho+\alpha+\alpha'-\gamma-\mu\xi-\nu\eta)\Gamma(1-\rho+\alpha+\beta'-\gamma-\mu\xi-\nu\eta)\Gamma(1-\rho-\beta-\mu\xi-\nu\eta)}{\Gamma(1-\rho-\mu\xi-\nu\eta)\Gamma(1-\rho+\alpha+\alpha'+\beta'-\gamma-\mu\xi-\nu\eta)\Gamma(1-\rho+\alpha-\beta-\mu\xi-\nu\eta)}\,d\xi\,d\eta.$$

By re-interpreting the Mellin-Barnes contour integral in terms of I-function of two variables defined by (1.1) , we obtain the right hand side of (2.5) after little simplifications. This completes proof of ***Theorem 2.2.***
In view of the relation (1.17), we get following corollary concerning right-sided Saigo fractional integral operator [24].

***Corollary* 2.3.** *Let $\alpha,\beta,\gamma,\rho \in C$, $z_1,z_2 \in C$, $Re(\alpha) > 0$, $\mu,\nu \in R_+$. Further let the constants $m_1,n_1,p,q \in N_0$, $a_j,b_j,a_{ji},b_{ji} \in C$, $\alpha_j,\beta_j,\alpha_{ji},\beta_{ji} \in R_+ (i = 1,\cdots,p_i^{(1)};\ j = 1,\cdots,q_i^{(1)})$, $c_j,d_j,c_{ji},d_{ji} \in C$, $\gamma_j,\delta_j,\gamma_{ji},\delta_{ji} \in R_+ (i = 1,\cdots,p_i^{(2)};\ j = 1,\cdots,q_i^{(2)})$, $|arg\,z_1| < \frac{A_i\pi}{2}$, $|arg\,z_2| < \frac{B_i\pi}{2}$, $A_i > 0$, $B_i > 0$ and satisfy the condition*

$$Re(\rho) + \mu\max_{1\le j\le n_2}\left[\tfrac{Re(a_j)-1}{\alpha_j}\right] + \nu\max_{1\le j\le n_3}\left[\tfrac{Re(c_j)-1}{\gamma_j}\right] < 1 + min\left[Re(\beta), Re(\gamma)\right].$$

Then the fractional integration $I_-^{\alpha,\beta,\gamma}$ of the I-function of two variables exists and the following relation holds:

$$\left\{I_-^{\alpha,\beta,\gamma}t^{\rho-1}I_{p,q:Q}^{m_1,n_1:P}\left[\begin{array}{c|c} z_1t^{\mu} & [(e_p:E_p,E_p')]:U \\ z_2t^{\nu} & [(f_q:F_q,F_q')]:V \end{array}\right]\right\}(x) \tag{2.7}$$
$$= x^{\rho-\beta-1}I_{p+2,q+2:Q}^{m_1+2,n_1:P}\left[\begin{array}{c|c} z_1x^{\mu} & [(e_p:E_p,E_p')],[(1-\rho:\mu,\nu)],[(1-\rho+\gamma:\mu,\nu)],:U \\ z_2x^{\nu} & [(1-\rho+\beta:\mu,\nu)],[(1-\rho+\gamma:\mu,\nu)][(f_q:F_q,F_q')]:V \end{array}\right].$$

Further, if we set $\beta = -\alpha$ in (2.7), we get following corollary concerning right-sided Riemann Liouville fractional integral operator [24].

***Corollary* 2.4.** *Let $\alpha,\rho \in C$, $z_1,z_2 \in C$, $Re(\alpha) > 0$, $\mu,\nu \in R_+$. Further let the constants $m_1,n_1,p,q \in N_0$, $a_j,b_j,a_{ji},b_{ji} \in C$, $\alpha_j,\beta_j,\alpha_{ji},\beta_{ji} \in R_+ (i = 1,\cdots,p_i^{(1)};\ j = 1,\cdots,q_i^{(1)})$, $c_j,d_j,c_{ji},d_{ji} \in C$, $\gamma_j,\delta_j,\gamma_{ji},\delta_{ji} \in R_+ (i = 1,\cdots,p_i^{(2)};\ j = 1,\cdots,q_i^{(2)})$, $|arg\,z_1| < \frac{A_i\pi}{2}$, $|arg\,z_2| < \frac{B_i\pi}{2}$, $A_i > 0$, $B_i > 0$ and satisfy the condition*

$$Re(\alpha) + Re(\rho) + \mu\max_{1\le j\le n_2}\left[\tfrac{Re(a_j)-1}{\alpha_j}\right] + \nu\max_{1\le j\le n_3}\left[\tfrac{Re(c_j)-1}{\gamma_j}\right] < 1.$$

Then the fractional integration I_-^{α} of the I-function of two variables exists and the following relation holds:

$$\left\{I_-^{\alpha}t^{\rho-1}I_{p,q:Q}^{m_1,n_1:P}\left[\begin{array}{c|c} z_1t^{\mu} & [(e_p:E_p,E_p')]:U \\ z_2t^{\nu} & [(f_q:F_q,F_q')]:V \end{array}\right]\right\}(x) \tag{2.8}$$

$$= x^{\rho-\alpha-1} I^{m_1+1,n_1:P}_{p+1,q+1:Q}\left[\begin{array}{c|c} z_1x^{\mu} & [(e_p:E_p,E_p^{'})],[(1-\rho:\mu,\nu)]:U \\ z_2x^{\nu} & [(1-\rho-\alpha:\mu,\nu)],[(f_q:F_q,F_q^{'})]:V \end{array}\right].$$

Theorem 2.3. *Let $\alpha,\alpha',\beta,\beta',\gamma,\rho\in C$, $z_1,z_2\in C$, $Re(\gamma)>0$, $\mu,\nu\in R_+$. Further let the constants $m_1,n_1,p,q\in N_0$, $a_j,b_j,a_{ji},b_{ji}\in C$, $\alpha_j,\beta_j,\alpha_{ji},\beta_{ji}\in R_+(i=1,\cdots,p_i^{(1)};\ j=1,\cdots,q_i^{(1)})$, $c_j,d_j,c_{ji},d_{ji}\in C$, $\gamma_j,\delta_j,\gamma_{ji},\delta_{ji}\in R_+(i=1,\cdots,p_i^{(2)};\ j=1,\cdots,q_i^{(2)})$, $|arg\,z_1|<\frac{A_i\pi}{2}$, $|arg\,z_2|<\frac{B_i\pi}{2}$, $A_i>0$, $B_i>0$ and satisfy the condition*

$$Re(\rho)+\mu\min_{1\le j\le m_2} Re\left(\frac{b_j}{\beta_j}\right)+\nu\min_{1\le j\le m_3} Re\left(\frac{d_j}{\delta_j}\right)>max\left[0,Re(-\alpha-\alpha'-\beta'+\gamma),Re(\beta-\alpha)\right]$$

Then the fractional derivative $D^{\alpha,\alpha',\beta,\beta',\gamma}_{0+}$ of the I-function of two variables exists and the following relation holds:

$$\left\{D^{\alpha,\alpha',\beta,\beta',\gamma}_{0+} t^{\rho-1} I^{m_1,n_1:P}_{p,q:Q}\left[\begin{array}{c|c} z_1t^{\mu} & [(e_p:E_p,E_p^{'})]:U \\ z_2t^{\nu} & [(f_q:F_q,F_q^{'})]:V \end{array}\right]\right\}(x) \tag{2.9}$$

$$= x^{\rho+\alpha+\alpha^{'}-\gamma-1} I^{m_1,n_1+3:P}_{p+3,q+3:Q}\left[\begin{array}{c|c} z_1x^{\mu} & X_5,[(e_p:E_p,E_p^{'})]:U \\ z_2x^{\nu} & [(f_q:F_q,F_q^{'})],X_6:V \end{array}\right],$$

where

$$X_5=[(1-\rho:\mu,\nu)],[(1-\rho-\alpha-\alpha^{'}-\beta'+\gamma:\mu,\nu)],[(1-\rho-\alpha+\beta:\mu,\nu)],$$
$$X_6=[(1-\rho-\alpha-\beta'+\gamma:\mu,\nu)],[(1-\rho+\beta:\mu,\nu)],[(1-\rho-\alpha-\alpha'+\gamma:\mu,\nu)].$$

Proof. To prove the fractional differential formula (2.9) we express I-function of two variables occurring on the left hand side of (2.9) in terms of double Mellin-Barnes contour integral with the help of equations (1.1),we obtain the following form after little simplification:

$$\left\{D^{\alpha,\alpha',\beta,\beta',\gamma}_{0+} t^{\rho-1} I^{m_1,n_1:P}_{p,q:Q}\left[\begin{array}{c|c} z_1t^{\mu} & [(e_p:E_p,E_p^{'})]:U \\ z_2t^{\nu} & [(f_q:F_q,F_q^{'})]:V \end{array}\right]\right\}(x) \tag{2.10}$$

$$= \frac{d^k}{dx^k}\left\{I^{-\alpha',-\alpha,-\beta'+k,-\beta,-\gamma+k}_{0+} t^{\rho-1} I^{m_1,n_1:P}_{p,q:Q}\left[\begin{array}{c|c} z_1t^{\mu} & [(e_p:E_p,E_p^{'})]:U \\ z_2t^{\nu} & [(f_q:F_q,F_q^{'})]:V \end{array}\right]\right\}(x),$$

$$= \frac{1}{(2\pi\omega)^2}\int_{L_1}\int_{L_2}\phi_1(\xi)\,\phi_2(\eta)\,\psi(\xi,\eta)\,z_1^{\xi}\,z_2^{\eta}\,\frac{d^k}{dx^k}(I^{-\alpha',-\alpha,-\beta'+k,-\beta,-\gamma+k}_{0+}\ t^{\rho+\mu\xi+\nu\eta-1})(x)\,d\xi\,d\eta,$$

where $k=[Re(\gamma)+1]$

Applying ***Lemma* 1.1** to (2.10),we obtain

$$= \frac{1}{(2\pi\omega)^2}\int_{L_1}\int_{L_2}\phi_1(\xi)\,\phi_2(\eta)\,\psi(\xi,\eta)\,z_1^{\xi}\,z_2^{\eta}$$
$$\times\frac{\Gamma(\rho+\mu\xi+\nu\eta)\Gamma(\rho+\mu\xi+\nu\eta-\gamma+\alpha'+\alpha+\beta')\Gamma(\rho+\mu\xi+\nu\eta-\beta+\alpha)}{\Gamma(\rho+\mu\xi+\nu\eta+\alpha'+\alpha-\gamma+k)\Gamma(\rho+\mu\xi+\nu\eta-\gamma+\alpha+\beta')\Gamma(\rho+\mu\xi+\nu\eta-\beta)}$$
$$\times\frac{d^k}{dx^k}x^{\rho+\mu\xi+\nu\eta+\alpha'+\alpha-\gamma+k-1}\,d\xi\,d\eta,$$

Using $\frac{d^n}{dx^n}x^m=\frac{\Gamma(m+1)}{\Gamma(m-n+1)}x^{m-n}$ where $m\ge n$ in the above expression, we obtain

$$= x^{\rho+\alpha+\alpha'-\gamma-1}\frac{1}{(2\pi\omega)^2}\int_{L_1}\int_{L_2}\phi_1(\xi)\,\phi_2(\eta)\,\psi(\xi,\eta)\,(z_1x)^{\xi}\,(z_2x)^{\eta}$$

$$\times \frac{\Gamma(\rho+\mu\xi+\nu\eta)\Gamma(\rho+\mu\xi+\nu\eta-\gamma+\alpha'+\alpha+\beta')\Gamma(\rho+\mu\xi+\nu\eta-\beta+\alpha)}{\Gamma(\rho+\mu\xi+\nu\eta-\gamma+\alpha+\beta')\Gamma(\rho+\mu\xi+\nu\eta-\beta)\Gamma(\rho+\mu\xi+\nu\eta+\alpha'+\alpha-\gamma)}\, d\xi\, d\eta.$$

By re-interpreting the Mellin-Barnes contour integral in terms of I-function of two variables defined by (1.1), we obtain the right hand side of (2.9) after little simplifications. This completes proof of ***Theorem* 2.3**.

In view of the relation(1.18), we get following corollary concerning left-sided Saigo fractional derivative operator [24].

***Corollary* 2.5.** *Let* $\alpha,\beta,\gamma,\rho \in C$, $z_1, z_2 \in C$, $Re(\alpha) > 0$, $\mu,\nu \in R_+$. *Further let the constants* $m_1, n_1, p, q \in N_0$, $a_j, b_j, a_{ji}, b_{ji} \in C$, $\alpha_j,\beta_j,\alpha_{ji},\beta_{ji} \in R_+(i = 1,\cdots,p_i^{(1)};\ j = 1,\cdots,q_i^{(1)})$, $c_j, d_j, c_{ji}, d_{ji} \in C$, $\gamma_j,\delta_j,\gamma_{ji},\delta_{ji} \in R_+(i = 1,\cdots,p_i^{(2)};\ j = 1,\cdots,q_i^{(2)})$, $|arg\, z_1| < \frac{A_i\pi}{2}$, $|arg\, z_2| < \frac{B_i\pi}{2}$, $A_i > 0$, $B_i > 0$ *and satisfy the condition*

$$Re(\rho) + \mu \min_{1\le j\le m_2} Re\left(\tfrac{b_j}{\beta_j}\right) + \nu \min_{1\le j\le m_3} Re\left(\tfrac{d_j}{\delta_j}\right) > max\,[0, Re(-\alpha-\beta-\gamma)].$$

Then the fractional derivative $D_{0+}^{\alpha,\beta,\gamma}$ *of the I-function of two variables exists and the following relation holds:*

$$(2.11)\quad \left\{D_{0+}^{\alpha,\beta,\gamma} t^{\rho-1} I_{p,q:Q}^{m_1,n_1:P}\left[\begin{array}{c|c} z_1 t^{\mu} & [(e_p : E_p, E_p')] : U \\ z_2 t^{\nu} & [(f_q : F_q, F_q')] : V \end{array}\right]\right\}(x)$$
$$= x^{\rho+\beta-1} I_{p+2,q+2:Q}^{m_1,n_1+2:P}\left[\begin{array}{c|c} z_1 x^{\mu} & [(1-\rho:\mu,\nu)], [(1-\rho-\alpha-\beta-\gamma:\mu,\nu)], [(e_p : E_p, E_p')] : U \\ z_2 x^{\nu} & [(f_q : F_q, F_q')], [(1-\rho-\gamma:\mu,\nu)], [(1-\rho-\beta:\mu,\nu)] : V \end{array}\right].$$

Next, if we set $\beta = -\alpha$ in the above result, we obtain following result concerning left-sided Riemann-Liouville fractional derivative operator [24].

***Corollary* 2.6.** *Let* $\alpha,\rho \in C$, $z_1, z_2 \in C$, $Re(\alpha) > 0$, $\mu,\nu \in R_+$. *Further let the constants* $m_1, n_1, p, q \in N_0$, $a_j, b_j, a_{ji}, b_{ji} \in C$, $\alpha_j,\beta_j,\alpha_{ji},\beta_{ji} \in R_+(i = 1,\cdots,p_i^{(1)};\ j = 1,\cdots,q_i^{(1)})$, $c_j, d_j, c_{ji}, d_{ji} \in C$, $\gamma_j,\delta_j,\gamma_{ji},\delta_{ji} \in R_+(i = 1,\cdots,p_i^{(2)};\ j = 1,\cdots,q_i^{(2)})$, $|arg\, z_1| < \frac{A_i\pi}{2}$, $|arg\, z_2| < \frac{B_i\pi}{2}$, $A_i > 0$, $B_i > 0$ *and satisfy the condition*

$$Re(\rho) + \mu \min_{1\le j\le m_2} Re\left(\tfrac{b_j}{\beta_j}\right) + \nu \min_{1\le j\le m_3} Re\left(\tfrac{d_j}{\delta_j}\right) > 0.$$

Then the fractional derivative D_{0+}^{α} *of the I-function of two variables exists and the following relation holds:*

$$(2.12)\quad \left\{D_{0+}^{\alpha} t^{\rho-1} I_{p,q:Q}^{m_1,n_1:P}\left[\begin{array}{c|c} z_1 t^{\mu} & [(e_p : E_p, E_p')] : U \\ z_2 t^{\nu} & [(f_q : F_q, F_q')] : V \end{array}\right]\right\}(x)$$
$$= x^{\rho+\alpha-1} I_{p+1,q+1:Q}^{m_1,n_1+1:P}\left[\begin{array}{c|c} z_1 x^{\mu} & [(1-\rho:\mu,\nu)], [(e_p : E_p, E_p')] : U \\ z_2 x^{\nu} & [(f_q : F_q, F_q')], [(1-\rho-\alpha:\mu,\nu)] : V \end{array}\right].$$

***Theorem* 2.4.** *Let* $\alpha,\alpha',\beta,\beta',\gamma,\rho \in C$, $z_1, z_2 \in C$, $Re(\gamma) > 0$, $\mu,\nu \in R_+$. *Further let the constants* $m_1, n_1, p, q \in N_0$, $a_j, b_j, a_{ji}, b_{ji} \in C$, $\alpha_j,\beta_j,\alpha_{ji},\beta_{ji} \in R_+(i = 1,\cdots,p_i^{(1)};\ j = 1,\cdots,q_i^{(1)})$, $c_j, d_j, c_{ji}, d_{ji} \in C$, $\gamma_j,\delta_j,\gamma_{ji},\delta_{ji} \in R_+(i = 1,\cdots,p_i^{(2)};\ j = 1,\cdots,q_i^{(2)})$, $|arg\, z_1| < \frac{A_i\pi}{2}$, $|arg\, z_2| < \frac{B_i\pi}{2}$, $A_i > 0$, $B_i > 0$ *and satisfy the condition*

$$Re(\rho) + \mu \max_{1\le j\le n_2}\left[\tfrac{Re(a_j)-1}{\alpha_j}\right] + \nu \max_{1\le j\le n_3}\left[\tfrac{Re(c_j)-1}{\gamma_j}\right] < 1 + min\,[Re(\beta'), Re(-\alpha-\alpha'+\gamma), Re(-\alpha'-\beta+\gamma)].$$

Then the fractional derivative $D_{-}^{\alpha,\alpha',\beta,\beta',\gamma}$ *of the I-function of two variables exists and the following relation holds:*

$$(2.13)\quad \left\{D_{-}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1} I_{p,q:Q}^{m_1,n_1:P}\left[\begin{array}{c|c} z_1 t^{\mu} & [(e_p : E_p, E_p')] : U \\ z_2 t^{\nu} & [(f_q : F_q, F_q')] : V \end{array}\right]\right\}(x)$$
$$= x^{\rho+\alpha+\alpha'-\gamma-1} I_{p+3,q+3:Q}^{m_1+3,n_1:P}\left[\begin{array}{c|c} z_1 x^{\mu} & [(e_p : E_p, E_p')], X_7 : U \\ z_2 x^{\nu} & X_8, [(f_q : F_q, F_q')] : V \end{array}\right],$$

where

$$X_7 = [(1-\rho : \mu,\nu)], [(1-\rho-\alpha-\alpha'-\beta+\gamma : \mu,\nu)], [(1-\rho-\alpha'+\beta' : \mu,\nu)],$$
$$X_8 = [(1-\rho-\alpha'-\beta+\gamma : \mu,\nu)], [(1-\rho+\beta' : \mu,\nu)], [(1-\rho-\alpha-\alpha'+\gamma : \mu,\nu)].$$

Proof. To prove the fractional differential formula (2.13) we express I-function of two variables occurring on the left hand side of (2.13) in terms of double Mellin-Barnes contour integral with the help of equations (1.1),we obtain the following form after little simplification:

$$(2.14)\quad \left\{D_{-}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1} I_{p,q:Q}^{m_1,n_1:P}\left[\begin{array}{c|c} z_1 t^{\mu} & [(e_p : E_p, E_p')] : U \\ z_2 t^{\nu} & [(f_q : F_q, F_q')] : V \end{array}\right]\right\}(x)$$
$$= (-1)^k \frac{d^k}{dx^k}\left\{I_{-}^{-\alpha',-\alpha,-\beta',-\beta+k,-\gamma+k} t^{\rho-1} I_{p,q:Q}^{m_1,n_1:P}\left[\begin{array}{c|c} z_1 t^{\mu} & [(e_p : E_p, E_p')] : U \\ z_2 t^{\nu} & [(f_q : F_q, F_q')] : V \end{array}\right]\right\}(x)$$
$$= (-1)^k \frac{1}{(2\pi\omega)^2}\int_{L_1}\int_{L_2} \phi_1(\xi)\,\phi_2(\eta)\,\psi(\xi,\eta)\, z_1^{\xi} z_2^{\eta} \frac{d^k}{dx^k}(I_{-}^{-\alpha',-\alpha,-\beta',-\beta+k,-\gamma+k}\, t^{\rho+\mu\xi+\nu\eta-1})(x)\, d\xi\, d\eta,$$

where $k = [Re(\gamma) + 1]$

Applying ***Lemma* 1.2** to (2.14),we obtain

$$= \frac{1}{(2\pi\omega)^2}\int_{L_1}\int_{L_2} \phi_1(\xi)\,\phi_2(\eta)\,\psi(\xi,\eta)\, z_1^{\xi} z_2^{\eta}$$
$$\times\frac{\Gamma(1-\rho-\alpha-\alpha'+\gamma-k-\mu\xi-\nu\eta)\Gamma(1-\rho-\alpha'-\beta+\gamma-\mu\xi-\nu\eta)\Gamma(1-\rho-\beta'-\mu\xi-\nu\eta)}{\Gamma(1-\rho-\mu\xi-\nu\eta)\Gamma(1-\rho-\alpha-\alpha'-\beta+\gamma-\mu\xi-\nu\eta)\Gamma(1-\rho-\alpha'+\beta'-\mu\xi-\nu\eta)}$$
$$\times(-1)^k \frac{d^k}{dx^k} x^{\rho+\mu\xi+\nu\eta+\alpha'+\alpha-\gamma+k-1}\, d\xi\, d\eta,$$
$$= \frac{1}{(2\pi\omega)^2}\int_{L_1}\int_{L_2} \phi_1(\xi)\,\phi_2(\eta)\,\psi(\xi,\eta)\, z_1^{\xi} z_2^{\eta}$$
$$\times\frac{\Gamma(1-\rho-\alpha-\alpha'+\gamma-k-\mu\xi-\nu\eta)\Gamma(1-\rho-\alpha'-\beta+\gamma-\mu\xi-\nu\eta)\Gamma(1-\rho+\beta'-\mu\xi-\nu\eta)}{\Gamma(1-\rho-\mu\xi-\nu\eta)\Gamma(1-\rho-\alpha-\alpha'-\beta+\gamma-\mu\xi-\nu\eta)\Gamma(1-\rho-\alpha'+\beta'-\mu\xi-\nu\eta)}$$
$$\times(1-\rho-\alpha-\alpha'+\gamma-k-\mu\xi-\nu\eta)_k\, x^{\rho+\mu\xi+\nu\eta+\alpha'+\alpha-\gamma-1}\, d\xi\, d\eta,$$
$$= \frac{1}{(2\pi\omega)^2}\int_{L_1}\int_{L_2} \phi_1(\xi)\,\phi_2(\eta)\,\psi(\xi,\eta)\, z_1^{\xi} z_2^{\eta}$$
$$\times\frac{\Gamma(1-\rho-\alpha'-\beta+\gamma-\mu\xi-\nu\eta)\Gamma(1-\rho+\beta'-\mu\xi-\nu\eta)\Gamma(1-\rho-\alpha-\alpha'+\gamma-\mu\xi-\nu\eta)}{\Gamma(1-\rho-\mu\xi-\nu\eta)\Gamma(1-\rho-\alpha-\alpha'-\beta+\gamma-\mu\xi-\nu\eta)\Gamma(1-\rho-\alpha'+\beta'-\mu\xi-\nu\eta)}$$
$$\times x^{\rho+\mu\xi+\nu\eta+\alpha'+\alpha-\gamma-1}\, d\xi\, d\eta.$$

By re-interpreting the Mellin-Barnes contour integral in terms of I-function of two variables defined by (1.1) , we obtain the right hand side of (2.13) after little simplifications. This completes proof of ***Theorem* 2.4**.

In view of the relation (1.19), we get following corollary concerning right-sided Saigo fractional derivative operator [24].

***Corollary* 2.7.** *Let* $\alpha,\beta,\gamma,\rho \in C$, $z_1,z_2 \in C$, $Re(\alpha) > 0$, $\mu,\nu \in R_+$. *Further let the constants* $m_1,n_1,p,q \in N_0$, $a_j,b_j,a_{ji},b_{ji} \in C$, $\alpha_j,\beta_j,\alpha_{ji},\beta_{ji} \in R_+(i = 1,\cdots,p_i^{(1)};\ j = 1,\cdots,q_i^{(1)})$, $c_j,d_j,c_{ji},d_{ji} \in C$, $\gamma_j,\delta_j,\gamma_{ji},\delta_{ji} \in R_+(i = 1,\cdots,p_i^{(2)};\ j = 1,\cdots,q_i^{(2)})$, $|arg\, z_1| < \frac{A_i\pi}{2}$, $|arg\, z_2| < \frac{B_i\pi}{2}$, $A_i > 0$, $B_i > 0$ *and satisfy the condition*

$$Re(\rho) + \mu \max_{1\le j\le n_2}\left[\frac{Re(a_j)-1}{\alpha_j}\right] + \nu \max_{1\le j\le n_3}\left[\frac{Re(c_j)-1}{\gamma_j}\right] < 1 + min\left[Re(-\beta), Re(\alpha+\gamma)\right].$$

Then the fractional derivative $D_-^{\alpha,\beta,\gamma}$ *of the I-function of two variables exists and the following relation holds:*

$$(2.15)\quad \left\{D_-^{\alpha,\beta,\gamma} t^{\rho-1} I_{p,q:Q}^{m_1,n_1:P}\left[\begin{array}{c|c} z_1 t^{\mu} & [(e_p:E_p,E_p^{'})]:U \\ z_2 t^{\nu} & [(f_q:F_q,F_q^{'})]:V \end{array}\right]\right\}(x)$$

$$= x^{\rho+\beta-1} I_{p+2,q+2:Q}^{m_1+2,n_1:P}\left[\begin{array}{c|c} z_1 x^{\mu} & [(e_p:E_p,E_p^{'})],[(1-\rho:\mu,\nu)],[(1-\rho-\beta+\gamma:\mu,\nu)]:U \\ z_2 x^{\nu} & [(1-\rho+\alpha+\gamma:\mu,\nu)],[(1-\rho-\beta':\mu,\nu)],[(f_q:F_q,F_q^{'})]:V \end{array}\right].$$

Further, if we set $\beta = -\alpha$ in (2.15), we obtain following corollary concerning right-sided Riemann-Liouville derivative operator [24].

***Corollary* 2.8.** *Let* $\alpha,\rho \in C$, $z_1,z_2 \in C$, $Re(\alpha) > 0$, $\mu,\nu \in R_+$. *Further let the constants* $m_1,n_1,p,q \in N_0$, $a_j,b_j,a_{ji},b_{ji} \in C$, $\alpha_j,\beta_j,\alpha_{ji},\beta_{ji} \in R_+(i = 1,\cdots,p_i^{(1)};\ j = 1,\cdots,q_i^{(1)})$, $c_j,d_j,c_{ji},d_{ji} \in C$, $\gamma_j,\delta_j,\gamma_{ji},\delta_{ji} \in R_+(i = 1,\cdots,p_i^{(2)};\ j = 1,\cdots,q_i^{(2)})$, $|arg\, z_1| < \frac{A_i\pi}{2}$, $|arg\, z_2| < \frac{B_i\pi}{2}$, $A_i > 0$, $B_i > 0$ *and satisfy the condition*

$$Re(\rho) + Re(\alpha) + \mu \max_{1\le j\le n_2}\left[\frac{Re(a_j)-1}{\alpha_j}\right] + \nu \max_{1\le j\le n_3}\left[\frac{Re(c_j)-1}{\gamma_j}\right] < 0.$$

Then the fractional derivative D_-^{α} *of the I-function of two variables exists and the following relation holds:*

$$(2.16)\quad \left\{D_-^{\alpha} t^{\rho-1} I_{p,q:Q}^{m_1,n_1:P}\left[\begin{array}{c|c} z_1 t^{\mu} & [(e_p:E_p,E_p^{'})]:U \\ z_2 t^{\nu} & [(f_q:F_q,F_q^{'})]:V \end{array}\right]\right\}(x)$$

$$= x^{\rho+\alpha-1} I_{p+1,q+1:Q}^{m_1+1,n_1:P}\left[\begin{array}{c|c} z_1 x^{\mu} & [(e_p:E_p,E_p^{'})],[(1-\rho:\mu,\nu)]:U \\ z_2 x^{\nu} & [(1-\rho-\alpha:\mu,\nu)],[(f_q:F_q,F_q^{'})]:V \end{array}\right].$$

3 Special Cases

The I-function of two variables is a most generalized form of special functions, consequently it can be reduced in a large number of special functions (or product of such functions) by suitably specializing the parameters involved in the function. Here we provide a few special cases of our main results.

(i) If we set $m_1 = n_1 = p = q = 0$ in ***Theorem* 2.1** then we have following known result given by Saxena *et al* [26], p.637, eq.(3.3) in terms of product of I-function of one variable introduced by Saxena [25].

$$(3.1)\quad \left\{I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1} I_{p_i^{(1)},q_i^{(1)}:r}^{m_2,n_2}\left[z_1 t^{\mu}\left|\begin{array}{c} (a_j,\alpha_j)_{1,n_2},(a_{ji},\alpha_{ji})_{n_2+1,p_i^{(1)}} \\ (b_j,\beta_j)_{1,m_2},(b_{ji},\beta_{ji})_{m_2+1,q_i^{(1)}} \end{array}\right.\right]\right.$$

$$\left.\times I_{p_i^{(2)},q_i^{(2)}:r}^{m_3,n_3}\left[z_2 t^{\nu}\left|\begin{array}{c} (c_j,\gamma_j)_{1,n_3},(c_{ji},\gamma_{ji})_{n_3+1,p_i^{(2)}} \\ (d_j,\delta_j)_{1,m_3},(d_{ji},\delta_{ji})_{m_3+1,q_i^{(2)}} \end{array}\right.\right]\right\}(x)$$

$$= x^{\rho+\gamma-\alpha-\alpha^{'}-1} I_{3,3:p_i^{(1)},q_i^{(1)};p_i^{(2)},q_i^{(2)}:r}^{0,3:m_2,n_2;m_3,n_3}\left[\begin{array}{c|c} z_1 x^{\mu} & X_1,\ldots:U \\ z_2 x^{\nu} & X_2,\ldots:V \end{array}\right],$$

where X_1 and X_2 are same as given in ***Theorem* 2.1**. The conditions of validity of the above result easily follow from ***Theorem* 2.1**.

(ii) If we set $m_1 = 0$ and $r = 1$ in ***Theorem* 2.1**, the I-function of two variables occurring in L.H.S. reduces into H-function of two variables [27] then we have following known result given by Dinesh Kumar [11], p.1128, eq.(4.2).

$$\left\{I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1} H^{0,n_1:m_2,n_2;m_3,n_3}_{p,q:p_1^{(1)},q_1^{(1)};p_1^{(2)},q_1^{(2)}}\left[\begin{array}{c} z_1 t^{\mu} \\ z_2 t^{\nu} \end{array}\middle| \begin{array}{c} [(e_p : E_p, E_p')] : T_1 \\ [(f_q : F_q, F_q')] : T_2 \end{array}\right]\right\}(x) \tag{3.2}$$

$$= x^{\rho+\gamma-\alpha-\alpha'-1} H^{0,n_1+3:m_2,n_2;m_3,n_3}_{p+3,q+3:p_1^{(1)},q_1^{(1)};p_1^{(2)},q_1^{(2)}}\left[\begin{array}{c} z_1 x^{\mu} \\ z_2 x^{\nu} \end{array}\middle| \begin{array}{c} X_1, [(e_p : E_p, E_p')] : T_1 \\ [(f_q : F_q, F_q')], X_2 : T_2 \end{array}\right],$$

where

$$T_1 = [(a_j, \alpha_j)_{1,p_1^{(1)}}]; [(c_j, \gamma_j)_{1,p_1^{(2)}}], \quad T_2 = [(b_j, \beta_j)_{1,q_1^{(1)}}]; [(d_j, \delta_j)_{1,q_1^{(2)}}].$$

Also X_1 and X_2 are same as given in ***Theorem* 2.1**. The conditions of validity of the above result easily follow from ***Theorem* 2.1**.

(iii) If we set $m_1 = 0$ and $r = 2$ in ***Theorem* 2.1**, then we obtain a result in terms of a particular case of the I-function of two variables.

$$\left\{I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1} I^{0,n_1:m_2,n_2;m_3,n_3}_{p,q:p_i^{(1)},q_i^{(1)};p_i^{(2)},q_i^{(2)}:2}\left[\begin{array}{c} z_1 t^{\mu} \\ z_2 t^{\nu} \end{array}\middle| \begin{array}{c} [(e_p : E_p, E_p')] : U \\ [(f_q : F_q, F_q')] : V \end{array}\right]\right\}(x) \tag{3.3}$$

$$= x^{\rho+\gamma-\alpha-\alpha'-1} I^{0,n_1+3:m_2,n_2;m_3,n_3}_{p+3,q+3:p_i^{(1)},q_i^{(1)};p_i^{(2)},q_i^{(2)}:2}\left[\begin{array}{c} z_1 x^{\mu} \\ z_2 x^{\nu} \end{array}\middle| \begin{array}{c} X_1, [(e_p : E_p, E_p')] : U \\ [(f_q : F_q, F_q')], X_2 : V \end{array}\right],$$

Also X_1 and X_2 are same as given in ***Theorem* 2.1**. The conditions of validity of the above result easily follow from ***Theorem* 2.1**.

(iv) If we set $m_1 = n_1 = p = q = 0$ and $r = 1$ in ***Theorem* 2.1**, then we have following known result given by J. Ram and D. Kumar [21], p.36, eq.(17) in terms of product of H-functions

$$\left\{I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1} H^{m_2,n_2}_{p_1^{(1)},q_1^{(1)}}\left[z_1 t^{\mu}\middle| \begin{array}{c} (a_j, \alpha_j)_{1,p_1^{(1)}} \\ (b_j, \beta_j)_{1,q_1^{(1)}} \end{array}\right] \times H^{m_3,n_3}_{p_1^{(2)},q_1^{(2)}}\left[z_2 t^{\nu}\middle| \begin{array}{c} (c_j, \gamma_j)_{1,p_1^{(2)}} \\ (d_j, \delta_j)_{1,q_1^{(2)}} \end{array}\right]\right\}(x) \tag{3.4}$$

$$= x^{\rho+\gamma-\alpha-\alpha'-1} H^{0,3:m_2,n_2;m_3,n_3}_{3,3:p_1^{(1)},q_1^{(1)};p_1^{(2)},q_1^{(2)}}\left[\begin{array}{c} z_1 x^{\mu} \\ z_2 x^{\nu} \end{array}\middle| \begin{array}{c} X_1 \ldots : T_1 \\ X_2 \ldots : T_2 \end{array}\right],$$

where X_1 and X_2 are same as given in ***Theorem* 2.1**, T_1 and T_2 are also same as given in (3.2). The conditions of validity of the above result easily follow from ***Theorem* 2.1**.

(v) On putting $m_1 = n_1 = p = q = 0, r = 1, \mu = 1, p_1^{(1)} = 0, m_2 = q_1^{(1)} = 1, b_1 = 0$ and $\beta_1 = 1$ in ***Theorem* 2.1** then by virtue of the relation $H^{1,0}_{0,1}[z_1 t|(0,1)] = e^{-z_1 t}$ we have following known result given by Saxena *et al.* [26], p.643, eq.(5.1).

$$\left\{I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1} e^{-z_1 t} H^{m_3,n_3}_{p_1^{(2)},q_1^{(2)}}\left[z_2 t^{\nu}\middle| \begin{array}{c} (c_j, \gamma_j)_{1,p_1^{(2)}} \\ (d_j, \delta_j)_{1,q_1^{(2)}} \end{array}\right]\right\}(x) \tag{3.5}$$

$$= x^{\rho+\gamma-\alpha-\alpha'-1} H^{0,3:1,0;m_3,n_3}_{3,3:0,1;p_1^{(2)},q_1^{(2)}} \left[\begin{array}{c} z_1 x \\ z_2 x^{\nu} \end{array} \middle| \begin{array}{c} X_9 ... : -; (c_j, \gamma_j)_{1,p_1^{(2)}} \\ X_{10} ... : (0,1); (d_j, \delta_j)_{1,q_1^{(2)}} \end{array} \right],$$

where

$$X_9 = [(1-\rho : 1, \nu)], [(1-\rho+\alpha+\alpha' + \beta - \gamma : 1, \nu)], [(1-\rho+\alpha' - \beta' : 1, \nu)],$$
$$X_{10} = [(1-\rho+\alpha+\alpha' - \gamma : 1, \nu)], [(1-\rho+\alpha' + \beta - \gamma : 1, \nu)], [(1-\rho-\beta' : 1, \nu)].$$

The conditions of validity of the above result easily follow from ***Theorem* 2.1**.
(vi) On setting $z_1 = 0$ in (3.5), we have following result.

(3.6)
$$\left\{ I^{\alpha,\alpha',\beta,\beta',\gamma}_{0+} t^{\rho-1} H^{m_3,n_3}_{p_1^{(2)},q_1^{(2)}} \left[z_2 t^{\nu} \middle| \begin{array}{c} (c_j, \gamma_j)_{1,p_1^{(2)}} \\ (d_j, \delta_j)_{1,q_1^{(2)}} \end{array} \right] \right\}(x)$$
$$= x^{\rho+\gamma-\alpha-\alpha'-1} H^{m_3,n_3+3}_{p_1^{(2)}+3,q_1^{(2)}+3} \left[z_2 x^{\nu} \middle| \begin{array}{c} X_{11}, (c_j, \gamma_j)_{1,p_1^{(2)}} \\ (d_j, \delta_j)_{1,q_1^{(2)}}, X_{12} \end{array} \right],$$

where

$$X_{11} = [(1-\rho : \nu)], [(1-\rho+\alpha+\alpha' + \beta - \gamma : \nu)], [(1-\rho+\alpha' - \beta' : \nu)],$$
$$X_{12} = [(1-\rho+\alpha+\alpha' - \gamma : \nu)], [(1-\rho+\alpha' + \beta - \gamma : \nu)], [(1-\rho-\beta' : \nu)].$$

The conditions of validity of the above result easily follow from ***Theorem* 2.1.**
(vi) Further on reducing H-function to Wright generalized hypergeometric function in (3.6) due to the relation

$${}_p\psi_q \left[\begin{array}{c} (c_1, \gamma_1), \cdots, (c_p, \gamma_p); \\ (d_1, \delta_1), \cdots, (d_q, \delta_q); \end{array} \middle| z \right] = \sum_{k=0}^{\infty} \frac{\prod_{j=1}^{p} \Gamma(c_j + \gamma_j k)}{\prod_{j=1}^{q} \Gamma(d_j + \delta_j k)} \frac{z^k}{k!}$$
$$= H^{1,p}_{p,q+1} \left[-z \middle| \begin{array}{c} (1-c_1, \gamma_1), \cdots, \left(1-c_p, \gamma_p\right) \\ (0,1), (1-d_1, \delta_1), \cdots, \left(1-d_q, \delta_q\right) \end{array} \right].$$

We obtain following result

(3.7)
$$\left\{ I^{\alpha,\alpha',\beta,\beta',\gamma}_{0+} t^{\rho-1} {}_p\psi_q \left[z t^{\nu} \middle| \begin{array}{c} (c_j, \gamma_j)_{1,p} \\ (d_j, \delta_j)_{1,q} \end{array} \right] \right\}(x)$$
$$= x^{\rho+\gamma-\alpha-\alpha'-1} {}_{p+3}\psi_{q+3} \left[z x^{\nu} \middle| \begin{array}{c} (c_j, \gamma_j)_{1,p}, X_{13} \\ (d_j, \delta_j)_{1,q}, X_{14} \end{array} \right],$$

where

$$X_{13} = (\rho, \nu), (\rho - \alpha - \alpha' - \beta + \gamma, \nu), (\rho - \alpha' + \beta', \nu),$$
$$X_{14} = (\rho - \alpha - \alpha' + \gamma, \nu), (\rho - \alpha' - \beta + \gamma, \nu), (\rho + \beta', \nu).$$

The conditions of validity of the above result easily follow from ***Theorem* 2.1**.
Acknowledgement. Acknowledgement We are very much thankful to the Editor and Referee for their valuable suggestions to bring the paper in its present form.

References

[1] Praveen Agarwal, Generalized fractional integration of the $\overline{H}$ function, *Le Matematiche*, **LXVII**-Fasc. II, (2012), 107-118.

[2] D. Baleanu and O. G. Mustafa, On the global existence of solutions to a class of fractional differential equations, *Comput. Math. Appl.*, **59(5)** (2010), 1835-1841.

[3] D. Baleanu and O.G. Mustafa and R.P. Agarwal, On the solution set for a class of sequential fractional differential equations, *J. Phys. A Math. Theor.*, **43(38)** (2010), Article ID 385209.

[4] R. C. Singh Chandel, Fractional integration and integral representations of certain generalized hypergeometric functions of several variables, *Jñānābha* **Sect. A, I** (1971), 45-56.

[5] R. C. Singh Chandel and P. K. Vishwakarma, Multidimensional fractional derivatives of multiple hypergeometric functions of several variables, *Jñānābha* , **24** (1994), 19-27.

[6] R. C. Singh Chandel and Vandana Gupta, On some generalized fractional integrals involving generalized special functions of several variables, *Jñānābha* , 43(2013), 128-148.

[7] K.C. Gupta, K. Gupta and A. Gupta, Generalized fractional integration of the product of two H-functions, *J. Rajasthan Acad. Phys. Sci.*, **9(3)** (2010), 203-212.

[8] Anil Goyal and R.D.Agrawal, Integral involving the product of I-function of two variables, *Journal of Maulana Azad College of Technology*, **28** (1995), 147-155.

[9] V. Kiryakova, *Generalized Fractional Calculus and Applications,* Pitman Res. Notes Math. Ser., **301**, Longman Scientific & Technical , Harlow, 1994.

[10] Dinesh Kumar, Fractional calculus formulas involving $\bar{H}$-function and srivastava polynomials, *Commun. Korean Math. Soc.*, **31 (4)** (2016), 827-844.

[11] Dinesh Kumar, Generalized fractional differintegral operators of the Aleph-function of two variables, *Jour. of Chem., Bio. and Phy. Sci.*, **6 (3)** (2016), 1116-1131.

[12] D. Kumar, S. D. Purohit and J. Choi, Generalized fractional integrals involving product of multivariable H-function and a general class of polynomials, *J. Nonlinear Sci. Appl.*, **9(1)** (2016) , 8-21.

[13] Hemant Kumar, On convergence properties and applications of two variable generalized Mittag-Leffler function, *Jñānābha*, **47 (1)** (June 2017), 125-138.

[14] Hemant Kumar, R. C. Singh Chandel and Harish Srivastava, On a fractional non linear biological madel problem and its approximate solutions through Volterra integral equation, *Jñānābha*, **47 (1)** (June 2017), 143-154.

[15] Hemant Kumar, M. A. Pathan and Shilesh Kumari, Identities for generalized fractional integral operators associated with product of analogous to Dirichlet averages and special function,*IJEST* **2 (5)** (2010), 149-161.

[16] K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Differential Equations*, A Wiley Interscience Publication, John Wiley and Sons Inc., New York, 1993.

[17] A. M. Mathai,R. K. Saxena and H.J. Haubold, *The H-function: Theory and Applications,* Springer, New York, 2010.

[18] M. A. Pathan, Hemant Kumar, Harish Srivastava and R. C. Singh Chandel, Summability and numerical approximation of the series involving Lauricella's triple hypergeometric functions, *Jñānābha*, **46** (2016), 91-104.

[19] S. D. Purohit and S. L. Kalla, On fractional partial differential equations related to quantum mechanics, *J. Phys. A Math. Theor.*, **44(4)** (2011), Article ID 045202.

[20] S. D. Purohit, Solutions of fractional partial differential equations of quantum mechanics, *Adv. Appl. Math Mech.*, **5(5)** (2013), 639-651.

[21] J. Ram and D. Kumar, Generalized fractional integration involving Appell hypergeometric of the product of two H-functions, *Vijanana Parishad Anusandhan Patrika,* **54(B) (3)** (2011), 33-43.

[22] S. G. Samko, A. A. Kilbas and O. I. Marichev, *Fractional Integrals and Derivatives, Theory and Applications*, Gordon and Breach, Yverdon et alibi, 1993.

[23] M. Saigo and N.Maeda, *More generalization of fractional calculus, Transform Methods and Special Functions,* Varna, Bulgaria, (1996), 386-400.

[24] M. Saigo, A remark on integral operators involving the Gauss hypergeometric function, *Math. Rep. Kyushu Univ.*, **11 (2)**, (1977/78), 135-143.

[25] V. P. Saxena, *The I-Function.* Anamaya publishers, New Delhi, 2008.

[26] R. K. Saxena, J. Ram and D. Kumar, Generalized fractional integral of the product of two Aleph-functions, *App. and App. Math.*. **8** (2013), 631-646.

[27] H.M.Srivastava, K. C. Gupta and S. P. Goyal, *The H-Function of One and Two Variables with Applications.* South Asian publications. New Delhi,Madras. 1982.

[28] H. M. Srivastava, R. K. Saxena and J. Ram, Some multidimensional fractional integral operators involving a general class of polynomials, *J. Math. Anal. Appl.*, **193 (2)**(1995), 373-389.

[29] H. M. Srivastava, R. C. Singh Chandel and P. K. Vishwakarma, Multidimensional fractional derivatives of multiple hypergeometric functions of several variables, Jour. Math. Anal. Appl., **184** (1994), 19-27.

Jñānābha, Vol. 50(1) (2020), 179-188
(Dedicated to Honor Professor H.M. Srivastava on His 80th Birth Anniversary Celebrations)

HYPERGEOMETRIC REPRESENTATIONS OF SOME MATHEMATICAL FUNCTIONS VIA MACLAURIN SERIES

By
M. I. Qureshi, Shakir Hussain Malik⋆ and Tafaz ul Rahman Shah
Department of Applied Sciences and Humanities
Faculty of Engineering and Technology
Jamia Millia Islamia (A Central University), New Delhi -110025, India.
Email:miqureshi_delhi@yahoo.co.in,⋆malikshakir774@gmail.com, tafazuldiv@gmail.com

(*Received : May* 27, 2020 *; Revised: June* 07, 2020)
DOI: https://doi.org/10.58250/jnanabha.2020.50118

Abstract

In this paper, by using Maclaurin series of given mathematical function and expressing the coefficient of the general term of corresponding Maclaurin series in the form of Pochhammer symbols, we obtain the hypergeometric forms of following functions:

$$\frac{\sin^{-1}(x)}{\sqrt{(1-x^2)}},\ [\sin^{-1}(x)]^2,\ \sin^{-1}(x),\ \frac{\sinh^{-1}(x)}{\sqrt{(1+x^2)}},\ [\sinh^{-1}(x)]^2,\ \sinh^{-1}(x) \text{ and } \ell n\{e(1-x)^{\frac{1}{x}}\}^{-\frac{2}{x}}.$$

2010 Mathematics Subject Classifications: 33Cxx, 34A35, 41A58, 33B10.
Keywords and phrases: Hypergeometric functions, Leibnitz theorem, Maclaurin series, Pochhammer symbol.

1 Introduction and Preliminaries

In our investigations, we shall use the following standard notations:
$\mathbb{N} := \ \{1,2,3,\cdots\}\,;\mathbb{N}_0 := \mathbb{N}\bigcup\{0\}\,;\mathbb{Z}_0^- := \mathbb{Z}^-\bigcup\{0\} = \{0,-1,-2,-3,\cdots\}\,.$

The symbols $\mathbb{C},\mathbb{R},\mathbb{N},\mathbb{Z},\mathbb{R}^+$ and $\mathbb{R}^-$ denote the sets of complex numbers, real numbers, natural numbers, integers, positive and negative real numbers respectively.

Pochhammer symbol:
The Pochhammer symbol (or the *shifted* factorial) $(\lambda)_\nu$ $(\lambda,\nu\in\mathbb{C})$[13, p.22 eq(1), p.32 Q.N.(8) and Q.N.(9)], see also [15, p.23, eq(22) and eq(23)], is defined by

$$(\lambda)_\nu := \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} = \begin{cases} 1 & (\nu=0;\ \lambda\in\mathbb{C}\backslash\{0\}) \\ \prod\limits_{j=0}^{n-1}(\lambda+j) & (\nu=n\in\mathbb{N};\ \lambda\in\mathbb{C}) \\ \frac{(-1)^k n!}{(n-k)!} & (\lambda=-n;\ \nu=k;\ n,k\in\mathbb{N}_0;\ 0\leqq k\leqq n) \\ 0 & (\lambda=-n;\ \nu=k;\ n,k\in\mathbb{N}_0;\ k>n) \\ \frac{(-1)^k}{(1-\lambda)_k} & (\nu=-k;\ k\in\mathbb{N};\ \lambda\in\mathbb{C}\backslash\mathbb{Z}), \end{cases}$$

it being understood conventionally that $(0)_0 = 1$ and assumed tacitly that the Gamma quotient exists.

Generalized hypergeometric function of one variable:
A natural generalization of the Gaussian hypergeometric series ${}_2F_1[\alpha,\beta;\gamma;z]$, is accomplished by

introducing any arbitrary number of numerator and denominator parameters. Thus, the resulting series

$$(1.1)\qquad {}_pF_q\left[\begin{matrix}(\alpha_p);\\ \\ (\beta_q);\end{matrix}\; z\right] = {}_pF_q\left[\begin{matrix}\alpha_1,\alpha_2,\ldots,\alpha_p;\\ \\ \beta_1,\beta_2,\ldots,\beta_q;\end{matrix}\; z\right] = \sum_{n=0}^{\infty}\frac{(\alpha_1)_n(\alpha_2)_n\ldots(\alpha_p)_n}{(\beta_1)_n(\beta_2)_n\ldots(\beta_q)_n}\,\frac{z^n}{n!}\,,$$

is known as the generalized hypergeometric series, or simply, the generalized hypergeometric function. Here p and q are positive integers or zero and we assume that the variable z, the numerator parameters $\alpha_1,\alpha_2,\ldots,\alpha_p$ and the denominator parameters $\beta_1,\beta_2,\ldots,\beta_q$ take on complex values, provided that

$$\beta_j \neq 0,-1,-2,\ldots\;;\; j = 1,2,\ldots,q.$$

Supposing that none of the numerator and denominator parameters is zero or a negative integer, we note that the ${}_pF_q$ series defined by equation (1.1):

(i) converges for $|z| < \infty$, if $p \leq q$,
(ii) converges for $|z| < 1$, if $p = q+1$,
(iii) diverges for all z, $z \neq 0$, if $p > q+1$,
(iv) converges absolutely for $|z| = 1$, if $p = q+1$, and $\Re(\omega) > 0$,
(v) converges conditionally for $|z| = 1 (z \neq 1)$, if $p = q+1$ and $-1 < \Re(\omega) \leqq 0$,
(vi) diverges for $|z| = 1$, if $p = q+1$ and $\Re(\omega) \leqq -1$,

where by convention, a product over an empty set is interpreted as 1 and

$$(1.2)\qquad \omega := \sum_{j=1}^{q}\beta_j - \sum_{j=1}^{p}\alpha_j,$$

$\Re(\omega)$ being the real part of complex number ω.

Relation between inverse hyperbolic and inverse trigonometric functions:

$$(1.3)\qquad \sin^{-1}(i\theta) = i\sinh^{-1}(\theta)\;\; or\;\; \sinh^{-1}(i\theta) = i\sin^{-1}(\theta).$$

Leibnitz theorem:

The *nth* derivative of the product of two functions, is given by

$$(1.4)\quad D^n[U(x)\,T(x)] = ({}^nC_0)(D^n\,U)(D^0\,T) + ({}^nC_1)(D^{n-1}\,U)(D^1\,T)$$
$$+ ({}^nC_2)(D^{n-2}\,U)(D^2\,T) + \cdots + +({}^nC_{n-1})(D^1\,U)(D^{n-1}\,T) + ({}^nC_n)(D^0\,U)(D^n\,T).$$

Maclaurin series :

Suppose *nth* derivative of $y(x)$, w.r.t. x is denoted by $D^ny = \frac{d^ny}{dx^n} = y_n$.

Then

$$y(x) = (y)_0 + x(y_1)_0 + \frac{x^2}{2!}(y_2)_0 + \frac{x^3}{3!}(y_3)_0 + \frac{x^4}{4!}(y_4)_0 + \frac{x^5}{5!}(y_5)_0 + \cdots$$

$$(1.5)\qquad = \sum_{n=0}^{\infty}\frac{x^n}{n!}(y_n)_0,$$

$$(1.6)\qquad = \sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}(y_{2n})_0 + \sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!}(y_{2n+1})_0,$$

where,

$$(y_m)_0 = \left(\frac{d^m y}{dx^m}\right)_{x=0}.$$

(1.7) $$(\alpha)_{2n} = 2^{2n}\left(\frac{\alpha}{2}\right)_n\left(\frac{\alpha+1}{2}\right)_n.$$

The present article is organized as follows. In ***Section*** **3**, we have derived the hypergeometric forms of some functions involving arcsine function and logarithmic function by using Maclaurin series. In ***Section*** **4**, we have given hypergeometric forms of some inverse hyperbolic sine function as special cases. For hypergeometric forms of other mathematical functions and functions of mathematical physics, one can refer the literature [1],[2],[3],[4],[5],[6],[7],[8],[14] and [16], where the proof of hypergeometric forms of related functions are not given. So we are interested to give the proof of hypergeometric forms of the functions mentioned in ***Section*** **2**. For some recent related work, the interested readers can consult the papers by Qureshi, et al.[9, 10, 11, 12].

2 Some Hypergeometric Forms Involving Arcsine Function and Logarithmic Function

When $|x| < 1$, then following hypergeometric forms hold true:

(2.1) $$\frac{\sin^{-1}(x)}{\sqrt{(1-x^2)}} = x\ {}_2F_1\left[\begin{matrix} 1,\ 1; \\ \frac{3}{2}; \end{matrix}\ x^2\right].$$

(2.2) $$[\sin^{-1}(x)]^2 = x^2\ {}_3F_2\left[\begin{matrix} 1,\ 1,\ 1; \\ 2,\ \frac{3}{2}; \end{matrix}\ x^2\right].$$

(2.3) $$\sin^{-1}(x) = x\ {}_2F_1\left[\begin{matrix} \frac{1}{2},\ \frac{1}{2}; \\ \frac{3}{2}; \end{matrix}\ x^2\right].$$

(2.4) $$\ell n\{e(1-x)^{\frac{1}{x}}\}^{-\frac{2}{x}} = {}_2F_1\left[\begin{matrix} 1,\ 2; \\ 3; \end{matrix}\ x\right].$$

3 Proof of Hypergeometric Forms

Proof of hypergeometric form (2.1)***:***

Consider the following function

(3.1) $$y = y(x) = \frac{\sin^{-1}(x)}{\sqrt{(1-x^2)}},$$

that is

$$\sqrt{(1-x^2)}\ y = \sin^{-1}(x).$$

Put $x = 0$ in the equation (3.1), we get

(3.2) $$(y)_0 = 0\ .$$

Differentiate the equation (3.1) w.r.t. x and use product rule, after simplification we get

(3.3) $$(1-x^2)y_1 - xy = 1,$$

(3.4) $$(y_1)_0 = 1\ .$$

Again differentiate the equation (3.3) w.r.t. x and use product rule, after simplification we have

$$(1-x^2)y_2 - 3xy_1 - y = 0, \tag{3.5}$$

$$(y_2)_0 = 0\,. \tag{3.6}$$

Now differentiate the equation (3.3) n-times w.r.t. x and apply Leibnitz theorem we obtain

$$D^n[(1-x^2)y_1] - D^n[xy] = D^n[1];\ n \geq 2, \tag{3.7}$$

$$(1-x^2)y_{n+1} - (2n+1)xy_n - n^2 y_{n-1} = 0;\ n \geq 2\,.$$

Put $x = 0$ in the equation (**??**), we get

$$(y_{n+1})_0 = n^2 (y_{n-1})_0;\ n \geq 2\,. \tag{3.8}$$

Put $n = 2, 3, 4, 5, 6, 7, 8, 9$ in the equation (3.8), we get

$$(y_3)_0 = (2)^2(1), \tag{3.9}$$

$$(y_5)_0 = (4)^2(2)^2(1), \tag{3.10}$$

$$(y_7)_0 = (6)^2(4)^2(2)^2(1), \tag{3.11}$$

$$(y_9)_0 = (8)^2(6)^2(4)^2(2)^2(1). \tag{3.12}$$

Using the equation (3.8), we can write the recurrence relation:

$$(y_m)_0 = (m-1)^2(y_{m-2})_0; m \geq 2\,. \tag{3.13}$$

When $m = 2n$, then

$$(y_{2n})_0 = (2n-1)^2(y_{2n-2})_0 = 0\,. \tag{3.14}$$

When $m = 2n+1$, then

$$\begin{aligned}(y_{2n+1})_0 &= (2n)^2(y_{2n-1})_0 \\ &= (2n)^2(2n-2)^2(2n-4)^2 \cdots (8)^2(6)^2(4)^2(2)^2(1) \\ &= \{2^n(1 \times 2 \times 3 \times 4 \times \cdots \times n)\}^2 \\ &= 4^n(n!)^2.\end{aligned} \tag{3.15}$$

Now using Maclaurin series, we get

$$y = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}(y_{2n})_0 + \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}(y_{2n+1})_0\,.$$

The function $\frac{\sin^{-1}(x)}{\sqrt{(1-x^2)}}$ is an odd function so that the even coefficients of its Maclaurin expansion vanish. That is

$$\begin{aligned}y &= 0 + \sum_{n=0}^{\infty} \frac{x^{2n+1}\ 4^n\ (n!)^2}{(2n+1)!} \\ &= x \sum_{n=0}^{\infty} \frac{4^n\ (1)_n\ (1)_n\ x^{2n}}{(1)_{2n+1}} \\ &= x \sum_{n=0}^{\infty} \frac{4^n\ (1)_n\ (1)_n\ x^{2n}}{(2)_{2n}} \\ &= x \sum_{n=0}^{\infty} \frac{(1)_n\ (1)_n\ x^{2n}}{(\frac{3}{2})_n\ n!}.\end{aligned}$$

On using the definition (1.1), we get the required hypergeometric form (2.1).

Proof of hypergeometric form* (2.2)*:

Consider the following function

$$y = y(x) = [\sin^{-1}(x)]^2. \tag{3.16}$$

Put $x = 0$ in the equation (3.16), we get

$$(y)_0 = 0\,. \tag{3.17}$$

Differentiate the equation (3.16) w.r.t. x, we get

$$\sqrt{(1-x^2)}\,y_1 = 2[\sin^{-1}(x)], \tag{3.18}$$

$$(y_1)_0 = 0\,. \tag{3.19}$$

Differentiate the equation (3.18) w.r.t. x and use product rule, after simplification we get

$$(1-x^2)y_2 - xy_1 = 2, \tag{3.20}$$

$$(y_2)_0 = 2. \tag{3.21}$$

Again differentiate the equation (3.20) w.r.t. x and use product rule, after simplification we have

$$(1-x^2)y_3 - 3xy_2 - y_1 = 0. \tag{3.22}$$

Now differentiate the equation (3.20) n-times w.r.t. x and apply Leibnitz theorem we obtain

$$D^n[(1-x^2)y_2] - D^n[xy_1] = D^n[2];\, n \geq 2, \tag{3.23}$$

$$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2 y_n = 0;\, n \geq 2.$$

Put $x = 0$ in the equation (3.23), we get

$$(y_{n+2})_0 = n^2 (y_n)_0;\, n \geq 2\,. \tag{3.24}$$

Put $n = 2, 3, 4, 5, 6, 7, 8$ in the equation (3.24), we get

$$(y_4)_0 = (2)^2(2), \tag{3.25}$$

$$(y_6)_0 = (4)^2(2)^2(2), \tag{3.26}$$

$$(y_8)_0 = (6)^2(4)^2(2)^2(2), \tag{3.27}$$

$$(y_{10})_0 = (8)^2(6)^2(4)^2(2)^2(2)\,. \tag{3.28}$$

Using the equation (3.24), we can write the recurrence relation:

$$(y_m)_0 = (m-2)^2 (y_{m-2})_0;\, m \geq 2\,. \tag{3.29}$$

When $m = 2n$, then

$$\begin{aligned}(y_{2n})_0 &= (2n-2)^2 (y_{2n-2})_0 \\ &= (2n-2)^2(2n-4)^2 \cdots (8)^2(6)^2(4)^2(2)^2(2) \\ &= 2\{2^{n-1}\,(n-1)!\}^2 \\ &= (2)^{2n-1}\,\{(n-1)!\}^2.\end{aligned} \tag{3.30}$$

When $m = 2n+1$, then

$$(y_{2n+1})_0 = (2n-1)^2 (y_{2n-1})_0 = 0. \tag{3.31}$$

Now using Maclaurin series, we get

$$y = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}(y_{2n})_0 + \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}(y_{2n+1})_0.$$

The function $[\sin^{-1}(x)]^2$ is an even function so that the odd coefficients of its Maclaurin expansion vanish. That is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}(y_{2n})_0 + 0 \\ &= 0 + \sum_{n=1}^{\infty} \frac{x^{2n}}{(2n)!}(y_{2n})_0 \\ &= \sum_{n=1}^{\infty} \frac{x^{2n}\,(2)^{2n-1}\{(n-1)!\}^2}{(2n)!}. \end{aligned} \tag{3.32}$$

Replacing n by $n+1$ in equation (3.32), we get

$$\begin{aligned} y &= \sum_{n=0}^{\infty} \frac{x^{2n+2}\,(2)^{2n+1}\{n!\}^2}{(2n+2)!} \\ &= x^2 \sum_{n=0}^{\infty} \frac{(1)_n(1)_n\,(2)^{2n+1}\,x^{2n}}{(1)_{2n+2}} \\ &= x^2 \sum_{n=0}^{\infty} \frac{(1)_n(1)_n\,(2)^{2n+1}\,x^{2n}}{(1)_2\,(3)_{2n}} \\ &= x^2 \sum_{n=0}^{\infty} \frac{(1)_n(1)_n(1)_n\,x^{2n}}{(2)_n\,(\frac{3}{2})_n\,n!}, \end{aligned}$$

on using the definition (1.1), we get the required hypergeometric form (2.2).

Proof of hypergeometric form (2.3)***:***

Consider the following function

$$y = y(x) = \sin^{-1}(x). \tag{3.33}$$

Put $x = 0$ in the equation (3.33), we get

$$(y)_0 = 0. \tag{3.34}$$

Differentiate the equation (3.33) w.r.t. x, we get

$$y_1 = y_1(x) = \frac{1}{\sqrt{(1-x^2)}},$$

that is

$$\sqrt{(1-x^2)}\,y_1 = 1, \tag{3.35}$$

$$(y_1)_0 = 1\,. \tag{3.36}$$

Differentiate the equation (3.35) w.r.t. x and use product rule, after simplification we get

$$(1-x^2)y_2 - xy_1 = 0, \tag{3.37}$$

$$(y_2)_0 = 0. \tag{3.38}$$

Again differentiate the equation (3.37) w.r.t. x and use product rule, after simplification we have

$$(1-x^2)y_3 - 3xy_2 - y_1 = 0, \tag{3.39}$$

$$(y_3)_0 = 1\ . \tag{3.40}$$

Now differentiate the equation (3.37) n-times w.r.t. x, and apply Leibnitz theorem we obtain

$$D^n[(1-x^2)y_2] - D^n[xy_1] = 0; n \geq 2, \tag{3.41}$$

$$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2 y_n = 0; n \geq 2\ .$$

Put $x = 0$ in the equation (**??**), we get

$$(y_{n+2})_0 = n^2 (y_n)_0; n \geq 2\ . \tag{3.42}$$

Put $n = 2, 3, 4, 5, 6, 7, 8$ in the equation (3.42),we get

$$(y_5)_0 = (3)^2(1), \tag{3.43}$$

$$(y_7)_0 = (5)^2(3)^2(1), \tag{3.44}$$

$$(y_9)_0 = (7)^2(5)^2(3)^2(1). \tag{3.45}$$

Using the equation (3.42), we can write the recurrence relation:

$$(y_m)_0 = (m-2)^2(y_{m-2})_0; m \geq 2\ . \tag{3.46}$$

When $m = 2n$, then

$$(y_{2n})_0 = (2n-2)^2(y_{2n-2})_0 = 0\ . \tag{3.47}$$

When $m = 2n + 1$, then

$$\begin{aligned}(y_{2n+1})_0 &= (2n-1)^2(y_{2n-1})_0 \\ &= (2n-1)^2(2n-3)^2(2n-5)^2 \cdots (7)^2(5)^2(3)^2(1) \\ &= \{(1)(3)(5)(7)\cdots(2n-5)(2n-3)(2n-1)\}^2.\end{aligned} \tag{3.48}$$

Now divide and Multiply R.H.S. of the equation (3.48) by $[(2)(4)(6)\cdots(2n-4)(2n-2)(2n)]^2$, we get

$$\begin{aligned}(y_{2n+1})_0 &= \frac{\{(1)(2)(3)(4)(5)(6)(7)\cdots(2n-5)(2n-4)(2n-3)(2n-2)(2n-1)(2n)\}^2}{[(2)(4)(6)\cdots(2n-4)(2n-2)(2n)]^2} \\ &= \frac{\{(2n)!\}^2}{[2^n(1\times 2\times 3\times\cdots\times n)]^2} \\ &= \frac{(2n)!\ (2n)!}{4^n\ (n!)^2}.\end{aligned} \tag{3.49}$$

Now using Maclaurin series, we get

$$y = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}(y_{2n})_0 + \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}(y_{2n+1})_0\ .$$

The function $\sin^{-1}(x)$ is an odd function so that the even coefficients of its Maclaurin expansion vanish. That is

$$y = 0 + \sum_{n=0}^{\infty} \frac{x^{2n+1}\ (2n)!\ (2n)!}{(2n+1)!\ 4^n\ (n!)^2}$$

$$= x \sum_{n=0}^{\infty} \frac{(1)_{2n}\,(1)_{2n}\,x^{2n}}{(1)_{2n+1}\,4^n\,(1)_n\,n!}$$
$$= x \sum_{n=0}^{\infty} \frac{(1)_{2n}\,(1)_{2n}\,x^{2n}}{(2)_{2n}\,4^n\,(1)_n\,n!}$$
$$= x \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n\,(\frac{1}{2})_n\,x^{2n}}{(\frac{3}{2})_n\,n!},$$

On using the definition (1.1), we get the required hypergeometric form (2.3).

Proof of hypergeometric form* (2.4)*:

Let,

$$y = \ell n\{e(1-x)^{\frac{1}{x}}\}^{-\frac{2}{x}}$$
$$= -\frac{2}{x}\ell n\{e(1-x)^{\frac{1}{x}}\}$$
$$= -\frac{2}{x}\{\ell n\, e + \ell n(1-x)^{\frac{1}{x}}\}$$
$$= -\frac{2}{x}\{1 + \frac{1}{x}\ell n(1-x)\}$$
$$= 1 + 2(\frac{x}{3} + \frac{x^2}{4} + \frac{x^3}{5} + \cdots)$$
$$= 1 + 2\sum_{n=1}^{\infty} \frac{x^n}{(n+2)}$$
$$= 1 + 2\sum_{n=1}^{\infty} \frac{(2)_n\,x^n}{(2)_n\,(2+n)}$$
$$= \sum_{n=0}^{\infty} \frac{(1)_n\,(2)_n\,x^n}{(3)_n\,n!},$$

on using the definition (1.1), we get the required hypergeometric form (2.4).

4 Hypergeometric Forms Involving Inverse Hyperbolic Sine Function

Replacing x by (ix) in both sides of equations (2.1), (2.2), (2.3) and using the relation (1.3), we obtain the following hypergeometric forms of the functions involving inverse hyperbolic sine function.

When $|x| < 1$, then following hypergeometric forms hold true:

$$\frac{\sinh^{-1}(x)}{\sqrt{(1+x^2)}} = x\,{}_2F_1\left[\begin{array}{ll} 1,\ 1; & \\ & -x^2 \\ \frac{3}{2}; & \end{array}\right]. \tag{4.1}$$

$$[\sinh^{-1}(x)]^2 = x^2\,{}_3F_2\left[\begin{array}{ll} 1,\ 1,\ 1; & \\ & -x^2 \\ 2,\ \frac{3}{2}; & \end{array}\right]. \tag{4.2}$$

$$\sinh^{-1}(x) = x\,{}_2F_1\left[\begin{array}{ll} \frac{1}{2},\ \frac{1}{2}; & \\ & -x^2 \\ \frac{3}{2}; & \end{array}\right]. \tag{4.3}$$

5 Conclusion

In our present investigation, we derived the hypergeometric forms of some functions involving arcsine function, inverse hyperbolic sine function and logarithmic function by using Maclaurin series. Moreover, the results derived in this paper are expected to have useful applications in wide range of problems of Mathematics, Statistics and Physical sciences. Similarly, we can derive the hypergeometric forms of other functions in an analogous manner.

Acknowledgement. The authors are very much thankful to the Editor and referee for their valuable suggestions (use of even and odd function in Maclaurin expansion) to improve the paper in its present form.

References

[1] M. Abramowitz and I.A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables*,Reprint of the 1972 Edition, Dover Publications, Inc., New York, 1992.

[2] G.E. Andrews, R. Askey and R. Roy, *Special Functions*, Cambridge University Press, Cambridge, UK, 1999.

[3] L.C. Andrews, *Special Functions for Engineers and Applied Mathematicians*, Macmillan Publishing Company, New York, 1985.

[4] L.C. Andrews, *Special Functions of Mathematics for Engineers*, Reprint of the 1992 Second Edition, SPIE Optical Engineering Press, Bellingham, W.A, Oxford University Press, Oxford, 1998.

[5] A. Erdélyi, W. Magnus, F. Oberhettinger and F.G. Tricomi, *Higher Transcendental Functions*, Vol. I, McGraw-Hill, New York /Toronto / London, 1953.

[6] I.S. Gradshteyn and I.M. Ryzhik, *Table of Integrals, Series and Products*, 5thEd. Academic Press, New York, 1994.

[7] W. Magnus, F. Oberhettinger and R.P. Soni, *Some Formulas and Theorems for the Special Functions of Mathematical Physics*, Third Enlarged Edition, Springer-Verlag, New York, 1966.

[8] A.P. Prudnikov, Yu. A. Brychkov and O.I. Marichev, *Integrals and Series: More special functions*, volume 3. Nauka, Moscow, 1986 (in Russian); (Translated from the Russian by G. G. Gould), Gordon and Breach Science Publishers, New York, Philadelphia. London, Paris, Montreux, Tokyo, Melbourne, 1990.

[9] M.I. Qureshi and M.S. Baboo, Power series and hypergeometric representations with positive integral powers of arctangent function, *South Asian Journal of Mathematics*, **8**(1) (2018), 11-17.

[10] M.I. Qureshi and M.S. Baboo, Power series and hypergeometric representations associated with positive integral powers, *South Asian Journal of Mathematics*, **8**(3) (2018), 144-150.

[11] M.I. Qureshi, I. H. Khan and M.P. Chaudhary, Hypergeometric forms of well known partial fraction expansions of some meromorphic functions, *Global Journal of Science Frontier Research*, **11**(6) (2011), 45-52.

[12] M. I. Qureshi, S. Porwal, D. Ahamad and K. A. Quraishi, Successive differentiations of tangent, cotangent, secant, cosecant functions and related hyperbolic functions (A hypergeometric approach), *International Journal of Mathematics Trends and Technology*, **65**(7) (2019), 352-367.

[13] E. D. Rainville, *Special Functions*, The Macmillan Co. Inc., New York 1960, Reprinted by Chelsea publ. Co., Bronx, New York, 1971.
[14] H. M. Srivastava and P. W. Karlsson, *Multiple Gaussian Hypergeometric Series*, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley, New York/ Chichester/ Brisbane/ Toronto, 1985.
[15] H. M. Srivastava and H. L. Manocha, *A Treatise on Generating Functions*, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester Brisbane, Toronto, 1984.
[16] G. Szegö, *Orthogonal Polynomials*, Amer. Math. Soc., Colloquium publ., New York, 1939.

Jñānābha, Vol. 50(1) (2020), 189-206
(Dedicated to Honor Professor H.M. Srivastava on His 80th Birth Anniversary Celebrations)

MODELING AND ANALYSIS THE EFFECT OF GLOBAL WARMING ON THE SPREAD OF CARRIER DEPENDENT INFECTIOUS DISEASES

By
Shikha Singh[1] and Ashish Omar[2]
Department of Mathematics, PPN Postgraduate College, CSJM University
Kanpur -208001, Uttar Pradesh, India.
Email:sshikha22976@yahoo.co.in and ashishomar999@gmail.com
(*Received : May* 07, 2019 *; Revised: June* 09, 2020)
DOI: https://doi.org/10.58250/jnanabha.2020.50119

Abstract

In this paper, a nonlinear SIS model is considered to study the effect of global warming on the spread of carrier dependent infectious diseases. In the model, we deal with five dependent variables, namely, the density of susceptibles, the density of infectives, the density of carrier population, the amount of carbon dioxide in the environment causing global warming and the global warming temperature. In the model, it is assumed that the density of carriers increases with constant growth rate as well as proportional to the global warming temperature. The amount of CO_2 in the environment increases due to human population activities as well as natural factors. It is also assumed that the global warming temperature is proportional to the amount of CO_2 in the environment. The model is studied and investigated with the help of stability theory of differential equations and numerical simulation. The investigation shows that, if global warming temperature increases, then the density of carrier population and the spread of carrier dependent infectious diseases increase.
2010 Mathematics Subject Classifications: 37C75, 92B05
Keywords and phrases: Global Warming, SIS Model, Stability, Carriers.

1 Introduction

Global warming is the most crucial subject matter to be taken under scrutiny by researchers in this century. Certain greenhouse gases in the Earth's atmosphere, like carbon dioxide (CO_2) and methane (CH_4) trap the sun's heat and do not let it escape. The increase of greenhouse gases in the atmosphere over long period leads to an increase in Earth's surface temperature causing global warming. Over the past 130 years, the world has warmed up to 0.85^0C (approx.). The last three decades have been progressively warmer than any preceding decades since 1850, IPCC [10]. Further, it is important to note that detrimental activities of growing human population such as deforestation, throwing waste in rivers and oceans and releasing smoke from factories and car fumes into air etc., has discharged enough amount of carbon dioxide (CO_2) in the environment. Because of the global warming, carriers such as ticks, flies, mites, mosquitoes, cockroaches survive in the warmer environment and can cause the spread of carrier dependent infectious diseases, Zhou et al. [18]. The worldwide distribution of carrier dependent infectious diseases shows the impact of climate change, IPCC [8, 9]. The prediction of World Health Organization is that, between 2030 and 2050 there will be an additional 0.25 million deaths per year from many infectious diseases as measles, smallpox, mumps, malaria, diarrhea etc., due to global warming, WHO [15-17].

It is noteworthy that the effect of global warming on the carrier population with a constant growth rate and it's role on the spread of infectious diseases has not been done till now. Many researchers have studied the spread of infectious diseases by using mathematical models in the past, as Anderson and May [1], Bailey [2], Ghosh et al. [3, 4], Greenhalgh [5] , Hethcote [6, 7], May and Anderson [11], Singh [12] and Singh et al. [13, 14]. In this paper, therefore the effect of global warming on carrier population with constant growth rate and its role on the spread of infectious diseases is modeled and studied.

The following assumptions are made in the modeling process:

(i) The rate of density of carrier population increases with a constant rate.

(ii) The rate of density of carrier population is proportional to the global warming temperature.

(iii) The rate of amount of CO_2 in the environment increases by human activities as well as natural factors.

(iv) The rate of global warming temperature is proportional to amount of CO_2 in the environment.

2 SIS Model

Let $N(t)$ be the total human population density which is divided into two categories, namely susceptible human population density $X(t)$ and infective human population density $Y(t)$. Let $C_r(t)$ be the carrier population density which grows with a constant growth rate coefficient "s". Let $C(t)$ be the amount of CO_2 in the environment and $T(t)$ be the global warming temperature of the environment due to discharge of carbon dioxide(CO_2). The model is governed by the following non linear differential equations:

$$
\begin{aligned}
\frac{dX}{dt} &= A - \beta XY - \lambda XC_r - dX + \nu Y, \\
\frac{dY}{dt} &= \beta XY + \lambda XC_r - (\nu + \alpha + d)Y, \\
\frac{dC_r}{dt} &= s + s_1(T - T_0) - s_0 C_r, \\
\frac{dC}{dt} &= Q_o + \alpha_1(A - dN) - \alpha_o C, \\
\frac{dT}{dt} &= \theta(C - C_o) - \theta_0(T - T_o),
\end{aligned}
\tag{2.1}
$$

with $X + Y = N$, where $X(0) = X_0 > 0, Y(0) = Y_0 \geq 0, N(0) = N_0 > 0, C_r(0) = C_{r0} \geq 0, C(0) = C_0 > 0$ and $T(0) = T_0 > 0$ and take $C_0 = \frac{Q_o}{\alpha_o}$.

In the above model system (2.1), the used parameters are positive real numbers, described as follows:

A : The constant immigration rate of human population from outside the region
β : Coefficient of transmission by infective human population density
λ : Coefficient of transmission by carrier population density
d : Natural death rate constant of human population
α : Coefficient of death rate of infective human population due to disease related factors
ν : Coefficient of recovery rate of infective human population density
Q_0 : Discharge rate of CO_2 from natural factors
α_1 : Discharge rate coefficient of CO_2 from man- made sources
α_0 : Natural reduction rate coefficient of CO_2
θ : Growth rate coefficient of temperature of the region due to rise in amount of CO_2
θ_0 : Natural reduction rate coefficient of temperature in the region
s_0 : Reduction rate coefficient of carriers
s : Constant growth rate coefficient of carrier population density
s_1 : Growth rate of carriers due to global warming temperature
T_0 : The equilibrium level of global warming temperature of the environment
C_0 : The equilibrium amount of CO_2

For analyzing the model (2.1), we take the following reduced system by using $X = N - Y$,

$$\frac{dY}{dt} = \beta(N-Y)Y + \lambda(N-Y)C_r - (\nu+\alpha+d)Y, \tag{2.2}$$

$$\frac{dN}{dt} = A - dN - \alpha Y, \tag{2.3}$$

$$\frac{dC_r}{dt} = s + s_1(T-T_0) - s_0C_r, \tag{2.4}$$

$$\frac{dC}{dt} = Q_o + \alpha_1(A-dN) - \alpha_o C, \tag{2.5}$$

$$\frac{dT}{dt} = \theta(C-C_o) - \theta_0(T-T_o), \tag{2.6}$$

with initial conditions

$$Y(0) = Y_0 \geq 0, N(0) = N_0 > 0, C_r(0) = C_{r0} \geq 0, C(0) = C_0 > 0 \text{ and } T(0) = T_0 > 0.$$

Region of Attraction

The region of attraction of the model system (2.2) - (2.6) is given by the set

$$\Omega = \{(Y, N, C_r, C, T) \in R^5_+ : 0 \leq Y \leq N \leq \frac{A}{d}, 0 \leq C_r \leq C_{rm}, 0 < C \leq C_m, 0 < T \leq T_m\},$$

where, $C_{rm} = \dfrac{s\theta_0\alpha_0 + s_1\theta\alpha_1 A}{s_0\theta_0\alpha_0}$, $C_m = C_0 + \dfrac{\alpha_1 A}{\alpha_o}$ and $T_m = T_o + \dfrac{\theta\alpha_1 A}{\theta_0\alpha_o}$.

It attracts all the solution initiating in the interior of the positive octant of the region.

3 Equilibrium Analysis

The model system (2.2) - (2.6) has only one non-negative equilibria:

(i) $E(Y^*, N^*, C_r^*, C^*, T^*)$.

Proof. To find the equilibrium point $E(Y^*, N^*, C_r^*, C^*, T^*)$ we solve the following set of equations

$$\beta(N-Y)Y + \lambda(N-Y)C_r - (\nu+\alpha+d)Y = 0, \tag{3.1}$$

$$A - dN - \alpha Y = 0, \tag{3.2}$$

$$s + s_1(T-T_0) - s_0C_r = 0, \tag{3.3}$$

$$(3.4) \qquad Q_o + \alpha_1(A - dN) - \alpha_o C = 0,$$

$$(3.5) \qquad \theta(C - C_o) - \theta_0(T - T_o) = 0.$$

Using above equations we get $C_r = \dfrac{s\theta_0\alpha_0 + s_1\theta\alpha_1\alpha Y}{s_0\theta_0\alpha_0}$.

Now, using the above value of C_r and equations (3.1), (3.2), we get Y^* as the root of the following equation

$$(3.6) \qquad F(Y) = \frac{\beta Y[A - (\alpha + d)Y]}{d} + \frac{\lambda[A - (\alpha + d)Y]}{d}\left(\frac{s\theta_0\alpha_0 + s_1\theta\alpha_1\alpha Y}{s_0\theta_0\alpha_0}\right) - (\nu + \alpha + d)Y = 0.$$

So we have, $F(0) = \dfrac{\lambda A s}{d s_0} > 0$ and $F\left(\dfrac{A}{\alpha + d}\right) = -(\nu + \alpha + d)\left(\dfrac{A}{\alpha + d}\right) < 0$

i.e. at least one root of the equation $F(Y) = 0$ lies in the range, $0 < Y < \dfrac{A}{\alpha + d}$.

Rewriting (3.6) as follows

$$(3.7) \qquad F(Y) = \frac{\beta Y[A - (\alpha + d)Y]}{d} + \frac{\lambda[A - (\alpha + d)Y]}{d}(a_0 + b_0 Y) - (\nu + \alpha + d)Y = 0,$$

where $a_0 = \frac{s}{s_0}$ and $b_0 = \frac{s_1\theta\alpha_1\alpha}{s_0\theta_0\alpha_0}$.

On differentiating (3.7) with respect to Y, we get

$$F'(Y) = \beta\frac{[A - (\alpha + d)Y]}{d} - \beta Y\left(\frac{\alpha + d}{d}\right) + \frac{\lambda[A - (\alpha + d)Y]}{d}b_0 - \lambda(a_0 + b_0 Y)\left(\frac{\alpha + d}{d}\right) - (\nu + \alpha + d).$$

Then, for $Y > 0$

$$YF'(Y) = \beta\frac{[A - (\alpha + d)Y]}{d}Y - \beta Y^2\left(\frac{\alpha + d}{d}\right) + \frac{\lambda[A - (\alpha + d)Y]}{d}b_0 Y - \lambda Y(a_0 + b_0 Y)\left(\frac{\alpha + d}{d}\right) - (\nu + \alpha + d)Y.$$

$$YF'(Y) = -\beta Y^2\left(\frac{\alpha + d}{d}\right) - \frac{\lambda(\alpha + d)}{d}b_0 Y^2 - a_0\frac{\lambda A}{d} < 0.$$

Hence the equation $F(Y) = 0$ has unique root in the interval $0 < Y < \frac{A}{(\alpha + d)}$

***Remark* 3.1.** *Here we noted that* $\frac{dY}{dQ_0}|_E > 0$

From the model system (2.2) - (2.6)*, we have*

$$(3.8) \qquad \left[\frac{\beta Y(\alpha + d)}{d} - \frac{\beta}{d}\left\{A - (\alpha + d)Y\right\} + \frac{\lambda(\alpha + d)C_r}{d} + (\nu + \alpha + d)\right] \times \frac{dY}{dQ_0} = \frac{\lambda[A - (\alpha + d)Y]}{d}\frac{dC_r}{dQ_0},$$

$$(3.9) \qquad s_1\frac{dT}{dQ_0} = s_0\frac{dC_r}{dQ_0},$$

$$(3.10) \qquad \frac{dC}{dQ_0} = \frac{1}{\alpha_o} + \frac{\alpha_1\alpha}{\alpha_o}\frac{dY}{dQ_0},$$

$$(3.11) \qquad \theta\frac{dC}{dQ_0} = \theta_0\frac{dT}{dQ_0}.$$

On writing (3.8) with the help of (3.9), (3.10) *and* (3.11)*, we get*

$$(3.12) \qquad \frac{dY}{dQ_0} = \frac{\lambda s_1\theta[A - (\alpha + d)Y]}{d s_0\theta_0\alpha_o\left[\frac{\beta Y(\alpha+d)}{d} + \frac{\lambda(\alpha+d)C_r}{d} - \left\{\frac{\beta\{A-(\alpha+d)Y\}}{d} + \frac{\lambda}{d}b_0\{A - (\alpha + d)Y\} - (\nu + \alpha + d)\right\}\right]}.$$

By equation (3.7)*, we have*

$$\left[\frac{\beta\{A-(\alpha+d)Y\}}{d}+\frac{\lambda\{A-(\alpha+d)Y\}}{d}b_0-(\nu+\alpha+d)\right]=-\frac{\lambda[A-(\alpha+d)Y]}{dY}a_0,$$

which on substituting in equation (3.12)*, gives*

$$\frac{dY}{dQ_0}=\frac{\lambda s_1\theta[A-(\alpha+d)Y]}{ds_0\theta_0\alpha_o\left[\frac{\beta Y(\alpha+d)}{d}+\frac{\lambda(\alpha+d)C_r}{d}+\frac{\lambda\{A-(\alpha+d)Y\}}{dY}a_0\right]}. \tag{3.13}$$

So we have, $\frac{dY}{dQ_0}|_E > 0$*, which shows that the density of infective human population density increases as the discharge rate of* CO_2 *from natural factors increases at the equilibrium point E.*

Since $C_r=\dfrac{s\theta_0\alpha_0+s_1\theta\alpha_1\alpha Y}{s_0\theta_0\alpha_0}=\dfrac{s}{s_0}+\dfrac{s_1\theta\alpha_1\alpha}{s_0\theta_0\alpha_0}Y=a_0+b_0Y$ *, using in* (3.8)*, we get*

$$\left[\frac{\beta Y(\alpha+d)}{d}-\frac{\beta}{d}\{A-(\alpha+d)Y\}+\frac{\lambda(\alpha+d)(a_0+b_0Y)}{d}+(\nu+\alpha+d)\right]\frac{dY}{dQ_0}=\frac{\lambda[A-(\alpha+d)Y]}{d}\frac{dC_r}{dQ_0}. \tag{3.14}$$

By equation (3.7)*, we have*

$$-\frac{\beta\{A-(\alpha+d)Y\}}{d}+\frac{\lambda(\alpha+d)}{d}(a_0+b_0Y)+(\nu+\alpha+d)=\frac{\lambda A}{dY}(a_0+b_0Y),$$

which on substituting in (3.14)*, we get*

$$\left[\frac{\beta Y(\alpha+d)}{d}+\frac{\lambda A}{dY}(a_0+b_0Y)\right]\frac{dY}{dQ_0}=\frac{\lambda[A-(\alpha+d)Y]}{d}\frac{dC_r}{dQ_0}$$

Since $\frac{dY}{dQ_0}|_E > 0$ *therefore* $\frac{dC_r}{dQ_0}|_E > 0$*. Thus, we have* $\frac{dC_r}{dQ_0}|_E > 0$ *, which shows that the density of carrier population increases as the discharge rate of* CO_2*from natural factors increases at the equilibrium point E.*

Remark 3.2. *It is also noted that* $\frac{dY}{d\theta}|_E > 0$

By equation (3.7)*, we have*

$$\frac{\beta Y[A-(\alpha+d)Y]}{d}+\frac{\lambda[A-(\alpha+d)Y]}{d}(a_0+d_0\theta Y)-(\nu+\alpha+d)Y=0, \tag{3.15}$$

where $d_0=\frac{s_1\alpha_1\alpha}{s_0\theta_0\alpha_0}$.

On differentiating (3.15) *with respect to* θ*, we get*

$$\frac{dY}{d\theta}\Big[\frac{\beta Y(\alpha+d)}{d}-\frac{\beta\{A-(\alpha+d)Y\}}{d}-\frac{\lambda d_0\theta\{A-(\alpha+d)Y\}}{d}+\frac{\lambda(\alpha+d)(a_0+d_0\theta Y)}{d}$$
$$+(\nu+\alpha+d)\Big]=\frac{\lambda d_0Y[A-(\alpha+d)Y]}{d},$$

$$\frac{dY}{d\theta}=\frac{\lambda d_0Y[A-(\alpha+d)Y]}{d[\frac{\beta Y(\alpha+d)}{d}-\frac{\beta\{A-(\alpha+d)Y\}}{d}-\frac{\lambda d_0\theta\{A-(\alpha+d)Y\}}{d}+\frac{\lambda(\alpha+d)(a_0+d_0\theta Y)}{d}+(\nu+\alpha+d)]}. \tag{3.16}$$

By equation (3.15)*, we have*

$$\frac{\beta Y(\alpha+d)}{d}+\frac{\lambda(\alpha+d)}{d}(a_0+d_0\theta Y)+(\nu+\alpha+d)=\frac{\beta A}{d}+\frac{\lambda A}{dY}(a_0+d_0\theta Y).$$

On putting in (3.16)*, we get*

$$\frac{dY}{d\theta}=\frac{\lambda d_0Y[A-(\alpha+d)Y]}{[\beta(\alpha+d)Y+\lambda d_0\theta(\alpha+d)Y+\frac{\lambda Aa_0}{Y}]}.$$

So we have, $\frac{dY}{d\theta}|_E > 0$ *which shows that the density of infective human population increases as the growth rate coefficient of temperature of the region due to rise in amount of* CO_2 *in the environment increases at the equilibrium point E.*

4 Stability Analysis

In this section, we study the stability behavior of the equilibrium point E. The results are stated in the following theorems.

***Theorem* 4.1.** *The equilibrium point $E(Y^*, N^*, C_r^*, C^*, T^*)$ is locally asymptotically stable provided the following conditions are satisfied*

$$a_1a_2a_3 - a_3^2 - a_1^2a_4 > 0, \tag{4.1}$$

$$(a_1a_4 - a_5)(a_1a_2a_3 - a_3^2 - a_1^2a_4) - a_5(a_1a_2 - a_3)^2 - a_1a_5^2 > 0, \tag{4.2}$$

where

$$\begin{aligned}
a_1 &= (\beta Y^* + \frac{\lambda C_r^* N^*}{Y^*}) + (s_0 + \alpha_0 + \theta_0) + d,\\
a_2 &= (\beta Y^* + \frac{\lambda C_r^* N^*}{Y^*})(d + s_0 + \alpha_0 + \theta_0) + \alpha(\beta Y^* + \lambda C_r^*) + d(s_0 + \alpha_0 + \theta_0),\\
&\quad + (s_0\alpha_0 + \alpha_0\theta_0 + s_0\theta_0),\\
a_3 &= (\beta Y^* + \frac{\lambda C_r^* N^*}{Y^*})\{d(s_0 + \alpha_0 + \theta_0) + (s_0\alpha_0 + \alpha_0\theta_0 + s_0\theta_0)\},\\
&\quad + \alpha(\beta Y^* + \lambda C_r^*)(s_0 + \alpha_0 + \theta_0) + d(s_0\alpha_0 + \alpha_0\theta_0 + s_0\theta_0) + s_0\theta_0\alpha_0,\\
a_4 &= (\beta Y^* + \frac{\lambda C_r^* N^*}{Y^*})\{d(s_0\alpha_0 + \alpha_0\theta_0 + s_0\theta_0) + s_0\theta_0\alpha_0\},\\
&\quad + \alpha(\beta Y^* + \lambda C_r^*)(s_0\alpha_0 + \alpha_0\theta_0 + s_0\theta_0) + ds_0\theta_0\alpha_0,\\
a_5 &= \left[(\beta Y^* + \lambda C_r^*)(\alpha + d) + \frac{\lambda d(N^* - Y^*)}{Y^*}\right] s_0\theta_0\alpha_0.
\end{aligned}$$

Here it is noted that $a_i > 0,\ \forall i = 1, 2, 3, 4, 5.$

Proof. See the Appendix A.

***Theorem* 4.2.** *The equilibrium point $E(Y^*, N^*, C_r^*, C^*, T^*)$ is globally asymptotically stable in Ω provided the following conditions are satisfied*

$$\alpha\lambda^2 C_{rm}^2 < \beta^2 d Y^{*2}, \tag{4.3}$$

$$4\alpha d\lambda^2\alpha_1^2\theta^2 s_1^2(N^* - Y^*)^2 < \beta^2 Y^{*2} s_0^2\theta_0^2\alpha_0^2, \tag{4.4}$$

where C_{rm} is the maximum value of C_r, which is given by $C_{rm} = \dfrac{s\theta_0\alpha_0 + s_1\theta\alpha_1 A}{s_0\theta_0\alpha_0}$.

Proof. See the Appendix B.

5 Numerical Simulation

Here we discuss the existence and stability of non-trivial equilibrium point $E(Y^*, N^*, C_r^*, C^*, T^*)$ by taking the following set of parameters and using *MAPLE*.

$$\begin{aligned}
&A = 20, d = 0.0004, \alpha = 0.0005, \alpha_0 = 0.016, \alpha_1 = 0.6 \times 10^{-3},\\
&\beta = 6 \times 10^{-7}, \nu = 0.012, \lambda = 2 \times 10^{-8}, s_0 = 0.3, T_0 = 14,\\
&Q_0 = 6, \theta = 0.1, \theta_0 = 0.19, C_0 = 375, s = 2.4 \times 10^4,\\
&s_1 = 2 \times 10^4
\end{aligned}$$

The Jacobian matrix $J(E)$for the above values of parameters at $E(Y^*, N^*, C_r^*, C^*, T^*)$ is

$$\begin{bmatrix} -0.01259970927 & 0.01037645075 & 0.0003558913828 & 0 & 0 \\ -0.0005 & -0.0004 & 0 & 0 & 0 \\ 0 & 0 & -0.3 & 0 & 20000 \\ 0 & -2.40 \times 10^{-7} & 0 & -0.016 & 0 \\ 0 & 0 & 0 & 0.1 & -0.19 \end{bmatrix}$$

The characteristic roots of the Jacobian matrix corresponding to the equilibrium point $E(Y^*, N^*, C_r^*, C^*, T^*)$ are:

$-0.00083244611, \quad -0.01219714633, \quad -0.01596815206,$
$-0.1899940766, \quad -0.3000078875$

Since all characteristic roots are negative real numbers, therefore $E(Y^*, N^*, C_r^*, C^*, T^*)$ is locally stable.

For above values of parameters, the non-trivial equilibrium point $E(Y^*, N^*, C_r^*, C^*, T^*)$ corresponding to (2.2) - (2.6) is obtained as follows:

$Y^* = 14313.52483, \quad N^* = 32108.09397, \quad C_r^* = 89416.79265,$
$C^* = 375.2683786, \quad T^* = 14.14125189$

Eliminating C_r, C, T from (3.1) - (3.5), we get

$$\beta(N - Y)Y + \lambda(N - Y)\left(\frac{s\theta_0\alpha_0 + s_1\theta\alpha_1\alpha Y}{s_0\theta_0\alpha_0}\right) - (\nu + \alpha + d)Y = 0, \tag{5.1}$$

$$A - dN - \alpha Y = 0. \tag{5.2}$$

For above values of parameters the equations (5.1) and (5.2) are plotted in Y-N plane (***Fig.*5.1**), and the intersection point is (Y^*, N^*).

Existence of (Y^*, N^*)

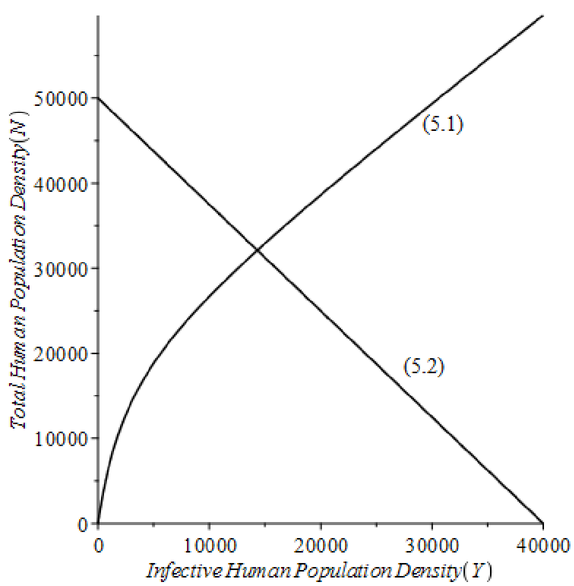


***Figure* 5.1**

With the above mentioned set of values of parameters, we plot the graphs from ***Fig.*5.1** - ***Fig.*5.13**. ***Fig.*5.2**, shows the nonlinear stability behavior between N and Y with different initial conditions tending to equilibrium point (Y^*, N^*) as time increases.

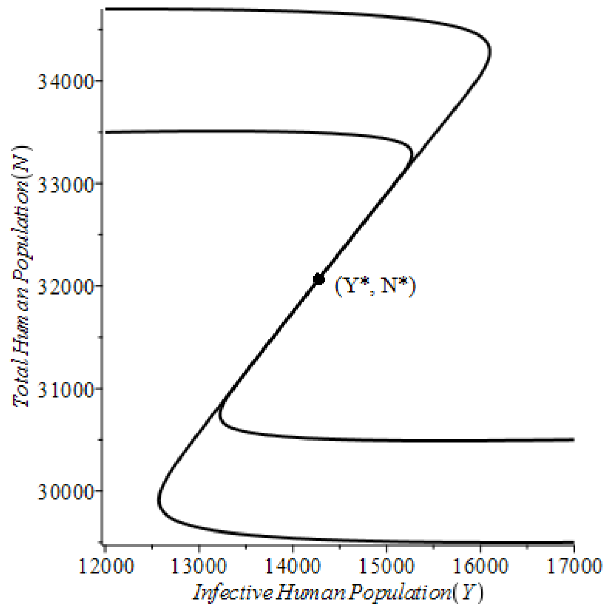


***Figure* 5.2:** Nonlinear stability of (Y^*, N^*) in the Y-N plane.

For solving the system of non-linear ODE, we use Runge-Kutta method in *MAPLE*.
The initial conditions for various quantities are given below:
$C(0) = 375, \quad C_r(0) = 80000, \quad T(0) = 14, \quad N(0) = 31000, \quad Y(0) = 13000$

***Fig.* 5.3** - ***Fig.* 5.13** show the effect of various parameters on infective human population density, carrier population density and amount of carbon dioxide. Every figure contains three curves with the same initial conditions as given above but with different values of corresponding parameter.

From ***Fig.* 5.3**, it is seen that the density of carriers increases if growth rate of carriers due to global warming temperature (s_1) increases. ***Fig.* 5.4** shows that the amount of CO_2 in the environment increases if the discharge rate coefficient of CO_2 from man-made sources(α_1) increases. From ***Fig.* 5.5**, it is seen that the density of infective human population increases as the growth rate coefficient of temperature of the region due to rise in amount of $CO_2(\theta)$ increases. ***Fig.* 5.6** shows that the density of infective human population increases if the discharge rate coefficient of CO_2 from man-made sources (α_1) increases. ***Fig.* 5.7** shows that the density of infective human population decreases if reduction rate coefficient of carriers(s_0) increases. ***Fig.* 5.8** shows that the density of infective human population increases if growth rate of carriers due to global warming temperature (s_1) increases. ***Fig.* 5.9** shows that the density of infective human population increases if constant growth rate coefficient of carrier population density (s) increases. ***Fig.* 5.10** shows that the density of infective human population increases if coefficient of transmission by carrier population density(λ) increases. ***Fig.* 5.11** shows that the density of infective human population increases if constant immigration rate of human population from outside the region under study (A) increases. ***Fig.* 5.12** shows that the density of infective human population increases if the discharge rate of CO_2 from natural factors (Q_0) increases. ***Fig.* 5.13**, shows that the density of carrier population increases if the discharge rate of CO_2 from natural factors (Q_0) increases.

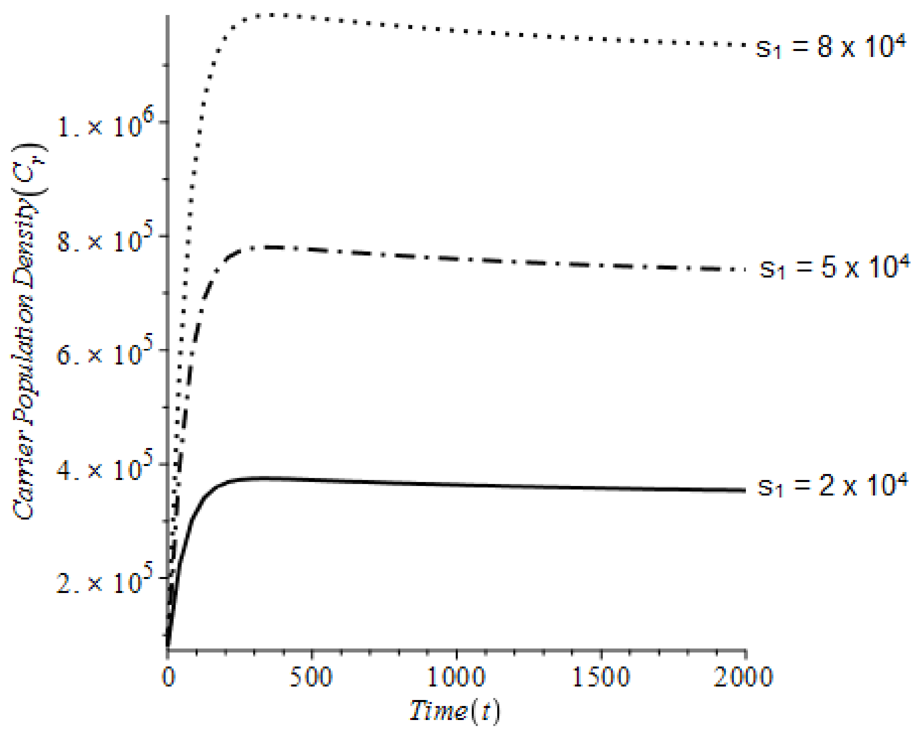


***Figure* 5.3:** Plot between carrier population density C_r and time t for various values of s_1.

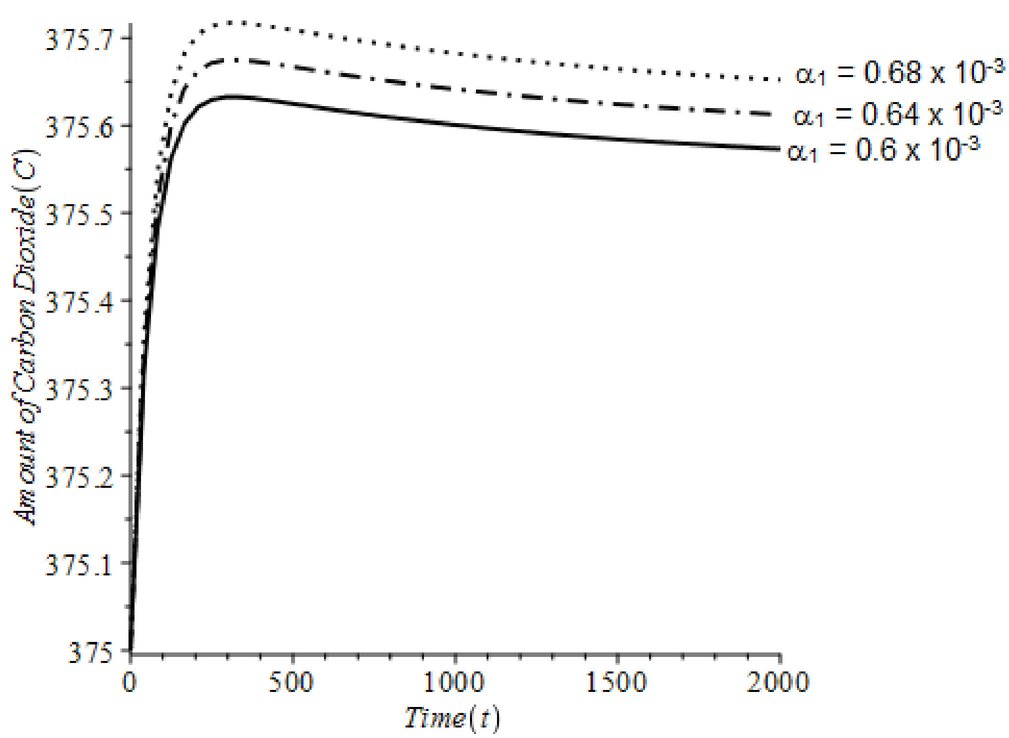


***Figure* 5.4:** Plot between amount of CO_2 and time t for various values of α_1.

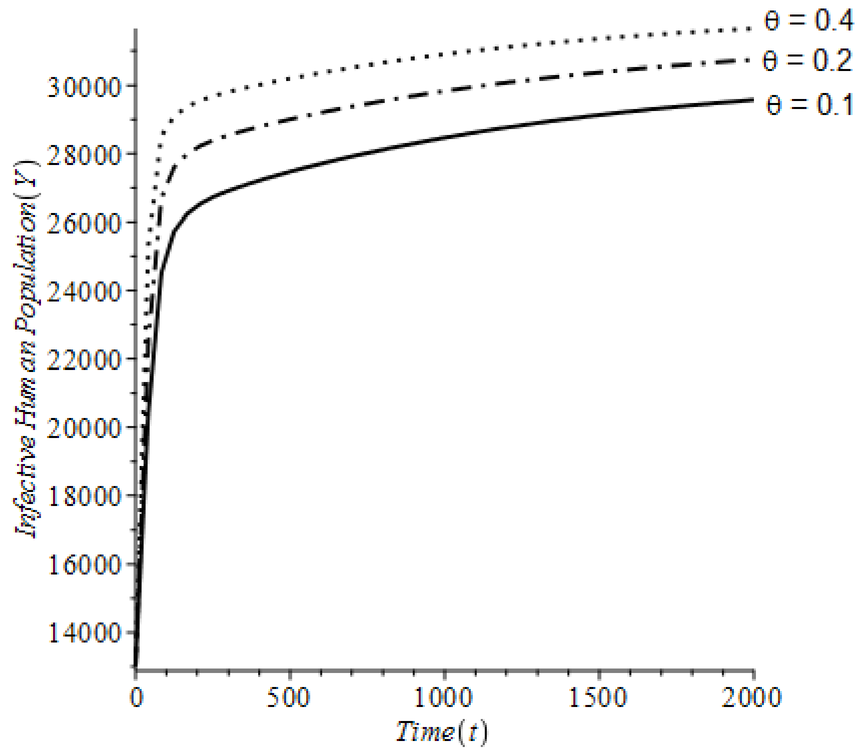


***Figure* 5.5:** Plot between infective human population Y and time t for various values of θ.

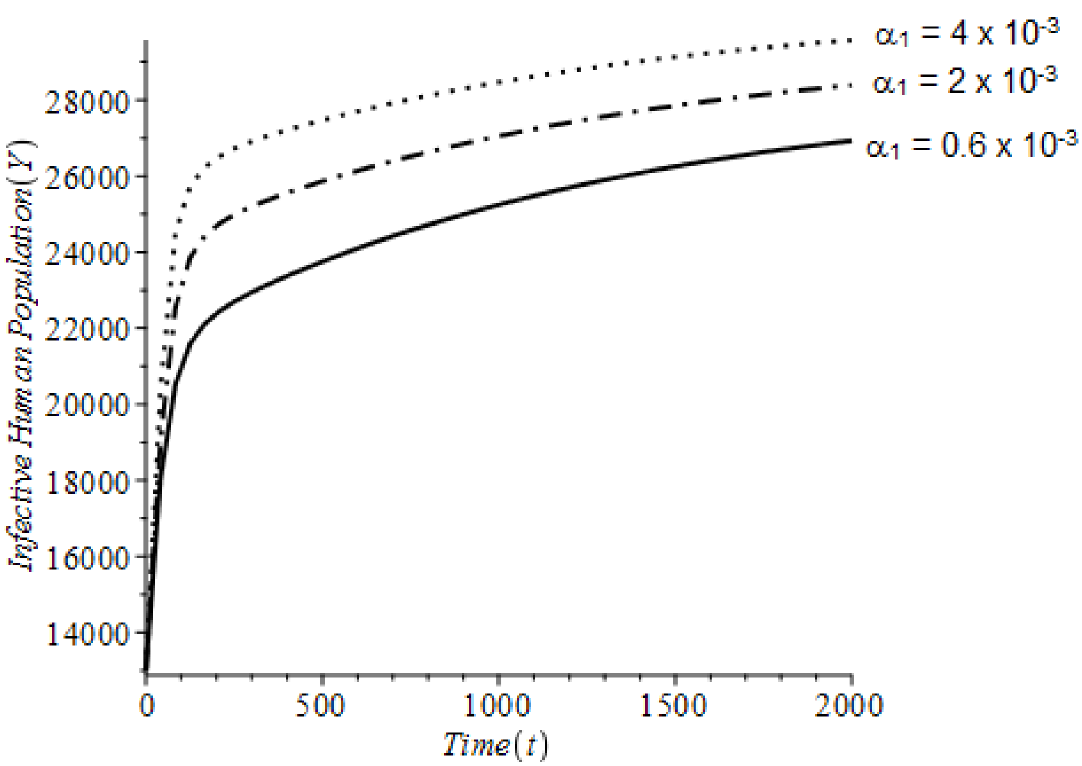


***Figure* 5.6:** Plot between infective human population Y and time t for various values of α_1.

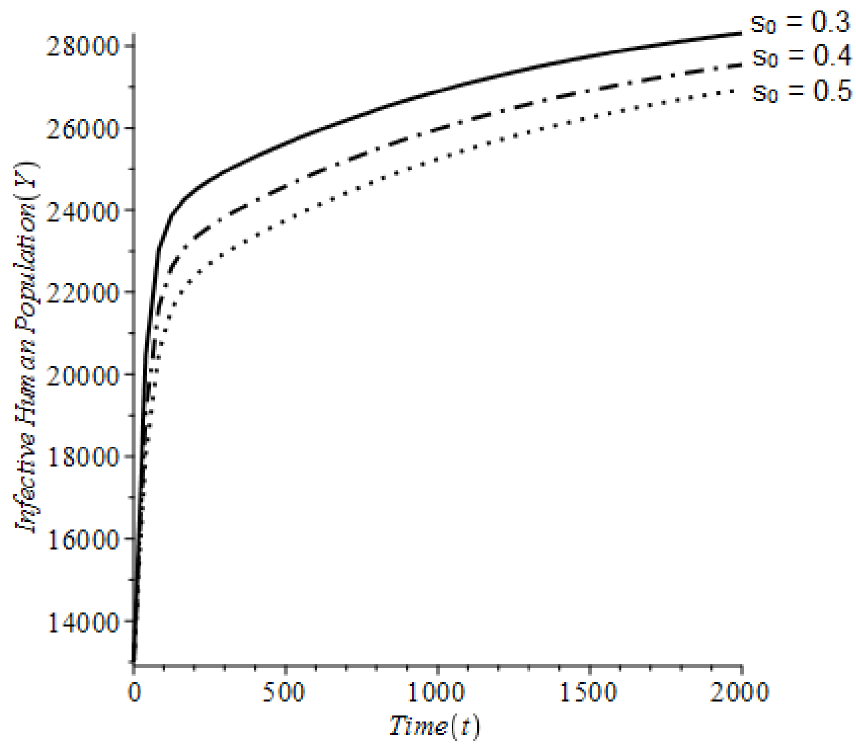


***Figure* 5.7:** Plot between infective human population Y and time t for various values of s_0.

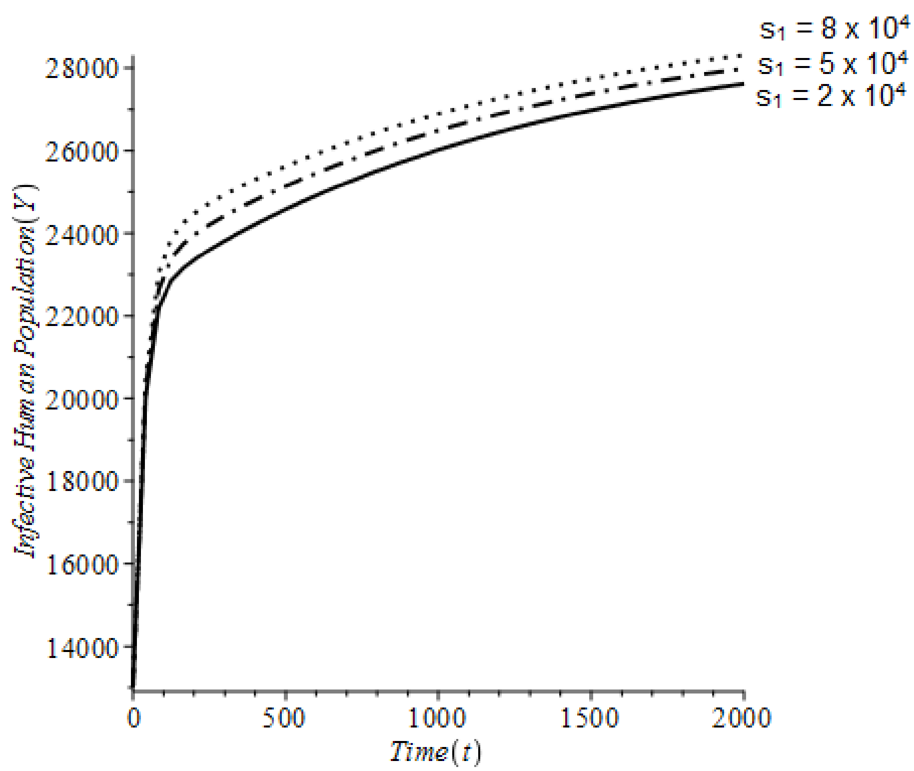


***Figure* 5.8:** Plot between infective human population Y and time t for various values of s_1.

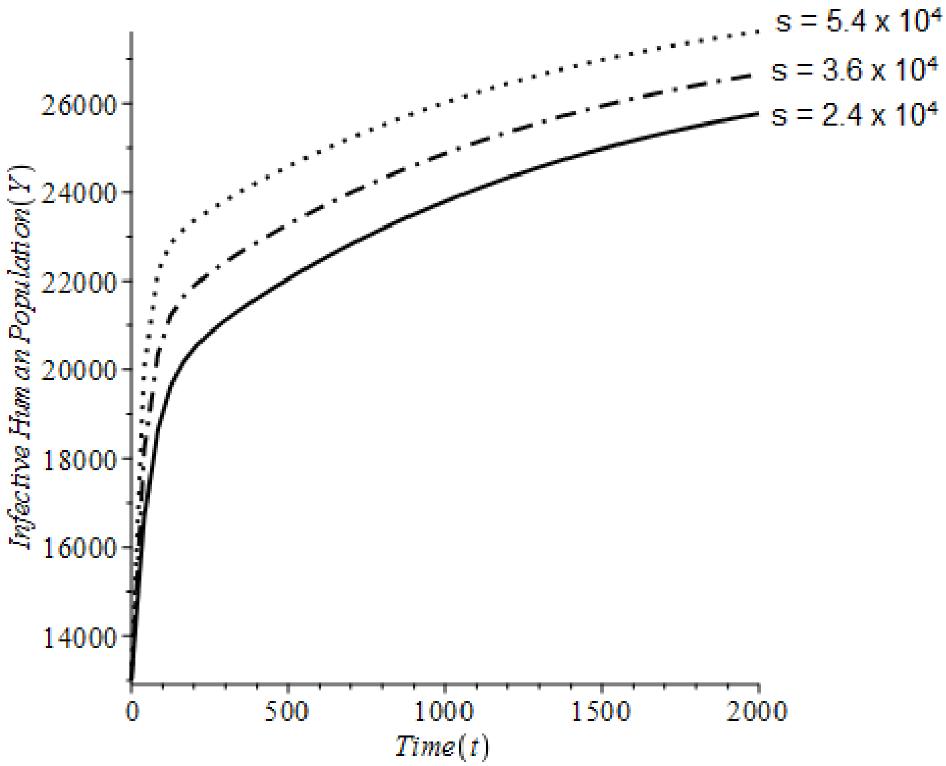


***Figure* 5.9:** Plot between infective human population Y and time t for various values of s.

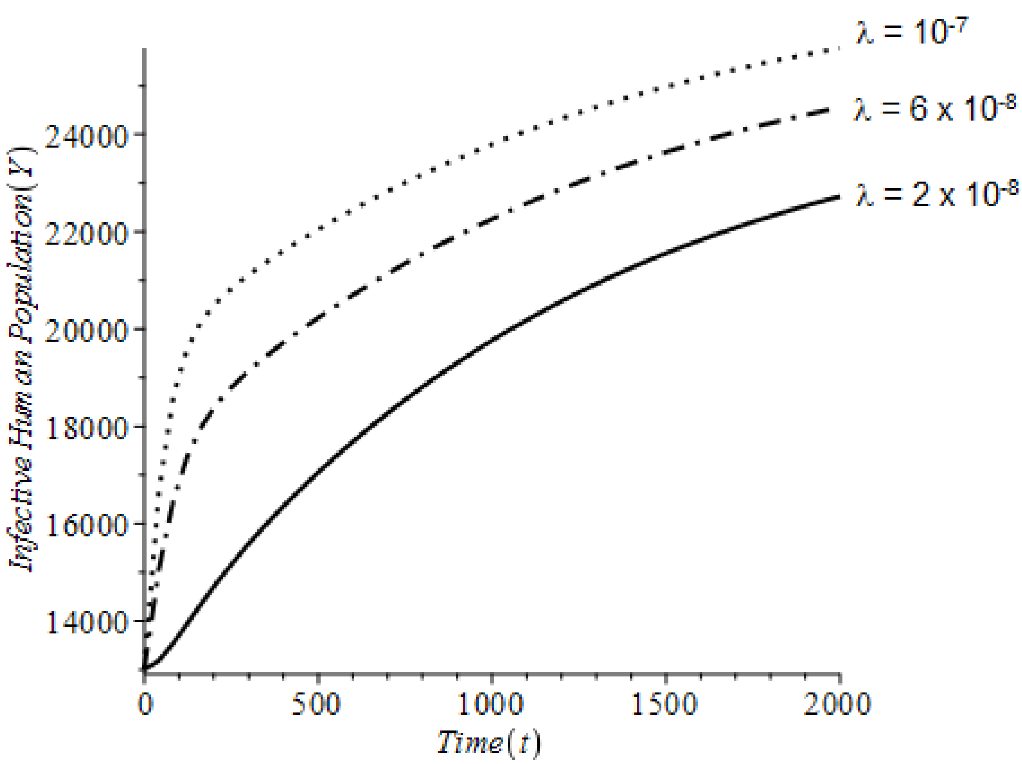


***Figure* 5.10:** Plot between infective human population Y and time t for various values of λ.

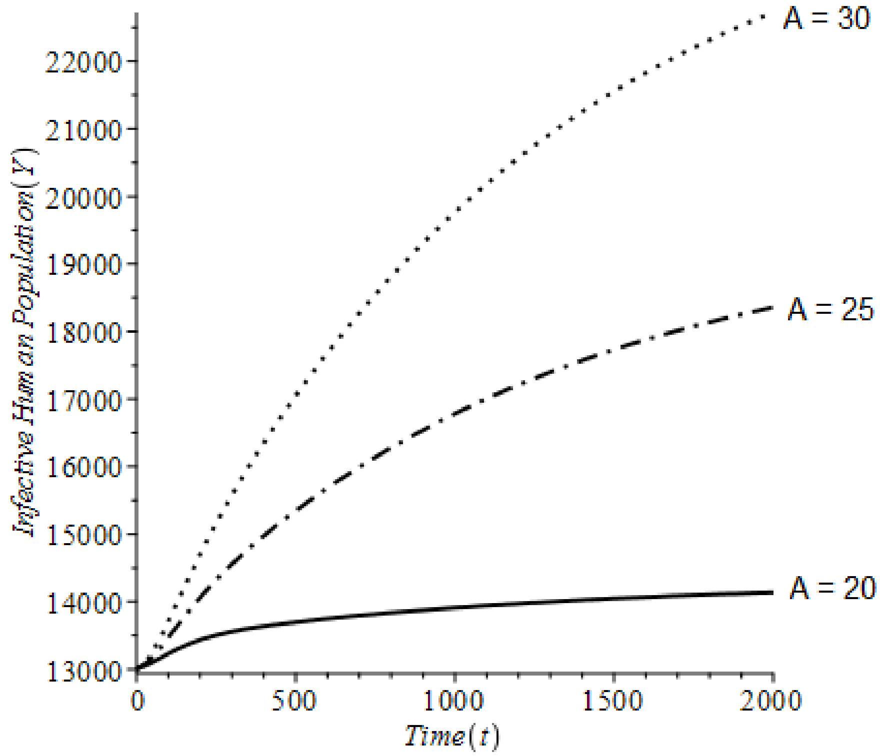


***Figure* 5.11:** Plot between infective human population Y and time t for various values of A.

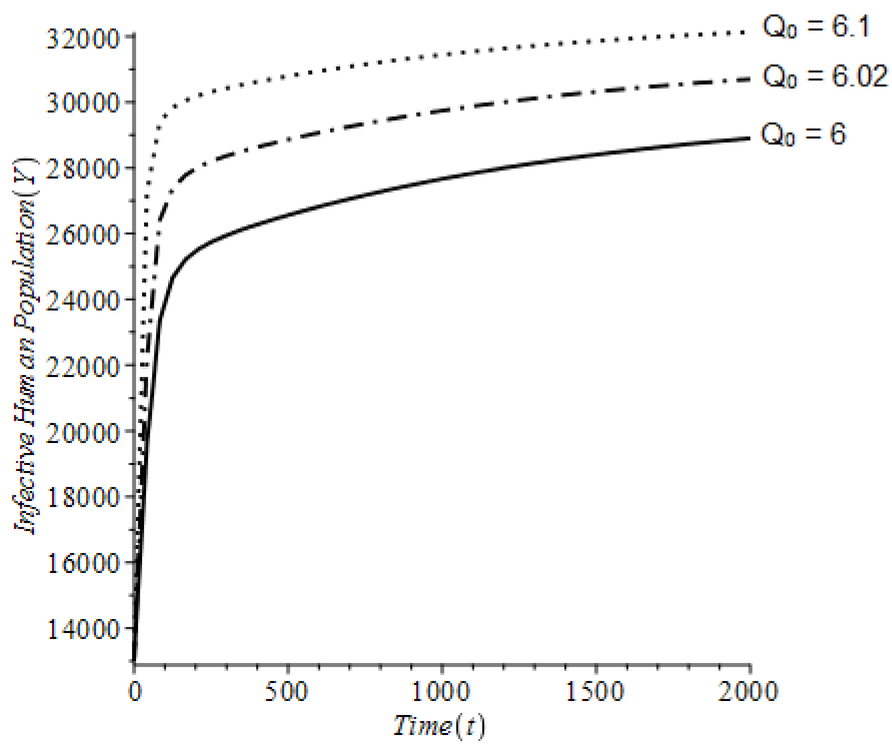


***Figure* 5.12:** Plot between infective human population Y and time t for various values of Q_0.

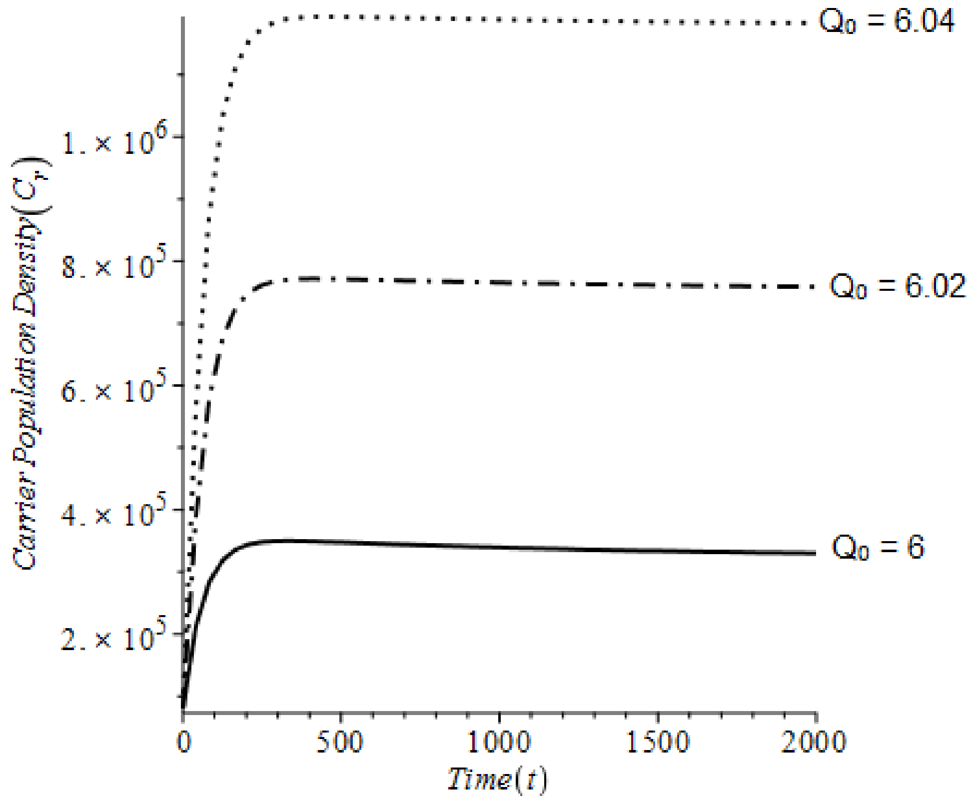


***Figure* 5.13:** Plot between carrier population density C_r and time t for various values of Q_0.

6 Conclusions

The larger amount of carbon dioxide (CO_2) in the environment is accountable for global warming. In this paper, we have studied the effect of global warming on the growth of carrier population with a constant growth rate and its role on the spread of infectious diseases. The model has five dependent variables namely, the density of susceptibles, the density of infectives, the density of carrier population, the amount of CO_2 in the environment causing global warming and the global warming temperature of the environment due to discharge of CO_2 . We have assumed that the density of carriers increases due to global warming temperature. The global warming temperature has been assumed to be proportional to the amount of CO_2 in the environment.

The proposed mathematical model has been studied with the help of stability theory of differential equations and numerical simulation. The local stability and the global stability conditions for non-trivial equilibrium point have been derived. For a set of parameters, numerical simulation proves the analytical results.

It has been found out that as the discharge rate of carbon dioxide from natural factors and man-made sources increases, the number of infectives increases. Also, the infectives and carrier population increase as the global warming temperature increases and thus the prevalence of carrier dependent infectious diseases increases in the environment.

Appendix A. Proof of the *Theorem* 4.1

Proof. The Jacobian matrix for the system (2.2) - (2.6) is

$$J(E) = \begin{bmatrix} a_{11} & a_{12} & a_{13} & 0 & 0 \\ -\alpha & -d & 0 & 0 & 0 \\ 0 & 0 & -s_0 & 0 & s_1 \\ 0 & -\alpha_1 d & 0 & -\alpha_o & 0 \\ 0 & 0 & 0 & \theta & -\theta_0 \end{bmatrix},$$

where

$$a_{11} = \beta(N^* - Y^*) - \beta Y^* - \lambda C_r^* - (\nu + \alpha + d), a_{12} = \beta Y^* + \lambda C_r^*, a_{13} = \lambda(N^* - Y^*).$$

The characteristic equation of above Jacobian matrix is as follows:

(A.1)

$$\begin{aligned}
&x^5 + \{-a_{11} + (s_0 + \alpha_0 + \theta_0) + d\}x^4 \\
&\quad + \{-a_{11}(d + s_0 + \alpha_0 + \theta_0) + a_{12}\alpha + d(s_0 + \alpha_0 + \theta_0) + (s_0\alpha_0 + \alpha_0\theta_0 + s_0\theta_0)\}x^3 \\
&\quad + [-a_{11}\{d(s_0 + \alpha_0 + \theta_0) + (s_0\alpha_0 + \alpha_0\theta_0 + s_0\theta_0)\} + a_{12}\alpha(s_0 + \alpha_0 + \theta_0) + d(s_0\alpha_0 + \alpha_0\theta_0 + s_0\theta_0) + s_0\theta_0\alpha_0]x^2 \\
&\quad + [-a_{11}\{d(s_0\alpha_0 + \alpha_0\theta_0 + s_0\theta_0) + s_0\theta_0\alpha_0\} + a_{12}\alpha(s_0\alpha_0 + \alpha_0\theta_0 + s_0\theta_0) + ds_0\theta_0\alpha_0]x \\
&\quad + (-a_{11}ds_0\theta_0\alpha_0 + a_{12}\alpha s_0\theta_0\alpha_0 - a_{13}\alpha ds_1\theta\alpha_1) = 0.
\end{aligned}$$

By (2.2), we have

(A.2)
$$\begin{aligned}
&\beta(N^* - Y^*)Y^* + \lambda(N^* - Y^*)C_r^* - (\nu + \alpha + d)Y^* = 0 \\
&\Rightarrow \{\beta(N^* - Y^*) - (\nu + \alpha + d)\} = -\frac{\lambda(N^* - Y^*)C_r^*}{Y^*}.
\end{aligned}$$

Now writing the value of a_{11}, with the help of (A.2), we get

(A.3)
$$a_{11} = -\left(\frac{\lambda N^* C_r^*}{Y^*} + \beta Y^*\right) < 0.$$

Also we have

(A.4)
$$C_r^* = \frac{s\theta_0\alpha_0 + s_1\theta\alpha_1\alpha Y^*}{s_0\theta_0\alpha_0} \Rightarrow s_1\theta\alpha_1\alpha = \frac{s_0\theta_0\alpha_0(C_r^* - 1)}{Y^*}.$$

Use (A.4) and the values of a_{11}, a_{12}, a_{13} in constant term of (A.1), we get

$$-a_{11}ds_0\theta_0\alpha_0 + a_{12}\alpha s_0\theta_0\alpha_0 - a_{13}\alpha ds_1\theta\alpha_1 = s_0\theta_0\alpha_0[(\beta Y^* + \lambda C_r^*)(\alpha + d) + \frac{d\lambda(N^* - Y^*)}{Y^*}].$$

Hence the constant term of (A.1) is positive.

Since $a_{11} = -(\frac{\lambda N^* C_r^*}{Y^*} + \beta Y^*) < 0$ and constant term of (A.1) is positive, therefore all coefficients of (A.1) are positive, so the characteristic equation (A.1) can be written as

(A.5)
$$x^5 + a_1x^4 + a_2x^3 + a_3x^2 + a_4x + a_5 = 0,$$

where

$$\begin{aligned}
a_1 &= (\beta Y^* + \frac{\lambda C_r^* N^*}{Y^*}) + (s_0 + \alpha_0 + \theta_0) + d, \\
a_2 &= (\beta Y^* + \frac{\lambda C_r^* N^*}{Y^*})(d + s_0 + \alpha_0 + \theta_0) + \alpha(\beta Y^* + \lambda C_r^*) + d(s_0 + \alpha_0 + \theta_0), \\
&\quad + (s_0\alpha_0 + \alpha_0\theta_0 + s_0\theta_0), \\
a_3 &= (\beta Y^* + \frac{\lambda C_r^* N^*}{Y^*})\{d(s_0 + \alpha_0 + \theta_0) + (s_0\alpha_0 + \alpha_0\theta_0 + s_0\theta_0)\},
\end{aligned}$$

$$+ \alpha(\beta Y^* + \lambda C_r^*)(s_0 + \alpha_0 + \theta_0) + d(s_0\alpha_0 + \alpha_0\theta_0 + s_0\theta_0) + s_0\theta_0\alpha_0,$$

$$a_4 = (\beta Y^* + \frac{\lambda C_r^* N^*}{Y^*})\{d(s_0\alpha_0 + \alpha_0\theta_0 + s_0\theta_0) + s_0\theta_0\alpha_0\},$$

$$+ \alpha(\beta Y^* + \lambda C_r^*)(s_0\alpha_0 + \alpha_0\theta_0 + s_0\theta_0) + d s_0\theta_0\alpha_0,$$

$$a_5 = [(\beta Y^* + \lambda C_r^*)(\alpha + d) + \frac{\lambda d(N^* - Y^*)}{Y^*}] s_0\theta_0\alpha_0.$$

Hence by Routh-Hurwitz criterion the equilibrium point $E(Y^*, N^*, C_r^*, C^*, T^*)$is locally asymptotically stable provided the following conditions are satisfied

$$a_1a_2a_3 - a_3^2 - a_1^2a_4 > 0, \tag{A.6}$$

$$(a_1a_4 - a_5)(a_1a_2a_3 - a_3^2 - a_1^2a_4) - a_5(a_1a_2 - a_3)^2 - a_1a_5^2 > 0. \tag{A.7}$$

Appendix B. Proof of the *Theorem* 4.2

Proof. For the proof of this theorem we use Lyapunov direct method. For this first we let the following positive definite Lyapunov function

$$W = K_0(Y - Y^* - Y^* \ln \frac{Y}{Y^*}) + \frac{K_1}{2}(N - N^*)^2 + \frac{K_2}{2}(C_r - C_r^*)^2 + \frac{K_3}{2}(C - C^*)^2 + \frac{K_4}{2}(T - T^*)^2, \tag{B.1}$$

where K_o, K_1, K_2, K_3 and K_4 are positive constants to be chosen appropriately.

Differentiate (B.1) with respect to t, we get

$$\dot{W} = K_o\frac{\dot{Y}}{Y}(Y - Y^*) + K_1(N - N^*)\dot{N} + K_2(C_r - C_r^*)\dot{C}_r + K_3(C - C^*)\dot{C} + K_4(T - T^*)\dot{T}. \tag{B.2}$$

Using model system (2.2) - (2.6), we get

$$\begin{aligned}\dot{W} =& - K_o\frac{C_r\lambda N}{YY^*}(Y - Y^*)^2 - K_o\beta(Y - Y^*)^2 - K_1 d(N - N^*)^2 - K_2 s_0(C_r - C_r^*)^2\\ &- K_3\alpha_0(C - C^*)^2 - K_4\theta_0(T - T^*)^2 + (K_o\beta - K_1\alpha)(N - N^*)(Y - Y^*)\\ &+ K_o\frac{C_r\lambda}{Y^*}(N - N^*)(Y - Y^*) + K_0\lambda(\frac{N^* - Y^*}{Y^*})(C_r - C_r^*)(Y - Y^*)\\ &+ K_2 s_1(C_r - C_r^*)(T - T^*) - K_3\alpha_1 d(N - N^*)(C - C^*) + K_4\theta(T - T^*)(C - C^*).\end{aligned}$$

Here we take the constants K_0 and K_1 such that $K_0\,\beta - K_1\alpha = 0$.

Further we take

$$K_1 = 1 \text{ then } K_0 = \frac{\alpha}{\beta}. \tag{B.3}$$

We have

$$\begin{aligned}\dot{W} =& - K_o\frac{C_r\lambda N}{YY^*}(Y - Y^*)^2 - \Big[\frac{K_o\beta}{2}(Y - Y^*)^2 - \frac{K_0\lambda C_r}{Y^*}(N - N^*)(Y - Y^*) + \frac{K_1 d}{2}(N - N^*)^2\Big]\\ &- \Big[\frac{K_1 d}{2}(N - N^*)^2 + K_3\alpha_1 d(N - N^*)(C - C^*) + \frac{K_3\alpha_0}{2}(C - C^*)^2\Big]\\ &- \Big[\frac{K_o\beta}{2}(Y - Y^*)^2 - K_0\lambda(\frac{N^* - Y^*}{Y^*})(C_r - C_r^*)(Y - Y^*) + \frac{K_2 s_0}{2}(C_r - C_r^*)^2\Big]\\ &- \Big[\frac{K_2 s_0}{2}(C_r - C_r^*)^2 - K_2 s_1(C_r - C_r^*)(T - T^*) + \frac{K_4\theta_0}{2}(T - T^*)^2\Big]\\ &- \Big[\frac{K_3\alpha_0}{2}(C - C^*)^2 - K_4\theta(T - T^*)(C - C^*) + \frac{K_4\theta_0}{2}(T - T^*)^2\Big].\end{aligned}$$

The derivative $\dot{W}$ is negative definite if the following conditions are satisfied:

(B.4) $$\alpha\lambda^2 C_{rm}^2 < d\beta^2 Y^{*2},$$

where C_{rm} is the maximum value of C_r

(B.5) $$K_3 < \frac{\alpha_o}{d\alpha_1^2},$$

(B.6) $$K_2 > \frac{\alpha\lambda^2(N^* - Y^*)^2}{s_0\beta^2 Y^{*2}},$$

(B.7) $$K_2 < \frac{s_0\theta_0}{s_1^2}K_4,$$

(B.8) $$K_4\theta^2 < K_3\alpha_o\theta_0.$$

Combining the inequalities (B.5), (B.6), (B.7) and (B.8), we get

(B.9) $$4\alpha d\lambda^2\alpha_1^2\theta^2 s_1^2(N^* - Y^*)^2 < \beta^2 Y^{*2} s_0^2\theta_0^2\alpha_0^2.$$

The inequalities (B.4) and (B.9) are the required conditions.

Acknowledgement. We are very much thankful to wrothy refree and Editor for their valuable suggestions to improve the paper in its present form

References

[1] R.M. Anderson and R.M. May, population biology of infectious diseases, Part I, *Nature*, **280** (1979), 361- 367.

[2] N.T.J. Bailey, T*he Mathematical Theory of Epidemics*. Griffin, London, 1957.

[3] M. Ghosh, P. Chandra, P. Sinha and J.B. Shukla, Modelling the Spread of Carrier dependent infectious diseases with environmental effect, *Appl. Math. Comput.*, **158** (2004), 385-402.

[4] M. Ghosh, J.B. Shukla, P. Chandra and P. Sinha, An epidemiological model for carrier dependent infectious diseases with environmental effect, *Int. J. Appl. Sci. Comput.*, **7(3)** (2000), 188-204.

[5] D. Greenhalgh, some threshold and stability results for epidemic models with a density dependent death rate, *Theor. Pop. Bio.,* **42** (1990), 130-151.

[6] H.W. Hethcote, Qualitative analysis of communicable disease models, *Math. Bio- Sciences,* **28** (1976), 335-356.

[7] H.W. Hethcote, The Mathematics of infectious diseases, *SIAM REVIEW* , **42(4)** (2000), 599-653.

[8] IPCC Summary for Policymakers. In: S. Solomon, D. Qin, M. Manning, Z. Chen, M. Marquis, K.B. Averyt, M. Tignor, H.L. Miller (eds) *Climate change* 2007: *the physical science basis. Contribution of working group I to the fourth assessment report of the intergovernmental panel on climate change.* Cambridge University Press, Cambridge, (2007a).

[9] IPCC Technical Summary. In: S. Solomon, D. Qin, M. Manning, Z. Chen, M. Marquis, K.B. Averyt, M. Tignor, H.L. Miller (eds) *Climate change* 2007: *the physical science basis. Contribution of working group I to the fourth assessment report of the intergovernmental panel on climate change.* Cambridge University Press, Cambridge, (2007b).

[10] IPCC Summary for Policymakers. In: O. Edenhofer, R. Pichs-Madruga, Y. Sokona, E. Farahani, S. Kadner, K. Seyboth, A. Adler, I. Baum, S. Brunner, P. Eickemeier, B. Kriemann, J. Savolainen, S. Schlomer, C.von stechow, T. Zwickel and J.C. Minx (eds.). *Climate change* 2014, *Mitigation of Climate Change. Contribution of working group III to the fifth assessment report of the intergovernmental panel on climate change.* Cambridge University Press, Cambridge, (2014).

[11] R.M. May and R.M. Anderson, Population biology of infectious diseases, Part II, *Nature*, **280** (1979), 455- 461.

[12] S. Singh, Modelling the role of ecology in the spread of carrier dependent infectious diseases: An SIS Model, *Jñānābha* , **44** (2014), 25 - 41.

[13] S. Singh, P. Chandra and J.B. Shukla, Modelling and analysis of the spread of carrier dependent infectious diseases with environmental effects, *J. Biol. Syst.,* **11** (2003), 325- 335.

[14] S. Singh, J.B. Shukla and P. Chandra, Modelling and analysis of spread of malaria: Environmental and ecological effects, *J. Biol. Syst.,* **13(1)** (2005), 1-11.

[15] WHO Global health risks: mortality and burden of disease attributable to the selected major risks. http://www.who.int/healthinfo/global_burden_disease/Global HealthRisks_report_full.pdf. Accessed 15 Apr 2015, (2009).

[16] WHO *Quantitative risk assessment of the effects of climate change on selected causes of death, 2030s and 2050s.* World Health Organisation, Geneva, (2014).

[17] WHO *Climate change and health. Factsheet.* http://www.who.int/mediacentre/ factsheets/fs266/en/, (2016).

[18] X.N. Zhou, G.J. Yang, K. Yang, X.H. Wang, Q.B. Hong, L.P. Sun et al., Potential impact of climate change on Schistosomiasis transmission in China. *Am J Trop Med Hyg,* **78(2)** (2008), 188-194.

Jñānābha, Vol. 50(1) (2020), 207-217
(Dedicated to Honor Professor H.M. Srivastava on His 80th Birth Anniversary Celebrations)

ANALYTICAL SOLUTIONS FOR TIME-FRACTIONAL CAUCHY REACTION-DIFFUSION EQUATIONS USING ITERATIVE LAPLACE TRANSFORM METHOD

By

R. K. Bairwa, Ajay Kumar and Karan Singh
Department of Mathematics, University of Rajasthan, Jaipur-302004, Rajasthan, India.
Email:dr.rajendra.maths@gmail.com, jangir.kmrajay@gmail.com,
karansinghmath@gmail.com

(*Received : May* 12, 2020 *; Revised: June* 10, 2020)
DOI: https://doi.org/10.58250/jnanabha.2020.50120

Abstract

In the present work, the iterative Laplace transform method *(ILTM)* is implemented to derive approximate analytical solutions for the time-fractional Cauchy reaction-diffusion equations *(CRDEs)* within the Caputo fractional derivative. The proposed technique is an elegant amalgam of the Iterative method and the Laplace transform method. The *ILTM* produces the solution in a rapid convergent series which may lead to the solution in a closed form. The obtained analytical outcomes with the help of the proposed technique are examined graphically.

2010 Mathematics Subject Classifications: 35A20; 35A22; 34A08; 33E12
Keywords and phrases: Caputo fractional derivative,Cauchy reaction-diffusion equations, Laplace transform, Mittag-Leffler function, Iterative Laplace transform method, fractional partial differential equations.

1 Introduction

Fractional calculus is a branch of mathematical analysis which is concerned with derivatives and integrals of arbitrary orders.It has attracted the great attention of scientists and engineers from a long time ago and has resulted in many applications being created. Since the 1990's, fractional calculus has been rediscovered and adapted in a growing number of fields such as biology, mathematical physics, electrochemistry, signal processing, chemical physics, electromagnetics, acoustics, viscoelasticity, material science, probability and statistics, engineering, physics fluid dynamics and other areas of sciences and technology. In recent years, many researchers have paid attention to investigating solutions of fractional differential equations using numerous techniques such as the Laplace decomposition method *(LDM)* [8], the Homotopy Analysis method *(HAM)* [12], the Homotopy perturbation method *(HPM)* [18], the Homotopy perturbation transform method *(HPTM)* [10, 11], Sumudu transform iterative method *(STIM)* [17], q-homotopy analysis transform method *(q-HATM)* [16] etc.

In 2006, Daftardar-Gejji and Jafari introduced the iterative technique for numerically examining nonlinear functional equations [6, 7]. Since then, iterative technique has been used to find a solution for several non-linear differential equations of arbitrary orders [3] and the viewing of fractional Boundary Value problems [5]. Jafari et al. [9] used Laplace transform together with iterative method, became a well-known technique called iterative Laplace transform method for solving a system of partial differential equations of fractional order. Recently, fractional heat and wave like equations [15], fractional Navier-Stokes equations [2] and fractional Zakharov-Kuznetsov Equations [1] are solved successfully using the iterative Laplace transform method.

In the present study, the following time-fractional Cauchy reaction-diffusion equation in operator form is considered as[11]

$$D_t^{\alpha} w(\xi,t) = \upsilon \frac{\partial^2 w(\xi,t)}{\partial \xi^2} + p(\xi,t)w(\xi,t), \xi \in \mathbb{R}, t > 0, 0 < \alpha \leq 1, \tag{1.1}$$

with initial condition $w(\xi,0) = w_0(\xi)$, where $\upsilon > 0$ is diffusion coefficient, $w(\xi,t)$ and $p(\xi,t)$ denote the concentration and the reaction parameter, respectively. In particular for $\alpha = 1$, time-fractional Cauchy reaction-diffusion equation reduces to classical Cauchy reaction-diffusion equation.
The main objective of the present paper is to extend the work of the *ILTM* technique to investigate approximate analytical solutions for the time-fractional Cauchy reaction-diffusion equations and to conclude accuracy, efficiency, simplicity of the proposed technique.

2 Preliminaries and Basic Definitions

In this section, we give certain basic definitions, notations and properties of fractional calculus with Laplace transform theory, which are used further in this paper.

***Definition* 2.1.** *The fractional derivative of a function in the sense of the Caputo is presented as* [4]

$$\begin{aligned} D_t^{\alpha} w(\xi,t) &= \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\rho)^{m-\alpha-1} w^{(m)}(\xi,\rho)d\rho, \; m-1 < \alpha \leq m, m \in \mathbb{N}, \\ &= J_t^{m-\alpha} D^m w(\xi,t). \end{aligned} \tag{2.1}$$

Here $D^m \equiv \dfrac{d^m}{dt^m}$ and J_t^{α} stands for the Riemann-Liouville fractional integral operator of order $\alpha > 0$, defined as [13]

$$J_t^{\alpha} w(\xi,t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\rho)^{\alpha-1} w(\xi,\rho)d\rho, \; \rho > 0, \tag{2.2}$$

where $\Gamma(.)$ is the well-known Gamma function.

***Definition* 2.2.** *The Laplace transform of a function $f(\xi)$, $\xi > 0$ is expressed as* [13, 14]

$$L[f(\xi)] = F(s) = \int_0^{\infty} e^{-s\xi} f(\xi)d\xi, \tag{2.3}$$

where s is real or complex number.

***Definition* 2.3.** *The Laplace transform of Caputo fractional derivative is presented in following manner* [13, 14]

$$L[D_t^{\alpha} w(\xi,t)] = s^{\alpha} L[w(\xi,t)] - \sum_{k=0}^{m-1} w^{(k)}(\xi,0)s^{\alpha-k-1}, m-1 < \alpha \leq m, m \in \mathbb{N}, \tag{2.4}$$

where $w^{(k)}(\xi,0)$ is the k-order derivative of $w(\xi,t)$ with respect to t at $t = 0$.

***Definition* 2.4.** *The Mittag-Leffler function $E_{\alpha}(z)$ is defined by the following series representation as* [13]

$$E_{\alpha}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \alpha \in \mathbb{C}, Re(\alpha) > 0. \tag{2.5}$$

3 Basic Idea of Iterative Laplace Transform Method

In order to elucidate the solution procedure of this method [9], we take the subsequent fractional partial differential equation having the prescribed initial conditions can be expressed in the form of an operator as

$$D_t^{\alpha} w(\xi,t) + Rw(\xi,t) + Nw(\xi,t) = g(\xi,t),\ m-1<\alpha\le m, m\in\mathbb{N}, \tag{3.1}$$

$$w^{(k)}(\xi,0) = h_k(\xi),\ k=0,1,2,...,m-1, \tag{3.2}$$

where $D_t^{\alpha} w(\xi,t)$ is the Caputo fractional derivative of order $\alpha, m-1<\alpha\le m$, defined by equation (2.1), R is a linear operator and may include other fractional derivatives of order less than α, N is a non-linear operator which may include other fractional derivatives of order less than α and $g(\xi,t)$ is a known analytic function.

Applying the Laplace transform on both sides of equation (3.1), we have

$$L[D_t^{\alpha} w(\xi,t)] + L[Rw(\xi,t) + Nw(\xi,t)] = L[g(\xi,t)]. \tag{3.3}$$

Using equation (2.4), we obtain

$$L[w(\xi,t)] = \frac{1}{s^{\alpha}}\sum_{k=0}^{m-1} s^{\alpha-1-k} w^k(\xi,0) + \frac{1}{s^{\alpha}} L[g(\xi,t)] - \frac{1}{s^{\alpha}} L[Rw(\xi,t) + Nw(\xi,t)]. \tag{3.4}$$

On taking inverse Laplace transform on equation (3.4), we have

$$w(\xi,t) = L^{-1}\Big[\frac{1}{s^{\alpha}}\Big(\sum_{k=0}^{m-1} s^{\alpha-1-k} w^k(\xi,0) + L[g(\xi,t)]\Big)\Big] - L^{-1}\Big[\frac{1}{s^{\alpha}} L[Rw(\xi,t) + Nw(\xi,t)]\Big]. \tag{3.5}$$

Further, we apply the iterative method introduced by Daftardar-Gejji and Jafari [6], which represents a solution $w(\xi,t)$ in infinite series of components

$$w(\xi,t) = \sum_{i=0}^{\infty} w_i(\xi,t). \tag{3.6}$$

As R is a linear operator, so we have

$$R\Big(\sum_{i=0}^{\infty} w_i(\xi,t)\Big) = \sum_{i=0}^{\infty} R[w_i(\xi,t)], \tag{3.7}$$

and the non-linear operator N is decomposed as

$$N\Big(\sum_{i=0}^{\infty} w_i(\xi,t)\Big) = N[w_0(\xi,t)] + \sum_{i=0}^{\infty}\Big[N\Big(\sum_{k=0}^{i} w_k(\xi,t)\Big) - N\Big(\sum_{k=0}^{i-1} w_k(\xi,t)\Big)\Big], \tag{3.8}$$

Substituting the results given by equations from (3.6) to (3.8) in the equation (3.5), we get

$$\begin{aligned}\sum_{i=0}^{\infty} w_i(\xi,t) &= L^{-1}\Big[\frac{1}{s^{\alpha}}\Big(\sum_{k=0}^{m-1} s^{\alpha-1-k} w^k(\xi,0) + L[g(\xi,t)]\Big)\Big]\\ &- L^{-1}\Big[\frac{1}{s^{\alpha}} L\Big[\sum_{i=0}^{\infty} R[w_i(\xi,t)] + N[w_0(\xi,t)] + \sum_{i=1}^{\infty}\Big(N\Big(\sum_{k=0}^{i} w_k(\xi,t)\Big) - N\Big(\sum_{k=0}^{i-1} w_k(\xi,t)\Big)\Big)\Big]\Big].\end{aligned} \tag{3.9}$$

We have defined the recurrence relations as

$$\begin{cases} w_0(\xi,t) &= L^{-1}\Big[\frac{1}{s^{\alpha}}\Big(\sum_{k=0}^{m-1} s^{\alpha-1-k} w^k(\xi,0) + L[g(\xi,t)]\Big)\Big] \\ w_1(\xi,t) &= -L^{-1}\Big[\frac{1}{s^{\alpha}} L\Big[\sum_{i=0}^{\infty} R[w_i(\xi,t)] + N[w_0(\xi,t)]\Big]\Big] \\ w_{m+1}(\xi,t) &= L^{-1}\Big[\frac{1}{s^{\alpha}} L\Big[R[w_m(\xi,t)] - \Big(N\Big(\sum_{k=0}^{m} w_k(\xi,t)\Big) \\ & -N\Big(\sum_{k=0}^{m-1} w_k(\xi,t)\Big)\Big)\Big]\Big], m\ge 1 \end{cases} \tag{3.10}$$

Proceeding in the same manner the rest of components of the *ILTM* solution can be obtained. Finally, we approximate the analytical solution $w(\xi, t)$ in truncated series form is given by

$$w(\xi, t) \cong \lim_{N\to\infty} \sum_{m=0}^{N} w_m(\xi, t). \tag{3.11}$$

The above series solutions generally converge very rapidly. A classical approach of convergence of this type of series is already presented by Jafari [7] and Daftardar-Gejji and Jafari [6]

4 Solutions of the time-fractional Cauchy Reaction-Diffusion equations

In this section, we apply the *ILTM* technique for solving time-fractional Cauchy reaction-diffusion equations with initial conditions.

***Example* 4.1.** In this example, the following linear time-fractional Cauchy reaction-diffusion equation is considered as [11]

$$D_t^{\alpha} w(\xi, t) = \frac{\partial^2 w(\xi, t)}{\partial x^2} - w(\xi, t), 0 < \alpha \le 1, \tag{4.1}$$

subject to the initial condition

$$w(\xi, 0) = e^{-\xi} + \xi. \tag{4.2}$$

Taking the Laplace transform on the both sides of equation (4.1), and making use of the result given by equation (4.2), we have

$$L[w(\xi, t)] = \frac{(e^{-\xi} + \xi)}{s} + \frac{1}{s^{\alpha}} L\Big[\frac{\partial^2 w}{\partial \xi^2} - w\Big]. \tag{4.3}$$

Operating with inverse Laplace transform on both sides of equation (4.3) gives

$$w(\xi, t) = e^{-\xi} + \xi + L^{-1}\Big[\frac{1}{s^{\alpha}} L\Big[\frac{\partial^2 w}{\partial \xi^2} - w\Big]\Big]. \tag{4.4}$$

Now, applying the iterative method, substituting the equations (3.6) to (3.8) into equation (4.4) and applying equation (3.10), we determine the components of the *ILTM* solution as follows

$$w_0(\xi, t) = e^{-\xi} + \xi, \tag{4.5}$$

$$w_1(\xi, t) = L^{-1}\Big[\frac{1}{s^{\alpha}} L\Big[\frac{\partial^2 w_0}{\partial \xi^2} - w_0\Big]\Big] = \xi \frac{(-t^{\alpha})}{\Gamma(\alpha + 1)}, \tag{4.6}$$

$$w_2(\xi, t) = L^{-1}\Big[\frac{1}{s^{\alpha}} L\Big[\frac{\partial^2 w_1}{\partial \xi^2} - w_1\Big]\Big] = \xi \frac{(-t^{\alpha})^2}{\Gamma(2\alpha + 1)}, \tag{4.7}$$

$$w_3(\xi, t) = L^{-1}\Big[\frac{1}{s^{\alpha}} L\Big[\frac{\partial^2 w_2}{\partial \xi^2} - w_2\Big]\Big] = \xi \frac{(-t^{\alpha})^3}{\Gamma(3\alpha + 1)}. \tag{4.8}$$

Proceeding in the same manner the rest of components $w_m(\xi, t)$ for $m \ge 4$ can be obtained. Thus, the approximate analytical solution in the series form can be obtained as

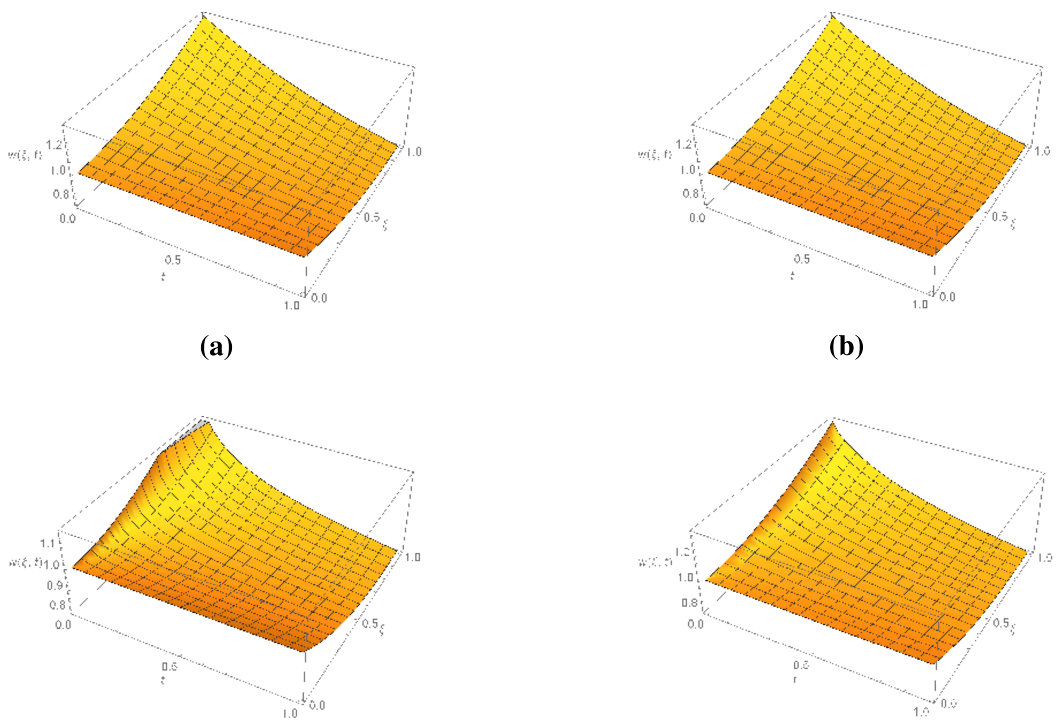
$$\begin{aligned} w(\xi, t) \cong \lim_{N\to\infty} \sum_{m=0}^{N} w_m(\xi, t) &= e^{-\xi} + \xi + \xi \frac{(-t^{\alpha})}{\Gamma(\alpha + 1)} + \xi \frac{(-t^{\alpha})^2}{\Gamma(2\alpha + 1)} + \xi \frac{(-t^{\alpha})^3}{\Gamma(3\alpha + 1)} + ..., \\ &= e^{-\xi} + \xi E_{\alpha}(-t)^{\alpha}. \end{aligned} \tag{4.9}$$

which is the same result was obtained by Kumar [11] using *HPTM*.

***Remark* 4.1.** For $\alpha = 1$, the result in equation (4.9) reduces to the following exact form

$$w(\xi, t) = e^{-\xi} + \xi e^{-t}. \tag{4.10}$$

This result was achieved earlier by Yildirim [18] using the *HPM* method.



(a) (b) (c) (d)

***Figure* 4.1:** The surface shows the solution $w(\xi, t)$ for ***Example* 4.1.**, when (a) The exact solution, (b) The approximate solution for $\alpha = 1$, (c) The approximate solution for $\alpha = 0.5$, (d) The approximate solution for $\alpha = 0.75$.

***Example* 4.2.** Consider the following time fractional linear Cauchy reaction-diffusion equation [11]

$$D_t^{\alpha} w(\xi, t) = \frac{\partial^2 w(\xi, t)}{\partial \xi^2} - (1 + 4\xi^2) w(\xi, t), 0 < \alpha \leq 1, \tag{4.11}$$

with the initial condition

$$w(\xi, 0) = e^{\xi^2}. \tag{4.12}$$

Taking the Laplace transform of the equation (4.11), and making use of the result given by equation (4.12), we have

$$L[w(\xi, t)] = \frac{e^{\xi^2}}{s} + \frac{1}{s} L\Big[\frac{\partial^2 w}{\partial \xi^2} - (1 + 4\xi^2) w\Big]. \tag{4.13}$$

Applying inverse Laplace transform to the equation (4.13), we obtain

$$w(\xi, t) = e^{\xi^2} + L^{-1}\Big[\frac{1}{s^{\alpha}} L\Big[\frac{\partial^2 w}{\partial \xi^2} - (1 + 4\xi^2) w\Big]\Big]. \tag{4.14}$$

Now,applying the iterative method, substituting the equations (3.6) to (3.8) into equation (4.14) and applying equation (3.10), we determine the components of the *ILTM* solution as follows

$$w_0(\xi, t) = e^{\xi^2}, \tag{4.15}$$

$$(4.16)\qquad w_1(\xi,t) = L^{-1}\Big[\frac{1}{s^\alpha}L\Big[\frac{\partial^2 w_0}{\partial\xi^2} - (1+4\xi^2)w_0\Big]\Big] = e^{\xi^2}\frac{t^\alpha}{\Gamma(\alpha+1)},$$

$$(4.17)\qquad w_2(\xi,t) = L^{-1}\Big[\frac{1}{s^\alpha}L\Big[\frac{\partial^2 w_1}{\partial\xi^2} - (1+4\xi^2)w_1\Big]\Big] = e^{\xi^2}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)},$$

$$(4.18)\qquad w_3(\xi,t) = L^{-1}\Big[\frac{1}{s^\alpha}L\Big[\frac{\partial^2 w_2}{\partial\xi^2} - (1+4\xi^2)w_2\Big]\Big] = e^{\xi^2}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}.$$

Proceeding in the same manner the rest of components $w_m(\xi,t)$ for $m \geq 4$ can be obtained. Thus the approximate analytical solution in the series form can be obtained as

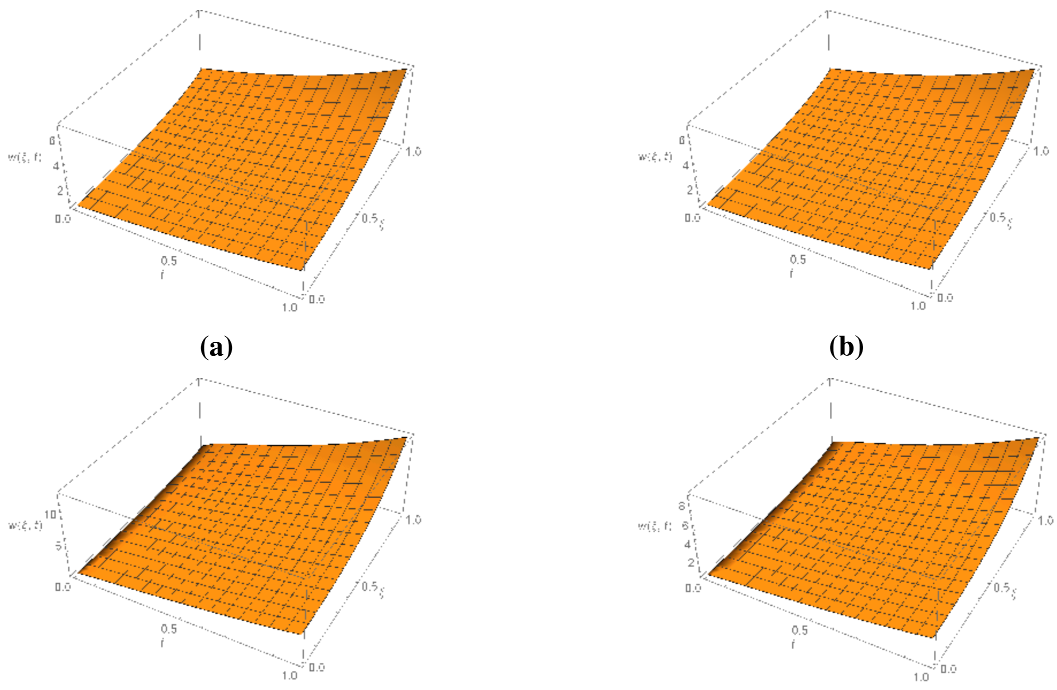
$$(4.19)\qquad w(\xi,t) \cong \lim_{N\to\infty}\sum_{m=0}^{N} w_m(\xi,t) = e^{\xi^2} + e^{\xi^2}\frac{t^\alpha}{\Gamma(\alpha+1)} + e^{\xi^2}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + e^{\xi^2}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \ldots.$$
$$= e^{\xi^2}E_\alpha(t^\alpha).$$

which is the same result was obtained by Kumar [11] using *HPTM*.

***Remark* 4.2.** For $\alpha = 1$, the result in equation (4.19) reduces to the following exact form

$$(4.20)\qquad w(\xi,t) = e^{\xi^2+t}.$$

This result was achieved earlier by Yildirim [18] using the *HPM* method.



(a) **(b)** **(c)** **(d)**

***Figure* 4.2:** The surface shows the solution $w(\xi,t)$ for ***Example* 4.2**, when (a) The exact solution, (b) The approximate solution for $\alpha = 1$, (c) The approximate solution for $\alpha = 0.5$, (d) The approximate solution for $\alpha = 0.75$.

***Example* 4.3.** Consider the following linear Cauchy reaction-diffusion equation involving time-fractional derivative as [11]

$$(4.21)\qquad D_t^\alpha w(\xi,t) = \frac{\partial^2 w(\xi,t)}{\partial\xi^2} - (4\xi^2 - 2t + 2)w(\xi,t), 0 < \alpha \leq 1,$$

subject to initial condition

$$w(\xi, 0) = e^{\xi^2}. \tag{4.22}$$

Taking the Laplace transform of the equation (4.21), and making use of the result given by equation (4.22), we have

$$L[w(\xi, t)] = \frac{e^{\xi^2}}{s} + \frac{1}{s^\alpha} L\Big[\frac{\partial^2 w}{\partial \xi^2} - (4\xi^2 - 2t + 2)w\Big]. \tag{4.23}$$

Applying inverse Laplace transform to the equation (4.23), we obtain

$$w(\xi, t) = e^{\xi^2} + L^{-1}\Big[\frac{1}{s^\alpha} L\Big[\frac{\partial^2 w}{\partial \xi^2} - (4\xi^2 - 2t + 2)w\Big]\Big]. \tag{4.24}$$

Now, applying the the iterative method, substituting the equations (3.6) to (3.8) into equation (4.24) and applying equation (3.10), we determine the components of the *ILTM* solution as follows

$$w_0(\xi, t) = e^{\xi^2}, \tag{4.25}$$

$$w_1(\xi, t) = L^{-1}\Big[\frac{1}{s^\alpha} L\Big[\frac{\partial^2 w_0}{\partial \xi^2} - (4\xi^2 - 2t + 2)w_0\Big]\Big] = 2e^{\xi^2} \frac{t^{\alpha+1}}{\Gamma(\alpha + 2)}, \tag{4.26}$$

$$w_2(\xi, t) = L^{-1}\Big[\frac{1}{s^\alpha} L\Big[\frac{\partial^2 w_1}{\partial \xi^2} - (4\xi^2 - 2t + 2)w_1\Big]\Big] = 4e^{\xi^2} \frac{(\alpha + 2)t^{2\alpha+2}}{\Gamma(2\alpha + 3)}, \tag{4.27}$$

$$w_3(\xi, t) = L^{-1}\Big[\frac{1}{s^\alpha} L\Big[\frac{\partial^2 w_2}{\partial \xi^2} - (4\xi^2 - 2t + 2)w_2\Big]\Big] = 8e^{\xi^2} \frac{(\alpha + 2)(2\alpha + 3)t^{3\alpha+3}}{\Gamma(3\alpha + 4)}, \tag{4.28}$$

$$w_4(\xi, t) = L^{-1}\Big[\frac{1}{s^\alpha} L\Big[\frac{\partial^2 w_2}{\partial \xi^2} - (4\xi^2 - 2t + 2)w_2\Big]\Big] = 16e^{\xi^2} \frac{(\alpha + 2)(2\alpha + 3)(3\alpha + 4)t^{4\alpha+4}}{\Gamma(4\alpha + 5)}. \tag{4.29}$$

Proceeding in the same manner the rest of components $w_m(\xi, t)$ for $m \geq 5$ can be obtained. Thus, the approximate analytical solution in the series form can be obtained as

$$\begin{aligned} w(\xi, t) \cong \lim_{N\to\infty} \sum_{m=0}^{N} w_m(\xi, t) &= e^{\xi^2} + 2e^{\xi^2} \frac{t^{\alpha+1}}{\Gamma(\alpha + 2)} + 4e^{\xi^2} \frac{(\alpha + 2)t^{2\alpha+2}}{\Gamma(2\alpha + 3)} \\ &+ 8e^{\xi^2} \frac{(\alpha + 2)(2\alpha + 3)t^{3\alpha+3}}{\Gamma(3\alpha + 4)} + 16e^{\xi^2} \frac{(\alpha + 2)(2\alpha + 3)(3\alpha + 4)t^{4\alpha+4}}{\Gamma(4\alpha + 5)} + \ldots \end{aligned} \tag{4.30}$$

which is the same result was obtained by Kumar [11] using *HPTM*.

***Remark* 4.3.** For $\alpha = 1$, the result in equation (4.30) reduces to the following exact form

$$w(\xi, t) = e^{\xi^2 + t^2}. \tag{4.31}$$

This result was obtained earlier by Yildirim [18] by using the method of *HPM*.

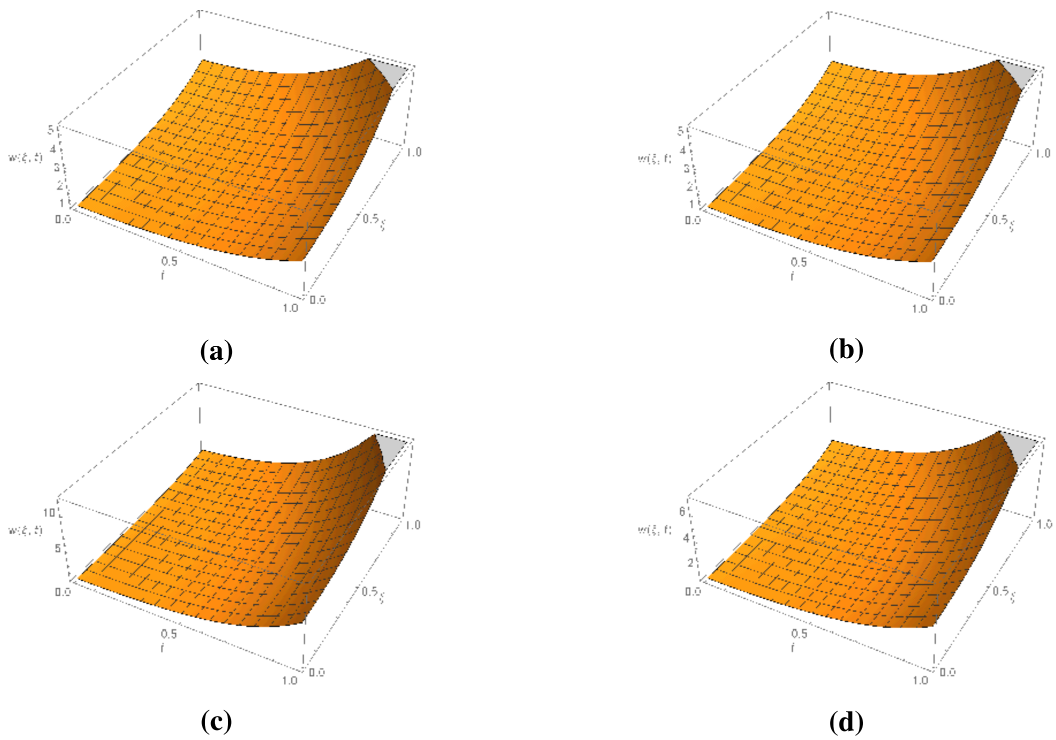


***Figure* 4.3:** The surface shows the solution $w(\xi, t)$ for ***Example* 4.3**, when (a) The exact solution, (b) The approximate solution for $\alpha = 1$, (c) The approximate solution for $\alpha = 0.5$, (d) The approximate solution for $\alpha = 0.75$.

***Example* 4.4.** Finally, we consider the following time-fractional linear Cauchy reaction-diffusion equation as [11]

$$D_t^{\alpha} w(\xi, t) = \frac{\partial^2 w(\xi, t)}{\partial \xi^2} + 2tw(\xi, t), 0 < \alpha \leq 1, \tag{4.32}$$

with the given initial condition

$$w(\xi, 0) = e^{\xi}. \tag{4.33}$$

Taking the Laplace transform of the equation (4.32), and making use of the result given by equation (4.33), we have

$$L[w(\xi, t)] = \frac{e^{\xi}}{s} + \frac{1}{s^{\alpha}} L\Big[\frac{\partial^2 w}{\partial \xi^2} + 2tw\Big]. \tag{4.34}$$

Applying inverse Laplace transform to the equation (4.34), we obtain

$$w(\xi, t) = e^{\xi} + L^{-1}\Big[\frac{1}{s^{\alpha}} L\Big[\frac{\partial^2 w}{\partial \xi^2} + 2tw\Big]\Big]. \tag{4.35}$$

Now, applying the iterative method, substituting the equations (3.6) to (3.8) into equation (4.35) and applying equation (3.10), we determine the components of the *ILTM* solution as follows

$$w_0(\xi, t) = e^{\xi}, \tag{4.36}$$

$$w_1(\xi, t) = L^{-1}\Big[\frac{1}{s^{\alpha}} L\Big[\frac{\partial^2 w_0}{\partial \xi^2} + 2tw_0\Big]\Big] = e^{\xi}\Big(\frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{2t^{\alpha+1}}{\Gamma(\alpha+2)}\Big), \tag{4.37}$$

$$w_2(\xi, t) = L^{-1}\Big[\frac{1}{s^{\alpha}} L\Big[\frac{\partial^2 w_1}{\partial \xi^2} + 2tw_1\Big]\Big] = e^{\xi}\Big(\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{2(\alpha+2)t^{2\alpha+1}}{\Gamma(2\alpha+2)} + \frac{4(\alpha+2)t^{2\alpha+2}}{\Gamma(2\alpha+3)}\Big), \tag{4.38}$$

$$(4.39)\qquad w_3(\xi,t) = L^{-1}\Big[\frac{1}{s^{\alpha}}L\Big[\frac{\partial^2 w_2}{\partial \xi^2} + 2tw_2\Big]\Big] = e^{\xi}\Big(\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \frac{6(\alpha+1)t^{3\alpha+1}}{\Gamma(3\alpha+2)} + \frac{4(\alpha+2)(2\alpha+3)t^{3\alpha+2}}{\Gamma(3\alpha+3)} + \frac{8(\alpha+2)(2\alpha+3)t^{3\alpha+3}}{\Gamma(3\alpha+4)}\Big).$$

Proceeding in the same manner the rest of components $w_m(\xi,t)$ for $m \geq 4$ can be obtained. Thus,the approximate analytical solution in the series form can be obtained as

$$(4.40)\qquad w(\xi,t) \cong \lim_{N\to\infty}\sum_{m=0}^{N} w_m(\xi,t) = e^{\xi} + e^{\xi}\Big(\frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{2t^{\alpha+1}}{\Gamma(\alpha+2)}\Big) + e^{\xi}\Big(\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{2(\alpha+2)t^{2\alpha+1}}{\Gamma(2\alpha+2)} + \frac{4(\alpha+2)t^{2\alpha+2}}{\Gamma(2\alpha+3)}\Big) + e^{\xi}\Big(\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \frac{6(\alpha+1)t^{3\alpha+1}}{\Gamma(3\alpha+2)} + \frac{4(\alpha+2)(2\alpha+3)t^{3\alpha+2}}{\Gamma(3\alpha+3)} + \frac{8(\alpha+2)(2\alpha+3)t^{3\alpha+3}}{\Gamma(3\alpha+4)}\Big) + \ldots$$

which is the same result was obtained by Kumar [11] using *HPTM*.

***Remark* 4.4.** For $\alpha = 1$, the result in equation (4.40) reduces to the following exact form

$$(4.41)\qquad w(\xi,t) = e^{\xi+t+t^2}.$$

This result was obtained earlier by Yildirim [18] by using the method of *HPM*.

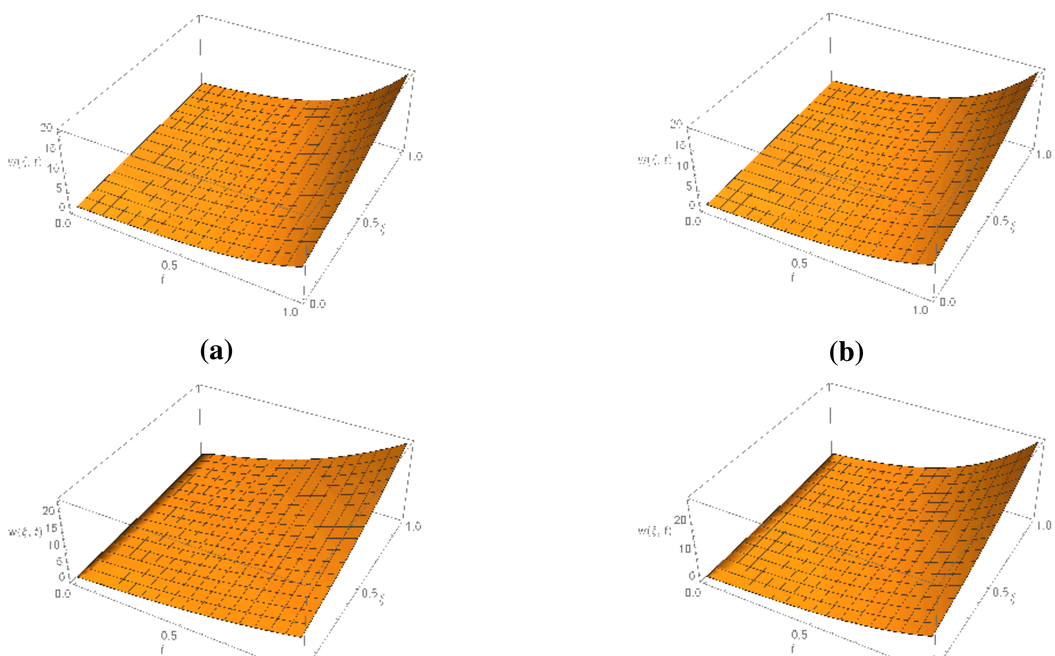


***Figure* 4.4:** The surface shows the solution $w(\xi,t)$ for ***Example* 4.4**, when (a) The exact solution, (b) The approximate solution for $\alpha = 1$, (c) The approximate solution for $\alpha = 0.5$, (d) The approximate solution for $\alpha = 0.75$.

5 Conclusion

In this study, the approximate analytical solutions for the time-fractional Cauchy reaction-diffusion equations were determined by using an effective and straight procedure of the iterative Laplace transform method *(ILTM)*. The fractional derivatives were described in the Caputo Sense. The graphical representation of the obtained solutions has been done successfully. The present method has proved to be an effective and straightforward procedure as compared with other analytical and numerical techniques to find approximate analytical solutions of the fractional partial differential equations and it can be utilized to investigate analytical solutions of more problems of the partial differential equations of fractional order.

Acknowledgements. The one of the authors Karan Singh is grateful to UGC for granting him Junior Research Fellowship *(JRF)* for this project. Authors are also thankful to the Editor and Referee for their suggestions to bring the paper in its present form.

References

[1] R.K. Bairwa and A. Kumar, Analytical study of nonlinear fractional Zakharov-Kuznetsov equations arising in ion-acoustic waves, *J. Raj. Acad. Phys. Sci.*, **18(3-4)**(2019), 127-138.

[2] R.K. Bairwa and J. Singh, Analytical approach to fractional Navier-Stokes equations by Iterative Laplace Transform Method, *Mathematical Modelling, Applied Analysis and Computation, Springer Proceeding in Mathematics and Statistics, Singapore*, **272**(2018).

[3] S. Bhalekar and V. Daftardar-Gejji, Solving evolution equations using a new iterative method, *Numerical Methods for Partial Differential Equations*, **26(4)**(2010), 906-916.

[4] M. Caputo, *Elasticita e Dissipazione*, Zani-Chelli, Bologna, 1969.

[5] V. Daftardar-Gejji and S. Bhalekar, Solving fractional boundary value problems with Dirichlet boundary conditions using a new iterative method, Comp. *Math. Appli.*, **59(5)**(2010), 1801-1809.

[6] V. Daftardar-Gejji and H. Jafari, An iterative method for solving non-linear functional equations, *J. Math. Anal. Appli.*, **316 (2)**(2006), 753-763.

[7] H. Jafari, *Iterative methods for solving system of fractional differential equations* [Ph.D. thesis], Pune University, 2006.

[8] H. Jafari, C.M. Khalique, and M. Nazari, Application of the Laplace decomposition method for solving linear and nonlinear fractional diffusion-wave equations, *Appl. Math. Lett.*, **24(11)**(2011), 1799-1805.

[9] H. Jafari, M. Nazari, D. Baleanu and C.M. Khalique, A new approach for solving a system of fractional partial differential equations, *Comp. Math. Appli.*, **66(5)**(2013), 838-843.

[10] Y. Khan and Q. Wu, Homotopy perturbation transform method for nonlinear equations using Hes polynomials, *Comp. Math. Appli.*, **61(8)**(2011), 1963-1967.

[11] S. Kumar, A new fractional modeling arising in engineering sciences and its analytical approximate solution., *Alexandria Engineering J.*, **52(4)**(2013), 813-819.

[12] S.J. Liao, On the homotopy analysis method for nonlinear problems, *Appl. Math. Comput.*, **147**(2004), 499-513.

[13] K.S. Miller and B. Ross, An introduction to the fractional calculus and fractional differential equations, *John Wiley and Sons New York, USA*, 1993.

[14] I. Podlubny, *Fractional differential equations*, Academic Press, New York, USA, **198**, 1999.

[15] S.C. Sharma and R.K. Bairwa, Iterative Laplace transform method for solving fractional heat and wave-like equation, *Res. J. Math. Stat. Sci.*, **3(2)**(2015), 4-9.

[16] P. Veeresha, D.G. Prakasha and H.M. Baskonus, Novel simulations to the time-fractional Fishers equation, *Mathematical Sciences*, **13(1)**(2019), 33-42.

[17] K. Wang and S. Liu, A new Sumudu transform iterative method for time-fractional Cauchy reaction-diffusion equation, *Springer Plus*, **5** (2016), 865.

[18] A. Yildirim, Application of He's homotopy perturbation method for solving the Cauchy reaction-diffusion problem, *Comp.Math. Appli.*, **57**(2009), 612-618.

Jñānābha, Vol. 50(1) (2020), 218-231
(Dedicated to Honor Professor H.M. Srivastava on His 80th Birth Anniversary Celebrations)

GROWTHS OF COMPOSITE ENTIRE FUNCTIONS DEPENDING ON GENERALIZED RELATIVE LOGARITHMIC ORDER

By
Chinmay Ghosh[1], Sanjib Kumar Datta[2], Sutapa Mondal[3], Subhadip Khan[4]
[1,3]Department of Mathematics
Kazi Nazrul University, Nazrul Road, P.O.- Kalla C.H., Asansol-713340, West Bengal, India.
[2]Department of Mathematics
University of Kalyani, P.O.-Kalyani, Dist-Nadia, PIN-741235, West Bengal, India.
[4]Jotekamal High School, Village- Jotekamal, Block- Raghunathganj 2,
Post- Jangipur, Dist- Murshidabad, Pin- 742213, West Bengal, India.
Email:[1]chinmayarp@gmail.com, [2]sanjibdatta05@gmail.com,
[3]sutapapinku92@gmail.com, [4]subhadip204@gmail.com
(*Received : June* 12, 2020 ; *Revised: June* 15, 2020)
DOI: https://doi.org/10.58250/jnanabha.2020.50121

Abstract

In this paper we introduce the concept of generalized relative logarithmic order and generalized relative lower logarithmic order of an entire function. We investigate some newly developed results on the growth rates of composite entire functions depending on generalized relative logarithmic orders and generalized relative lower logarithmic orders.
2010 Mathematics Subject Classifications: 30D20, 30D30, 30D35.
Keywords and phrases: Entire function, generalized relative logarithmic order, generalized relative lower logarithmic order.

1 Introduction

The maximum modulus $M_f(r) = \max\{|f(z)| : |z| \leq r\}$ of an entire function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is nondecreasing function of $r > 0$. The order $\rho(f)$ and lower order $\lambda(f)$ of the entire function f are

$$\rho(f) = \limsup_{r\to\infty} \frac{\log\log M_f(r)}{\log r}$$

and

$$\lambda(f) = \liminf_{r\to\infty} \frac{\log\log M_f(r)}{\log r}$$

respectively.

Here we use the following notations:

$$\begin{aligned} \log^{[k]} x &= \log\left(\log^{[k-1]} x\right), \text{ for } k = 1, 2, 3, ... \\ \log^{[0]} x &= x \end{aligned}$$

and

$$\begin{aligned} \exp^{[k]} x &= \exp\left(\exp^{[k-1]} x\right), \text{ for } k = 1, 2, 3, ... \\ \exp^{[0]} x &= x. \end{aligned}$$

***Definition* 1.1.** [3] *Also for a entire function f with order zero, the logarithmic order $\rho_{\log}(f)$ and lower logarithmic order $\lambda_{\log}(f)$ are defined as*

$$\rho_{\log}(f) = \limsup_{r\to\infty} \frac{\log\log M_f(r)}{\log\log r}, \tag{1.1}$$

$$\lambda_{\log}(f) = \liminf_{r\to\infty} \frac{\log\log M_f(r)}{\log\log r}. \tag{1.2}$$

Now the maximum term $\mu_f(r)$ of the function f is defined as

$$\mu_f(r) = \max_{n\geq 0} |a_n| r^n.$$

For $0 \leq r < R$, we have [7]

$$\mu_f(r) \leq M_f(r) \leq \frac{R}{R-r}\mu_f(R). \tag{1.3}$$

***Definition* 1.2.** *Using the maximum term we can define $\rho_{\log}(f)$ and $\lambda_{\log}(f)$ as*

$$\rho_{\log}(f) = \limsup_{r\to\infty} \frac{\log\log \mu_f(r)}{\log\log r} \tag{1.4}$$

and

$$\lambda_{\log}(f) = \liminf_{r\to\infty} \frac{\log\log \mu_f(r)}{\log\log r}. \tag{1.5}$$

Since maximum modulus M_f of a nonconstant entire function f is continuous and strictly increasing, there exists

$$M_f^{-1} : (|f(0)|, \infty) \to (0, \infty)$$

such that $\lim_{s\to\infty} M_f^{-1}(s) = \infty$.

Bernel [1] introduced the following concept of relative order of an entire function.

***Definition* 1.3.** *The relative order of f with respect to g is defined as*

$$\begin{aligned} \rho_g(f) &= \inf\left\{\mu > 0 : M_f(r) < M_g(r^{\mu}) \text{ for all } r > r_0(\mu) > 0\right\} \\ &= \limsup_{r\to\infty} \frac{\log M_g^{-1} M_f(r)}{\log r}. \end{aligned}$$

and the relative lower order of f with respect to g is defined as

$$\lambda_g(f) = \liminf_{r\to\infty} \frac{\log M_g^{-1} M_f(r)}{\log r}.$$

Lahiri and Banerjee [6] gave us a generalized concept of relative order.

***Definition* 1.4.** *If $p \geq 1$ is a positive integer, then the p-th generalized relative order of f with respect to g, denoted by $\rho_g^{[p]}(f)$ and is defined as*

$$\begin{aligned} \rho_g^{[p]}(f) &= \inf\left\{\mu > 0 : M_f(r) < M_g(\exp^{[p-1]} r^{\mu}) \text{ for all } r > r_0(\mu) > 0\right\} \\ &= \limsup_{r\to\infty} \frac{\log^{[p]} M_g^{-1} M_f(r)}{\log r}. \end{aligned}$$

and the generalized relative lower order of f with respect to g, denoted by $\lambda_g^{[p]}(f)$ and is defined as

$$\lambda_g^{[p]}(f) = \liminf_{r\to\infty} \frac{\log^{[p]} M_g^{-1} M_f(r)}{\log r}.$$

Datta and Maji [5] defined relative order of an entire function in terms of its maximum term as:

Definition **1.5.** *The relative order and relative lower order of an entire function f with respect to g are defined as*

$$\begin{aligned} \rho_g(f) &= \limsup_{r\to\infty} \frac{\log \mu_g^{-1}\mu_f(r)}{\log r}, \\ \lambda_g(f) &= \liminf_{r\to\infty} \frac{\log \mu_g^{-1}\mu_f(r)}{\log r}. \end{aligned}$$

Again in terms of maximum term, ***Definition*** 1.4 can be rewritten as

Definition **1.6.** *If $p \geq 1$ is a positive integer, then $\rho_g^{[p]}(f)$ and $\lambda_g^{[p]}(f)$ are defined as:*

$$\begin{aligned} \rho_g^{[p]}(f) &= \limsup_{r\to\infty} \frac{\log^{[p]} \mu_g^{-1}\mu_f(r)}{\log r}, \\ \lambda_g^{[p]}(f) &= \liminf_{r\to\infty} \frac{\log^{[p]} \mu_g^{-1}\mu_f(r)}{\log r}. \end{aligned}$$

In view of ***Definitions*** 1.1 and 1.4, here we define the generalized relative logarithmic order as:

Definition **1.7.** *If $p \geq 1$ is a positive integer, then the p-th generalized relative logarithmic order of f with respect to g, denoted by $\rho_{\log g}^{[p]}(f)$, is defined by*

$$\rho_{\log g}^{[p]}(f) = \limsup_{r\to\infty} \frac{\log^{[p]} M_g^{-1} M_f(r)}{\log\log r},$$

and the p-th generalized relative lower logarithmic order of f with respect to g, denoted by $\lambda_{\log g}^{[p]}(f)$ and is defined as:

$$\lambda_{\log g}^{[p]}(f) = \liminf_{r\to\infty} \frac{\log^{[p]} M_g^{-1} M_f(r)}{\log\log r}.$$

Also we define the generalized relative logarithmic order and generalized relative lower logarithmic order by using maximum term as:

Definition **1.8.** *If $p \geq 1$ is a positive integer, then $\rho_{\log g}^{[p]}(f)$ and $\lambda_{\log g}^{[p]}(f)$ are defined as:*

$$\rho_{\log g}^{[p]}(f) = \limsup_{r\to\infty} \frac{\log^{[p]} \mu_g^{-1}\mu_f(r)}{\log\log r},$$

and

$$\lambda_{\log g}^{[p]}(f) = \liminf_{r\to\infty} \frac{\log^{[p]} \mu_g^{-1}\mu_f(r)}{\log\log r}.$$

Biswas et. al. [2] established some results on the growth rates of composite entire functions depending on their generalized relative orders and generalized relative lower orders. In this paper we will prove some results of [2] on the basis of their generalized relative logarithmic orders and generalized relative lower logarithmic orders.

From Valiron [8] we get the general theory of entire functions and so we do not explain them in details.

2 Lemmas

In this section we present some lemmas which will be needed to prove our results.

***Lemma* 2.1.** [4] *If f and g are two entire functions, then for all sufficiently large values of r,*

$$M_f\left(\frac{1}{8}M_g\left(\frac{r}{2}\right)-|g(0)|\right)\leq M_{f\circ g}(r)\leq M_f\left(M_g(r)\right).$$

***Lemma* 2.2.** [7] *If f and g are any two entire functions. Then for every $\alpha>1$ and $0<r<R$,*

$$\mu_{f\circ g}(r)\leq\frac{\alpha}{\alpha-1}\mu_f\left(\frac{\alpha R}{R-r}\mu_g(R)\right).$$

***Lemma* 2.3.** [7] *If f and g are two entire functions with $g(0)=0$, then for all sufficiently large values of r,*

$$\mu_{f\circ g}(r)\geq\frac{1}{2}\mu_f\left(\frac{1}{8}\mu_g(\frac{r}{4})-|g(0)|\right).$$

***Lemma* 2.4.** [1] *If f is an entire function and $\alpha>1, 0<\beta<\alpha$. Then for all sufficiently large r,*

$$M_f(\alpha r)\geq\beta M_f(r).$$

***Lemma* 2.5.** [5] *If f is an entire function and $\alpha>1, 0<\beta<\alpha$. Then for all sufficiently large r,*

$$\mu_f(\alpha r)\geq\beta\mu_f(r).$$

3 Main results:

In this section we will present the main results of this paper.

***Theorem* 3.1.** *Let f and h be any two entire functions such that $0<\lambda^{[p]}_{\log h}(f)\leq\rho^{[p]}_{\log h}(f)<\infty$ and g be an entire function with $\lambda^{[q]}_{\log}(g)>0$ where p,q are any integers with $p>1$ and $q>2$. Then for every positive constant δ and every real number α,*

$$\lim_{r\to\infty}\frac{\log^{[p]}M_h^{-1}M_{f\circ g}(r)}{\left[\log^{[p]}M_h^{-1}M_f\left(\{\exp^{[q-3]}r\}^{\delta}\right)\right]^{1+\alpha}}=\infty.$$

Proof. If α is such that $1+\alpha\leq 0$, then the theorem is trivial. So we suppose that $1+\alpha>0$. Since $M_h^{-1}(r)$ is an increasing function of r, it follows from the first part of ***Lemma* 2.1** for all sufficiently large values of r that

$$\begin{aligned}\log^{[p]}M_h^{-1}M_{f\circ g}(r) &\geq \left(\lambda^{[p]}_{\log h}(f)-\varepsilon\right)\log\log\left(\frac{1}{8}M_g(\frac{r}{2})-|g(0)|\right)\\ &\geq \left(\lambda^{[p]}_{\log h}(f)-\varepsilon\right)\log\log M_g\left(\frac{r}{2}\right)+O(1)\\ &\geq \left(\lambda^{[p]}_{\log h}(f)-\varepsilon\right)\exp^{[q-3]}(\log r)^{\lambda^{[q]}_{\log}(g)-\varepsilon}.\end{aligned}\tag{3.1}$$

Choose ε, $0<\varepsilon<\min\left(\lambda^{[p]}_{\log h}(f),\lambda^{[q]}_{\log}(g)\right)$.

Again for sufficiently large values of r we have from the definition of $\rho^{[p]}_{\log h}(f)$,

$$\begin{aligned}\left[\log^{[p]}M_h^{-1}M_f\left(\left(\exp^{[q-3]}r\right)^{\delta}\right)\right]^{1+\alpha} &\leq \left[\left(\rho^{[p]}_{\log h}(f)+\varepsilon\right)\log\log\left(\exp^{[q-3]}r\right)^{\delta}\right]^{1+\alpha}\\ &= \left(\rho^{[p]}_{\log h}(f)+\varepsilon\right)^{1+\alpha}\left[\log\left\{\delta\exp^{[q-4]}r\right\}\right]^{1+\alpha}\end{aligned}\tag{3.2}$$

$$\leq \left(\rho^{[p]}_{\log h}(f)+\varepsilon\right)^{1+\alpha}\left[\exp^{[q-5]} r\right]^{1+\alpha}+O(1).$$

From (3.2) and (3.3) ,

$$\frac{\log^{[p]} M_h^{-1} M_{f\circ g}(r)}{\left[\log^{[p]} M_h^{-1} M_f\left((\exp^{[q-2]} r)^{\delta}\right)\right]^{1+\alpha}} \geq \frac{\left(\lambda^{[p]}_{\log h}(f)-\varepsilon\right)\exp^{[q-3]}(\log r)^{\lambda^{[q]}_{\log}(g)-\varepsilon}}{\left(\rho^{[p]}_{\log h}(f)+\varepsilon\right)^{1+\alpha}\left[\exp^{[q-5]} r\right]^{1+\alpha}+O(1)}.$$

Since $\lim\limits_{r\to\infty} \frac{\exp^{[q-3]}(\log r)^{\lambda^{[q]}_{\log}(g)-\varepsilon}}{\left[\exp^{[q-5]} r\right]^{1+\alpha}} \to \infty$, the theorem follows.

In view of ***Theorem* 3.1**, one can easily prove the following theorem.

***Theorem* 3.2.** *Let f, g, h and k be any four entire functions with $\lambda^{[p]}_{\log h}(f) > 0, \lambda^{[q]}_{\log}(g) > 0$ and $\rho^{[m]}_{\log k}(g) < \infty$ where $p(> 1), q(> 2)$ and $m(> 1)$ be any three integers. Then for every $\delta > 0$ and for every real number α,*

$$\lim_{r\to\infty} \frac{\log^{[p]} M_h^{-1} M_{f\circ g}(r)}{\left[\log^{[m]} M_k^{-1} M_g\left(\{\exp^{[q-3]} r\}^{\delta}\right)\right]^{1+\alpha}} = \infty.$$

Using ***Lemma* 2.3** and ***Definition* 1.8**, the following theorems can be proved in view of ***Theorem* 3.1** and ***Theorem* 3.2**,

***Theorem* 3.3.** *Let f and h be any two entire functions such that $0 < \lambda^{[p]}_{\log h}(f) \leq \rho^{[p]}_{\log h}(f) < \infty$ and g be an entire function with $\lambda^{[q]}_{\log}(g) > 0$ where $p(> 1)$ and $q(> 2)$ be any two integers. Then for every positive constant δ and every real number α,*

$$\lim_{r\to\infty} \frac{\log^{[p]} \mu_h^{-1} \mu_{f\circ g}(r)}{\left[\log^{[p]} \mu_h^{-1} \mu_f\left(\{\exp^{[q-3]} r\}^{\delta}\right)\right]^{1+\alpha}} = \infty.$$

***Theorem* 3.4.** *Let f, g, h and k be any four entire functions with $\lambda^{[p]}_{\log h}(f) > 0, \lambda^{[q]}_{\log}(g) > 0$ and $\rho^{[m]}_{\log k}(g) < \infty$ where $p(> 1), q(> 2)$ and $m(> 1)$ be any three integers. Then for every $\delta > 0$ and for every real number α,*

$$\lim_{r\to\infty} \frac{\log^{[p]} \mu_h^{-1} \mu_{f\circ g}(r)}{\left[\log^{[m]} \mu_k^{-1} \mu_g\left(\{\exp^{[q-3]} r\}^{\delta}\right)\right]^{1+\alpha}} = \infty.$$

***Remark* 3.1.** *If we consider $0 < \lambda^{[p]}_{\log h}(f) < \infty$ instead of $0 < \lambda^{[p]}_{\log h}(f) \leq \rho^{[p]}_{\log h}(f) < \infty$ in **Theorems 3.1** and **3.3** and the other conditions remain the same, the conclusion of **Theorems 3.1** and **3.3** remain valid with "limit superior" is replaced by "limit".*

***Remark* 3.2.** *If we consider $0 < \lambda^{[m]}_{\log k}(g) < \infty$ instead of $\rho^{[m]}_{\log k}(g) < \infty$ in **Theorems 3.2** and **3.4** and the other conditions remain the same, the conclusion of **Theorems 3.2** and **3.4** remain valid with "limit superior" is replaced by "limit".*

***Theorem* 3.5.** *Let f, g and h be any three entire functions such that $0 < \lambda^{[p]}_{\log h}(f) \leq \rho^{[p]}_{\log h}(f) < \infty$ and $\rho^{[q]}_{\log}(g) < \infty$ where p, q are integers with $p > 1$ and $q > 2$. Then for every $\delta > 0$ and each $\alpha \in (-\infty, \infty)$,*

$$\lim_{r\to\infty} \frac{\left[\log^{[p]} M_h^{-1} M_{f\circ g}(r)\right]^{1+\alpha}}{\log^{[p]} M_h^{-1} M_f\left(\exp^{[q-1]}(\log r)^{\delta}\right)} = 0,$$

where $\delta > (1+\alpha)\rho^{[q]}_{\log}(g)$.

Proof. For $1+\alpha \le 0$, there is nothing to proof. Let us consider $1+\alpha > 0$. As $M_h^{-1}(r)$ is an increasing function, then from the last part of ***Lemma* 2.1** we have for large values of r,

$$\begin{aligned}\log^{p} M_h^{-1} M_{f\circ g}(r) &\le \log^{p} M_h^{-1} M_f\left(M_g(r)\right)\\ &\le \left(\rho_{\log h}^{[p]}(f)+\varepsilon\right)\log\log\left(M_g(r)\right)\\ &\le \left(\rho_{\log h}^{[p]}(f)+\varepsilon\right)\exp^{[q-3]}(\log r)^{\rho_{\log}^{[q]}(g)+\varepsilon}.\end{aligned}\tag{3.3}$$

Also for sufficiently large values of r we have from the definition of $\lambda_{\log h}^{[p]}(f)$,

$$\begin{aligned}\log^{[p]} M_h^{-1} M_f\left(\exp^{[q-1]}(\log r)^{\delta}\right) &\ge \left(\lambda_{\log h}^{[p]}(f)-\varepsilon\right)\log\log\left(\exp^{[q-1]}(\log r)^{\delta}\right)\\ &= \left(\lambda_{\log h}^{[p]}(f)-\varepsilon\right)\exp^{[q-3]}(\log r)^{\delta}.\end{aligned}\tag{3.4}$$

For sequence of values of r tending to infinity we have from (3.4) and (3.5) ,

$$\begin{aligned}\frac{\left[\log^{[p]} M_h^{-1} M_{f\circ g}(r)\right]^{1+\alpha}}{\log^{[p]} M_h^{-1} M_f\left(\exp^{[q-1]}(\log r)^{\delta}\right)} &\le \frac{\left[\left(\rho_{\log h}^{[p]}(f)+\varepsilon\right)\exp^{[q-3]}(\log r)^{\rho_{\log}^{[q]}(g)+\varepsilon}\right]^{1+\alpha}}{\left(\lambda_{\log h}^{[p]}(f)-\varepsilon\right)\exp^{[q-3]}(\log r)^{\delta}}\\ &= \frac{\left(\rho_{\log h}^{[p]}(f)+\varepsilon\right)^{1+\alpha}\exp^{[q-3]}(\log r)^{\left(\rho_{\log}^{[q]}(g)+\varepsilon\right)(1+\alpha)}}{\left(\lambda_{\log h}^{[p]}(f)-\varepsilon\right)\exp^{[q-3]}(\log r)^{\delta}},\end{aligned}\tag{3.5}$$

where we choose $0<\varepsilon<\min\left\{\lambda_{\log h}^{[p]}(f), \frac{\delta}{1+\alpha}-\rho_{\log}^{[q]}(g)\right\}$. So from (3.6) we get

$$\lim_{r\to\infty}\frac{\exp^{[q-3]}(\log r)^{\left(\rho_{\log}^{[q]}(g)+\varepsilon\right)(1+\alpha)}}{\exp^{[q-3]}(\log r)^{\delta}}=0.$$

Therefore the theorem is proved.

In the line of ***Theorem* 3.5**, one may state the following Theorem with the similar proof.

***Theorem* 3.6.** *Let f, g, h and k be any four entire functions such that $\rho_{\log h}^{[p]}(f)<\infty$, $\rho_{\log}^{[q]}(g)<\infty$ and $\lambda_{\log k}^{[m]}(g)>0$ where p, q, m are integers with $p>1$, $q>2$ and $m>1$. Then for every $\delta>0$ and each $\alpha\in(-\infty,\infty)$,*

$$\lim_{r\to\infty}\frac{\left[\log^{[p]} M_h^{-1} M_{f\circ g}(r)\right]^{1+\alpha}}{\log^{[m]} M_k^{-1} M_g\left(\exp^{[q-1]}(\log r)^{\delta}\right)}=0,$$

where $\delta>(1+\alpha)\rho_{\log}^{[q]}(g)$.

In view of ***Theorem* 3.5** and ***Theorem* 3.6**, the following two Theorems can be proved by using ***Lemma* 2.2**, ***Lemma* 2.5** and ***Definition* 1.8** and hence their proofs are omitted.

***Theorem* 3.7.** *Let f, g and h be any three entire functions such that $0<\lambda_{\log h}^{[p]}(f)\le\rho_{\log h}^{[p]}(f)<\infty$ and $\rho_{\log}^{[q]}(g)<\infty$ where p, q are integers with $p>1$ and $q>2$. Then for every $\delta>0$ and each $\alpha\in(-\infty,\infty)$,*

$$\lim_{r\to\infty}\frac{\left[\log^{[p]}\mu_h^{-1}\mu_{f\circ g}(r)\right]^{1+\alpha}}{\log^{[p]}\mu_h^{-1}\mu_f\left(\exp^{[q-1]}(\log r)^{\delta}\right)}=0,$$

where $\delta>(1+\alpha)\rho_{\log}^{[q]}(g)$.

***Theorem* 3.8.** *Let f, g, h and k be any four entire functions such that $\rho^{[p]}_{\log h}(f) < \infty$, $\rho^{[q]}_{\log}(g) < \infty$ and $\lambda^{[m]}_{\log k}(g) > 0$ where p, q, m are integers with $p > 1$, $q > 2$ and $m > 1$. Then for every $\delta > 0$ and each $\alpha \in (-\infty, \infty)$,*

$$\lim_{r\to\infty} \frac{\left[\log^{[p]} \mu_h^{-1}\mu_{f\circ g}(r)\right]^{1+\alpha}}{\log^{[m]} \mu_k^{-1}\mu_g\left(\exp^{[q-1]}(\log r)^{\delta}\right)} = 0,$$

where $\delta > (1+\alpha)\rho^{[q]}_{\log}(g)$.

***Remark* 3.3.** *If we consider $0 < \rho^{[p]}_{\log h}(f) < \infty$ instead of $0 < \lambda^{[p]}_{\log h}(f) \le \rho^{[p]}_{\log h}(f) < \infty$ in **Theorems** **3.5** and **3.7** and the other conditions remain the same, the conclusion of **Theorems** **3.5** and **3.7** remain valid with "limit inferior" is replaced by "limit".*

***Remark* 3.4.** *If we consider $\rho^{[m]}_{\log k}(g) > 0$ instead of $\lambda^{[m]}_{\log k}(g) > 0$ in **Theorems** **3.6** and **3.8** and the other conditions remain the same, the conclusion of **Theorems** **3.6** and **3.8** remain valid with "limit inferior" is replaced by "limit".*

***Theorem* 3.9.** *Let f, g and h be any three entire functions such that $\rho^{[p]}_{\log h}(f) < \infty$ and $\lambda^{[p]}_{\log h}(f\circ g) = \infty$ where p is any integer > 1. Then for every $A(> 0)$,*

$$\lim_{r\to\infty} \frac{\log^{[p]} M_h^{-1}M_{f\circ g}(r)}{\log^{[p]} M_h^{-1}M_f\left((\log r)^{A}\right)} = \infty.$$

Proof. Let us consider the contrarary part, i.e $\lim_{r\to\infty} \frac{\log^{[p]} M_h^{-1}M_{f\circ g}(r)}{\log^{[p]} M_h^{-1}M_f\left((\log r)^{A}\right)}$ is finite, then there exists a constant B such that for a sequence of values of r tending to infinity we have,

$$\log^{[p]} M_h^{-1}M_{f\circ g}(r) \le B.\log^{[p]} M_h^{-1}M_f\left((\log r)^{A}\right) \tag{3.6}$$

Also for sufficiently large values of r we have from the definition of $\rho^{[p]}_{\log h}(f)$,

$$\begin{aligned} \log^{[p]} M_h^{-1}M_f\left((\log r)^{A}\right) &\le \left(\rho^{[p]}_{\log h}(f)+\varepsilon\right)\log\log(\log r)^{A} \\ &< \left(\rho^{[p]}_{\log h}(f)+\varepsilon\right)\log(\log r)^{A} \\ &= \left(\rho^{[p]}_{\log h}(f)+\varepsilon\right)A\log\log r. \end{aligned} \tag{3.7}$$

For sequence of values of r tending to infinity we have from (3.6) and (3.8) ,

$$\log^{[p]} M_h^{-1}M_{f\circ g}(r) \le B.A.\left(\rho^{[p]}_{\log h}(f)+\varepsilon\right)\log\log r,$$

i.e,

$$\lambda^{[p]}_{\log h}(f\circ g) \le B.A.\left(\rho^{[p]}_{\log h}(f)+\varepsilon\right),$$

which contradicts the fact that $\lambda^{[p]}_{\log h}(f\circ g)$ is infinite. So our assumption is wrong and hence the theorem follows.

One can prove the following theorem in view of ***Theorem* 3.9**,

***Theorem* 3.10.** *Let f, g and h be any three entire functions such that $\rho^{[p]}_{\log h}(f) < \infty$ and $\lambda^{[p]}_{\log h}(f\circ g) = \infty$ where $p(> 1)$ is any integer. Then for every $A(> 0)$ we have*

$$\lim_{r\to\infty} \frac{\log^{[p]} \mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[p]} \mu_h^{-1}\mu_f\left((\log r)^{A}\right)} = \infty.$$

***Remark* 3.5.** *If we replace "limit" by "limit superior" and $\lambda^{[p]}_{\log h}(f \circ g) = \infty$ by $\rho^{[p]}_{\log h}(f \circ g) = \infty$ in* ***Theorem* 3.9** *and* ***Theorem* 3.10** *then they are also valid.*

Corollary* 3.1.** *Under the assumption of* ***Theorems* 3.9** *and* **3.10,*

$$\begin{aligned} \lim_{r\to\infty} \frac{M_h^{-1} M_{f\circ g}(r)}{M_h^{-1} M_f\left((\log r)^A\right)} &= \infty \\ \textit{and } \lim_{r\to\infty} \frac{\mu_h^{-1} \mu_{f\circ g}(r)}{\mu_h^{-1} \mu_f\left((\log r)^A\right)} &= \infty \end{aligned}$$

hold.

Proof. For all sufficiently large values of r and for $K > 1$ we have from ***Theorem* 3.9**,

$$\begin{aligned} \log^{[p]} M_h^{-1} M_{f\circ g}(r) &\geq K.\log^{[p]} M_h^{-1} M_f\left((\log r)^A\right) \\ \log^{[p-1]} M_h^{-1} M_{f\circ g}(r) &\geq \left[\log^{[p-1]} M_h^{-1} M_f\left((\log r)^A\right)\right]^K . \end{aligned}$$

Therefore first part is proved.

Similarly from ***Theorem* 3.10**, we get the second part.

Corollary* 3.2.** *Under the assumption of* ***Remark* 3.5, one can prove the following results.*

$$\begin{aligned} \limsup_{r\to\infty} \frac{M_h^{-1} M_{f\circ g}(r)}{M_h^{-1} M_f\left((\log r)^A\right)} &= \infty \\ \textit{and } \limsup_{r\to\infty} \frac{\mu_h^{-1} \mu_{f\circ g}(r)}{\mu_h^{-1} \mu_f\left((\log r)^A\right)} &= \infty. \end{aligned}$$

Using ***Theorems* 3.9** and **3.10, *Remark* 3.5**, ***Corollaries* 3.1** and **3.2**, one may also state the following theorems and corollaries without their proofs.

***Theorem* 3.11.** *Let f, g and k be any three entire functions such that $\rho^{[m]}_{\log k}(g) < \infty$ and $\rho^{[m]}_{\log k}(f \circ g) = \infty$ where $m(> 1)$ is any integer. Then for every $B(> 0)$ we have*

$$\limsup_{r\to\infty} \frac{\log^{[m]} M_k^{-1} M_{f\circ g}(r)}{\log^{[m]} M_k^{-1} M_g\left((\log r)^B\right)} = \infty.$$

***Theorem* 3.12.** *Let f, g and k be any three entire functions such that $\rho^{[m]}_{\log k}(g) < \infty$ and $\rho^{[m]}_{\log k}(f \circ g) = \infty$ where $m(> 1)$ is any integer. Then for every $B(> 0)$ we have*

$$\limsup_{r\to\infty} \frac{\log^{[m]} \mu_k^{-1} \mu_{f\circ g}(r)}{\log^{[m]} \mu_k^{-1} \mu_g\left((\log r)^B\right)} = \infty.$$

***Remark* 3.6.** *If we replace "limit superior" by "limit" and $\rho^{[m]}_{\log k}(f \circ g) = \infty$ by $\lambda^{[m]}_{\log k}(f \circ g) = \infty$ then* ***Theorems* 3.11** *and* **3.12** *also hold.*

Corollary* 3.3.** *Again under the assumption of* ***Theorems* 3.11** *and* **3.12, we get*

$$\begin{aligned} \limsup_{r\to\infty} \frac{M_k^{-1} M_{f\circ g}(r)}{M_k^{-1} M_g\left((\log r)^B\right)} &= \infty, \\ \limsup_{r\to\infty} \frac{\mu_k^{-1} \mu_{f\circ g}(r)}{\mu_k^{-1} \mu_g\left((\log r)^B\right)} &= \infty. \end{aligned}$$

***Corollary* 3.4.** *Also under the assumption of **Remark** 3.6 we have*

$$\lim_{r\to\infty}\frac{M_k^{-1}M_{f\circ g}(r)}{M_k^{-1}M_g\left((\log r)^B\right)} = \infty,$$

$$\lim_{r\to\infty}\frac{\mu_k^{-1}\mu_{f\circ g}(r)}{\mu_k^{-1}\mu_g\left((\log r)^B\right)} = \infty.$$

***Theorem* 3.13.** *Let f and h be any two entire functions such that $0 < \lambda^{[p]}_{\log h}(f) \leq \rho^{[p]}_{\log h}(f) < \infty$, $p(> 1)$ be any integer. Also suppose g be an entire function with $0 < \delta < \rho^{[q]}_{\log}(g) \leq \infty$, $q(> 2)$ be any integer. Then for a sequence of values of r tending to infinity,*

$$M_h^{-1}M_{f\circ g}(r) > M_h^{-1}M_f\left(\exp^{[q-1]}(\log r)^{\delta}\right).$$

Proof. Since $M_h^{-1}(r)$ is an increasing function, then from ***Lemma* 2.1**, for a sequence of values of r tending to infinity,

$$\begin{aligned}\log^{[p]} M_h^{-1}M_{f\circ g}(r) &\geq \left(\lambda^{[p]}_{\log h}(f)-\varepsilon\right)\log\log\left(\frac{1}{8}M_g\left(\frac{r}{2}\right)\right)\\ &\geq \left(\lambda^{[p]}_{\log h}(f)-\varepsilon\right)\exp^{[q-3]}\left(\log\frac{r}{2}\right)^{\rho^{[q]}_{\log}(g)-\varepsilon}+O(1).\end{aligned}\tag{3.8}$$

Also for sufficiently large values of r we have from the definition of $\rho^{[p]}_{\log h}(f)$,

$$\begin{aligned}\log^{[p]} M_h^{-1}M_f\left(\exp^{[q-1]}(\log r)^{\delta}\right) &\leq \left(\rho^{[p]}_{\log h}(f)+\varepsilon\right)\log\log\left(\exp^{[q-1]}(\log r)^{\delta}\right)\\ &= \left(\rho^{[p]}_{\log h}(f)+\varepsilon\right)\exp^{[q-3]}(\log r)^{\delta}.\end{aligned}\tag{3.9}$$

For sequence of values of r tending to infinity we have from (3.9) and (3.10) ,

$$\frac{\log^{[p]} M_h^{-1}M_{f\circ g}(r)}{\log^{[p]} M_h^{-1}M_f\left(\exp^{[q-1]}(\log r)^{\delta}\right)} \geq \frac{\left(\lambda^{[p]}_{\log h}(f)-\varepsilon\right)\exp^{[q-3]}\left(\log\frac{r}{2}\right)^{\rho^{[q]}_{\log}(g)-\varepsilon}+O(1)}{\left(\rho^{[p]}_{\log h}(f)+\varepsilon\right)\exp^{[q-3]}(\log r)^{\delta}}.\tag{3.10}$$

As $\delta < \rho^{[q]}_{\log}(g)$, choose $\varepsilon(> 0)$ in such that

$$\delta < \rho^{[q]}_{\log}(g)-\varepsilon.\tag{3.11}$$

Using (3.11) in (3.10) we get

$$\limsup_{r\to\infty}\frac{\log^{[p]} M_h^{-1}M_{f\circ g}(r)}{\log^{[p]} M_h^{-1}M_f\left(\exp^{[q-1]}(\log r)^{\delta}\right)} = \infty.\tag{3.12}$$

Therefore we have for $K > 1$ and for a sequence of values of r tending to infinity,

$$M_h^{-1}M_{f\circ g}(r) > M_h^{-1}M_f\left(\exp^{[q-1]}(\log r)^{\delta}\right).$$

Hence the theorem is proved.

***Theorem* 3.14.** *Let f and h be any two entire functions with $0 < \lambda^{[p]}_{\log h}(f) \leq \rho^{[p]}_{\log h}(f) < \infty$, $p(> 1)$ be any integer. Also suppose g and k be any two entire functions such that $\rho^{[m]}_{\log k}(g) < \infty$ and $0 < \delta < \rho^{[q]}_{\log}(g)$ where $q(> 1)$ and $m(> 2)$ are two integers. Then for a sequence of values of r tending to infinity,*

$$\log^{[p-1]} M_h^{-1}M_{f\circ g}(r) > \log^{[m-1]} M_k^{-1}M_g\left(\exp^{[q-1]}(\log r)^{\delta}\right).$$

Proof. Let us consider $0 < \delta < \delta_0 < \rho^{[q]}_{\log}(g)$ and for a sequence of values of r tending to infinity we get from (3.9),

$$\log^{[p]} M_h^{-1} M_f\left(\exp^{[q-1]}(\log r)^{\delta}\right) > \left(\lambda^{[p]}_{\log h}(f) - \varepsilon\right)\exp^{[q-3]}(\log r)^{\delta_0}. \tag{3.13}$$

Also for sufficiently large values of r we have from the definition of $\rho^{[m]}_{\log k}(g)$,

$$\begin{aligned}\log^{[m]} M_k^{-1} M_g\left(\exp^{[q-1]}(\log r)^{\delta}\right) &\leq \left(\rho^{[m]}_{\log k}(g) + \varepsilon\right)\log\log\left(\exp^{[q-1]}(\log r)^{\delta}\right)\\ &= \left(\rho^{[m]}_{\log k}(g) + \varepsilon\right)\exp^{[q-3]}(\log r)^{\delta}.\end{aligned} \tag{3.14}$$

For sequence of values of r tending to infinity we have from (3.13) and (3.15),

$$\frac{\log^{[p]} M_h^{-1} M_{f\circ g}(r)}{\log^{[m]} M_k^{-1} M_g\left(\exp^{[q-1]}(\log r)^{\delta}\right)} \geq \frac{\left(\lambda^{[p]}_{\log h}(f) - \varepsilon\right)\exp^{[q-3]}(\log r)^{\delta_0}}{\left(\rho^{[m]}_{\log k}(g) + \varepsilon\right)\exp^{[q-3]}(\log r)^{\delta}}. \tag{3.15}$$

Since $\delta < \delta_0$, then from (3.15), we have

$$\limsup_{r\to\infty} \frac{\log^{[p]} M_h^{-1} M_{f\circ g}(r)}{\log^{[m]} M_k^{-1} M_g\left(\exp^{[q-1]}(\log r)^{\delta}\right)} = \infty. \tag{3.16}$$

Hence the theorem is proved.

Again in view of ***Theorem*** **3.13** and **3.14**, the following two theorems can be proved by using ***Lemma*** **2.3** and ***Definition*** **1.8**. Here we state these theorems without their proofs.

Theorem **3.15.** *Let f and h be any two entire functions such that $0 < \lambda^{[p]}_{\log h}(f) \leq \rho^{[p]}_{\log h}(f) < \infty$, $p(> 1)$ be any integer. Also suppose g be an entire function with $0 < \delta < \rho^{[q]}_{\log}(g) \leq \infty$, $q(> 2)$ be any integer. Then for a sequence of values of r tending to infinity,*

$$\mu_h^{-1}\mu_{f\circ g}(r) > \mu_h^{-1}\mu_f\left(\exp^{[q-1]}(\log r)^{\delta}\right).$$

Theorem **3.16.** *Let f and h be any two entire functions with $0 < \lambda^{[p]}_{\log h}(f) \leq \rho^{[p]}_{\log h}(f) < \infty$, $p(> 1)$ be any integer. Also suppose g and k be any two entire functions such that $\rho^{[m]}_{\log k}(g) < \infty$ and $0 < \delta < \rho^{[q]}_{\log}(g)$ where $q(> 1)$ and $m(> 2)$ are two integers. Then for a sequence of values of r tending to infinity,*

$$\log^{[p-1]}\mu_h^{-1}\mu_{f\circ g}(r) > \log^{[m-1]}\mu_k^{-1}\mu_g\left(\exp^{[q-1]}(\log r)^{\delta}\right).$$

Theorem **3.17.** *Let f, g and h be any three entire functions such that $0 < \lambda^{[p]}_{\log h}(f) < \rho^{[p]}_{\log h}(f) < \infty$ and $\lambda^{[q]}_{\log}(g) < \delta < \infty$ where $p(> 1)$ and $q(> 2)$ are any two integers. Then for a sequence of values of r tending to infinity,*

$$M_h^{-1} M_{f\circ g}(r) < M_h^{-1} M_f\left(\exp^{[q-1]}(\log r)^{\delta}\right).$$

Proof. Since $M_h^{-1}(r)$ is an increasing function of r, then for a sequence of values of r tending to infinity we have from the last part of ***Lemma*** **2.1**,

$$\begin{aligned}\log^{p} M_h^{-1} M_{f\circ g}(r) &\leq \left(\rho^{[p]}_{\log h}(f) + \varepsilon\right)\log\log M_g(r)\\ &\leq \left(\rho^{[p]}_{\log h}(f) + \varepsilon\right)\exp^{[q-3]}(\log r)^{\lambda^{[q]}_{\log}(g)+\varepsilon}.\end{aligned} \tag{3.17}$$

Now for sequence of values of r tending to infinity we have from (3.5) and (3.18),

$$\frac{\log^{[p]} M_h^{-1} M_f\left(\exp^{[q-1]}(\log r)^{\delta}\right)}{\log^{p} M_h^{-1} M_{f\circ g}(r)} \geq \frac{\left(\lambda^{[p]}_{\log h}(f) - \varepsilon\right)\exp^{[q-3]}(\log r)^{\delta}}{\left(\rho^{[p]}_{\log h}(f) + \varepsilon\right)\exp^{[q-3]}(\log r)^{\lambda^{[q]}_{\log}(g)+\varepsilon}}. \tag{3.18}$$

Since $\lambda^{[q]}_{\log}(g) < \delta$, we choose $\varepsilon(>0)$ such that

$$\lambda^{[q]}_{\log}(g) + \varepsilon < \delta < \rho^{[q]}_{\log}(g). \tag{3.19}$$

Then from (3.18) and (3.19) ,we have

$$\limsup_{r\to\infty} \frac{\log^{[p]} M_h^{-1} M_f\left(\exp^{[q-1]}(\log r)^{\delta}\right)}{\log^{p} M_h^{-1} M_{f\circ g}(r)} = \infty. \tag{3.20}$$

Therefore we have for $K > 1$ and for a sequence of values of r tending to infinity,

$$M_h^{-1} M_f\left(\exp^{[q-1]}(\log r)^{\delta}\right) > M_h^{-1} M_{f\circ g}(r).$$

Hence the theorem is proved.

In view of ***Theorem* 3.17**, one can get the following theorem.

***Theorem* 3.18.** *Let f, g, h and k be any four entire functions such that $\lambda^{[m]}_{\log k}(g) > 0$ and $\rho^{[p]}_{\log h}(f) < \infty$ where $p(>1), m(>1)$ are two integers. Also $\lambda^{[q]}_{\log}(g) < \delta < \infty$ where $q(>2)$ be any two integers. Then for a sequence of values of r tending to infinity,*

$$\log^{[p-1]} M_h^{-1} M_{f\circ g}(r) < \log^{[m-1]} M_k^{-1} M_g\left(\exp^{[q-1]}(\log r)^{\delta}\right).$$

In view of ***Theorems* 3.17** and **3.18**, the following two theorems can be proved in similar way,

***Theorem* 3.19.** *Let f, g and h be any three entire functions such that $0 < \lambda^{[p]}_{\log h}(f) < \rho^{[p]}_{\log h}(f) < \infty$ and $\lambda^{[q]}_{\log}(g) < \delta < \infty$ where $p(>1)$ and $q(>2)$ are any two integers. Then for a sequence of values of r tending to infinity,*

$$\mu_h^{-1} \mu_{f\circ g}(r) < \mu_h^{-1} \mu_f\left(\exp^{[q-1]}(\log r)^{\delta}\right).$$

***Theorem* 3.20.** *Let f, g, h and k be any four entire functions such that $\lambda^{[m]}_{\log k}(g) > 0$ and $\rho^{[p]}_{\log h}(f) < \infty$ where $p(>1), m(>1)$ are two integers. Also $\lambda^{[q]}_{\log}(g) < \delta < \infty$ where $q(>2)$ be any two integers. Then for a sequence of values of r tending to infinity,*

$$\log^{[p-1]} \mu_h^{-1} \mu_{f\circ g}(r) < \log^{[m-1]} \mu_k^{-1} \mu_g\left(\exp^{[q-1]}(\log r)^{\delta}\right).$$

We may state the following theorem in view of ***Theorems* 3.13** and **3.17**,

***Theorem* 3.21.** *Let f, g and h be any three entire functions such that $0 < \lambda^{[p]}_{\log h}(f) \le \rho^{[p]}_{\log h}(f) < \infty$ and $\lambda^{[q]}_{\log}(g) < \delta < \rho^{[q]}_{\log}(g)$ where $p(>1)$ and $q(>2)$ are any two integers. Then we get,*

$$\liminf_{r\to\infty} \frac{M_h^{-1} M_{f\circ g}(r)}{M_h^{-1} M_f\left(\exp^{[q-1]}(\log r)^{\delta}\right)} \le 1 \le \limsup_{r\to\infty} \frac{M_h^{-1} M_{f\circ g}(r)}{M_h^{-1} M_f\left(\exp^{[q-1]}(\log r)^{\delta}\right)}.$$

Proof. The proof is omitted.

Again in view of ***Theorems* 3.14** and **3.18**, we obtain the following theorem

***Theorem* 3.22.** *Let f, g, h and k be any four entire functions such that $0 < \lambda^{[p]}_{\log h}(f) \le \rho^{[p]}_{\log h}(f) < \infty$, $0 < \lambda^{[m]}_{\log k}(g) \le \rho^{[m]}_{\log k}(g) < \infty$ and $0 < \lambda^{[m]}_{\log}(g) < \delta < \rho^{[m]}_{\log}(g) < \infty$ where $p(>1), q(>2)$ and $m(>1)$ are any three integers. Then we get,*

$$\liminf_{r\to\infty} \frac{\log^{[p-1]} M_h^{-1} M_{f\circ g}(r)}{\log^{[m-1]} M_k^{-1} M_g\left(\exp^{[q-1]}(\log r)^{\delta}\right)} \le 1 \le \limsup_{r\to\infty} \frac{\log^{[p-1]} M_h^{-1} M_{f\circ g}(r)}{\log^{[m-1]} M_k^{-1} M_g\left(\exp^{[q-1]}(\log r)^{\delta}\right)}.$$

Proof. The proof is omitted.

Similarly one can state the following two theorems without their proofs using the ***Theorems* 3.15, 3.19** and the ***Theorems* 3.16, 3.20** respectively.

***Theorem* 3.23.** *Let f, g and h be any three entire functions such that $0 < \lambda^{[p]}_{\log h}(f) \leq \rho^{[p]}_{\log h}(f) < \infty$ and $\lambda^{[q]}_{\log}(g) < \delta < \infty$ where $p(> 1)$ and $q(> 2)$ are any two integers. Then for a sequence of values of r tending to infinity,*

$$\liminf_{r\to\infty} \frac{\mu_h^{-1}\mu_{f\circ g}(r)}{\mu_h^{-1}\mu_f\left(\exp^{[q-1]}(\log r)^{\delta}\right)} \leq 1 \leq \limsup_{r\to\infty} \frac{\mu_h^{-1}\mu_{f\circ g}(r)}{\mu_h^{-1}\mu_f\left(\exp^{[q-1]}(\log r)^{\delta}\right)}.$$

***Theorem* 3.24.** *Let f, g, h and k be any four entire functions such that $0 < \lambda^{[p]}_{\log h}(f) \leq \rho^{[p]}_{\log h}(f) < \infty$, $0 < \lambda^{[m]}_{\log k}(g) \leq \rho^{[m]}_{\log k}(g) < \infty$ and $0 < \lambda^{[m]}_{\log}(g) < \delta < \rho^{[m]}_{\log}(g) < \infty$ where $p(> 1), q(> 2)$ and $m(> 1)$ are any three integers. Then we get,*

$$\liminf_{r\to\infty} \frac{\log^{[p-1]}\mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[m-1]}\mu_k^{-1}\mu_g\left(\exp^{[q-1]}(\log r)^{\delta}\right)} \leq 1 \leq \limsup_{r\to\infty} \frac{\log^{[p-1]}\mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[m-1]}\mu_k^{-1}\mu_g\left(\exp^{[q-1]}(\log r)^{\delta}\right)}.$$

***Theorem* 3.25.** *Let f, g, h, k, l and b be any six entire functions such that $\lambda^{[m]}_{\log b}(l) > 0, \rho^{[p]}_{\log h}(f) < \infty$ and $\rho^{[q]}_{\log}(g) < \lambda^{[n]}_{\log}(k)$ where p, q, m, n are all positive integers with $p \geq 1$, $m \geq 1$ and $n \geq q \geq 2$. Then*

$$(i)\ \lim_{r\to\infty} \frac{M_b^{-1}M_{l\circ k}(r)}{M_h^{-1}M_{f\circ g}(r)} = \infty \text{ if } p = m$$

$$(ii)\ \lim_{r\to\infty} \frac{M_b^{-1}M_{l\circ k}(r)}{\log^{[p-m]}M_h^{-1}M_{f\circ g}(r)} = \infty \text{ if } p > m$$

and

$$(iii)\ \lim_{r\to\infty} \frac{\log^{[m-p]}M_b^{-1}M_{l\circ k}(r)}{M_h^{-1}M_{f\circ g}(r)} = \infty \text{ if } p < m.$$

Proof. Since $M_b^{-1}(r)$ is an increasing function of r, then for a sequence of values of r tending to infinity we have from the first part of ***Lemma* 2.1**,

$$\log^{[m]}M_b^{-1}M_{l\circ k}(r) \geq \left(\lambda^{[m]}_{\log b}(f) - \varepsilon\right)\exp^{[n-3]}\left(\log \frac{r}{2}\right)^{\lambda^{[n]}_{\log}(k)-\varepsilon} + O(1). \tag{3.21}$$

Since $\rho^{[q]}_{\log}(g) < \lambda^{[n]}_{\log}(k)$, choose $\varepsilon(> 0)$ such that

$$\rho^{[q]}_{\log}(g) + \varepsilon < \lambda^{[n]}_{\log}(k) - \varepsilon. \tag{3.22}$$

Case I: Suppose $p = m$. Now combining (3.4) and (3.21) and in view of (3.22) we get for all sufficiently large values of r that,

$$\frac{M_b^{-1}M_{l\circ k}(r)}{M_h^{-1}M_{f\circ g}(r)} \geq \frac{\exp^{[m]}\left[\left(\lambda^{[m]}_{\log b}(f) - \varepsilon\right)\exp^{[n-3]}\left(\log \frac{r}{2}\right)^{\lambda^{[n]}_{\log}(k)-\varepsilon} + O(1)\right]}{\exp^{[m]}\left[\left(\rho^{[m]}_{\log h}(f) + \varepsilon\right)\exp^{[q-3]}(\log r)^{\rho^{[q]}_{\log}(g)+\varepsilon}\right]}$$

then

$$\lim_{r\to\infty} \frac{M_b^{-1}M_{l\circ k}(r)}{M_h^{-1}M_{f\circ g}(r)} = \infty.$$

(i) is proved.
Case II: Suppose $p > m$.Now combining (3.4) and (3.21) and in view of (3.22) we get for all sufficiently large values of r that,

$$\frac{M_b^{-1} M_{l\circ k}(r)}{\log^{[p-m]} M_h^{-1} M_{f\circ g}(r)} \geq \frac{\exp^{[m]}\left[\left(\lambda_{\log b}^{[m]}(f) - \varepsilon\right)\exp^{[n-3]}\left(\log \frac{r}{2}\right)^{\lambda_{\log}^{[n]}(k)-\varepsilon} + O(1)\right]}{\exp^{[m]}\left[\left(\rho_{\log h}^{[p]}(f) + \varepsilon\right)\exp^{[q-3]}(\log r)^{\rho_{\log}^{[q]}(g)+\varepsilon}\right]}$$

then

$$\lim_{r\to\infty} \frac{M_b^{-1} M_{l\circ k}(r)}{\log^{[p-m]} M_h^{-1} M_{f\circ g}(r)} = \infty.$$

(ii) is proved.
Case III: Suppose $p < m$. Similarly combining (3.4) and (3.21) and in view of (3.22) we get for all sufficiently large values of r that,

$$\frac{\log^{[m-p]} M_b^{-1} M_{l\circ k}(r)}{M_h^{-1} M_{f\circ g}(r)} \geq \frac{\exp^{[p]}\left[\left(\lambda_{\log b}^{[m]}(f) - \varepsilon\right)\exp^{[n-3]}\left(\log \frac{r}{2}\right)^{\lambda_{\log}^{[n]}(k)-\varepsilon} + O(1)\right]}{\exp^{[p]}\left[\left(\rho_{\log h}^{[p]}(f) + \varepsilon\right)\exp^{[q-3]}(\log r)^{\rho_{\log}^{[q]}(g)+\varepsilon}\right]}$$

then

$$\lim_{r\to\infty} \frac{\log^{[m-p]} M_b^{-1} M_{l\circ k}(r)}{M_h^{-1} M_{f\circ g}(r)} = \infty.$$

(iii) is proved.

***Theorem* 3.26.** *Let f, g, h, k, l and b be any six entire functions such that $\lambda_{\log b}^{[m]}(l) > 0$, $\rho_{\log h}^{[p]}(f) < \infty$ and $\rho_{\log}^{[q]}(g) < \lambda_{\log}^{[n]}(k)$ where p, q, m, n are all positive integers with $p \geq 1$, $m \geq 1$ and $n \geq q \geq 2$. Then*

$$(i)\ \lim_{r\to\infty} \frac{\mu_b^{-1}\mu_{l\circ k}(r)}{\mu_h^{-1}\mu_{f\circ g}(r)} = \infty \text{ if } p = m$$

$$(ii)\ \lim_{r\to\infty} \frac{\mu_b^{-1}\mu_{l\circ k}(r)}{\log^{[p-m]} \mu_h^{-1}\mu_{f\circ g}(r)} = \infty \text{ if } p > m$$

and

$$(iii)\ \lim_{r\to\infty} \frac{\log^{[m-p]} \mu_b^{-1}\mu_{l\circ k}(r)}{\mu_h^{-1}\mu_{f\circ g}(r)} = \infty \text{ if } p < m.$$

Proof. We can prove the theorem similarly as ***Theorem* 3.26** and with the help of ***Lemmas* 2.2,2.3** and **2.5**.

***Remark* 3.7.** *If we replace "limit" by "limit superior" and $\rho_{\log}^{[q]}(g) < \lambda_{\log}^{[n]}(k)$ by $\rho_{\log}^{[q]}(g) < \rho_{\log}^{[n]}(k)$ then **Theorems** **3.25** and **3.26** also hold.*

Acknowledgement. Authors are very much grateful to the reviewer and Editor for their suggestions to bring the paper in its preset form.

References

[1] L. Bernal, Orden relative de crecimiento de funciones enteras, *Collect. Math.* **39** (1988), 209-229.

[2] T. Biswas, S. K. Datta, and C. Ghosh, Generalized relative order-dependent growths of composite entire functions, Commun. *Korean Math. Soc.*, **32(3)** (2017), 619-635.

[3] P. T. Y. Chern, Om meromorphic functions with finite logarithmic order, *Trans. Amer. Math. Soc.,* **358(2)** (2006), 473-489.

[4] J. Clunie, *The composition of entire and meromorphic functions,* Mathematical Essays Dedicated to A.J. Macintyre, pp. 7592, Ohio Univ. Press, Athens, Ohio, 1970..

[5] S. K. Datta, and A. R. Maji, Relative order of entire functions in terms of their maximum terms, *Int. J. Math. Analy.,* **5(41-44)** (2011), 2119-2126.

[6] B. K. Lahiri and D. Banerjee, Generalized relative order of entire functions, *Proc. Nat. Acad. Sci. India,* **72(A)(4)** (2002), 271-351.

[7] A. P. Singh, On maximum term of composition of entire functions, *Proc. Nat. Acad. Sci. India*, **59(A)(1)** (1989), 103–115.

[8] G. Valiron, *Lectures on the General Theory of Integral Functions,* Chelsea Publishing Company, 1949.

Jñānābha, Vol. 50(1) (2020), 232-242
(Dedicated to Honor Professor H.M. Srivastava on His 80th Birth Anniversary Celebrations)

CONTOUR INTEGRAL REPRESENTATIONS OF TWO VARIABLE GENERALIZED HYPERGEOMETRIC FUNCTION OF SRIVASTAVA AND DAOUST WITH THEIR APPLICATIONS TO INITIAL VALUE PROBLEMS OF ARBITRARY ORDER

By
R. C. Singh Chandel and Hemant Kumar*
Former Head, Department of Mathematics
D. V. Postgraduate College Orai- 285001, Uttar Pradesh, India
Email:rc_chandel@yahoo.com
*Department of Mathematics
D. A-V. Postgraduate College Kanpur- 208001, Uttar Pradesh, India
Email:palhemant2007@rediffmail.com

(*Received : June* 01, 2020 *; Revised: June* 21, 2020)
DOI: https://doi.org/10.58250/jnanabha.2020.50122

Abstract

In this paper, we establish two contour integral representations involving Mittag - Leffler functions (i) for a two variable generalized hypergeometric function of Srivastava and Daoust and (ii) a sum of the Kummer's confluent hypergeometric functions. Then, we make their appeal to obtain the contour integrals for many generating functions and bilateral generating relations. Further, in development and extensions of fractional calculus, we obtain various relations of contour integrals with fractional derivatives and integral operators to use them in solving of any order initial value problems.

2010 Mathematics Subject Classification: 33C15, 33C20, 33C60, 33E12, 26A33.
Keywords and phrases: Srivastava and Daoust two variable function, Kummer's confluent hypergeometric function, contour integral representations, Mittag - Leffler functions, arbitrary order initial value problems.

1 Introduction

Recently, Pathan and Kumar [27] studied and solved the generalized Cauchy problem by representing the multi-parameter Mittag - Leffler functions ([13], [14]), in terms of two variable generalized hypergeometric function due to Srivastava and Daoust ([24], [28], [29], [30])

$$(1.1)\qquad S\,{}^{A\,:\,B;B}_{C\,:\,D;D'}\left(\begin{matrix}[(a):\theta,\vartheta]:[(b):\psi];[(b'):\psi'];\\ [(c):\delta,\kappa]:[(d):\varphi];[(d'):\varphi'];\end{matrix}z,w\right)=\sum_{m,n=0}^{\infty}\mathscr{H}\,{}^{A\,:\,B;B'}_{C\,:\,D;D'}(m,n)\frac{z^m}{m!}\frac{w^n}{n!},$$

where, $\mathscr{H}\,{}^{A\,:\,B;B'}_{C\,:\,D;D'}(m,n)=\dfrac{\prod_{j=1}^{A}\Gamma(a_j+m\theta_j+n\vartheta_j)}{\prod_{j=1}^{C}\Gamma(c_j+m\delta_j+n\kappa_j)}\dfrac{\prod_{j=1}^{B}\Gamma(b_j+m\psi_j)}{\prod_{j=1}^{D}\Gamma(d_j+m\varphi_j)}\dfrac{\prod_{j=1}^{B'}\Gamma(b'_j+n\psi'_j)}{\prod_{j=1}^{D'}\Gamma(d'_j+n\varphi'_j)}$.

The series (1.1) is convergent under the conditions

$$\sum_{j=1}^{C}\delta_j+\sum_{j=1}^{D}\varphi_j-\sum_{j=1}^{A}\theta_j-\sum_{j=1}^{B}\psi_j+1>0;\sum_{j=1}^{C}\kappa_j+\sum_{j=1}^{D'}\varphi_j{}'-\sum_{j=1}^{A}\vartheta_j-\sum_{j=1}^{B'}\psi'_j+1>0.$$

The Srivastava and Daoust function (1.1) is the generalization of the Kampé de Fériet function [33] including the Appell's functions, Horn's functions and Humbert's confluent hypergeometric functions of two variables (see for instance, ([1], [5], [6], [31], [32]).

Applications and detailed analysis of Euler type and Hankel's contour type integral representations of the Kampé de Fériet function and Appell's functions are studied in various fields of science and technology due to (Exton [5], Srivstava and Karlsson [31]).The Appell's functions are transformed into product of Gaussian and Kummer's confluent hypergeometric functions by many authors ([2], [31], [32]).

In this connection, presently, Fejzullahu [7] established a contour integral representation of the Kummer's confluent hypergeometric function [4] in the form

$$(1.2)\qquad M(\alpha,\beta+\nu,z)=\frac{z^{1-\beta}}{2\pi i}\int_{-\infty}^{(0^+,1^+)}e^{zt}t^{-\beta}\left(1-\frac{1}{t}\right)^{-\alpha}\gamma^*(\nu,zt)dt,\ \alpha,\beta,\nu\in\mathbb{C},|\arg(z)|<\frac{\pi}{2},$$

where, throughout the paper $i=\sqrt{(-1)}$, $\mathbb{C}$ being the set of complex numbers, $\gamma^*(\nu,z)=\dfrac{z^{-\nu}}{\Gamma(\nu)}\gamma(\nu,z)$, while

$$(1.3)\qquad \gamma(\nu,z)=\int_0^z u^{\nu-1}e^{-u}du,\Re(\nu)>0,$$

$$\frac{\Gamma(\nu+k)}{\Gamma(\nu)}=(\nu)_k=\begin{cases}1,k=0;\\ \nu(\nu+1)\dots(\nu+k-1),\forall k\in\mathbb{N};\end{cases}$$

$\mathbb{N}$ be the set of natural numbers, and

$$(1.4)\qquad M(\alpha,\beta,z)=\frac{z^{1-\beta}\Gamma(\beta)}{2\pi i}\int_{-\infty}^{(0^+,1^+)}e^{zt}t^{-\beta}\left(1-\frac{1}{t}\right)^{-\alpha}dt,\alpha,\beta\in\mathbb{C},\ |\arg(z)|<\frac{\pi}{2}.$$

The function $M(\alpha,\beta,z)$ is a Kummer's function ([4], [7], [32, p.36]) defined by

$$(1.5)\qquad M(\alpha,\beta,z)=\sum_{k=0}^{\infty}\frac{(\alpha)_k}{(\beta)_k}\frac{z^k}{k!}.$$

For further innovation and extensions of the results (1.2) - (1.5), we establish two contour integral representations involving

(i) the generalized Mittag - Leffler function, $E_{\nu,\rho}(z)$, of order $\dfrac{1}{\Re(\nu)}$, $\nu\in\mathbb{C},\Re(\nu)>0$, and

(ii) the Mittag - Leffler function, $E_{\frac{1}{Q}}(z)$ of order $Q\in\mathbb{N}^*=\{2,3,4,\dots\}$, for two variable Srivastava - Daoust function (1.1) and sum up of Kummer's functions (1.5), respectively.

Then, we make their applications to obtain various contour integral representations for generating functions and many bilateral generating relations.

The generalized Mittag - Leffler function, $E_{\nu,\rho}(z)$ ([10], [34]) is defined by

$$(1.6)\qquad E_{\nu,\rho}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\nu k+\rho)},\forall z,\nu,\rho\in\mathbb{C},\Re(\nu)>0,\Re(\rho)>0,$$

which for $\rho=1$, reduces to the Mittag - Leffler function, $E_\nu(z)$ ([25], [34]) such that $E_\nu(z)=E_{\nu,1}(z)$. We also have

$$(1.7)\qquad E_0(z)=\frac{1}{1-z};E_1(z)=e^z;E_1(1)=e;E_2(z)=\cosh(\sqrt{z});E_2(-z^2)=\cos z;\ \text{and}$$

$$E_{\frac{1}{2}}(z^{\frac{1}{2}})=e^z erf(-\sqrt{z}),\ \text{where, } erf(z)=\int_z^{\infty}e^{-u^2}du.$$

The Mittag-Leffler function arises naturally in the solution of fractional order integral equations or fractional order differential equations, and especially in the investigations of the fractional generalization of the kinetic equation, random walks, Lévy flights, super-diffusive transport and in the study of complex systems. (See for instance the literature of Diethelm [3], Hilfer [9], Kilbas et al. [11], Kiryakova [12], Oldham and Spanier [26]). To make our study more applicable in various diffusion and wave problems ([8], [15] - [22]), we transform the contour integrals into various fractional derivative and integral operators and use them in solving of any order initial value problems.

2 Contour integral representations of hypergeometric functions of one and two variables and bilateral generating relations

In this section, we establish two contour integrals involving Mittag - Leffler functions and make their applications to obtain the contour integral representations of generalized hypergeometric functions of two variables of Srivastava and Daoust (1.1) and to sum up of Kummer's functions defined in (1.5). Then, we establish some theorems involving integral representations for some bilateral generating functions:

***Theorem* 2.1.** *If for all $\alpha,\beta,\lambda,\rho,z,w\in\mathbb{C}, |\arg(w)|<\frac{\pi}{2}, \lambda\neq 0, \Re(\rho)>0, \nu\in\mathbb{R}^+, \mathbb{R}^+$ is the set of positive real numbers and then, there exists an integral representation*

$$\frac{1}{2\pi i}\int_{-\infty}^{(0^+,1^+)} E_{\nu,\rho}(\lambda z^\nu t^{-\nu})e^{wt}t^{-\beta}\left(1-\frac{1}{t}\right)^{-\alpha} dt,$$

such that it holds a relation for the Srivastava and Daoust function (1.1) *in the form*

$$\frac{1}{2\pi i}\int_{-\infty}^{(0^+,1^+)} E_{\nu,\rho}(\lambda z^\nu t^{-\nu})e^{wt}t^{-\beta}\left(1-\frac{1}{t}\right)^{-\alpha} dt = \frac{w^{\beta-1}}{\Gamma(\alpha)} S^{0:1;1}_{1:1;0}\left(\begin{matrix}[-:-,-]:[1:1];[\alpha:1];\\ [\beta:\nu,1]:[\rho:\nu];[-:-];\end{matrix}\lambda w^\nu z^\nu, w\right). \tag{2.1}$$

Proof. Making an application of (1.6) in left hand side of (2.1) and changing the order of integration and summation, we get

$$\frac{1}{2\pi i}\int_{-\infty}^{(0^+,1^+)} E_{\nu,\rho}(\lambda z^\nu t^{-\nu})e^{wt}t^{-\beta}\left(1-\frac{1}{t}\right)^{-\alpha} dt = \sum_{k=0}^{\infty}\frac{\lambda^k z^{\nu k} w^{\nu k+\beta-1}}{\Gamma(\nu k+\rho)\Gamma(\nu k+\beta)}\left\{\frac{w^{1-\nu k-\beta}\Gamma(\nu k+\beta)}{2\pi i}\int_{-\infty}^{(0^+,1^+)} e^{wt}t^{-\nu k-\beta}\left(1-\frac{1}{t}\right)^{-\alpha} dt\right\}. \tag{2.2}$$

Making an appeal to (1.5) and (1.6), the (2.2) gives

$$\frac{1}{2\pi i}\int_{-\infty}^{(0^+,1^+)} E_{\nu,\rho}(\lambda z^\nu t^{-\nu})e^{wt}t^{-\beta}\left(1-\frac{1}{t}\right)^{-\alpha} dt = \frac{w^{\beta-1}}{\Gamma(\alpha)}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{1}{\Gamma(\beta+\nu k+m)}\frac{\Gamma(1+k)\Gamma(\alpha+m)}{\Gamma(\rho+\nu k)}\frac{(\lambda w^\nu z^\nu)^k}{k!}\frac{w^m}{m!}, \tag{2.3}$$

which by an appeal to (1.1) gives the result (2.1).

***Theorem* 2.2.** *If all $\alpha,\beta,z,w\in\mathbb{C}, |arg(w+z)|<\frac{\pi}{2}$, then, there exists a new integral*

$$\frac{1}{2\pi i}\int_{-\infty}^{(0^+,1^+)} E_{\frac{1}{Q}}\left((zt)^{\frac{1}{Q}}\right)e^{wt}t^{-\beta}\left(1-\frac{1}{t}\right)^{-\alpha} dt, \forall Q\in N^*,$$

which gives an integral representation for the sum of Kummer's functions as

$$(2.4)\quad \frac{1}{2\pi i}\int_{-\infty}^{(0^+,1^+)} E_{\frac{1}{Q}}\left((zt)^{\frac{1}{Q}}\right)e^{wt}t^{-\beta}\left(1-\frac{1}{t}\right)^{-\alpha}dt$$
$$=\sum_{r=1}^{Q} X_r(w+z)^{\beta-1}\left(\frac{z}{(w+z)}\right)^{1-\frac{r}{Q}} M(\alpha,\beta+\frac{r}{Q}-1,w+z)\forall Q=2,3,4,\ldots,$$

where, $X_r=\begin{cases}1, 1\le r\le Q-1;\\ \frac{1}{\Gamma(\beta)}, r=Q.\end{cases}$

Proof. An appeal to the formula due to Mathai and Haubold [23, p.84]

$$E_{\frac{1}{Q}}((z)^{\frac{1}{Q}})=e^{z}\left[1+\sum_{r=1}^{Q-1}\frac{\gamma(1-\frac{r}{Q},z)}{\Gamma(1-\frac{r}{Q})}\right],\forall Q\in N^{*},$$

in the left hand side of (2.4) gives

$$(2.5)\quad \frac{1}{2\pi i}\int_{-\infty}^{(0^+,1^+)} E_{\frac{1}{Q}}\left((zt)^{\frac{1}{Q}}\right)e^{wt}t^{-\beta}\left(1-\frac{1}{t}\right)^{-\alpha}dt=\frac{1}{2\pi i}\int_{-\infty}^{(0^+,1^+)} e^{(w+z)t}t^{-\beta}\left(1-\frac{1}{t}\right)^{-\alpha}dt$$
$$+\sum_{r=1}^{Q-1}\frac{(z)^{1-\frac{r}{Q}}}{2\pi i}\int_{-\infty}^{(0^+,1^+)} e^{(w+z)t}t^{-(\beta+\frac{r}{Q}-1)}\left(1-\frac{1}{t}\right)^{-\alpha}\gamma^{*}(1-\frac{r}{Q},zt)dt$$
$$\Rightarrow \frac{1}{2\pi i}\int_{-\infty}^{(0^+,1^+)} E_{\frac{1}{Q}}\left((zt)^{\frac{1}{Q}}\right)e^{wt}t^{-\beta}\left(1-\frac{1}{t}\right)^{-\alpha}dt=\frac{(w+z)^{1-\beta}\Gamma(\beta)}{(w+z)^{1-\beta}\Gamma(\beta)2\pi i}\int_{-\infty}^{(0^+,1^+)} e^{(w+z)t}t^{-\beta}\left(1-\frac{1}{t}\right)^{-\alpha}dt$$
$$+\sum_{r=1}^{Q-1}\frac{(z)^{1-\frac{r}{Q}}}{2\pi i}\frac{(w+z)^{2-\beta-\frac{r}{Q}}}{(w+z)^{2-\beta-\frac{r}{Q}}}\int_{-\infty}^{(0^+,1^+)} e^{(w+z)t}t^{-(\beta+\frac{r}{Q}-1)}\left(1-\frac{1}{t}\right)^{-\alpha}\gamma^{*}(1-\frac{r}{Q},zt)dt.$$

Finally, making an appeal to (1.1) and (1.4) and (2.5), we obtain

$$(2.6)\quad \frac{1}{2\pi i}\int_{-\infty}^{(0^+,1^+)} E_{\frac{1}{Q}}\left((zt)^{\frac{1}{Q}}\right)e^{wt}t^{-\beta}\left(1-\frac{1}{t}\right)^{-\alpha}dt$$
$$=\frac{(w+z)^{\beta-1}}{\Gamma(\beta)}M(\alpha,\beta,w+z)+\sum_{r=1}^{Q-1}(w+z)^{\beta-1}\left(\frac{z}{(w+z)}\right)^{1-\frac{r}{Q}}M(\alpha,\beta+\frac{r}{Q}-1,w+z),\ \forall Q=2,3,4,\ldots.$$

By Eqn. (2.6), we easily obtain the result (2.4).

***Corollary* 2.1.** *If all conditions of the* ***Theorem* 2.1** *are satisfied along with set* $\nu=1,\rho=1$, *then, following result holds*

$$(2.7)\qquad \frac{1}{2\pi i}\int_{-\infty}^{(0^+,1^+)} E_{1,1}(\lambda z t^{-1})e^{wt}t^{-\beta}\left(1-\frac{1}{t}\right)^{-\alpha}dt=\frac{w^{\beta-1}}{\Gamma(\beta)}\sum_{k=0}^{\infty}\frac{(\lambda wz)^k}{(\beta)_k}M(\alpha,\beta+k,w).$$

***Corollary* 2.2.** *If all conditions of the* ***Theorem* 2.1** *are satisfied along with set* $\nu=1,\rho=1$, $\lambda=-\frac{1}{2}$, *and replace* z *by* w', w *by* $z+\frac{w'}{2}$, *then, by* (2.7) *following result holds*

$$(2.8)\quad \frac{1}{2\pi i}\int_{-\infty}^{(0^+,1^+)} E_{1,1}\left(-\frac{1}{2}w't^{-1}\right)e^{(z+\frac{w'}{2})t}t^{-\beta}\left(1-\frac{1}{t}\right)^{-\alpha}dt$$
$$=\frac{(z+\frac{w'}{2})^{\beta-1}}{\Gamma(\beta)}\sum_{k=0}^{\infty}\frac{(-\frac{1}{2}(zw'+\frac{(w')^2}{2}))^k}{(\beta)_k}M(\alpha,\beta+k,w').$$

***Corollary* 2.3.** *If all conditions of the* ***Theorem* 2.1** *are satisfied along with set* $\nu = 1, \rho = 1$, $\lambda = -\frac{1}{2}$, *and replace* z *by* w', w *by* $z + \frac{w'}{2}$, *then, by* (2.7) *following result holds*

$$\frac{1}{2\pi i}\int_{-\infty}^{(0^+,1^+)} \exp\left[\frac{w'}{2}(t - t^{-1})\right] e^{zt} t^{-\beta}\left(1 - \frac{1}{t}\right)^{-\alpha} dt = \sum_{n=-\infty}^{\infty} \frac{z^{\beta-n-1}}{\Gamma(\beta - n)} J_n(w') M(\alpha, \beta - n, z). \tag{2.9}$$

In the left hand side of equation (2.8), *set some rearrangements and apply the generating relation,* $\exp[\frac{x}{2}(t - t^{-1})] = \sum_{n=-\infty}^{\infty} J_n(x)t^n$, *where,* $J_n(x)$ *are the Bessel functions for all* $n \in \{0, \pm 1, \pm 2, \ldots\}$, *we get the contour integral representation for bilateral generating function* (2.9).

***Corollary* 2.4.** *If all conditions of the* ***Theorem* 2.1** *are satisfied along with set* $\nu = 1, \rho = 1$, *and again, replace in it,* $\lambda z = \eta t \log(1 + t) - xt^2$, *then, following identities hold*

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{-\infty}^{(0^+,1^+)} E_{1,1}(\log(1 + t)^{\eta} - xt) e^{wt} t^{-\beta}\left(1 - \frac{1}{t}\right)^{-\alpha} dt \\
&= \frac{1}{2\pi i}\int_{-\infty}^{(0^+,1^+)} (1 + t)^{\eta} e^{-xt} e^{wt} t^{-\beta}\left(1 - \frac{1}{t}\right)^{-\alpha} dt = \frac{1}{2\pi i}\int_{-\infty}^{(0^+,1^+)} (1 + t)^{\eta} e^{(w-x)t} t^{-\beta}\left(1 - \frac{1}{t}\right)^{-\alpha} dt.
\end{aligned} \tag{2.10}$$

Again on applying the relation $\sum_{n=0}^{\infty} L_n^{(\eta-n)}(x)t^n = (1 + t)^{\eta} e^{-xt}$ *where,* $L_n^{(\eta)}(x)$ *are the Laguerre polynomials* $\forall n \in \{0, 1, 2, \ldots\}$ *in second integral of* (2.10), *we get the contour integrals for a bilateral generating relation*

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{-\infty}^{(0^+,1^+)} E_{1,1}(\log(1 + t)^{\eta} - xt) e^{wt} t^{-\beta}\left(1 - \frac{1}{t}\right)^{-\alpha} dt \\
&= \frac{1}{2\pi i}\int_{-\infty}^{(0^+,1^+)} (1 + t)^{\eta} e^{-xt} e^{wt} t^{-\beta}\left(1 - \frac{1}{t}\right)^{-\alpha} dt = \sum_{n=0}^{\infty} \frac{w^{\beta-n-1}}{\Gamma(\beta - n)} L_n^{(\eta-n)}(x) M(\alpha, \beta - n, w).
\end{aligned} \tag{2.11}$$

Further by (2.10), *we find the contour integrals for a generating function*

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{-\infty}^{(0^+,1^+)} E_{1,1}(\log(1 + t)^{\eta} - xt) e^{wt} t^{-\beta}\left(1 - \frac{1}{t}\right)^{-\alpha} dt \\
&= \frac{1}{2\pi i}\int_{-\infty}^{(0^+,1^+)} (1 + t)^{\eta} e^{(w-x)t} t^{-\beta}\left(1 - \frac{1}{t}\right)^{-\alpha} dt = \sum_{n=0}^{\infty} \frac{(-\eta)_n(-1)^n}{n!} \frac{(w - x)^{\beta-n-1}}{\Gamma(\beta - n)} M(\alpha, \beta - n, w - x).
\end{aligned} \tag{2.12}$$

***Corollary* 2.5.** *If all conditions of the* ***Theorem* 2.1** *are satisfied along with set* $\nu = 1, \rho = 1$, *and replace* $\lambda z = 2xt^2 - t^3$, *then, by the generating relation* $\sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n = exp(2xt - t^2)$, *where,* $H_n(x)$ *are the Hermite polynomials* $\forall n \in \{0, 1, 2, \ldots\}$, *following identities hold*

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{-\infty}^{(0^+,1^+)} E_{1,1}(2xt - t^2) e^{wt} t^{-\beta}\left(1 - \frac{1}{t}\right)^{-\alpha} dt \\
&= \frac{1}{2\pi i}\int_{-\infty}^{(0^+,1^+)} exp(2xt - t^2) e^{wt} t^{-\beta}\left(1 - \frac{1}{t}\right)^{-\alpha} dt = \sum_{n=0}^{\infty} \frac{w^{\beta-n-1}}{n!\Gamma(\beta - n)} H_n(x) M(\alpha, \beta - n, w).
\end{aligned} \tag{2.13}$$

3 Extended contour integral representations for generalized hypergeometric functions of two variables

In this section, we extend the contour integral given in the ***Section* 2** and then, by properties of Mttag - Leffler functions (1.6) and (1.7), we obtain some more results for generalized hypergeometric functions of two variables.

***Theorem* 3.1.** *If for all $\alpha, \beta, \lambda, \rho, \sigma \in C, |arg(w)| < \frac{\pi}{2}, \lambda \neq 0, \Re(\rho) > 0, \Re(\sigma) > 0$, $\nu \in \mathbb{R}^+$ then, the contour integral $\frac{1}{2\pi i}\int_{-\infty}^{(0^+,1^+)} E_{\nu,\rho+\sigma}(\lambda z^\nu t^{-\nu}) e^{wt} t^{-\beta}\left(1-\frac{1}{t}\right)^{-\alpha} dt$ exists as*

$$\text{(3.1)}\quad \frac{1}{2\pi i}\int_{-\infty}^{(0^+,1^+)} E_{\nu,\rho+\sigma}(\lambda z^\nu t^{-\nu}) e^{wt} t^{-\beta}\left(1-\frac{1}{t}\right)^{-\alpha} dt$$
$$= \frac{w^{\beta-1}}{\Gamma(\alpha)} S^{0:1;1}_{1:1;0}\left(\begin{matrix}[-:-,-]:[1:1],;[\alpha:1];\\ [\beta:\nu,1]:[\rho+\sigma:\nu];[-:-];\end{matrix}\lambda w^\nu z^\nu, w\right).$$

Proof. Consider the contour integral

$$\frac{1}{2\pi i}\int_{-\infty}^{(0^+,1^+)} E_{\nu,\rho+\sigma}(\lambda z^\nu t^{-\nu}) e^{wt} t^{-\beta}\left(1-\frac{1}{t}\right)^{-\alpha} dt$$

and apply the result due to Mathai and Haubold [23, p.90],

$$z^{\beta+\mu-1}E_{\alpha,\beta+\mu}(\lambda z^\alpha) = \frac{1}{\Gamma(\mu)}\int_0^z u^{\beta-1}(z-u)^{\mu-1}E_{\alpha,\beta}(\lambda u^\alpha)du,$$

we get

$$\frac{1}{2\pi i}\int_{-\infty}^{(0^+,1^+)} E_{\nu,\rho+\sigma}(\lambda z^\nu t^{-\nu}) e^{wt} t^{-\beta}\left(1-\frac{1}{t}\right)^{-\alpha} dt$$
$$= \frac{1}{z^{\rho+\sigma-1}\Gamma(\sigma)2\pi i}\int_0^z u^{\rho-1}(z-u)^{\sigma-1}\int_{-\infty}^{(0^+,1^+)} t^{-\beta}\left(1-\frac{1}{t}\right)^{-\alpha} e^{wt}E_{\nu,\rho}(\lambda t^{-\nu}u^\nu)dtdu.$$

In right hand side of above equality, an appeal to the ***Theorem* 2.1** gives

$$\text{(3.2)}\quad \frac{1}{2\pi i}\int_{-\infty}^{(0^+,1^+)} E_{\nu,\rho+\sigma}(\lambda z^\nu t^{-\nu}) e^{wt} t^{-\beta}\left(1-\frac{1}{t}\right)^{-\alpha} dt$$
$$= \frac{w^{\beta-1}}{\Gamma(\alpha)z^{\rho+\sigma-1}\Gamma(\sigma)}\int_0^z u^{\rho-1}(z-u)^{\sigma-1} S^{0:1;1}_{1:1;0}\left(\begin{matrix}[-:-,-]:[1:1];[\alpha:1];\\ [\beta:\nu,1]:[\rho:\nu];[-:-];\end{matrix}\ \lambda w^\nu u^\nu, w\right)du$$
$$= \frac{w^{\beta-1}}{\Gamma(\alpha)} S^{0:1;1}_{1:1;0}\left(\begin{matrix}[-:-,-]:[1:1];[\alpha:1];\\ [\beta:\nu,1]:[\rho+\sigma:\nu];[-:-];\end{matrix}\ \lambda w^\nu z^\nu, w\right).$$

Corollary* 3.1.** *In the* ***Theorem* 3.1, when set $\nu = 1$, there exists a contour integral for Srivastava and Panda's generalized Kampé de Fériet function* [32]*, in the form*

$$\text{(3.3)}\quad \frac{1}{2\pi i}\int_{-\infty}^{(0^+,1^+)} E_{1,\rho+\sigma}(\lambda z t^{-1}) e^{wt} t^{-\beta}\left(1-\frac{1}{t}\right)^{-\alpha} dt = \frac{w^{\beta-1}}{\Gamma(\beta)\Gamma(\rho+\sigma)} F^{0:1;1}_{1:1;0}\left[\begin{matrix}(-):(1);(\alpha);\\ (\beta):(\rho+\sigma);(-);\end{matrix}\lambda wz, w\right].$$

4 Transformations of contour integrals in various fractional derivative and integral operators

In this section, we present some transformations of the contour integrals defined in the ***Sections* 2** and **3** into Riemann - Liouville fractional derivative and integral operators and in Caputo derivative operators. To obtain the transformation formulae, we define following fractional derivative and integral operators:

The Riemann - Liouville fractional derivative $D^\mu_{a^+}Y$ of order $\mu, \mu \in \mathbb{C}, \Re(\mu) > 0$, as (for instance see [3], [11], [23])

$$(4.1)\quad (D^{\mu}_{a^+}Y)(x) := \left(\frac{d}{dx}\right)^n (I^{n-\mu}_{a^+}Y)(x)$$
$$= \frac{1}{\Gamma(n-\mu)}\left(\frac{d}{dx}\right)^n \int_a^x (x-u)^{n-\mu-1}Y(u)du, (n = [\Re(\mu)]+1; x > a).$$

Here, in formula (4.1), $I^{\alpha}_a Y$ is the Riemann - Liouville integral of order $\alpha, \alpha \in \mathbb{C}, \Re(\alpha) > 0$, given by
$$(I^{\alpha}_a Y)(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-\xi)^{\alpha-1}Y(\xi)d\xi, a \leq x \leq b.$$

The Caputo derivative of the function $Y(t)$, denoted by ${}^{\mathsf{C}}_{t}D^{\alpha}_{0^+}Y(t)$ where, $m-1 < \alpha \leq m, \forall m \in \mathbb{N}$, is defined by
$$(4.2)\qquad ({}^{\mathsf{C}}_{t}D^{\alpha}_{a^+}Y)(t) = (I^{m-\alpha}_{a^+}Y^{(m)})(t), Y^{(m)}(t) = D^m_t Y(t) = \frac{d^m Y(t)}{dt^m} = \frac{d}{dt}\left(\frac{d^{m-1}}{dt^{m-1}}\right)Y(t).$$

***Theorem* 4.1.** *If for all $\alpha, \beta, \lambda, \rho, \sigma \in C, |arg(w)| < \frac{\pi}{2}, \lambda \neq 0, \Re(\rho) > 0, \Re(\sigma) > 0, \nu \in \mathbb{R}^+$ then, by the contour integral $\frac{1}{2\pi i}\int_{-\infty}^{(0^+,1^+)} E_{\nu,\rho+\sigma}(\lambda z^{\nu}t^{-\nu})e^{wt}t^{-\beta}\left(1-\frac{1}{t}\right)^{-\alpha}dt$, there exists following equalities among Riemann - Liouville fractional derivative, contour integral and Srivstava and Daoust function as*

$$(4.3)\quad \left(D^{\rho}_{0^+}\left\{\frac{1}{2\pi i}\int_{-\infty}^{(0^+,1^+)} u^{\rho+\sigma-1}E_{\nu,\rho+\sigma}(\lambda u^{\nu}t^{-\nu})e^{wt}t^{-\beta}\left(1-\frac{1}{t}\right)^{-\alpha}dt\right\}\right)(z)$$
$$= \frac{z^{\sigma-1}}{2\pi i}\int_{-\infty}^{(0^+,1^+)} E_{\nu,\sigma}(\lambda z^{\nu}t^{-\nu})e^{wt}t^{-\beta}\left(1-\frac{1}{t}\right)^{-\alpha}dt$$
$$= \frac{w^{\beta-1}z^{\sigma-1}}{\Gamma(\alpha)}S^{0:1;1}_{1:1;0}\left(\begin{matrix}[-:-,-]:[1:1];[\alpha:1];\\ [\beta:\nu,1]:[\rho:\nu];[-:-];\end{matrix}\lambda w^{\nu}z^{\nu}, w\right).$$

Proof. By the properties of the Mittag - Leffler functions $(I^{\alpha}_{0^+}u^{\beta-1}E_{\nu,\beta}(\lambda u^{\nu}))(x) = x^{\alpha+\beta-1}E_{\nu,\alpha+\beta}(\lambda x^{\nu})$ and the formula (4.1) the contour integral, given in the ***Theorem* 4.1**, is written as

$$(4.4)\quad \frac{1}{2\pi i}\int_{-\infty}^{(0^+,1^+)} z^{\rho+\sigma-1}E_{\nu,\rho+\sigma}(\lambda z^{\nu}t^{-\nu})e^{wt}t^{-\beta}\left(1-\frac{1}{t}\right)^{-\alpha}dt$$
$$= \left(I^{\rho}_{0^+}\left\{\frac{1}{2\pi i}\int_{-\infty}^{(0^+,1^+)} u^{\sigma-1}E_{\nu,\sigma}(\lambda u^{\nu}t^{-\nu})e^{wt}t^{-\beta}\left(1-\frac{1}{t}\right)^{-\alpha}dt\right\}\right)(z).$$

Therefore, by the property that $(D^{\rho}_{0^+}I^{\rho}_{0^+}Y)(z) = Y(z), Y(z) \in L_p(a,b), (1 \leq p \leq \infty), \forall z \in (a,b) \subset \mathbb{R}$, we have

$$\left(D^{\rho}_{0^+}\left\{\frac{1}{2\pi i}\int_{-\infty}^{(0^+,1^+)} u^{\rho+\sigma-1}E_{\nu,\rho+\sigma}(\lambda u^{\nu}t^{-\nu})e^{wt}t^{-\beta}\left(1-\frac{1}{t}\right)^{-\alpha}dt\right\}\right)(z)$$
$$= \left(D^{\rho}_{0^+}I^{\rho}_{0^+}\left\{\frac{1}{2\pi i}\int_{-\infty}^{(0^+,1^+)} u^{\sigma-1}E_{\nu,\sigma}(\lambda u^{\nu}t^{-\nu})e^{wt}t^{-\beta}\left(1-\frac{1}{t}\right)^{-\alpha}dt\right\}\right)(z)$$
$$= \frac{z^{\sigma-1}}{2\pi i}\int_{-\infty}^{(0^+,1^+)} E_{\nu,\sigma}(\lambda z^{\nu}t^{-\nu})e^{wt}t^{-\beta}\left(1-\frac{1}{t}\right)^{-\alpha}dt.$$

So that by (2.1), the equalities are given by

$$\left(D_{0^+}^{\rho}\left\{\frac{1}{2\pi i}\int_{-\infty}^{(0^+,1^+)} u^{\rho+\sigma-1}E_{\nu,\rho+\sigma}(\lambda u^{\nu}t^{-\nu})e^{wt}t^{-\beta}\left(1-\frac{1}{t}\right)^{-\alpha}dt\right\}\right)(z)$$
$$=\frac{z^{\sigma-1}}{2\pi i}\int_{-\infty}^{(0^+,1^+)} E_{\nu,\sigma}(\lambda z^{\nu}t^{-\nu})e^{wt}t^{-\beta}\left(1-\frac{1}{t}\right)^{-\alpha}dt$$
$$=\frac{w^{\beta-1}z^{\sigma-1}}{\Gamma(\alpha)}S^{0\,:\,1;1}_{1\,:\,1;0}\left(\begin{matrix}[-:-,-]:[1:1];[\alpha:1];\\ [\beta:\nu,1]:[\rho:\nu];[-:-];\end{matrix}\ \lambda w^{\nu}z^{\nu},w\right).$$

Hence, the ***Theorem*** **4.1** is followed.

In the similar manner, we obtain

$$\frac{z^{\sigma-1}}{2\pi i}\int_{-\infty}^{(0^+,1^+)} E_{\nu,\sigma}(\lambda z^{\nu}t^{-\nu})e^{wt}t^{-\beta}\left(1-\frac{1}{t}\right)^{-\alpha}dt$$
$$=\frac{1}{2\pi i}\int_{-\infty}^{(0^+,1^+)} z^{1+(\sigma-1)-1}E_{\nu,1+\sigma-1}(\lambda z^{\nu}t^{-\nu})e^{wt}t^{-\beta}\left(1-\frac{1}{t}\right)^{-\alpha}dt$$
$$=\left(I_{0^+}^{\sigma-1}\left\{\frac{1}{2\pi i}\int_{-\infty}^{(0^+,1^+)} E_{\nu}(\lambda u^{\nu}t^{-\nu})e^{wt}t^{-\beta}\left(1-\frac{1}{t}\right)^{-\alpha}dt\right\}\right)(z).$$

Therefore,

$$\left(D_{0^+}^{\sigma-1}\left\{\frac{1}{2\pi i}\int_{-\infty}^{(0^+,1^+)} u^{\sigma-1}E_{\nu,\sigma}(\lambda u^{\nu}t^{-\nu})e^{wt}t^{-\beta}\left(1-\frac{1}{t}\right)^{-\alpha}dt\right\}\right)(z)$$
$$=\frac{1}{2\pi i}\int_{-\infty}^{(0^+,1^+)} E_{\nu}(\lambda z^{\nu}t^{-\nu})e^{wt}t^{-\beta}\left(1-\frac{1}{t}\right)^{-\alpha}dt. \tag{4.5}$$

Finally, by an appeal to (4.3) and (4.5) we get

$$\left(D_{0^+}^{\sigma-1+\rho}\left\{\frac{1}{2\pi i}\int_{-\infty}^{(0^+,1^+)} u^{\rho+\sigma-1}E_{\nu,\rho+\sigma}(\lambda u^{\nu}t^{-\nu})e^{wt}t^{-\beta}\left(1-\frac{1}{t}\right)^{-\alpha}dt\right\}\right)(z)$$
$$=\frac{1}{2\pi i}\int_{-\infty}^{(0^+,1^+)} E_{\nu}(\lambda z^{\nu}t^{-\nu})e^{wt}t^{-\beta}\left(1-\frac{1}{t}\right)^{-\alpha}dt. \tag{4.6}$$

Again, by action of Caputo derivative (4.2) on the Mittag - Leffler functions (1.6, for $\rho = 1$), we obtain

$${}_z^{C}D_{0^+}^{\nu}\left(D_{0^+}^{\sigma-1+\rho}\left\{\frac{1}{2\pi i}\int_{-\infty}^{(0^+,1^+)} u^{\rho+\sigma-1}E_{\nu,\rho+\sigma}(\lambda u^{\nu}t^{-\nu})e^{wt}t^{-\beta}\left(1-\frac{1}{t}\right)^{-\alpha}dt\right\}(z)\right)$$
$$=\frac{\lambda}{2\pi i}\int_{-\infty}^{(0^+,1^+)} E_{\nu,1}(\lambda z^{\nu}t^{-\nu})e^{wt}t^{-\nu-\beta}\left(1-\frac{1}{t}\right)^{-\alpha}dt. \tag{4.7}$$

Thus,

$${}_z^{C}D_{0^+}^{n\nu}\left(D_{0^+}^{\sigma-1+\rho}\left\{\frac{1}{2\pi i}\int_{-\infty}^{(0^+,1^+)} u^{\rho+\sigma-1}E_{\nu,\rho+\sigma}(\lambda u^{\nu}t^{-\nu})e^{wt}t^{-\beta}\left(1-\frac{1}{t}\right)^{-\alpha}dt\right\}(z)\right)$$
$$=\frac{\lambda^{n}}{2\pi i}\int_{-\infty}^{(0^+,1^+)} E_{\nu}(\lambda z^{\nu}t^{-\nu})e^{wt}t^{-n\nu-\beta}\left(1-\frac{1}{t}\right)^{-\alpha}dt,\ \forall n\in\mathbb{N}. \tag{4.8}$$

5 Numerical Example

Consider the initial value diffusion and wave problem $\forall \rho \in \mathbb{C}$,

$$(D_{0^+}^{\rho} Y)(z) = \frac{z^{\sigma-1}}{2\pi i} \int_{-\infty}^{(0^+,1^+)} E_{\nu,\sigma}(\lambda z^{\nu} t^{-\nu}) e^{wt} t^{-\beta} \left(1 - \frac{1}{t}\right)^{-\alpha} dt, \tag{5.1}$$

where, the initial conditions are given by

$$\frac{d^{n-j}}{dz^{n-j}} (I_{0^+}^{n-\rho} Y)(z)\bigg|_{z=0^+} = 0, \; j = 1, 2, \ldots, n; \; \Re(\rho) > 0, n = [\Re(\rho)] + 1.$$

To solve the problem (5.1), we operate both sides of equation (5.1) by ρ order Riemann - Liouville fractional integral operator $I_{0^+}^{\rho}$ and apply the formula [11, Eqn. (2.1.44), p. 75]

$$(I_{0^+}^{\rho} D_{0^+}^{\rho} Y)(z) = Y(z) - \sum_{j=1}^{n} \frac{\frac{d^{n-j}}{dz^{n-j}} (I_{0^+}^{n-\rho} Y)(z)\Big|_{z=0^+}}{\Gamma(\rho - j + 1)} z^{\rho-j}, \Re(\rho) > 0, n = [\Re(\rho)] + 1,$$

and then, use the initial conditions given in (5.1), we get

$$Y(z) = \left(I_{0^+}^{\rho} \left\{ \frac{u^{\sigma-1}}{2\pi i} \int_{-\infty}^{(0^+,1^+)} E_{\nu,\sigma}(\lambda u^{\nu} t^{-\nu}) e^{wt} t^{-\beta} \left(1 - \frac{1}{t}\right)^{-\alpha} dt \right\} \right)(z). \tag{5.2}$$

Now, making an appeal to formula (4.4) in right hand side of (5.2), we derive

$$Y(z) = \frac{z^{\rho+\sigma-1}}{2\pi i} \int_{-\infty}^{(0^+,1^+)} E_{\nu,\rho+\sigma}(\lambda z^{\nu} t^{-\nu}) e^{wt} t^{-\beta} \left(1 - \frac{1}{t}\right)^{-\alpha} dt. \tag{5.3}$$

6 Conclusion

In the ***Sections*** **2** and **3**, we derive various relations and known functions by the contour integrals in form of some special functions, their generating functions and bilateral generating relations involving Bessel functions, Laguerre polynomials and Hermite polynomials. Then, in the ***Section*** **4**, by the action of Riemann - Liouville fractional derivatives and integrals and by the operation of the Caputo derivative on contour integrals, we derive some identities among other contour integrals, and special functions. In the end of our investigation, we present a simple initial value problem to find its solution in terms of contour integrals. The presented work is applicable in various diffusion and wave problems occurring of Mittag - Leffler functions seen in the literature ([3], [8], [9], [11], [12], [23], [26] among others).

Acknowledgement. Authors are very much thankful to the referees for their valuable suggestions and comments to make the paper in the present form.

References

[1] P. Appell, *Sur les Fonctions Hyperge ometriques de Plusieurs Variables,* Gauthier -Villars, Paris, 1925.

[2] W. N. Bailey, *Generalized Hypergeometric Series*, Cambridge University Press, New York, 1935.

[3] K. Diethelm, *The Analysis of Fractional Differential Equations: An Application-Oriented Exposition Using Differential Operators of Caputo Type*, Springer, New York, 2010.

[4] A. Erde'lyi, W. Magnus, F. Oberhettinger and F. G. Tricomi, *Higher Transcendental Functions*, Vol. I McGraw-Hill, New York, 1953.

[5] H. Exton, *Multiple Hypergeometric Functions and Applications*, John Wiley and Sons (Halsted Press), New York, 1976.

[6] H. Exton, *Handbook of Hypergeometric Integrals, Theory, Applications*, Tables, Computer Programs, John Wiley and Sons (Halsted Press), New York, 1978.

[7] B. Xh. Fejzullahu, On the integral representations for the confluent hypergeometric function. *Proc. R. Soc. A* **472** (2016), 1 - 8.

[8] R. Gorenflo and F. Mainardi, *Fractional Calculus: Integral and Differential Equations of Fractional Order*, 378 of CISM courses and lectures, Springer - Verlag, Berlin, 1997.

[9] R. Hilfer, *Application of Fractional Calculus in Physics*, World Scientific Publishing Co. Singapore, 2000.

[10] P. Humbert and R.P. Agarwal Sur la fonction de Mittag-Leffler et quelques unes de ses, generalizations, *Bull. Sci. Math.* (Ser. II) **77** (1953), 180-185.

[11] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations,* Vol. 204, (North-Holland Mathematics Studies), Elsevier, 2006.

[12] V. S. Kiryakova, *Generalized Fractional Calculus and Applications*, Pitman Research Notes in Mathematics, Vol. 301, Longman Sci. Tech. and J. Wiley, New York, 1994.

[13] H. Kumar, On convergence properties and applications of two variable generalized Mittag - Leffler function, *Jnānābha* **47** (1) (June 2017), 125 - 138.

[14] H. Kumar, Real zeros of generalized Mittag - Leffler functions of two variables through integral transforms, *Jnānābha* **48** (2) (2018), 32 - 45.

[15] H. Kumar, R. C. Singh Chandel and H. Srivastava, On a fractional non linear biological model problem and its approximate solutions through Volterra integral equation, *Jnānābha* **47** (1) (2017), 143 - 154.

[16] H. Kumar and M. A. Pathan, On some results involving statistical characteristics, operational formulae of special functions and anomalous diffusion*, ROMAI J.* **13** (1) (2017), 41 - 55.

[17] H. Kumar, M. A. Pathan and S. K. Rai, On certain solutions of a generalized Perl's vector equation involving fractional time derivative, *Montes Taurus J. Pure Appl. Math.***1** (2) (2019), 42-57.

[18] H. Kumar, M. A. Pathan and S. K. Rai, Obtaining Voigt functions via quadrature formula for the fractional in time diffusion and wave problem, *Kragujev. J. Math.*, **46** (5) (2022), 759 - 772.

[19] H. Kumar, M. A. Pathan and Harish Srivastava, A general solution of a space-time fractional anomalous diffusion problem using the series of bilateral eigenfunctions, *Commun. Korean Math. Soc.* **29** (1) (2014), 173-185.

[20] H. Kumar, M. A. Pathan and P. Yadav, Series solutions for initial - value problems of time fractional generalized anomalous diffusion equations, *Le Matematiche* **LXVII** (Fasc. II) (2012), 217 - 229.

[21] H. Kumar and S. K. Rai, On a fractional time derivative and multi-dimensional space evaluation Bessel Sturm - Liouville diffusion and wave problem, *Jñānābha* **Special Issue** (2018), 61 -71.

[22] H. Kumar, H. Srivastava and S. K. Rai, On a bi dimensional basis involving special functions for partial in space and time fractional wave mechanical problems and approximation, *Jñānābha* **47** (II) (2017), 291 -300.

[23] A. M. Mathai and H. J. Haubold, *Special Functions for Applied Scientists*, Springer, New York, 2008.

[24] Branislav Martic', Some results involving a generalized Kampé de Fériet function, *Publ. Inst. Math.* Nou. Ser. tome **29** (15) (1973), 101-104.

[25] G. M. Mittag-Leffler, Sur la nonvelle function E_α (x), *C. R. Acad.* Paris **137** (1903), 554-558.
[26] K. B. Oldham, and J. Spanier, *The Fractional Calculus: Theory and Application of Differentiation and Integration to Arbitrary Order,* Dover Books on Mathematics, 1974.
[27] M. A. Pathan and H. Kumar, Generalized multivariable Cauchy residue theorem and non-zero zeros of multivariable and multi-parameters generalized Mittag-Leffler functions, *Southeast Asian Bull. Math.* **43** (5) (2019), 733 -749.
[28] H. M. Srivastava and M. C. Daoust, On Eulerian integrals associated with Kampe' de Fe'riet's function, *Publ. Inst. Math.* (Beograd) (N. S. **9**) **23** (1969), 199 - 202.
[29] H. M. Srivastava and M. C. Daoust, Certain generalized Neumann expansions associated Kamp$e$ de F$e$riet function, *Indag. Math.* **31** (1969), 449 - 457.
[30] H. M. Srivastava and M. C. Daoust, A note on the convergence of Kamp$e$ de F$e$riet's double hypergeometric series, *Math. Nachr*. **53** (1972), 151 - 159.
[31] H. M. Srivastava and P. W. Karlsson, *Multiple Gaussian Hypergeometric Series*, Wiley Halsted, New York, 1985.
[32] H. M. Srivastava and H. L. Manocha, *A Treatise on Generating Functions*, John Wiley and Sons, New York, 1984.
[33] H. M. Srivastava and R. Panda, An integral representation for the product of two Jacobi Polynomials, *J. London Math. Soc.* **12** (2) (1976), 419 - 425.
[34] A. Wiman, Uber den fundamental Saiz in der Theorie der Funktionen E_α(x), *Acta. Math.*, **29** (1905), 191-201.

Jñānābha, Vol. 50(1) (2020), 243-252
(Dedicated to Honor Professor H.M. Srivastava on His 80th Birth Anniversary Celebrations)

TIME SERIES ANALYSIS OF RAINFALL USING HETEROSKEDASTICITY MODELS

By
Rashmi Bhardwaj and Varsha Duhoon
University School of Basic and Applied Sciences, Non-Linear Dynamics Research Lab, Guru Gobind Singh Indraprastha University, Dwarka, Delhi-110078, India.
Email:rashmib22@gmail.com, varshaduhoon5@gmail.com
(*Received : February* 04, 2020 *; Revised: June* 21, 2020)
DOI: https://doi.org/10.58250/jnanabha.2020.50123

Abstract

Weather forecasting is predicting present state of weather by the help of analyzing collected data such as temperature, humidity, wind etc. to analyze atmospheric processes and determine how weather condition is going to change in future. Weather forecasting is important not only for prediction but also to prepare for future coming events if any such as cyclone, heavy rainfall, hails which can cause harm to the agricultural production of the country and hence affect the livelihood of the farmers. Continuous change in variance of time series over time; such a process is termed as Volatility. Heteroskecdascity refers to increasing variance in a way, such as increase in trend, this property of series is termed as Heteroskecdascity. Objective of the paper is to analyze, model and predict Rainfall time series of Delhi region from January 01, 2017 to December 31, 2018 using Heteroskecdascity model such as *ARCH, GARCH, TARCH/ GJR-GARCH, EGARCH* models to select most suited model on basis of probability value of the model hence calculated. Further, to analyze and choose model has full filled required conditions the model is checked for Serial Correlation, *ARCH* Effect, Normal distribution of Residuals, *ARCH-LM* test is applied, *AIC*, *SIC* values are calculated. *GJR-GARCH* model is most suited model among all models tested for modeling and analyzing rainfall. Model selection is done based on *AIC* value and *SIC* value calculated.
2010 Mathematics Subject Classifications: 33C60, 33C45.
Keywords and phrases: Volatility, Heteroscedascity, *ARCH*, *GARCH*, *TARCH*, *EGARCH*, Time Series, Rainfall.

1 Introduction

Climate change has adverse effect on rainfall as, monsoons are shifted, rainfall as in average has decreased per year and agricultural sector of the economy is adversely affect in many parts of country due to change in pattern of rainfall causing drought and forcing the farmer to find an alternate for their living. In general, climate is highly non-linear phenomena in nature. Climate change is a serious environmental threat to human kind nowadays. Change in weather pattern has and is affecting livelihood of people, it can also be seen that there are longer summers and shifted monsoon every year. Over a long period of time it has been observed that every year temperature is raising by some degrees and overall rainfall expectancy has decreased. The objective in the paper is to test whether volatility which is applied in the field of economics to study the nature of time series can also be applied to weather data or not and to further see which model will give better forecasting of rainfall.

Ardia et al. [1] studied single regime & markov switching *GARCH* models by comparing forecasting efficiency in terms of risk management. Bhardwaj et al. ([2],[3],[4]) studied various methods for weather forecasting by using data mining techniques which included *MLP*, Gaussian Process, *SMO* Regression, Linear Regression among which best suited tool was selected using the statistical calculations *CC, MAE, RMSE, RRSE, RAE*; also studied *ANFIS-SUGENO* model using subtractive clustering technique for the formation of membership function to forecast minimum and maximum temperature; further studied behavior of temperature from 1981-2015 by calculating hurst exponent, fractal dimension, predictability index using R/S method ([2],[3],[4]). Bentes [5] studied GARCH, *IGARCH, FIGARCH* models for predicting volatility of gold returns. Bollerseley [6] studied generalized *ARCH(GARCH)* model for parsimonious representation of *ARCH*. Dickey [7] studied likelihood ratio test and also studied least square estimator of alpha. Engles ([8],[9]) studied *ARCH* and *GARCH* models for analysing time seriesdata for financial application and also studied symmetry effect in exchange rate data by applying *ARCH* and *GARCH* and *EGARCH* model for studying volatility. Gabriel [11] studied forecasting efficiency of GARCH model. Herwartz et al. [12] studied *GARCH* model to predict stock return. Huang et al. [13] studied *HAR-GARCH* model for modelling long memory volatility by the help of measures of volatility. Jordan et al. [14] studied *ARCH/GARCH* model to study volatility in price of field crops. Kobayashi et al. [15] studied *EGARCH* model and compared it to stochastic volatility models. Liu et al. [16] studied *GARCH* model for predicting stocks. D.N. Fente et al. [10] studied different weather parameter and used *LSTM* Technique for forecasting the weather parameters. N. Singh et al. [18] studied weather parameters and forecasted the parameters using machine learning algorithm such as random forest. K. Sushmitha et al. [19] studied weather parameters such as temperature, wind, humidity, rainfall. The parameters were studied and forecasted using *ARIMA*. B. Munmun et al. [17] studied and forecasted weather parameters using data mining technique such as Naïve Bayes and Chi Square algorithm.

GJR-GARCH, ARCH, GARCH, EGARCH models are studied step wise analyzing that whether residuals show that *ARCH* family models can be applied or not and so further Residuals are diagnosed by applying heteroscedascity test and best model is initially selected using the *AIC*, *SIC* values and further the diagnostic test to re check if selected model is actually full filling the conditions required.

2 Methodology

2.1 Time Series Analysis

Time series analysis is done in order to analysis sequence of observation of the time pattern. Analysis is done to study various pattern of the data such as data has a seasonal pattern or random pattern. This study of pattern of rainfall helps us know whether data has repetitive pattern or not. The time series process is analysed based on the past values to study the pattern if any and if the pattern exist then the time series is modelled and forecasting is done.

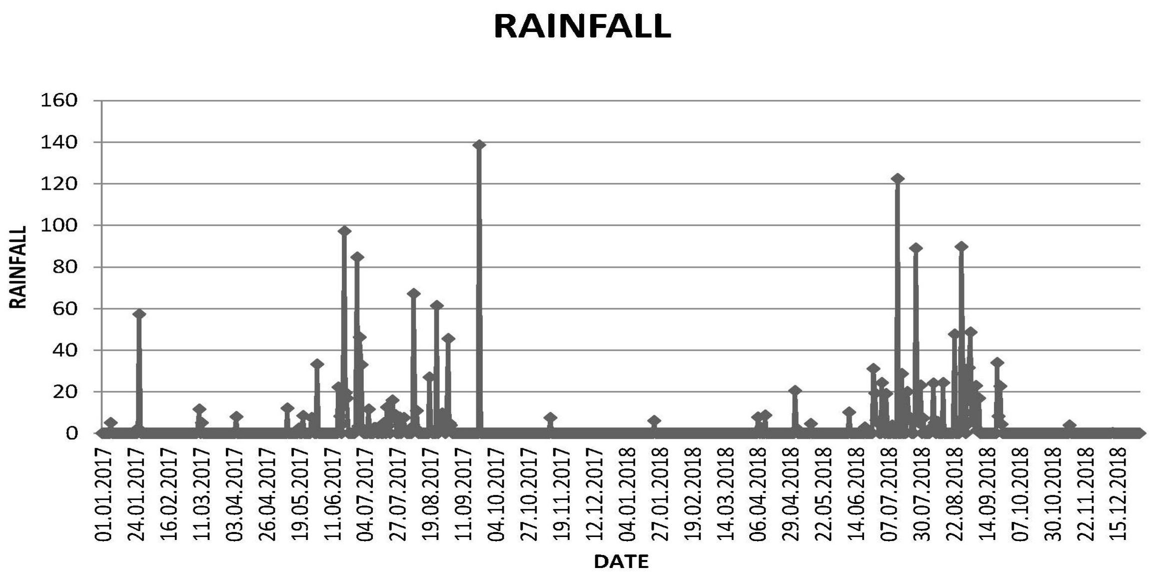


***Figure* 2.1:** Plot of rainfall values from January 01, 2017 till December 31, 2018

2.2 *ARCH*

Auto Regressive Conditional Heteroscedasticity *(ARCH)* model includes single or many data values in series further variance of existing error term is a function of actual size of previous time periods error terms. *ARCH* model models time series. *ARCH(q)* model is used to model time series by *ARCH* process, ϵ_t: error term,ϵ_t are split into stochastic piece, λ_t: standard deviation.

Step 1: consider a time series t, for all values from 1 to n.

Step 2: estimate best fit *AR(q)* by calculating: $g_t = x_0 + x_1 g_{t-1} + ... + x_q g_{t-q} + \epsilon_t$

Step 3: calculate the error term ϵ_t , with the help of formula:
$\epsilon_t = Z_t \lambda_t$, Z_t: white noise process.

Step 4: model time series by calculating $\lambda_t^2 = x_0 + x_1 \epsilon_{t-1}^2 + ... + x_q \epsilon_q^2]$, Such that is positive or equal to 0.

Further, *ARCH (q)* model is analyzedby ordinary least squares.

2.3 *GARCH*

ARMA model is used as error variance, Generalized Autoregressive Conditional Heteroscedasticity model. *GARCH(p, q)* model, p:order of *GARCH* terms λ^2& q:order of *ARCH* ϵ^2.

Step 1: consider time series t, for all values from 1 to n.

Step 2: estimate best fit *AR*(q) by calculating:$g_t = x_0 + x_1 g_{t-1} + ... + x_z gt - q + \epsilon_t$

Step 3: calculate $Y_t = X_t' b + \epsilon_t$, such that $\epsilon_t / \phi_t \ N(0, \lambda_t^2)$, where ϵ_t is the error term.

Step 4: now calculate standard deviation for time series t for order q with ϵ_t as error term

$$\lambda_t^2 = W + \alpha_1 \epsilon_{t-1}^2 + ... + \beta_1 \lambda_{t-1}^2 + ... = W + \sum_{i=1}^{q} \lambda_i \epsilon_{t-1}^2 + \sum_{i=1}^{q} \beta_i \epsilon_{t-1}^2$$

GARCH model is applied for reducing error in forecasting time series and further improve accuracy of forecast.

2.4 *EGARCH*

Exponential Generalized Autoregressive Conditional Heteroskedastic model is extension of *GARCH* model which studies time series values by calculating variance of present error term, such that "p'':degree of *GARCH* polynomial and "q'' is degree of *ARCH* & Leverage polynomial.

Formally, $EGARCH(p,q)$:

Step 1: consider time series t, for all values from 1 to n.

Step 2: estimate best fit $AR(q)$ by calculating: $g_t = x_0 + x_1 g_{t-1} + ... + x_q g_{t-q} + \epsilon_t$

Step 3: now, calculate $log\lambda_t^2 = W + \sum_{i=1}^{q} \beta_i(z_{t-i}^i) + \sum_{i=1}^{p} \alpha_i log\lambda_{t-i}^2$, $log\lambda_t^2$ can be negative, hence no restrictions on parameters.

Step 4: further calculate $g(z_t) = \phi z_t + \tau(|z_t| - E(|z_t|))$, λ_t^2, $g(z_t, z_t)$, $z_t$sign and magnitude have separate effect on volatility.

Above is conditional variance $w, \beta, \alpha, \phi, \tau$: coefficients,$z_t$:standard normal variable,also known as error distribution. Positive or negative sign is not a limitation in *GARCH* model. Test of *ARCH* and *GARCH* errors is important for studying time series.

2.5 *T-GARCH*

Threshold-*GARCH* model which has another name that is *GJR-GARCH* model. Model includes calculation of conditional Standard Deviation instead of calculating conditional variance. $T - GARCH(p,q)$ model is calculated as follows:

Step 1: consider time series t, for all values from 1 to n.

Step 2: estimate best fit $AR(q)$ by calculating:$g_t = x_0 + x_1 g_{t-1} + ... + x_q g_{t-q} + \epsilon_q$

Step 3: calculate error term using $\epsilon_{t-1}^+ = \epsilon_{t-1}$ if $\epsilon_{t-1}^+ = 00, \epsilon_{t-1} > 0$ and vice versa.

Step 4: calculate standard deviation using: $\lambda_t = K + \delta\lambda_{t-1} + ... + \alpha_i + \epsilon_{t-1}^+$

3 Resluts and Discussion

Daily data of Temperature, rainfall for Delhi with coordinates Longitude 770 09' 27" Latitude280 38 '23" N Altitude:228.61m from January 01, 2017 till December 31, 2018 is taken. The time series of daily values of rainfall have been taken and the parameters are set accordingly with respect to the resulting probability values and values of *AIC, SIC* the best fit model is selected. The best suited model is selected among *ARCH, GARCH, TARCH/ GJR-GARCH, EGARCH*.

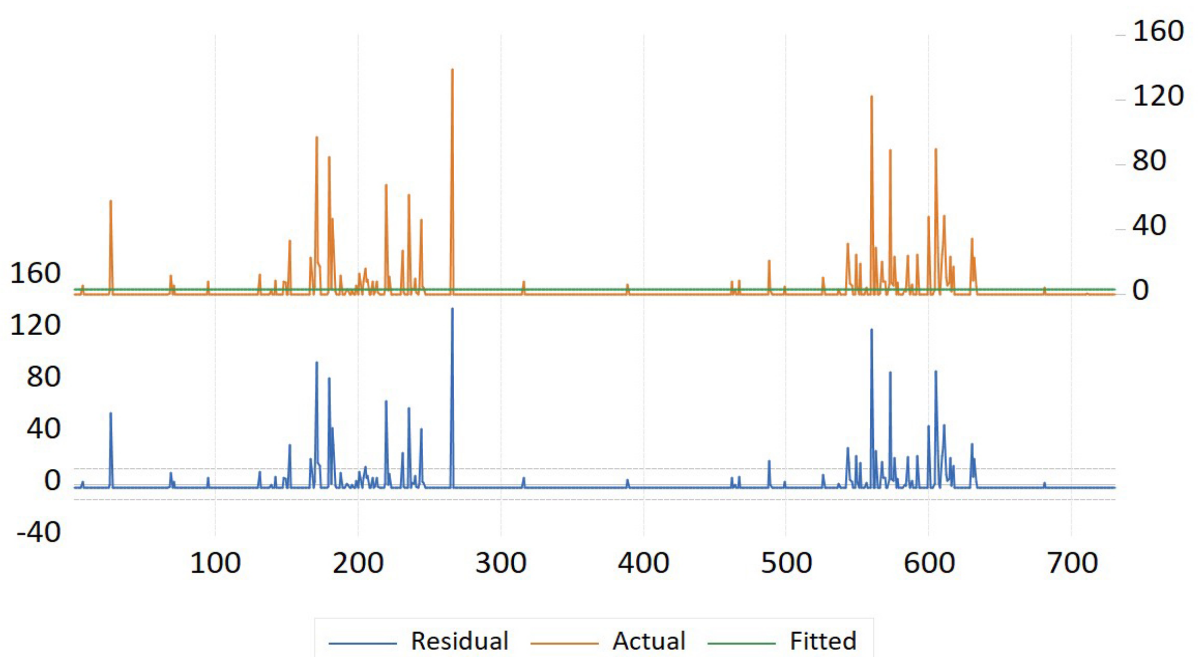


***Figure* 3.1:** Plot of rainfall values from January 01, 2017 till December 31, 2018

In ***Figure* 3.1**, residuals have been plotted of rainfall time series to understand pattern of time series, it shows that periods of low values is followed by high values and similarly further low values for longer duration is followed by the period of high values thus it implies that we can apply the *ARCH* family models.

Further, the residuals are diagnosed whether *ARCH* family models can be applied or not and so further Residuals are diagnosed by applying Heteroskecdascity test: *ARCH*, in which the following hypothesis is considered:

Null Hypothesis: Absence of *ARCH* effect. (If probability value < 5%)

Alternative Hypothesis: Presence of *ARCH* effect. (If probability value > 5%)

***Table* 3.1:** Residual Diagnostic using Heteroskecdascity test for Rainfall

Heteroskecdascity test	Observed R^2	Chi-probability
ARCH	**9.681242**	**0.0019 (< 5%)**

Now, it is clear from ***Table* 3.1**, that probability value comes 0.0019, that is lesser than 5%; hence rejecting the Null Hypothesis, absence ofARCHeffect. Now, there exists *ARCH* effect in time series of Rainfall therefore, *ARCH* family models such as *GARCH, TGARCH, EGARCH* models can be applied.

***Table* 3.2:** AIC, SIC values as per the model for Rainfall time series

Heteroscedascity test	Observed R^2	Chi-probability
ARCH	**0.04**	**0.82 (> 5%)**

The above ***Table* 3.2**, shows that *GJR-GARCH* or *TGARCH* model is best suitedon basis of *AIC*, SIC values calculated for each model. Now, *GJR-GARCH* model will be tested whether it full fills all the conditions required.

Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob*
		1	-0.000	-0.000	5.E-05	0.994
		2	-0.001	-0.001	0.0016	0.999
		3	0.004	0.004	0.0113	1.000
		4	-0.002	-0.002	0.0138	1.000
		5	0.002	0.002	0.0161	1.000
		6	-0.003	-0.003	0.0226	1.000
		7	-0.008	-0.008	0.0688	1.000
		8	-0.006	-0.006	0.0990	1.000
		9	0.024	0.024	0.5356	1.000
		10	-0.007	-0.007	0.5696	1.000
		11	0.003	0.003	0.5773	1.000
		12	-0.007	-0.008	0.6179	1.000
		13	0.036	0.036	1.5851	1.000
		14	-0.013	-0.013	1.7082	1.000
		15	-0.010	-0.010	1.7865	1.000
		16	-0.002	-0.002	1.7893	1.000
		17	0.011	0.011	1.8789	1.000
		18	-0.013	-0.014	2.0084	1.000
		19	-0.004	-0.004	2.0209	1.000
		20	-0.010	-0.010	2.0901	1.000
		21	-0.009	-0.009	2.1560	1.000
		22	0.007	0.005	2.1939	1.000
		23	-0.013	-0.012	2.3167	1.000
		24	-0.009	-0.009	2.3824	1.000
		25	-0.007	-0.006	2.4167	1.000
		26	-0.005	-0.007	2.4346	1.000
		27	-0.005	-0.004	2.4540	1.000
		28	-0.008	-0.008	2.5048	1.000
		29	-0.009	-0.009	2.5703	1.000
		30	0.058	0.057	5.1531	1.000
		31	-0.012	-0.011	5.2603	1.000
		32	0.013	0.014	5.3867	1.000
		33	-0.009	-0.010	5.4546	1.000
		34	-0.005	-0.004	5.4766	1.000
		35	-0.002	-0.003	5.4784	1.000
		36	-0.008	-0.007	5.5310	1.000

***Figure* 3.2:** Correlogram plot of GJR-GARCH Model

Now, ***Figure* 3.2**, shows probability values greater than 5% hence, we accept Null Hypothesis which is: Absence of Serial Correlation.

Hence, there is no Serial Correlation.

***Table* 3.3:** Diagnostic using ARCH-LM test for Rainfall for ARCH effect

Diagnostic using *ARCH-LM* test for Rainfall for *ARCH* effect

Model	***Equation***	***AIC***	***SIC***
***ARCH*(5)**	**X(2)+X(3)*RESID(-1)^2+ X(4) * RESID(-1)^@+X(5)*RESID(-3)^2+X(6)*RESID(-4)^2+X(7)*RESID(-5)^2**	**7.670149**	**7.714192**
***GARCH*(1,1)**	**X(2)+X(3)*RESID(-1)^2+X(4)*GARCH(-1)**	**7.233153**	**7.258321**
***GJR-GARCH/ GARCH*(1,1)**	**X(2)+X(3)*RESID(-1)^2+X(4)*RESID(-1)^2*(RESID(-1)<0)+X(5)*GARCH(-1)**	**6.489396**	**6.520855**
***EGARCH* (1,1)**	**X(2)+X(3)*ABS(RESID(-1)SQRT(GARCH(-1)))+X(4)*RESID(-1)SQRT(GARCH(-1)+X(5)*LOG(GARCH(-1))**	**7.493835**	**7.525294**

Null Hypothesis: *ARCH* Effect-NOT PRESENT. (prob >5%)
Alternative Hypothesis: *ARCH* effect-PRESENT. (prob < 5%)
Hence, Null Hypothesis is accepted since probability value is greater than 5%; no *ARCH* effect.
Now, checking if residuals are distributed normally or not using Histogram plot and checking the probability value of the thus calculated.

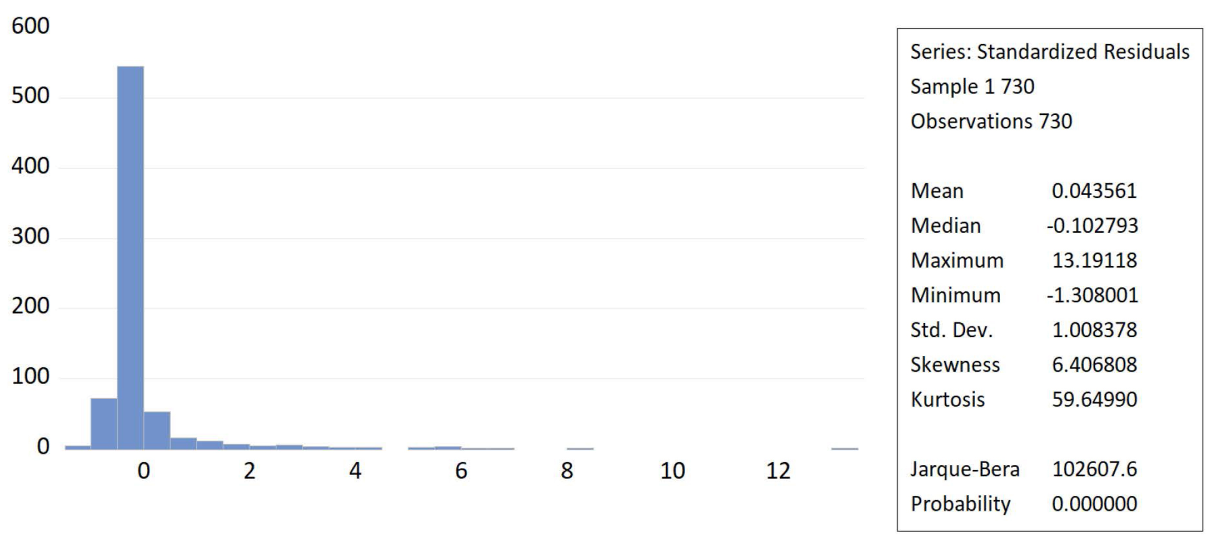


***Figure* 3.3:** Histogram plot to check the normal distribution and probability value

Now, since value of probability is less than5% thus, we reject Null Hypothesis (X_0)&accept

Alternative Hypothesis (X_1).

X_0: Normally Distributed.

X_1: Not Normally Distributed. Hence, distribution of residuals is not normal. As there is absence of *ARCH* effect therefore *GJR-GARCH* is to be considered.

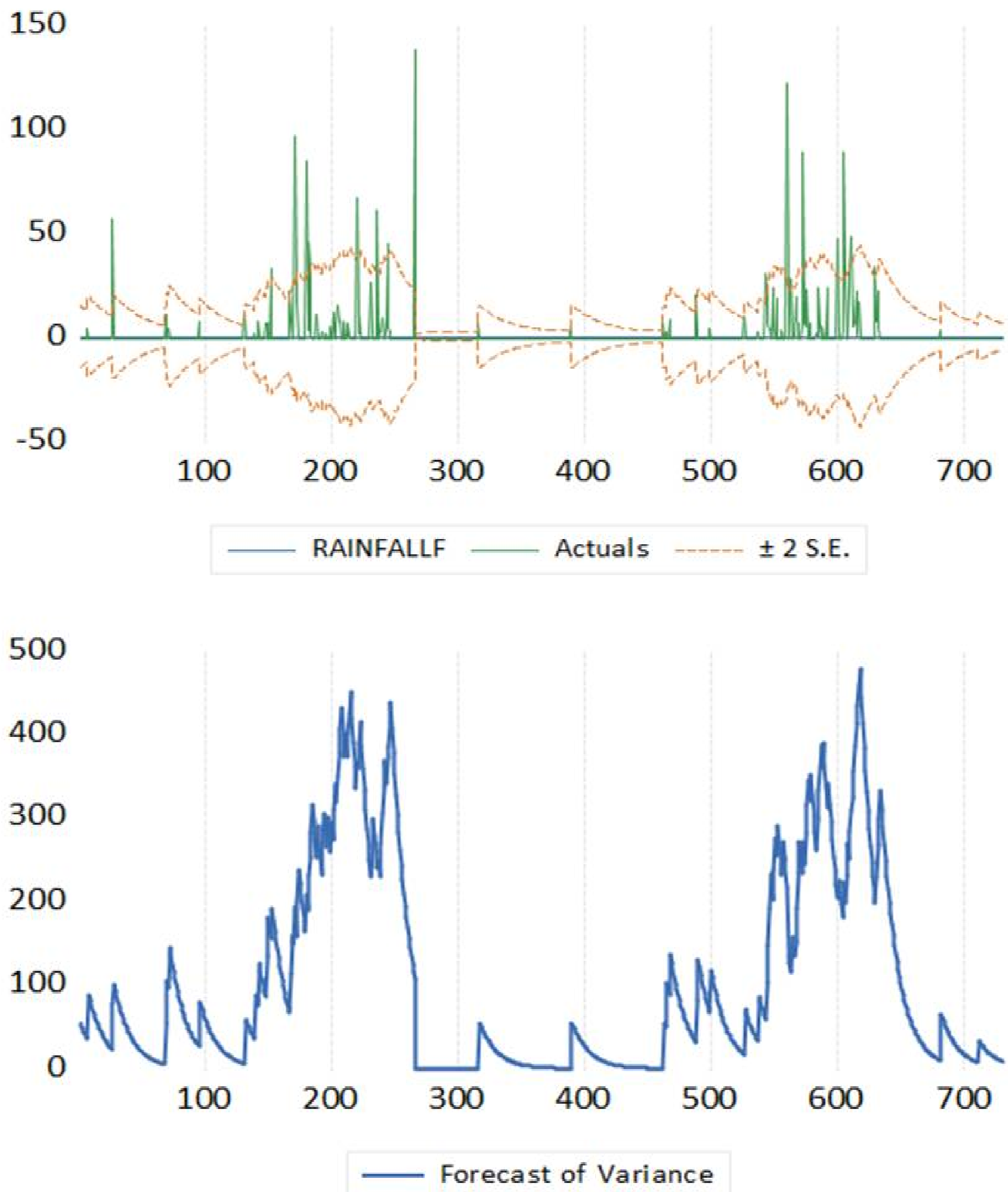


***Figure* 3.4:** Forecast of variance and Rainfall

***Figure* 3.4**, shows the forecast of variance of rainfall and standard error calculated. *GJR-GARCH* is considered on the basis of tests and *AIC, SIC* values and hence the above plot is obtained using the model.

4 Conclusion

The above study has been done to analyze, model and predict Rainfall time series of Delhi region using Heteroskecdascity model such as *ARCH, GARCH, TARCH/ GJR-GARCH, EGARCH* models by selecting most suited model on basis of calculations. Further, the objective was to analyze whether model has full filled required conditions by checking for Serial Correlation, *ARCH* Effect, Normal distribution of Residuals, *ARCH-LM* test, *AIC, SIC* values calculated. *GJR-GARCH* model is the most suited model among all models tested for modelling rainfall. Model selection is done based on *AIC* value and *SIC* value calculated and all other tests done. It can also be concluded that Heteroskecdascity model which is used in economics to study the volatility of time series can be applied in area of weather forecasting by studying the time series behavior of weather parameters and checking if the time series shows required behavior for application of such models as applied in the study above.

Hence, *GJR-GARCH/ TARCH* is the suited for forecasting daily rainfall. Since, the residuals have no *ARCH* effect also no Serial correlation thus its observed that *GJR-GARCH* model is the suited model.

5 Acknowledgements

The authors are thankful to Guru Gobind Singh Indraprastha University, Delhi (India) for providing research facilities and financial support. They are also thankful to the referee and Editor for their valuable suggestions to bring the paper in its present form.

References

[1] D. Ardia, K. Bluteau, K. Boudt and Leopoldo Catania, Forecasting risk with Markov-switching *GARCH* models: A large-scale performance study, *International Journal of Forecasting*, **34 (4)** (2018), 733-747

[2] R. Bhardwaj and V. Duhoon, Weather forcasting using Soft Computing Technique, International Conference on Computing, Power, and Communication Technologies (GUCON), IEEE Explore Digital Library, (2019), 1111-1115.

[3] R. Bhardwaj and V. Duhoon, Real Time Weather Parameter Forecasting Using Anfis-Sugeno, International Journal of Engineering and Advanced Technology *(IJEAT)*, Blue Eyes Intelligence Engineering & Sciences Publication *(BEIESP)*, **9(1)** (2019), 461-469.

[4] R. Bhardwaj and V. Duhoon, Time Series Analysis of Heat Stroke, Jnbha, Vijnana Parishad of India, **49(1)** (2019), 01-10.

[5] R. S. Bentes, Forecasting volatility in gold returns under the *GARCH, IGARCH* and *FI-GARCH* frameworks: New evidence, Physica A: Statistical Mechanics and its Applications, **438** (2015), 355-364

[6] T. Bollerslev, Generalized autoregressive conditional heteroscedasticity, *Journal of Econometrics*, **31(3)** (1986), 307-327.

[7] D. A. Dickey and W. A. Fuller, Likelihood ratio statistics for autoregressive time series with unit root, *Econometrica*, **49** (1981), 1057-1072.

[8] R. Engle, GARCH 101: The use of ARCH/GARCH models in applied econometrics, Journal of Economic Perspectives, **15(4)** (2001), 157-168.

[9] R. Engle, Risk and volatility: Econometric models and financial practice, *American Economic Review*, **94(3)** (2004), 405-416.

[10] D. N. Fente and D. Kumar Singh, Weather Forecasting Using Artificial Neural Network, *2018 Second International Conference on Inventive Communication and Computational Technologies (ICICCT)*, Coimbatore, (2018), 1757-1761.

[11] S. A. Gabriel, Evaluating the Forecasting Performance of *GARCH* Models, *Procedia - Social and Behavioral Sciences*, **62** (2012), 1006-1010

[12] H. Herwartz, Stock return prediction under GARCH - An empirical assessment, *International Journal of Forecasting*, **33 (3)** (2017), 569-580

[13] Z. Huang ,H. Liu and T. Wang , Modeling long memory volatility using realized measures of volatility: A realized *HAR GARCH* model, *Economic Modelling*, **52(B)** (2016), 812-821.

[14] H. Jordaan, B. Grove, A. Jooste, Z.G. Alemu, Measuring the price volatility of certain field crops in South Africa using *ARCH/GARCH* approach, **46(3)** (2007), 306-322.

[15] M. Kobayashi, X. Shi, Testing for *EGARCH* against stochastic volatility models, *Journal of Time Series Analysis*, **26** (2005), 135-150.

[16] C. H. Liu and C. J. Hung, Forecasting S&P-100 stock index volatility: The role of volatility asymmetry and distributional assumption in *GARCH* models, *Expert Systems with Applications*, **37 (7)** (2010), 4928-4934

[17] B. Munmun, D. Tanni and B. Sayantanu, Weather Forecast Prediction: An Integrated Approach for Analyzing and Measuring Weather Data. *International Journal of Computer Applications*, **182** (2018), 20-24. 10.5120/ijca2018918265.

[18] N. Singh, S. Chaturvedi and S. Akhter, Weather Forecasting Using Machine Learning Algorithm, 2019 International Conference on Signal Processing and Communication (ICSC), NOIDA, India, (2019), 171-174, doi: 10.1109/ICSC45622.2019.8938211

[19] K. Sushmitha and T. Shashikumar, A real-time weather forecasting and analysis, (2017), 1567-1570. 10.1109/ICPCSI.2017.8391974.

Jñānābha, Vol. 50(1) (2020), 253-264
(Dedicated to Honor Professor H.M. Srivastava on His 80th Birth Anniversary Celebrations)

MULTIPLE FRACTIONAL DIFFUSIONS VIA MULTIVARIABLE H- FUNCTION

By
Hemant Kumar and Surya Kant Rai*
D. A-V. Postgraduate College Kanpur - 208001, Uttar Pradesh, India.
Email:palhemant2007@rediffmail.com, suryakantrai@gmail.com*

(*Received : June* 14, 2020 *; Revised: June* 23, 2020)
DOI: https://doi.org/10.58250/jnanabha.2020.50124

Abstract

In this paper, we introduce a diffusion and wave equation consisting of multidimensional space Riesz-Feller fractional operators and Caputo time fractional derivative. Imposing certain boundary values, we obtain its solution in terms of multivariable H-function and finally making an appeal to our results, we evaluate various multiple fractional diffusions.

2010 Mathematics Subject Classification: 26A33, 33C20, 33C60, 33E12, 33E20, 33E30, 44A15, 60G18, 60J60.
Keywords and phrases: Multivariable H-function, Riesz-Feller fractional operators, Caputo time fractional derivative, diffusion and wave problems, multivariable Green functions.

1 Introduction, formulae and definitions

Since 1976, various researchers and authors studied the multivariable H-function ([19], [20]) as considering as an initial function to solve many physical and scientific problems, for example (see [1], [2]). This multivariable H-function is widely used by many researchers in the derivation of the results involving fractional derivatives and fractional integrals ([12], [15], [16], [18]). Also, the Fourier series representations are studied for the multivariable H-function (see [3]). In 2005, Mainardi et al. [10] employed Fox's H-function in fractional diffusion problems, formerly this H-function was introduced by [5] and contour integrals for H-function were appeared in [17].

To explore new ideas for enlarging to the field of fractional diffusions in multidimensional space, in the present investigation, we obtain a solution of multidimensional in space fractional, time fractional diffusion and wave problem in terms of the multivariable H-function involving a multiple contour integral of Mellin - Barnes type [9], defined by ([19], [20]) as

$$(1.1)\quad H[z_1,...,z_r] = H^{0,n:m_1,n_1;...;m_r,n_r}_{p,q:p_1,q_1;...;p_r,q_r}\left[\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix}\middle| \begin{matrix} (a_j:\alpha_j^{(1)},\dots,\alpha_j^{(r)})_{1,p}:(c_j^{(1)}:\sigma_j^{(1)})_{1,p_1};\dots;(c_j^{(r)}:\sigma_j^{(r)})_{1,p_r} \\ (b_j:\beta_j^{(1)},\dots,\beta_j^{(r)})_{1,q}:(d_j^{(1)}:\rho_j^{(1)})_{1,q_1};\dots;(d_j^{(r)}:\rho_j^{(r)})_{1,q_r} \end{matrix}\right]$$

$$= \frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\Psi(\varrho_1,\dots,\varrho_r)\left\{\prod_{i=1}^{r}\Phi_i(\varrho_i)(z_i)^{\varrho_i}\right\}d\varrho_1\dots d\varrho_r.$$

Here in Eqn. (1.1), $\omega = \sqrt{(-1)}$ throughout this paper, and also

$$(1.2)\quad \Psi(\varrho_1,\dots,\varrho_r) = \frac{\prod_{j=1}^{n}\Gamma(1-a_j+\sum_{i=1}^{r}\alpha_j^{(i)}\varrho_i)}{[\prod_{j=n+1}^{p}\Gamma(a_j-\sum_{i=1}^{r}\alpha_j^{(i)}\varrho_i)][\prod_{j=1}^{q}\Gamma(1-b_j+\sum_{i=1}^{r}\beta_j^{(i)}\varrho_i)]},$$

$$\Phi_i(\varrho_i) = \frac{[\prod_{j=1}^{m_i}\Gamma(d_j^{(i)}-\rho_j^{(i)}\varrho_i)][\prod_{j=1}^{n_i}\Gamma(1-c_j^{(i)}+\sigma_j^{(i)}\varrho_i)]}{[\prod_{j=n_i+1}^{p_i}\Gamma(c_j^{(i)}-\sigma_j^{(i)}\varrho_i)][\prod_{j=m_i+1}^{q_i}\Gamma(1-d_j^{(i)}+\rho_j^{(i)}\varrho_i)]},\ \forall i=1,2,\dots,r.$$

Again, $L_i = L_{\gamma_i\omega\infty}$ represents the contours which start at the point $\gamma_i - \omega\infty$ and goes to the point $\gamma_i + \omega\infty$ with $\gamma_i \in \mathbb{R} = (-\infty, \infty), \forall i = 1, \ldots, r$ such that all the poles of $\Gamma(d_j^{(i)} - \rho_j^{(i)}\varrho_i), \forall j = 1, \ldots, m_i; i = 1, \ldots, r$ are separated from those of $\Gamma(1 - c_j^{(i)} + \sigma_j^{(i)}\varrho_i)\ \forall j = 1, \ldots, n_i; i = 1, \ldots, r$ and $\Gamma(1 - a_j + \sum_{i=1}^{r} \alpha_j^{(i)}\varrho_i)\forall j = 1, \ldots, n$. Here, the integers n, p, q, m_i, n_i, p_i and q_i satisfy the inequalities $0 \le n \le p$; $q \ge 0; 1 \le m_i \le q_i$ and $1 \le n_i \le p_i, i = 1, \ldots, r$. The parameters $a_j, \forall j = 1, \ldots, p; c_j^{(i)}, \forall j = 1, \ldots, p_i; i = 1, \ldots, r; b_j, \forall j = 1, \ldots, q; d_j^{(i)}, \forall j = 1, \ldots, q_i; i = 1, \ldots, r$, are complex numbers and the associated coefficients $\alpha_j^{(i)}, \forall j = 1, \ldots, p; i = 1, \ldots, r; \sigma_j^{(i)}, \forall j = 1, \ldots, p_i; i = 1, \ldots, r$; $\beta_j^{(i)}, \forall j = 1, \ldots, q; i = 1, \ldots, r; \rho_j^{(i)}, \forall j = 1, \ldots, q_i; i = 1, \ldots, r$, are positive real numbers, such that, $\forall i = 1, \ldots, r$

$$\Delta_i = \sum_{j=1}^{p} \alpha_j^{(i)} + \sum_{j=1}^{p_i} \sigma_j^{(i)} - \sum_{j=1}^{q} \beta_j^{(i)} - \sum_{j=1}^{q_i} \rho_j^{(i)} \le 0; \tag{1.3}$$

and

$$\Omega_i = \sum_{j=1}^{n} \alpha_j^{(i)} - \sum_{j=n+1}^{p} \alpha_j^{(i)} - \sum_{j=1}^{q} \beta_j^{(i)} + \sum_{j=1}^{n_i} \sigma_j^{(i)} - \sum_{j=n_i+1}^{p_i} \sigma_j^{(i)} + \sum_{j=1}^{m_i} \rho_j^{(i)} - \sum_{j=m_i+1}^{q_i} \rho_j^{(i)} > 0. \tag{1.4}$$

In the integrand of (1.1), the poles are supposed to be simple. The integral (1.1) converges absolutely for $|arg\,(z_i)| < \frac{\pi}{2}\Omega_i, i = 1, \ldots, r$. The points $z_i = 0, i = 1, 2, \ldots, r$ and various exceptional parameter values being tacitly excluded.

Particularly, for $r = 1$, the Eqns. (1.1) - (1.4) reduce to Fox's H- function ([5], [10], [12, p.2], [17]).

By Eqns. (1.1) - (1.4) an approximation formula is given by

$$H[z_1, \ldots, z_r] = O(|z_1|^{\vartheta_1} \ldots |z_r|^{\vartheta_r}), \max_{1\le j\le r} \|z_j\| \to 0; \tag{1.5}$$

where $\vartheta_i = \min_{1\le j\le m_i}\left(\frac{\Re(d_j^{(i)})}{\rho_j^{(i)}}\right), i = 1, \ldots, r$.

Although for $n = 0$, another approximation formula holds

$$H\,[z_1, \ldots, z_r] = O\,(|z_1|^{\varsigma_1} \ldots |z_r|^{\varsigma_r})\,, \min_{1\le j\le r} \|z_j\| \to \infty; \tag{1.6}$$

where

$$\varsigma_i = \min_{1\le j\le n_i}\left(\frac{\Re(c_j^{(i)}) - 1}{\sigma_j^{(i)}}\right), i = 1, \ldots, r.$$

Again, for $n = p = q = 0$, the multivariable H-function (1.1) - (1.4) consists a relation with Fox's H-function ([5], [10], [17]) in the form (see [12, p.207])

$$H^{0,0:m_1,n_1;\ldots;m_r,n_r}_{0,0:p_1,q_1;\ldots;p_r,q_r}\left[\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix}\middle| \begin{matrix} - : (c_j^{(1)} : \sigma_j^{(1)})_{1,p_1}; \ldots; (c_j^{(r)} : \sigma_j^{(r)})_{1,p_r} \\ - : (d_j^{(1)} : \rho_j^{(1)})_{1,q_1}; \ldots; (d_j^{(r)} : \rho_j^{(r)})_{1,q_r} \end{matrix}\right] = \prod_{i=1}^{r} H^{m_i,n_i}_{p_i,q_i}\left[z_i\middle|\begin{matrix}(c_j^{(i)} : \sigma_j^{(i)})_{1,p_i} \\ (d_j^{(i)} : \rho_j^{(i)})_{1,q_i}\end{matrix}\right], \tag{1.7}$$

provided that $\Delta_i' = \sum_{j=1}^{p_i} \sigma_j^{(i)} - \sum_{j=1}^{q_i} \rho_j^{(i)} \le 0$; and

$$\Omega_i' = \sum_{j=1}^{n_i} \sigma_j^{(i)} - \sum_{j=n_i+1}^{p_i} \sigma_j^{(i)} + \sum_{j=1}^{m_i} \rho_j^{(i)} - \sum_{j=m_i+1}^{q_i} \rho_j^{(i)} > 0 \text{ with } |\arg(z_i)| < \frac{\pi}{2}\Omega_i', i = 1, \ldots, r.$$

Recently, series and analytic solutions of fractional in time and space - fractional anomalous diffusion problems are obtained and studied by many authors ([6], [7], [8], [13]). In [10] the solution of fractional diffusion is found in terms of Fox's H-function (see also [5], [12, p.2], [17]). Now, in the present paper, we point out a generalization of a diffusion equation introduced and studied by various researchers ([8], [10], [12, p.199 Eqn. (6.186)]) as

$$(1.8)\quad {}_t^{\mathrm{C}}D_{0^+}^{\alpha}u\,(x,t) = {}_xD_{\theta}^{\beta}u\,(x,t)\,, -\infty < x < +\infty, t \geq 0; 0 < \alpha \leq 2; |\theta| \leq \min\{\beta, 2-\beta\},$$
$$\text{where, } 0 < \beta \leq 2.$$

In Eqn. (1.8), the *Caputo derivative* of the function $f\,(t)$, denoted by ${}_t^{\mathrm{C}}D_{0^+}^{\alpha}f\,(t)$ where, $m-1 < \alpha \leq m, \forall m \in \mathbb{N}$, is defined by [4, p.49]

$$(1.9)\qquad ({}_t^{\mathrm{C}}D_{0^+}^{\alpha}f)(t) = (I^{m-\alpha}f^{(m)})(t),$$

where, $f^{(m)}(t) = D_t^m f(t), \{D_t^m \equiv \frac{d^m}{dt^m} = \frac{d}{dt}(\frac{d^{m-1}}{dt^{m-1}})\}, I^{m-\alpha}$ being the Riemann - Liouville fractional integral given by

$$(I^{m-\alpha}f)(t) = \begin{cases} \frac{1}{\Gamma(m-\alpha)}\int_0^t (t-\tau)^{m-\alpha-1}f(\tau)d\tau, t > 0, m-1 < \alpha \leq m, \\ f(t), \alpha = m, \forall m \in \mathbb{N}. \end{cases}$$

The Laplace transformation of a sufficiently well behaved function $v(t)$ is denoted and defined by $L\{v(t); s\} = V(s) = \int_0^{\infty} e^{-st}v(t)dt, s > 0$. In our researches, we use this Laplace transformation of the Caputo time fractional derivative (1.9) in the form [4, p.134]

$$(1.10)\qquad L\{({}_t^{\mathrm{C}}D_{0^+}^{\alpha}v)(t); s\} = s^{\alpha}V(s) - \sum_{k=0}^{m-1} s^{\alpha-1-k}v^{(k)}(0^+)\ \forall m-1 < \alpha \leq m.$$

Further in the x- domain, we are needful of an operation by *an integro - differential Riesz-Feller operator* ${}_xD_{\theta}^{\beta}$, on $f(x)$, denoted by ${}_xD_{\theta}^{\beta}f(x)$ where $|\theta| \leq \min\{\beta, 2-\beta\}, 0 < \beta \leq 2$, is given as [8], [12, p.192, Eqn. (6.150)]

$$(1.11)\quad {}_xD_{\theta}^{\beta}f(x) =$$
$$\frac{\Gamma(1+\beta)}{\pi}\left[\sin\frac{(\beta+\theta)\pi}{2}\int_0^{\infty}\frac{f(x+\varrho)-f(x)}{\varrho^{\beta+1}}d\varrho + \sin\frac{(\beta-\theta)\pi}{2}\int_0^{\infty}\frac{f(x-\varrho)-f(x)}{\varrho^{\beta+1}}d\varrho\right].$$

The Fourier transformation of a sufficiently well behaved function $f(x)$, denoted by $\bar{f}(\varrho) = \mathcal{F}\{f(x);\varrho\} = \int_{-\infty}^{+\infty} e^{+\omega\varrho x}f(x)dx, \varrho \in \mathbb{R}$, then, the Fourier transform of the integro - differential *Riesz-Feller* operator (1.11) is obtained as ([8], [12, p.191])

$$(1.12)\qquad \mathcal{F}\{{}_xD_{\theta}^{\beta}\, f(x);\varrho\} = -\psi_{\beta}^{\theta}\,(\varrho)\bar{f}(\varrho).$$

Here in (1.12), we have

$$(1.13)\qquad \psi_{\beta}^{\theta}(\varrho) = |\varrho|^{\beta}\exp[\omega(sign\ \varrho)\frac{\theta\pi}{2}],\ \text{where, } |\theta| \leq\ \min\{\beta, 2-\beta\}, 0 < \beta \leq 2.$$

To make extensions in the area of fractional calculus and its applications in various scientific fields, we refer the books ([4], [12], [13], [16]). Again, for enriching in the present investigation, we introduce a diffusion and wave equation consisting of Riesz - Feller fractional operators in multidimensional space and Caputo time - fractional operator. We solve this diffusion and wave equation on imposing certain boundary conditions and obtain its solution in terms of multivariable H-function given in Eqns. (1.1) - (1.4). Its estimations are found by the Eqns. (1.5) and (1.6). The action of Riesz - Feller operator on multiple space - variable function $U(x_1, \ldots, x_r)$ is given by

$$(1.14)\quad {}_{x_1}D^{\beta_1}_{\theta_1}U(x_1,...,x_r) = \frac{\Gamma(1+\beta_1)}{\pi}[\sin\frac{(\beta_1+\theta_1)\pi}{2}\int_0^\infty \frac{U(x_1+\xi,\ldots,x_r)-U(x_1,\ldots,x_r)}{\xi^{\beta_1+1}}d\xi + \sin\frac{(\beta_1-\theta_1)\pi}{2}\int_0^\infty \frac{U(x_1-\xi,\ldots,x_r)-U(x_1,\ldots,x_r)}{\xi^{\beta_1+1}}d\xi]$$

$$\vdots$$

$${}_{x_r}D^{\beta_r}_{\theta_r}U(x_1,...,x_r) = \frac{\Gamma(1+\beta_r)}{\pi}[\sin\frac{(\beta_r+\theta_r)\pi}{2}\int_0^\infty \frac{U(x_1,\ldots,x_r+\xi)-U(x_1,\ldots,x_r)}{\xi^{\beta_r+1}}d\xi + \sin\frac{(\beta_r-\theta_r)\pi}{2}\int_0^\infty \frac{U(x_1,\ldots,x_r-\xi)-U(x_1,\ldots,x_r)}{\xi^{\beta_r+1}}d\xi].$$

Motivated by above work, in this paper, we introduce to a diffusion and wave equation consisting of multidimensional in space Riesz-Feller fractional operators defined by (1.11) and Caputo time fractional derivative by (1.9). Then, we shall dissipate to the multiple function $u(x_1,\ldots,x_r,t)$ into the product of the functions involving of separate space and time variables as given in Eqn. (2.2) and on imposing certain boundary conditions (2.3) - (2.5), obtain a solution and some results in terms of multivariable H-function and thus on applying our results, we evaluate various multiple fractional diffusions.

2 A multiple generalized diffusion and wave equation and its degenerations into several equations of different variables

In this section, we make a generalization of the Eqn. (1.8) on introducing many space fractional Riesz - Feller derivatives as defined in Eqn. (1.11) and the time fractional derivative by Eqn. (1.9) and thus find a multiple generalized diffusion and wave equation in the form

$$(2.1)\quad {}^{\mathrm{C}}_{t}D^{\alpha}_{0^+}u(x_1,\ldots,x_r,t) = {}_{x_1}D^{\beta_1}_{\theta_1}u(x_1,\ldots,x_r,t) + \ldots + {}_{x_r}D^{\beta_r}_{\theta_r}u(x_1,\ldots,x_r,t), t>0, 0<\alpha\le 2;$$
$$|\theta_i| \le \min\{\beta_i, 2-\beta_i\}, 0<\beta_i\le 2; -\infty < x_i < +\infty; \forall i = 1,2,\ldots,r.$$

Now, we equip following initial and boundary conditions to degenerate of the equation (2.1) into several equations consisting of fractional in time and space fractional variables

$$(2.2)\qquad u(x_1,\ldots,x_r,t) = \prod_{i=1}^{r} v(x_i,t)\ \forall x_i\in\mathbb{R}, t\in\mathbb{R}^+,\ i=1,2,\ldots,r.$$

For all $x_i \in \mathbb{R}, i = 1,2,\ldots,r$; the initial conditions are followed by

$$(2.3)\quad u(x_1,\ldots,x_r,t) = \sum_{i=1}^{r}\left\{\frac{\prod_{j=1,j\ne i}^{r}(t-\tau_j)v(x_j,t)}{\prod_{j=1,j\ne i}^{r}(\tau_i-\tau_j)}\right\}\varphi(x_i),\ \text{at } t=\tau_i, \tau_i\to 0^+\forall i=1,2,3,\ldots,r;$$

and

$$(2.4)\qquad u_t(x_1,\ldots,x_r,0^+) = \lim_{t\to 0^+}\frac{\partial}{\partial t}u(x_1,\ldots,x_r,t) = 0.$$

For $t > 0$, the boundary conditions are given by

$$(2.5)\qquad \lim_{x_1,\ldots,x_r\to\pm\infty} u(x_1,\ldots,x_r,t) = 0.$$

***Theorem* 2.1.** *If the relation* (2.2) *is applied on both sides of Eqn.* (2.1)*, then the equation* (2.1) *degenerates into r-generalized diffusion and wave equations as*

$$(2.6)\qquad {}^{\mathrm{C}}_{t}D^{\alpha}_{0^+}v(x_i,t) = {}_{x_i}D^{\beta_i}_{\theta_i}v(x_i,t),\ \textit{where } 0<\alpha\le 2;\ \textit{and } |\theta_i|\le\min\{\beta_i,2-\beta_i\}, 0<\beta_i\le 2;$$
$$-\infty < x_i < +\infty,\ \forall i=1,2,...,r.$$

Along with the initial and boundary conditions

$$(2.7)\qquad v(x_i,\tau_i) = \varphi(x_i)\ \textit{at}\ \tau_i\to 0^+, v_t(x_i,0^+) = 0\ \forall i=1,2,\ldots,r;$$

and

$$\lim_{x_i\to\pm\infty} v(x_i,t) = 0\ \forall i = 1,2,\ldots,r.$$

Proof. Under the conditions $0 < \alpha \leq 2, |\theta_i| \leq \min\{\beta_i, 2-\beta_i\}, 0 < \beta_i \leq 2; -\infty \leq x_i \leq \infty$, use the formula (2.2) in the Eqn. (2.1), we find that

$$\sum_{i=1}^{r} \prod_{j=1, j\neq i}^{r} v(x_j, t)\, {}_t^C D_{0^+}^{\alpha} v(x_i, t) = \sum_{i=1}^{r} \prod_{j=1, j\neq i}^{r} v(x_j, t)\, {}_{x_i} D_{\theta_i}^{\beta_i} v(x_i, t). \tag{2.8}$$

Then, equating the i^{th} element in both sides of Eqn. (2.8), we obtain r-equations given in the Eqn. (2.6).

Again, expand the right hand side of Eqn. (2.2) by Lagrange's interpolation formula in regard of t, we get the identities

$$\begin{aligned} u(x_1, \ldots, x_r, t) = \prod_{i=1}^{r} v(x_i, t) &= \sum_{i=1}^{r} \left(\frac{\prod_{j=1, j\neq i}^{r} (t-\tau_j)}{\prod_{j=1, j\neq i}^{r} (\tau_i - \tau_j)} \right) v(x_i, \tau_i) \prod_{j=1, j\neq i}^{r} v(x_j, t) \\ &= \sum_{i=1}^{r} \left(\frac{\prod_{j=1, j\neq i}^{r} (t-\tau_j) v(x_j, t)}{\prod_{j=1, j\neq i}^{r} (\tau_i - \tau_j)} \right) v(x_i, \tau_i). \end{aligned} \tag{2.9}$$

Thus, use the Eqns. (2.3) and (2.9), we obtain the first condition given in Eqn. (2.7) as $v(x_i, \tau_i) = \varphi(x_i)$ at $\tau_i \to 0^+$. In the similar manner, other conditions of (2.7) are found as $v_t(x_i, 0^+) = 0\ \forall i = 1, 2, \ldots, r$; and $\lim_{x_i \to \pm\infty} v(x_i, t) = 0\ \forall i = 1, 2, \ldots, r$.

3 Solution of the problem (2.1)-(2.5)

Before finding out the solution of the problem (2.1)-(2.5), we first prove following Lemmas:

***Lemma* 3.1.** *If* $\Delta_i' = \sum_{j=1}^{p_i} \sigma_j^{(i)} - \sum_{j=1}^{q_i} \rho_j^{(i)} \leq 0$; *and* $\Omega_i' = \sum_{j=1}^{n_i} \sigma_j^{(i)} - \sum_{j=n_i+1}^{p_i} \sigma_j^{(i)} + \sum_{j=1}^{m_i} \rho_j^{(i)} - \sum_{j=m_i+1}^{q_i} \rho_j^{(i)} > 0$ *with* $|arg(z_i)| < \frac{\pi}{2}\Omega_i', i = 1, \ldots, r$, *then, there exists an equality of multivariable H-function as*

$$\begin{aligned} &H_{0,0:p_1,q_1;\ldots;p_r,q_r}^{0,0:m_1,n_1;\ldots;m_r,n_r} \left[\begin{matrix} z_1 \\ \cdot \\ \cdot \\ \cdot \\ z_r \end{matrix} \Bigg| \begin{matrix} - : \left(c_j^{(1)} : \sigma_j^{(1)}\right)_{1,p_1}; \ldots; \left(c_j^{(r)} : \sigma_j^{(r)}\right)_{1,p_r} \\ - : \left(d_j^{(1)} : \rho_j^{(1)}\right)_{1,q_1}; \ldots; \left(d_j^{(r)} : \rho_j^{(r)}\right)_{1,q_r} \end{matrix} \right] \\ &= H_{0,0:q_1,p_1;\ldots;q_r,p_r}^{0,0:n_1,m_1;\ldots;n_r,m_r} \left[\begin{matrix} (z_1)^{-1} \\ \cdot \\ \cdot \\ \cdot \\ (z_r)^{-1} \end{matrix} \Bigg| \begin{matrix} - : \left(1-d_j^{(1)} : \rho_j^{(1)}\right)_{1,q_1}; \ldots; \left(1-d_j^{(r)} : \rho_j^{(r)}\right)_{1,q_r} \\ - : \left(1-c_j^{(1)} : \sigma_j^{(1)}\right)_{1,p_1}; \ldots; \left(1-c_j^{(r)} : \sigma_j^{(r)}\right)_{1,p_r} \end{matrix} \right]. \end{aligned} \tag{3.1}$$

Proof. Consider the property of Fox's H-function given in ([10], [12, p.11])

$$H_{p,q}^{m,n} \left[z \Bigg| \begin{matrix} (a_j : A_j)_{1,p} \\ (b_j : B_j)_{1,q} \end{matrix} \right] = H_{q,p}^{n,m} \left[\frac{1}{z} \Bigg| \begin{matrix} (1-b_j : B_j)_{1,q} \\ (1-a_j : A_j)_{1,p} \end{matrix} \right]. \tag{3.2}$$

Then, use the result (3.2) in right hand side of the result (1.7), we obtain the equality (3.1).

***Lemma* 3.2.** *If theory and conditions of the* ***Theorem* 2.1** *are followed, then the fundamental solution of the problem* (2.1) - (2.5) *exists as*

$$u(x_1, \ldots, x_r, t) = \prod_{i=1}^{r} G(\alpha, \beta_i, \theta_i; t, x_i). \tag{3.3}$$

Here in Eqn. (3.3), $\forall i = 1, 2, \ldots, r$, *the functions* $G(\alpha, \beta_i, \theta_i; t, x_i)$ *satisfy the integral equations* $G(\alpha, \beta_i, \theta_i; t, x_i) = \int_{-\infty}^{\infty} G(\alpha, \beta_i, \theta_i; t, \zeta_i)\delta(\zeta_i - x_i) d\zeta_i$, $\delta(x)$ *is a well known Dirac - delta function.*

Proof. We make an appeal to the ***Theorem* 2.1**, to find r-generalized diffusion and wave equations with initial and boundary values given in the Eqns. (2.6) - (2.7). Now, take Laplace and Fourier transformation of the equations given in (2.6) - (2.7) and find the relations in following transformed form

$$\bar{V}(\varrho_i, s) = \frac{s^{\alpha-1}}{s^\alpha + \psi_{\beta_i}^{\theta_i}(\varrho_i)}\bar{\varphi}(\varrho_i)\forall i = 1, 2, \ldots, r. \tag{3.4}$$

Again, by Eqn. (3.4) consider that a function $\bar{G}(\alpha, \beta_i, \theta_i; s, \varrho_i) = \frac{s^{\alpha-1}}{s^\alpha+\psi_{\beta_i}^{\theta_i}(\varrho_i)}$, which by taking inverse Laplace transformation as on applying the formulae of [12, p.49, Eqns. (2.12) and (2.14)], and by Eqn. (1.13), we obtain

$$\bar{G}(\alpha, \beta_i, \theta_i; t, \varrho_i) = E_\alpha(-\psi_{\beta_i}^{\theta_i}(\varrho_i)t^\alpha) = E_\alpha(-(\varrho_i)^{\beta_i} \exp[\omega\frac{\theta_i\pi}{2}]t^\alpha)\forall i = 1, 2, \ldots, r. \tag{3.5}$$

Now, use Eqn. (3.5) in the Eqn. (3.4) and thus apply Fourier convolution, we obtain the integral equations

$$V(x_i, t) = \int_{-\infty}^{\infty} G(\alpha, \beta_i, \theta_i; t, \zeta_i)\varphi(\zeta_i - x_i)d\zeta_i. \tag{3.6}$$

Further, in Eqn. (3.6) if we take $\varphi(x_i) = \delta(x_i)\forall i = 1, 2, \ldots, r, (\delta(x)$ is well known Dirac delta function), then, $V(x_i, t) = G(\alpha, \beta_i, \theta_i; t, x_i)$, and then, we find another integral equations

$$G(\alpha, \beta_i, \theta_i; t, x_i) = \int_{-\infty}^{\infty} G(\alpha, \beta_i, \theta_i; t, \zeta_i)\delta(\zeta_i - x_i)d\zeta_i. \tag{3.7}$$

Since $\forall i = 1, 2, \ldots, r$, the functions $G(\alpha, \beta_i, \theta_i; t, x_i)$ satisfy all Eqns. and conditions given in ***Theorem* 2.1**, hence $G(\alpha, \beta_i, \theta_i; t, x_i)$ are general Green functions. Finally, by our assumption (2.2) and Eqns. (3.6) and (3.7), we obtain the fundamental solution (3.3).

***Lemma* 3.3.** *If $\forall i = 1, 2, \ldots, r, |\theta_i| \leq \min\{\beta_i, 2 - \beta_i\}, 0 < \beta_i \leq 2$. Then, for $t > 0, x_i > 0$, the Green functions $G(\alpha, \beta_i, \theta_i; t, x_i), i = 1, 2, \ldots, r$, defined in **Lemma 3.2**, are expressed by several Fox's H -functions as*

$$G(\alpha, \beta_i, \theta_i; t, x_i) = \begin{cases} \frac{1}{\beta_i x_i} H_{3,3}^{1,2}\left[(x_i)^{-1}(t)^{\frac{\alpha}{\beta_i}} \Big| \begin{matrix} (0, \frac{1}{\beta_i}), (0, 1), (0, \frac{\beta_i-\theta_i}{2\beta_i}) \\ (0, \frac{1}{\beta_i}), (0, \frac{\alpha}{\beta_i}), (0, \frac{\beta_i-\theta_i}{2\beta_i}) \end{matrix}\right], \\ \text{when } \beta_i < \alpha, \text{also, singular at } x_i = 0; \\ \frac{1}{\beta_i x_i} H_{3,3}^{2,1}\left[(x_i)(t)^{\frac{-\alpha}{\beta_i}} \Big| \begin{matrix} (1, \frac{1}{\beta_i}), (1, \frac{\alpha}{\beta_i}), (1, \frac{\beta_i-\theta_i}{2\beta_i}) \\ (1, \frac{1}{\beta_i}), (1, 1), (1, \frac{\beta_i-\theta_i}{2\beta_i}) \end{matrix}\right], \\ \text{when, } \beta_i > \alpha, \text{ also singular at } x_i \to \infty. \end{cases} \tag{3.8}$$

Also, when, $\alpha = \beta_i, G(\alpha, \beta_i, \theta_i; t, x_i)$ is singular at $x_i = t\forall i = 1, \ldots, r$.

Proof. In Eqn. (3.5), by the relation $\psi_{\beta_i}^{\theta_i}(\varrho_i) = \psi_{\beta_i}^{-\theta_i}(-\varrho_i)$, there is a symmetric relation $\psi_{\beta_i}^{\theta_i}(-x_i) = \psi_{\beta_i}^{-\theta_i}(x_i)\ \forall x_i > 0,\ i = 1, 2, \ldots, r$. Thus, on taking inverse Fourier transformation of the function $\bar{G}(\alpha, \beta_i, \theta_i; t, \varrho_i)$ of (3.5), we find (see also [11])

$$G(\alpha, \beta_i, \theta_i; t, x_i) = \begin{cases} \frac{1}{\pi}\int_0^\infty E_\alpha\left(-(\varrho_i)^{\beta_i}\exp\left[\omega\frac{\theta_i\pi}{2}\right]t^\alpha\right)\cos\varrho_i x_i d\varrho_i, \\ \frac{1}{\pi}\int_0^\infty E_\alpha\left(-(\varrho_i)^{\beta_i}\exp\left[\omega\frac{\theta_i\pi}{2}\right]t^\alpha\right)\sin\varrho_i x_i d\varrho_i. \end{cases} \tag{3.9}$$

Now, to achieve the result of above ***Lemma* 3.3**, we have to define the Mellin transformation of a sufficiently well behaved function $f(\varrho)$ (see [14]) as

$$\mathfrak{M}\{f(\varrho); s\} = f^*(s) = \int_0^{+\infty} f(\varrho)\varrho^{s-1}d\varrho, \gamma_1 < \Re(s) = \gamma < \gamma_2, \tag{3.10}$$

and the inverse Mellin transformation as

$$\mathfrak{M}^{-1}\{f^*(s);\varrho\} = f(\varrho) = \frac{1}{2\pi\omega}\int_{\gamma-\omega\infty}^{\gamma+\omega\infty} f^*(s)\varrho^{-s}ds, \varrho > 0. \tag{3.11}$$

Also, the properties of juxtaposition $\overset{\mathfrak{M}}{\leftrightarrow}$ of a function $f(\varrho)$ with its Mellin transform $f^*(s)$ are

$$f(a\varrho) \underset{\leftrightarrow}{\mathfrak{M}} a^{-s}f^*(s), a > 0, \tag{3.12}$$

$$(\varrho^p) \underset{\leftrightarrow}{\mathfrak{M}} \frac{1}{|p|}f^*(\frac{s}{p}), p \neq 0. \tag{3.13}$$

The Parseval's formula is given by

$$\int_0^{+\infty} f(\varrho)g(\varrho)d\varrho = \frac{1}{2\pi\omega}\int_{\gamma-\omega\infty}^{\gamma+\omega\infty} f^*(s)g^*(1-s)ds. \tag{3.14}$$

Then by (3.10), (3.11) and (3.12), for $x > 0$ there exist the formulae for trigonometric functions

$$\mathfrak{M}\{\sin(x\varrho);s\} = x^{-s}\Gamma(s)\sin(\frac{\pi s}{2}), -1 < \Re(s) < 1] \tag{3.15}$$

and

$$\mathfrak{M}\{\cos(x\varrho);s\} = x^{-s}\Gamma(s)\cos(\frac{\pi s}{2}),\ 0 < \Re(s) < 1. \tag{3.16}$$

By above formula (3.10) for $\varrho_i > 0,\ t > 0$, the Mellin transform of Eqn. (3.5) is written by

$$G^*(s_i) = \int_0^{+\infty} E_\alpha(-(\varrho_i)^{\beta_i}\exp[\omega\frac{\theta_i\pi}{2}]t^\alpha)(\varrho_i)^{s_i-1}d\varrho_i, \tag{3.17}$$

for $\gamma_{i,1} < \Re(s_i) < \gamma_{i,2}\ \forall i = 1,2,\ldots,r$.

Then, for $\gamma_{i,1} < \Re(s_i) < \gamma_{i,2}\ \forall i = 1,2,\ldots,r$, by the Mellin transforms of trigonometrical functions (3.15), (3.16) in the Eqn. (3.9), and using the Eqn. (3.17) and thus on applying the above Parseval's formula (3.14), there exists a relation

$$G(\alpha,\beta_i,\theta_i;t,x_i) = \begin{cases} \frac{1}{\pi x_i}\frac{1}{2\pi\omega}\int_{\gamma_i-\omega\infty}^{\gamma_i+\omega\infty} G^*(s_i)\Gamma(1-s_i)\sin(\frac{\pi s_i}{2})(x_i)^{s_i}ds_i, \\ \qquad x_i > 0, 0 < \gamma_i < 1; \\ \frac{1}{\pi x_i}\frac{1}{2\pi\omega}\int_{\gamma_i-\omega\infty}^{\gamma_i+\omega\infty} G^*(s_i)\Gamma(1-s_i)\cos(\frac{\pi s_i}{2})(x_i)^{s_i}ds_i, \\ \qquad x_i > 0, 0 < \gamma_i < 2. \end{cases} \tag{3.18}$$

Again, use above juxtaposition (3.13) in Eqn. (3.5), and apply the techniques of [11], we write

$$G^*(s_i) = \frac{1}{\beta_i}\frac{\Gamma(\frac{s_i}{\beta_i})\Gamma(1-\frac{s_i}{\beta_i})}{\Gamma(1-\frac{\alpha s_i}{\beta_i})}\exp[-\omega\frac{\pi s_i\theta_i}{2\beta_i}](t)^{\frac{-\alpha s_i}{\beta_i}}, \tag{3.19}$$

for $t > 0, |\theta_i| \leq \{2-\alpha\}, 0 < \Re(s_i) < \beta_i \forall i = 1,2,\ldots,\ r$. (See also [10]).

Therefore on applying (3.19) in the Eqns. of (3.18), we find a result for $x_i > 0, t > 0, |\theta_i| \leq \{2-\alpha\}, 0 < \Re(s_i) < \beta_i \forall i = 1,\ldots,r$, in the form

$$\begin{aligned} G(\alpha,\beta_i,\theta_i;t,x_i) = \\ &\frac{1}{\pi\beta_i x_i}\frac{1}{2\pi\omega}\int_{\gamma_i-\omega\infty}^{\gamma_i+\omega\infty}\frac{\Gamma\left(\frac{s_i}{\beta_i}\right)\Gamma\left(1-\frac{s_i}{\beta_i}\right)}{\Gamma\left(1-\frac{\alpha s_i}{\beta_i}\right)}\Gamma(1-s_i)\sin\left(\frac{\pi s_i}{2}\right)\cos\left(\frac{\pi\theta_i s_i}{2\beta_i}\right)\left(x_i(t)^{\frac{-\alpha}{\beta_i}}\right)^{s_i}ds_i \\ &-\frac{1}{\pi\beta_i x_i}\frac{1}{2\pi\omega}\int_{\gamma_i-\omega\infty}^{\gamma_i+\omega\infty}\frac{\Gamma\left(\frac{s_i}{\beta_i}\right)\Gamma\left(1-\frac{s_i}{\beta_i}\right)}{\Gamma\left(1-\frac{\alpha s_i}{\beta_i}\right)}\Gamma(1-s_i)\cos\left(\frac{\pi s_i}{2}\right)\sin\left(\frac{\pi\theta_i s_i}{2\beta_i}\right)\left(x_i(t)^{\frac{-\alpha}{\beta_i}}\right)^{s_i}ds_i. \end{aligned} \tag{3.20}$$

The Eqn. (3.20) gives us the formula, for $x_i > 0, t > 0, |\theta_i| \le \{2-\alpha\}, 0 < \Re(s_i) < \beta_i \forall i = 1, \ldots, r$, as

$$\text{(3.21)} \quad G(\alpha, \beta_i, \theta_i; t, x_i) = \frac{1}{\pi\beta_i x_i} \frac{1}{2\pi\omega} \int_{\gamma_i - \omega\infty}^{\gamma_i + \omega\infty} \frac{\Gamma(\frac{s_i}{\beta_i})\Gamma(1-\frac{s_i}{\beta_i})}{\Gamma(1-\frac{\alpha s_i}{\beta_i})} \Gamma(1-s_i) \sin\left(\left\{\frac{(\beta_i-\theta_i)\pi s_i}{2\beta_i}\right\}\right) (x_i(t)^{\frac{-\alpha}{\beta_i}})^{s_i} ds_i.$$

Now in Eqn. (3.21), set $\left\{\frac{(\beta_i-\theta_i)}{2\beta_i}\right\} = \lambda_i$, and then, use the property of Gamma function that $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$, we find the Green function for $x_i > 0, t > 0, |\theta_i| \le \{2-\alpha\}, 0 < \Re(s_i) < \beta_i$, in the Mellin - Barnes contour integrals $\forall i = 1, \ldots, r$, as

$$\text{(3.22)} \qquad G(\alpha, \beta_i, \theta_i; t, x_i) = \frac{1}{\beta_i x_i} \frac{1}{2\pi\omega} \int_{\gamma_i - \omega\infty}^{\gamma_i + \omega\infty} \frac{\Gamma\left(\frac{s_i}{\beta_i}\right)\Gamma\left(1-\frac{s_i}{\beta_i}\right)\Gamma(1-s_i)}{\Gamma\left(1-\frac{\alpha s_i}{\beta_i}\right)\Gamma(\lambda_i s_i)\Gamma(1-\lambda_i s_i)} \left(x_i(t)^{\frac{-\alpha}{\beta_i}}\right)^{s_i} ds_i.$$

Finally, in right hand side of Eqn. (3.22) make an application of the definitions (1.1) - (1.4) (for $r = 1$), and again use the relation (3.2), we obtain all results of Eqn. (3.8).

$\forall i = 1, 2, \ldots, r$, in the complex s_i- planes, the right hand side of Eqn. (3.22) shows that there is an extension of probability distribution in the ranges $\{0 < \beta_i \le 2\} \cap \{0 < \alpha \le 1\}$ and $\{1 < \alpha \le \beta_i \le 2\}$, thus we present following Main Theorem:

Theorem 3.1. *If* $|\theta_i| \le min\{\beta_i, 2-\beta_i\}, 0 < \beta_i \le 2; -\infty < x_i < +\infty; \forall i = 1, 2, \ldots, r$, *such that the ranges are* $\{0 < \beta_i \le 2\} \cap \{0 < \alpha \le 1\}$ *and* $\{1 < \alpha \le \beta_i \le 2\}$, *then the Eqn.* (2.1) *on imposing the conditions* (2.2)-(2.5)*, have following three solutions in terms of a multivariable Green function, defined by* $u(x_1, \ldots, x_r, t) = G^{\theta_1,\ldots,\theta_r}_{\alpha,\beta_1,\ldots,\beta_r}(x_1, \ldots, x_r, t) = \prod_{i=1}^{r} G(\alpha, \beta_i, \theta_i; t, x_i)$*, as*

Case 1 *If* $\beta_i < \alpha$ *and* $x_i > 0 \; \forall i = 1, 2, \ldots, r$*, then*

$$\text{(3.23)} \quad G^{\theta_1,\ldots,\theta_r}_{\alpha,\beta_1,\ldots,\beta_r}(x_1, \ldots, x_r, t) = \left\{\prod_{i=1}^{r} \frac{1}{\beta_i x_i}\right\}$$
$$\times H^{0,0:1,2;\ldots;1,2}_{0,0:3,3;\ldots;3,3} \left[\begin{matrix} (x_1)^{-1}(t)^{\frac{\alpha}{\beta_1}} \\ \cdot \\ \cdot \\ \cdot \\ (x_r)^{-1}(t)^{\frac{\alpha}{\beta_r}} \end{matrix} \left| \begin{matrix} - : \left(0, \frac{1}{\beta_1}\right), (0, 1), \left(0, \frac{\beta_1-\theta_1}{2\beta_1}\right); \ldots; \left(0, \frac{1}{\beta_r}\right), (0, 1), \left(0, \frac{\beta_r-\theta_r}{2\beta_r}\right) \\ - : \left(0, \frac{1}{\beta_1}\right), \left(0, \frac{\alpha}{\beta_1}\right), \left(0, \frac{\beta_1-\theta_1}{2\beta_1}\right); \ldots; \left(0, \frac{1}{\beta_r}\right), \left(0, \frac{\alpha}{\beta_r}\right), \left(0, \frac{\beta_r-\theta_r}{2\beta_r}\right) \end{matrix} \right.\right],$$

also, singular at $x_i = 0$.

Case 2 *If* $\beta_i > \alpha$ *and* $x_i > 0 \; \forall i = 1, 2, \ldots, r$*, then*

$$\text{(3.24)} \quad G^{\theta_1,\ldots,\theta_r}_{\alpha,\beta_1,\ldots,\beta_r}(x_1, \ldots, x_r, t) = \left\{\prod_{i=1}^{r} \frac{1}{\beta_i x_i}\right\}$$
$$\times H^{0,0:2,1;\ldots;2,1}_{0,0:3,3;\ldots;3,3} \left[\begin{matrix} x_1(t)^{\frac{-\alpha}{\beta_1}} \\ \cdot \\ \cdot \\ \cdot \\ x_r(t)^{\frac{-\alpha}{\beta_r}} \end{matrix} \left| \begin{matrix} - : \left(1, \frac{1}{\beta_1}\right), \left(1, \frac{\alpha}{\beta_1}\right), \left(1, \frac{\beta_1-\theta_1}{2\beta_1}\right); \ldots; \left(1, \frac{1}{\beta_r}\right), \left(1, \frac{\alpha}{\beta_r}\right), \left(1, \frac{\beta_r-\theta_r}{2\beta_r}\right) \\ - : \left(1, \frac{1}{\beta_1}\right), (1, 1), \left(1, \frac{\beta_1-\theta_1}{2\beta_1}\right); \ldots; \left(1, \frac{1}{\beta_r}\right), (1, 1), \left(1, \frac{\beta_r-\theta_r}{2\beta_r}\right) \end{matrix} \right.\right],$$

also singular at $x_i \to \infty$.

***Case* 3** *If* $\alpha = \beta_i\ \forall i = 1,\ldots,r,$

$$G^{\theta_1,\ldots,\theta_r}_{\alpha,\beta_1,\ldots,\beta_r}(x_1,\ldots,x_r,t) \text{ is singular at } x_i = t,\ \ \forall\ i = 1,2,...,r. \tag{3.25}$$

Proof. Consider the Eqn. (2.2) in the Eqns. (2.1), (2.3), (2.4) and (2.5) and thus use the theory and results obtained in the ***Lemmas* 3.2** and **3.3**, we obtain the Green functions $G(\alpha,\beta_i,\theta_i;t,x_i)\ \forall i = 1,2,\ldots,r$. Then, make an application of Eqn. (1.7) and ***Lemma* 3.1** in the statement of the ***Theorem* 3.1**, we find the multivariable Green functions in three cases given in the Eqns. (3.23), (3.24) and (3.25).

4 Various multiple diffusions

In Eqn. (2.1), if we set $\theta_i = 0\forall i = 1,\ldots,r,$ *the integro - differential Riesz-Feller operators*, ${}_{x_i}D^{\beta_i}_{\theta_i}$, $\forall i = 1,\ldots,r$, in the domain $(x_1,\ldots,x_r)$, become symmetric operators with respect to the variables $x_1,\ldots,x_r$, as

$$ {}_{x_i}D^{\beta_i}_{0} = -\left(-\frac{d^2}{dx_i^2}\right)^{\frac{\beta_i}{2}}\ \forall i = 1,\ldots,r, \tag{4.1}$$

and thus in Eqn. (4.1), we interpret that

$$-|\varrho|^{\beta_i} = -(\varrho^2)^{\frac{\beta_i}{2}}\forall i = 1,\ldots,r.$$

Now, on specializing the values of the parameters α,β_i and $\theta_i\forall i = 1,\ldots,r$, by above results of the ***Theorem* 3.1**, we discuss following multiple diffusions:

(I) In ***Theorem* 3.1**, if we set $\alpha = 1,\beta_i = 2$ and $\theta_i = 0,\ \forall i = 1,\ldots,r$, then there exists, a standard diffusion equation $\frac{\partial u}{\partial t} = \sum_{i=1}^{r}\frac{\partial^2 u}{\partial x_i^2},\ \ u = u(x_1,x_2,...,x_r,t)$, for $t > 0,\ x_i > 0,\ \forall i = 1,\ldots,r$, and thus in ***Case* 2** of this ***Theorem* 3.1**, the multivariable Green function $G^{\theta_1,\ldots,\theta_r}_{\alpha,\beta_1,\ldots,\beta_r}(x_1,\ldots,x_r,t)$ becomes as

$$G^{0,\ldots,0}_{1,2,\ldots,2}(x_1,\ldots,x_r,t) = \left\{\frac{(t)^{\frac{-1}{2}}}{2}\right\}^r H^{0,0:1,0;\ldots;1,0}_{0,0:1,1;\ldots;1,1}\left[\begin{matrix} x_1(t)^{\frac{-1}{2}} \\ \cdot \\ \cdot \\ \cdot \\ x_r(t)^{\frac{-1}{2}}\end{matrix}\left|\begin{matrix} - : \left(\frac{1}{2},\frac{1}{2}\right);\ldots;\left(\frac{1}{2},\frac{1}{2}\right) \\ - : (0,1);\ldots;(0,1)\end{matrix}\right.\right]. \tag{4.2}$$

Again as we are familiar with the results

$$H^{1,0}_{1,1}\left[(x(t)^{\frac{-1}{2}}\left|\begin{matrix}\left(\frac{1}{2},\frac{1}{2}\right)\\(0,1)\end{matrix}\right.\right] = \frac{1}{\sqrt{\pi}}e^{-\frac{\left(x(t)^{\frac{-1}{2}}\right)^2}{4}} = \frac{1}{\sqrt{\pi}}e^{-\frac{x^2}{4t}}. \tag{4.3}$$

Hence, use the results (1.7), (4.2) and (4.3), we find a multivariable normal distribution as

$$G^{0,\ldots,0}_{1,2,\ldots,2}(x_1,\ldots,x_r,t) = \left\{\frac{1}{2\sqrt{\pi t}}\right\}^r \exp\left[-\left\{\frac{(x_1)^2}{4t}+\ldots+\frac{(x_r)^2}{4t}\right\}\right]. \tag{4.4}$$

(II) When, $0 < \beta_i < 2, |\theta_i| \leq \min\{\beta_i, 2-\beta_i\}\ \forall i = 1,\ldots,r$; and $\alpha = 1$ then this case is called space *fractional diffusion*, in which two situations are arised for $x_i > 0, \forall i = 1,\ldots,r,$

(a) $0 < \beta_i < 1, |\theta_i| \leq \beta_i\forall i = 1,\ldots,r$, $\alpha = 1$, so that by ***Theorem* 3.1**, ***Case* 1**, we get

$$G^{\theta_1,\ldots,\theta_r}_{1,\beta_1,\ldots,\beta_r}(x_1,\ldots,x_r,t) = \left\{\prod_{i=1}^{r}\frac{(t)^{\frac{-1}{\beta_i}}}{\beta_i}\right\} \tag{4.5}$$

$$\times H^{0,0:1,1;\ldots;1,1}_{0,0:2,2;\ldots;2,2}\left[\begin{matrix}(x_1)^{-1}(t)^{\frac{1}{\beta_1}}\\ \cdot\\ \cdot\\ \cdot\\ (x_r)^{-1}(t)^{\frac{1}{\beta_r}}\end{matrix}\left|\begin{matrix}-:(1,1),\left(\frac{\beta_1-\theta_1}{2\beta_1},\frac{\beta_1-\theta_1}{2\beta_1}\right);\ldots;(1,1),\left(\frac{\beta_r-\theta_r}{2\beta_r},\frac{\beta_r-\theta_r}{2\beta_r}\right)\\ -:\left(\frac{1}{\beta_1},\frac{1}{\beta_1}\right),\left(\frac{\beta_1-\theta_1}{2\beta_1},\frac{\beta_1-\theta_1}{2\beta_1}\right);\ldots;\left(\frac{1}{\beta_r},\frac{1}{\beta_r}\right),\left(\frac{\beta_r-\theta_r}{2\beta_r},\frac{\beta_r-\theta_r}{2\beta_r}\right)\end{matrix}\right.\right].$$

(b) $1 < \beta_i < 2, |\theta_i| \leq \{2-\beta_i\ \forall i = 1, \ldots, r, \alpha = 1$, so that by ***Theorem* 3.1**, ***Case* 2**), we get

$$(4.6)\quad G^{\theta_1,\ldots,\theta_r}_{1,\beta_1,\ldots,\beta_r}(x_1,\ldots,x_r,t) = \left\{\prod_{i=1}^{r}\frac{(t)^{-\frac{1}{\beta_i}}}{\beta_i}\right\}$$

$$\times H^{0,0:1,1;\ldots;1,1}_{0,0:2,2;\ldots;2,2}\left[\begin{matrix}x_1(t)^{\frac{-1}{\beta_1}}\\ \cdot\\ \cdot\\ \cdot\\ x_r(t)^{\frac{-1}{\beta_r}}\end{matrix}\left|\begin{matrix}-:\left(\frac{\beta_1-1}{\beta_1},\frac{1}{\beta_1}\right),\left(\frac{\beta_1+\theta_1}{2\beta_1},\frac{\beta_1-\theta_1}{2\beta_1}\right);\ldots;\left(\frac{\beta_r-1}{\beta_r},\frac{1}{\beta_r}\right),\left(\frac{\beta_r+\theta_r}{2\beta_r},\frac{\beta_r-\theta_r}{2\beta_r}\right)\\ -:(0,1),\left(\frac{\beta_1+\theta_1}{2\beta_1},\frac{\beta_1-\theta_1}{2\beta_1}\right);\ldots;(0,1),\left(\frac{\beta_r+\theta_r}{2\beta_r},\frac{\beta_r-\theta_r}{2\beta_r}\right)\end{matrix}\right.\right].$$

(III) *When, $\beta_i = 2, \theta_i = 0, \forall i = 1, \ldots, r$; and $0 < \alpha < 2$, then, this case is called time fractional diffusion,* in which for $x_i > 0, \forall i = 1, \ldots, r$, ***Theorem* 3.1**, ***Case* 2**) arises and hence, we find
(4.7)

$$G^{0,\ldots,0}_{\alpha,2,\ldots,2}(x_1,\ldots,x_r,t) = \left\{\frac{(t)^{-\frac{\alpha}{2}}}{2}\right\}^r H^{0,0:1,0;\ldots;1,0}_{0,0:1,1;\ldots;1,1}\left[\begin{matrix}x_1(t)^{\frac{-\alpha}{2}}\\ \cdot\\ \cdot\\ \cdot\\ x_r(t)^{\frac{-\alpha}{2}}\end{matrix}\left|\begin{matrix}-:\left(\frac{2-\alpha}{2},\frac{\alpha}{2}\right);\ldots;\left(\frac{2-\alpha}{2},\frac{\alpha}{2}\right)\\ -:(0,1);\ldots;(0,1)\end{matrix}\right.\right].$$

(IV) As the case discussed in Eqn. (4.7), where put $r = 2$ and $0 < \alpha < 1$, then this becomes generalized anomalous diffusion of Kumar, Pathan and Yadav [7] of which another solution in the form of Green function is found by

$$(4.8)\qquad G^{0,0}_{\alpha,2,2}(x_1,x_2,t) = \left\{\frac{(t)^{-\alpha}}{4}\right\} H^{0,0:1,0;1,0}_{0,0:1,1;1,1}\left[\begin{matrix}x_1(t)^{\frac{-\alpha}{2}}\\ x_2(t)^{\frac{-\alpha}{2}}\end{matrix}\left|\begin{matrix}-:\left(\frac{2-\alpha}{2},\frac{\alpha}{2}\right);\left(\frac{2-\alpha}{2},\frac{\alpha}{2}\right)\\ -:(0,1);(0,1)\end{matrix}\right.\right].$$

In the similar manner, by ***Theorem* 3.1**, we also obtain the Green function solution of the Eqn. (2.1) of the case for $r = 2$, $0 < \alpha < 1, 0 < \beta_1 < 1, 1 < \beta_2 < 2$; of the anomalous diffusion problem due to Kumar, Pathan and Srivastava [6]. For further directions of the researches in this field, we omit them.

5 Special cases.

In this section, we specialize the values of the parameters involving in the results (4.2) to (4.7) of ***Section* 4** (where set $r = 1$, then take $\beta_1 = \beta$ and $\theta_1 = \theta$) and then, we obtain various diffusions as studied and derived by many authors to them (see [4], [8], [10] and [12]) given in followings:

(i) when $\theta = 0, \beta = 2, \alpha = 1$, by (4.2) to (4.4), there exists a normal diffusion.

(ii) when $\theta = 0, 0 < \beta < 2, \alpha = 1$, by (4.5) and (4.6), there exists a space fractional diffusion.

(iii) when $\theta = 0, 0 < \alpha < 2, \beta = 2$, by (4.7), there exists a time fractional diffusion.

(iv) when $\theta \leq min\{\beta, 2-\beta\}, 0 < \alpha = \beta < 2$, there exists a neutral fractional diffusion (see [10], Eqn. (4.3)).

6 Conclusions

A solution of multidimensional in space fractional and time fractional diffusion and wave problem, in terms of the multivariable H-function involving a multiple contour integral of Mellin - Barnes type [9], defined by ([19], [20]), is obtained by imposing certain conditions and the relations given in Eqns. (2.2) - (2.5) in the Eqn. (2.1). The obtained solution is converted into a classical multivariable Green function by which various multiple diffusions as particular cases are discussed in section 4 on specializing of the parameters involving in multidimensional space fractional operators with Caputo time fractional derivative, in which ***Case*** I) represents the standard diffusion, ***Case*** II) represents space fractional diffusion problem in which two cases are raised and in ***Case*** III) time diffusion problem is analyzed. In ***Case*** IV), the fundamental solution of anomalous diffusion problem is obtained. On putting $r = 1$, the special cases are checked by the results in one dimensional in space-time fractional derivatives of previous work of many researchers in the literature for example ([4], [6], [7], [8], [10], [12]).

Acknowledgement. Authors are very much grateful to the referees for their valuable suggestions and comments to prepare the paper in the present form.

References

[1] R. C. Singh Chandel, R. D. Agrawal and H. Kumar, The multivariable H-function of Srivastava and Panda and its applications in a problem on a electrostatic potential in spherical region, *Jour. Maulana Azad College Tech.* **23** (1990), 39 - 46.

[2] R. C. Singh Chandel, R. D. Agrawal and H. Kumar, An integral involving sine functions, exponential functions, the Kampé de Fériet function and the multivariable H-function of Srivastava and Panda and its application in a potential problem on a circular disk, *Pure Appl. Math. Sci.* **XXXV** (1-2) (March, 1992), 59 - 69.

[3] R. C. Singh Chandel, R. D. Agrawal and H. Kumar, Fourier series involving the multivariable H-function of Srivastava and Panda, *Indian J. Pure Appl. Math.* **23** (5) (May, 1992), 343 - 357.

[4] Kai Diethelm, *The Analysis of Fractional Differential Equations: An Application-Oriented Exposition Using Differential Operators of Caputo Type*, Springer, New York, 2010.

[5] C. Fox, The G and H functions as symmetrical Fourier kernels, *Trans. Amer. Math. Soc.* **98** (1961), 395 - 429.

[6] H. Kumar, M. A. Pathan and H. Srivastava, A general solution of a space-time fractional anomalous diffusion problem using the series of bilateral eigen-functions, *Commun. Korean Math. Soc*. **29** (1) (2014), 173 - 185.

[7] H. Kumar, M. A. Pathan and P. Yadav, Series solutions for initial - value problems of time fractional generalized anomalous diffusion equations, *Le Matematiche* **LXVII** (Fasc. II) (2012), 217 - 229.

[8] F. Mainardi, Yu. Luchko and G.Pagnini, The fundamental solution of the space-time fractional diffusion equation, *Fract.Cal. Appl. Anal*. **4** (2) (2001), 153-192.

[9] F. Mainardi and G.Pagnini, Salvatore Pincherle, The pioneer of the Mellin-Barnes integrals, *J. Comput. Appl. Math.* **153** (2003), 331-342.

[10] F. Mainardi, G. Pagnini and R. K. Saxena, Fox H functions in fractional diffusion, *J. Comput. Appl. Math.* **178** (2005), 321-331.

[11] O.I. Marichev, *Handbook of Integral Transforms of Higher Transcendental Functions, Theory and Algorithmic Tables*, Ellis Horwood, Chichester, 1983.

[12] A.M. Mathai, R.K. Saxena and H. J. Haubold, *The H-function; Theory and Applications*, Springer, New York, 2010.
[13] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.
[14] A. D. Poularikas, *"The Mellin Transforms"; The Handbook of Formulas and Tables for Signal Processing*, Boca Raton: CRC Press LLC, 1999.
[15] M. Saigo and R. K. Saxena, Unified fractional integral formulas for the multivariable H-function III, *J. Frac. Calc.* **20** (2001), 45 - 68.
[16] S. G. Samko, A. A. Kilbas and O. I. Marichev, *Fractional Integrals and Derivatives*, translated from the 1987 Russian original, Gordon and Breach, Yverdon, 1993.
[17] H. M. Srivastava, A contour integral involving Fox's H-function, *Indian J. Math.* **14** (1972), 1 - 6.
[18] H. M. Srivastava and M. A. Hussain, Fractional integration of H - function of several variables, *Computers Math. Appl.* **30** (1995), 73 - 85.
[19] H. M. Srivastava and R. Panda, Expansion theorems for the H - function of several complex variables, *J. Reine Angew Math.* **288** (1976 a), 129 - 145.
[20] H. M. Srivastava and R. Panda, Some expansion theorems and generating relations for the H-function of several complex variables II, *Comment Math. Univ. St. Paul* **25** (2) (1976 b), 167 - 197.

Jñānābha, Vol. 50(1) (2020), 265-278
(Dedicated to Honor Professor H.M. Srivastava on His 80th Birth Anniversary Celebrations)

ITERATION PRINCIPLE FOR IVPS OF NONLINEAR FIRST ORDER IMPULSIVE DIFFERENTIAL EQUATIONS

By
Bapurao C. Dhage
"Kasubai", Gurukul Colony, Thodga Road
Ahmedpur - 413515, Dist. Latur, Maharashtra, India.
Email:bcdhage@gmail.com
(*Received : April* 09, 2019 *; Revised: June* 23, 2020)
DOI: https://doi.org/10.58250/jnanabha.2020.50125

Abstract

In this paper we prove the existence and approximation theorems for the initial value problems of first order nonlinear impulsive differential equations under certain mixed partial Lipschitz and partial compactness type conditions. Our results are based on the Dhage monotone iteration principle embodied in a hybrid fixed point theorem of Dhage involving the sum of two monotone order preserving operators in a partially ordered Banach space. The novelty of the present approach lies the fact that we obtain an algorithm for the solution. Our abstract main result is also illustrated by indicating a numerical example.
2010 Mathematics Subject Classifications: 47H07, 47H10, 34A12, 34A45
Keywords and phrases: Impulsive differential equation; Dhage iteration method; hybrid fixed point principle; existence and approximate solution.

1 Introduction

It is well-known that many of the dynamical systems in the universe involve the jumps or discontinuities due to to impulses at finite number of places in a given period of time and impulsive differential equations are the mathematical models to describe such phenomena precisely. The existence and uniqueness theory for nonlinear impulsive differential equations have received much attention during the last decade, however the theory of approximation of the solutions to such equations is relatively rare in the literature. The dynamical systems, which involve the jumps or discontinuities at finite number of points are modeled on the nonlinear impulsive differential equations. The sudden changes in the dynamic systems for a short period of time can betterly be discussed with he help of impulsive differential equations. The importance of such impulsive differential equations in the dynamic systems as well as exhaustive account of various topics related to this problem may be found in the research monographs of Samoilenko and Perestyuk [21], Lakshmikantam et al [20] and the references therein. The existence theorems so far discussed in the literature for such impulsive differential equations involve either the use of usual Lipschitz or compactness type condition on the nonlinearities and which are considered to be very strong conditions in the subject of nonlinear analysis. Here in the present set up of new Dhage monotone iteration method, we do not need usual Lipschitz and compactness type conditions but require only partial Lipschitz and partial compactness type conditions of the nonlinearity and the existence as well as approximation of the solutions is obtained under certain monotonic conditions. We claim that the results of this paper are new to the literature on impulsive differential equations.

Let $\mathbb{R}$ be the real line and let $J = [0, T]$ be a closed and bounded interval in $\mathbb{R}$. Let $t_0, \dots t_{p+1}$ be the points in J such that $0 = t_0 < t_1 < \cdots, < t_p < t_{p+1} = T$ and let $J' = J \setminus \{t_1, \dots, t_p\}$. Denote

$J_j = (t_j, t_{j+1}) \subset J$ for $j = 1, 2, \dots, p$. By $X = C(J, \mathbb{R})$ and $L^1(J, \mathbb{R})$ we denote respectively the spaces of continuous and Lebesgue integrable real-valued functions defined on J.

Now, given a function $h \in L^1(J, \mathbb{R}_+)$, consider the initial value problem (in short IVP) for the first order impulsive differential equation (in short IDE)

$$\left.\begin{aligned} x'(t) + h(t)x(t) &= f(t, x(t)), \;\; t \in J \setminus \{t_1, \dots, t_p\}, \\ x(t_j^+) - x(t_j^-) &= \mathcal{I}_j(x(t_j)), \\ x(0) &= x_0 \in \mathbb{R}, \end{aligned}\right\} \tag{1.1}$$

where, the limits $x(t_j^+)$ and $x(t_j^-)$ are respectively the right and left limit of x at $t = t_j$ such that $x(t_j) = x(t_j^-)$, $\mathcal{I}_j \in C(\mathbb{R}, \mathbb{R})$, $\mathcal{I}_j(x(t_j))$ are the impulsive effects at the points $t = t_j$, $j = 1, \dots, p$ and $f : J \times \mathbb{R} \to \mathbb{R}$ is such that f is continuous on $J' = J - \{t_1, ..., t_p\}$, and there exist the limits

$$\lim_{t \to t_j^-} f(t, u) = f(t_j, u) \quad \text{ and } \quad \lim_{t \to t_{j+}} f(t, u), \;\; u \in \mathbb{R},$$

for each $j = 1, \dots, p$.

By a *impulsive solution* of the IDE (1.1) we mean a function $x \in PC^1(J, \mathbb{R})$ that satisfies the differential equation and the conditions in (1.1), where $PC^1(J, \mathbb{R})$ is the space of piecewise continuously differentiable real-valued functions defined on J.

The IDE (1.1) has already been discussed in the literature under continuity and compactness type conditions of the function f for various aspects of the solutions. The existence and unqueness theorems for the IDE (1.1) may be proved using the classical hybrid fixed point theorems Schauder and Banach given in Dhage [8] and references therein. Here in the present study, we discuss the IDE (1.1) for existence and approximate impulsive solution under partial Lipschit and partial compactness type conditions via Dhage iteration method based on a hybrid fixed point theorems of Dhage [3, 4].

2 Auxiliary Results

Throughout this paper, unless otherwise mentioned, let $(E, \preceq |\cdot\|)$ denote a partially ordered normed linear space. Two elements x and y in E are said to be **comparable** if either the relation $x \preceq y$ or $y \preceq x$ holds. A non-empty subset C of E is called a **chain** or **totally ordered** if all the elements of C are comparable. It is known that E is **regular** if $\{x_n\}$ is a nondecreasing (resp. nonincreasing) sequence in E such that $x_n \to x^*$ as $n \to \infty$, then $x_n \preceq x^*$ (resp. $x_n \succeq x^*$) for all $n \in \mathbb{N}$. The conditions guaranteeing the regularity of E may be found in Heikkilä and Lakshmikantham [19] and the references therein.

We need the following definitions (see Dhage [2, 3, 4] and the references therein) in what follows.

A mapping $\mathcal{T} : E \to E$ is called **isotone** or **monotone nondecreasing** if it preserves the order relation $\preceq$, that is, if $x \preceq y$ implies $\mathcal{T}x \preceq \mathcal{T}y$ for all $x, y \in E$. Similarly, $\mathcal{T}$ is called **monotone nonincreasing** if $x \preceq y$ implies $\mathcal{T}x \succeq \mathcal{T}y$ for all $x, y \in E$. Finally, $\mathcal{T}$ is called **monotonic** or simply **monotone** if it is either monotone nondecreasing or monotone nonincreasing on E. A mapping $\mathcal{T} : E \to E$ is called **partially continuous** at a point $a \in E$ if for given $\epsilon > 0$ there exists a $\delta > 0$ such that $\|\mathcal{T}x - \mathcal{T}a\| < \epsilon$ whenever x is comparable to a and $\|x - a\| < \delta$. $\mathcal{T}$ is called partially continuous on E if it is partially continuous at every point of it. It is clear that if $\mathcal{T}$ is partially continuous on E, then it is continuous on every chain C contained in E and vice-versa. A non-empty subset S of the partially ordered metric space E is called **partially bounded** if every chain C in S is bounded. A mapping $\mathcal{T}$ on a partially ordered metric space E into itself is called **partially bounded** if $\mathcal{T}(E)$ is a partially bounded subset of E. $\mathcal{T}$ is called **uniformly partially**

bounded if all chains C in $\mathcal{T}(E)$ are bounded by a unique constant. A non-empty subset S of the partially ordered metric space E is called **partially compact** if every chain C in S is a compact subset of E. A mapping $\mathcal{T} : E \to E$ is called **partially compact** if every chain C in $\mathcal{T}(E)$ is a relatively compact subset of E. $\mathcal{T}$ is called **uniformly partially compact** if $\mathcal{T}$ is a uniformly partially bounded and partially compact operator on E. $\mathcal{T}$ is called **partially totally bounded** if for any bounded subset S of E, $\mathcal{T}(S)$ is a partially totally bounded subset of E. If $\mathcal{T}$ is partially continuous and partially totally bounded, then it is called **partially completely continuous** on E.

***Remark* 2.1.** *Suppose that $\mathcal{T}$ is a monotone operator on E into itself. Then $\mathcal{T}$ is a partially bounded or partially compact on E if $\mathcal{T}(C)$ is a bounded or compact subset of E for each chain C in E.*

***Definition* 2.1** (Dhage [5, 6], Dhage and Dhage [15])**.** *The order relation $\preceq$ and the metric d on a non-empty set E are said to be $\mathcal{D}$-**compatible** if $\{x_n\}$ is a monotone sequence, that is, monotone nondecreasing or monotone nonincreasing sequence in E and if a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges to x^* implies that the original sequence $\{x_n\}$ converges to x^*. Similarly, given a partially ordered normed linear space $(E, \preceq, \|\cdot\|)$, the order relation $\preceq$ and the norm $\|\cdot\|$ are said to be $\mathcal{D}$-compatible if $\preceq$ and the metric d defined through the norm $\|\cdot\|$ are $\mathcal{D}$-compatible. A subset S of E is called **Janhavi set** if the order relation $\preceq$ and the metric d or the norm $\|\cdot\|$ are $\mathcal{D}$-compatible in S. In particular, if $S = E$, then E is called a **Janhavi metric space** or **Janhavi Banach space**.*

Clearly, the set $\mathbb{R}$ of real numbers with usual order relation $\leq$ and the norm defined by the absolute value function $|\cdot|$ has this property. Similarly, the finite dimensional Euclidean space $\mathbb{R}^n$ with usual componentwise order relation and the standard norm possesses the compatibility property and so is a **Janhavi Banach space**.

***Definition* 2.2.** *An upper semi-continuous and monotone nondecreasing function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is called a $\mathcal{D}$-function provided $\psi(0) = 0$. A monotone operator $\mathcal{T} : E \to E$ is called nonlinear partial $\mathcal{D}$-contraction if there exists a $\mathcal{D}$-function $\psi_{\mathcal{T}}$ such that*

$$\|\mathcal{T}x - \mathcal{T}y\| \leq \psi_{\mathcal{T}}(\|x - y\|) \tag{2.1}$$

for all comparable elements $x, y \in E$, where $0 < \psi_{\mathcal{T}}(r) < r$ for $r > 0$.

In particular, if $\psi_{\mathcal{T}}(r) = k\,r$, $k > 0$, $\mathcal{T}$ is called a partial Lipschitz operator with a Lipschitz constant k and moreover, if $0 < k < 1$, $\mathcal{T}$ is called a linear partial contraction on E with the contraction constant k.

The **Dhage monotone iteration principle** or **Dhage monotone iteration method** embodied in the following applicable hybrid fixed point theorems of Dhage [3] in a partially ordered normed linear space is used as a key tool for our work contained in this paper. The details of the Dhage monotone iteration principle or method are given in Dhage [5, 6, 7], Dhage *et al.* [13, 14], Dhage and Otrocol [17] and the references therein.

***Theorem* 2.1** (Dhage [3])**.** *Let $(E, \preceq, \|\cdot\|)$ be a partially ordered Banach space and let $\mathcal{T} : E \to E$ be a monotone nondecreasing and nonlinear partial $\mathcal{D}$-contraction. Suppose that there exists an element $x_0 \in E$ such that $x_0 \preceq \mathcal{T}x_0$ or $x_0 \succeq \mathcal{T}x_0$. If $\mathcal{T}$ is continuous or E is regular, then $\mathcal{T}$ has a unique comparable fixed point x^* and the sequence $\{\mathcal{T}^n x_0\}$ of successive iterations converges monotonically to x^*. Moreover, the fixed point x^* is unique if every pair of elements in E has a lower bound or an upper bound.*

***Theorem* 2.2** (Dhage [3]). *Let $(E, \preceq, \|\cdot\|)$ be a regular partially ordered complete normed linear space and let every compact chain C in E be Janhavi set. Let $\mathcal{A}, \mathcal{B} : E \to E$ be two monotone nondecreasing operators such that*

(a) $\mathcal{A}$ is partially bounded and nonlinear partial $\mathcal{D}$-contraction,
(b) $\mathcal{B}$ is partially continuous and partially compact, and
(c) there exists an element $x_0 \in E$ such that $x_0 \preceq \mathcal{A}x_0 + \mathcal{B}x_0$ or $x_0 \succeq \mathcal{A}x_0 + \mathcal{B}x_0$.

Then the hybrid operator equation $\mathcal{A}x + \mathcal{B}x = x$ has a solution x^* in E and the sequence $\{x_n\}$ of successive iterations defined by $x_{n+1} = \mathcal{A}x_n + \mathcal{B}x_n$, n=0,1,..., converges monotonically to x^*.

***Remark* 2.2.** *The condition that every compact chain of E is Janhavi set holds if every partially compact subset of E possesses the compatibility property with respect to the order relation $\preceq$ and the norm $\|\cdot\|$ in E.*

***Remark* 2.3.** *1 We remark that hypothesis (a) of Theorem 2.2 implies that the operator $\mathcal{A}$ is partially continuous and consequently both the operators $\mathcal{A}$ and $\mathcal{B}$ in the theorem are partially continuous on E. The regularity of E in above Theorems 2.1 and 2.2 may be replaced with a stronger continuity condition respectively of the operators $\mathcal{T}$ and $\mathcal{A}$ and $\mathcal{B}$ on E which are the results proved in Dhage [2, 3].*

3 Existence and Approximation Theorem

Let $X_j = C(J_j, \mathbb{R})$ denote the class of continuous real-valued functions on the interval $J_j = (t_j, t_{j+1})$. Denote by $PC(J, \mathbb{R})$ the space of piecewise continuous real-valued functions on J defined by

$$PC(J, \mathbb{R}) = \Big\{x \in X_j \mid x(t_j^-) \text{ and } x(t_j^+) \text{ exists for } j = 1, \dots, p; \quad \text{and } x(t_j^-) = x(t_j)\Big\}. \tag{3.1}$$

Define a supremum norm $\|\cdot\|$ in $PC(J, \mathbb{R})$ by

$$\|x\|_{PC} = \sup_{t \in J} |x(t)| \tag{3.2}$$

and define the order cone K in $PC(J, \mathbb{R})$ by

$$K = \{x \in PC(J, \mathbb{R}) \mid x(t) \geq 0 \text{ for all } t \in J\}, \tag{3.3}$$

which is obviously a normal cone in $PC(J, \mathbb{R})$. Now, define the order relation $\preceq$ in $PC(J, \mathbb{R})$ by

$$x \preceq y \iff y - x \in K \tag{3.4}$$

which is equivalent to

$$x \preceq y \iff x(t) \leq y(t) \text{ for all } t \in J.$$

Clearly, $(PC(J, \mathbb{R}), K)$ becomes a regular ordered Banach space with respect to the above norm and order relation in $PC(J, \mathbb{R})$ and every compact chain C in $PC(J, \mathbb{R})$ is Janhavi set in view of the following lemmas proved in Dhage [6, 7].

***Lemma* 3.1** (Dhage [6, 7]). *Every ordered Banach space (E, K) is regular.*

***Lemma* 3.2** (Dhage [6, 7]). *Every partially compact subset S of an ordered Banach space (E, K) is a Janhavi set in E.*

We need the following definition in what follows.

***Definition* 3.1.** *A function $u \in PC^1(J, \mathbb{R})$ is said to be a lower impulsive solution of the IDE (1.1) if it satisfies*

$$\left.\begin{aligned} u'(t) + h(t)u(t) \le f(t, u(t)),\ \ t \in J \setminus \{t_1, \dots, t_p\}, \\ u(t_j^+) - u(t_j^-) \le \mathcal{I}_j(u(t_j)), \\ u(0) \le x_0 \in \mathbb{R}, \end{aligned}\right\}$$

for $j = 1, 2, ..., p$. Similarly, a function $v \in PC^1(J, \mathbb{R})$ is called an upper impulsive solution of the IDE (1.1) if the above inequality is satisfied with reverse sign.

We consider the following set of assumptions in what follows:

(H_1) The impulsive functions $\mathcal{I}_j \in C(\mathbb{R}, \mathbb{R})$ are bounded on X with bounds $M_{\mathcal{I}_j}$ for each $j = 1, \dots, p.$,

(H_2) There exists a constants $L_{\mathcal{I}_j} > 0$ such that

$$0 \le \mathcal{I}_j x - \mathcal{I}_j y \le L_{\mathcal{I}_j}(x - y)$$

for all $x, y \in \mathbb{R}$, $x \ge y$, where $j = 1, \dots, p$.

(H_3) The function f is bounded on $J \times \mathbb{R}$ with bound M_f.

(H_4) $f(t, x)$ is nondecreasing in x for each $t \in J$.

(H_5) There exists a constant $L_f > 0$ such that

$$0 \le f(t, x) - f(t, y) \le L_f(x - y)$$

for all $t \in J$ and $x, y \in \mathbb{R}$, $x \ge y$.

(H_6) The IDE (1.1) has a lower impulsive solution $u \in PC^1(J, \mathbb{R})$.

Below we prove some useful results in what follows.

***Lemma* 3.3.** *Given $\sigma \in L^1(J, \mathbb{R})$, a function $x \in PC(J, \mathbb{R})$ is a impulsive solution to the IDE*

$$\left.\begin{aligned} x'(t) + h(t)x(t) = \sigma(t),\ \ t \in J \setminus \{t_1, \dots, t_p\}, \\ x(t_j^+) - x(t_j^-) = \mathcal{I}_j(x(t_j)), \\ x(0) = x_0, \end{aligned}\right\} \tag{3.5}$$

if and only if it is an impulsive solution of the impulsive integral equation

$$x(t) = x_0\, e^{-H(t)} + \sum_{0<t_j<t} k(t, t_j)\mathcal{I}_j(x(t_j)) + \int_0^t k(t, s)\sigma(s)\, ds,\ \ t \in J, \tag{3.6}$$

where the kernel function k is given by

$$k(t, s) = e^{-H(t)+H(s)} \quad \textit{and} \quad H(t) = \int_0^t h(s)\, ds. \tag{3.7}$$

Proof. First note that the integral in $H(t)$ is a continuous and nonnegative real-valued function on J. Therefore, we have $H(t) > 0$ on J provided h is not an identically zero function. Otherwise $H(t) \equiv 0$ on J. Moreover, we have $H(t^-) = H(t) = H(t^+)$ for all $t \in J$.

First suppose that x is an impulsive solution of the IDE (3.5) on J. Then, we have

$$\left.\begin{aligned} \left(e^{H(t)}x(t)\right)' = e^{H(t)}\sigma(t),\ \ t \in J \setminus \{t_1, \dots, t_p\}, \\ x(t_j^+) - x(t_j^-) = \mathcal{I}_j(x(t_j)), \\ x(0) = x_0 \in \mathbb{R}, \end{aligned}\right\} \tag{3.8}$$

for $j = 1, 2, \dots, p$.

From the theory of integral calculus, it follows that

$$\begin{aligned} e^{H(t_1^-)}x(t_1^-) - e^{H(0)}x(0) &= \int_0^{t_1} \left(e^{H(s)}x(s)\right)' \, ds \\ e^{H(t_2^-)}x(t_2^-) - e^{H(t_1^+)}x(t_1^+) &= \int_{t_1}^{t_2} \left(e^{H(s)}x(s)\right)' \, ds \\ &\vdots \\ e^{H(t)}x(t) - e^{H(t_p^+)}x(t_p^+) &= \int_{t_p}^{t} \left(e^{H(s)}x(s)\right)' \, ds. \end{aligned}$$

Summing up the above equations,

$$e^{H(t)}x(t) - \sum_{0<t_j<t} e^{H(t_j)}\mathcal{I}_j(x(t_j)) = x_0 + \int_0^t e^{H(s)}h(s)\, ds,$$

or

$$x(t) = x_0 e^{-H(t)} + \sum_{0<t_j<t} k(t, t_j)\mathcal{I}_j(x(t_j)) + \int_0^t k(t, s)\sigma(s)\, ds.$$

for $t \in J$.

Conversely, suppose that x is an impulsive solution of the impulsive integral equation (3.6). Obviously x satisfies the initial and jump conditions given in (3.5). By the definition of the kernel function k, we obtain

$$e^{H(t)}x(t) = x_0 + \sum_{0<t_j<t} e^{H(t_j)}\mathcal{I}_j(x(t_j)) + \int_0^t e^{H(s)}\sigma(s)\, ds \tag{3.9}$$

for all $t \in J$. Since $\sigma \in L^1(J, \mathbb{R})$, one has $\int_0^t e^{H(s)}\sigma(s)\, ds \in AC(J, \mathbb{R})$. So, a direct differentiation of (3.8) yields,

$$\left(e^{H(t)}x(t)\right)' = e^{H(t)}\sigma(t),$$

or

$$x'(t) + h(t)x(t) = \sigma(t),$$

for $t \in J$ satisfying $x(0) = x_0$ and (3.3). The proof of the lemma is complete.

***Remark* 3.1.** *We note that the kernel function $k(t, s)$ is continuous and nonnegative real-valued function on $J \times J$. Moreover,* $\sup_{t>s} k(t, s) \le 1$.

***Lemma* 3.4.** *Given $\sigma \in L^1(J, \mathbb{R})$, if there is a function $u \in PC(J, \mathbb{R})$ satisfying the impulsive differential inequality*

$$\left.\begin{aligned} u'(t) + h(t)u(t) \le \sigma(t),\ \ t \in J \setminus \{t_1, \dots, t_p\}, \\ u(t_j^+) - u(t_j^-) \le \mathcal{I}_j(u(t_j)), \\ u(0) \le x_0, \end{aligned}\right\} \tag{3.10}$$

then it satisfies the impulsive integral inequality

$$u(t) \le x_0 e^{-H(t)} + \sum_{0<t_j<t} k(t, t_j)\mathcal{I}_j(u(t_j)) + \int_0^t k(t, s)\sigma(s)\, ds,\ \ t \in J, \tag{3.11}$$

where the kernel function k is defined by the expression (3.7) *on $J \times J$.*

Proof. Proceeding as in the proof of Lemma 3.3, we obtain

$$\left.\begin{aligned} \left(e^{H(t)}u(t)\right)' \le e^{H(t)}\sigma(t), \quad t \in J \setminus \{t_1, \dots, t_p\}, \\ u(t_j^+) - u(t_j^-) \le \mathcal{I}_j(u(t_j)), \\ u(0) \le x_0, \end{aligned}\right\}$$

for $j = 1, 2, \dots, p$.

From the theory of integral calculus, it follows that

$$\begin{aligned} e^{H(t_1^-)}u(t_1^-) - e^{H(0)}u(0) &= \int_0^{t_1} \left(e^{H(s)}u(s)\right)' ds \\ e^{H(t_2^-)}u(t_2^-) - e^{H(t_1^+)}u(t_1^+) &= \int_{t_1}^{t_2} \left(e^{H(s)}u(s)\right)' ds \\ &\vdots \\ e^{H(t)}u(t) - e^{H(t_p^+)}u(t_p^+) &= \int_{t_p}^{t} \left(e^{H(s)}u(s)\right)' ds. \end{aligned}$$

Summing up the above equations,

$$e^{H(t)}u(t) - \sum_{0<t_j<t} e^{H(t_j)}\mathcal{I}_j(u(t_j)) \le u_0 + \int_0^t e^{H(s)}h(s)\,ds,$$

or

$$u(t) \le x_0 e^{-H(t)} + \sum_{0<t_j<t} k(t, t_j)\mathcal{I}_j(u(t_j)) + \int_0^t k(t, s)\sigma(s)\,ds$$

for $t \in J$ and the proof of the lemma is complete.

Similarly, we have the following useful result concerning the impulsive differential inequality with reverse sign.

***Lemma* 3.5.** *Given $\sigma \in L^1(J, \mathbb{R})$, if there is a function $v \in PC(J, \mathbb{R})$ satisfying the impulsive differential inequality*

$$\left.\begin{aligned} v'(t) + h(t)v(t) \ge \sigma(t), \quad t \in J \setminus \{t_1, \dots, t_p\}, \\ v(t_j^+) - v(t_j^-) \ge \mathcal{I}_j(v(t_j)), \\ v(0) \ge x_0, \end{aligned}\right\} \tag{3.12}$$

then it satisfies the impulsive integral inequality

$$v(t) \ge x_0 e^{-H(t)} + \sum_{0<t_j<t} k(t, t_j)\mathcal{I}_j(v(t_j)) + \int_0^t k(t, s)\sigma(s)\,ds, \quad t \in J, \tag{3.13}$$

where the kernel function k is defined by the expression (3.7) *on $J \times J$.*

***Theorem* 3.1.** *Suppose that the hypotheses (H_1) through (H_4) and (H_6) hold. Furthermore, if $\sum_{j=1}^{p} L_{\mathcal{I}_j} < 1$, then the IDE (1.1) has a impulsive solution x^* defined on J and the sequence $\{x_n\}$ of successive approximations defined by*

$$\begin{aligned} x_0(t) &= u(t), \\ x_{n+1}(t) &= x_0 e^{-H(t)} + \sum_{0<t_j<t} k(t, t_j)\mathcal{I}_j(x_n(t_j)) \\ &\quad + \int_0^t k(t, s) f(s, x_n(s))\,ds \end{aligned} \tag{3.14}$$

for all $t \in J$, converges monotonically to x^*.

Proof. Set $E = PC(J,\mathbb{R})$. Then, by Lemma 3.2, every compact chain C in E possesses the compatibility property with respect to the norm $\|\cdot\|_{PC}$ and the order relation $\preceq$ so that every compact chain C is a Janhavi set in E.

Now, by Lemma 3.3, the IDE (1.1) is equivalent to the nonlinear impulsive integral equation

$$x(t) = x_0 e^{-H(t)} + \sum_{0<t_j<t} k(t,t_j)\mathcal{I}_j(x(t_j)) + \int_0^t k(t,s)f(s,x(s))\,ds \tag{3.15}$$

for all $t \in J$.

Define two operators $\mathcal{A}$ and $\mathcal{B}$ on E by

$$\mathcal{A}x(t) = \sum_{0<t_j<t} k(t,t_j)\mathcal{I}_j(x(t_j)),\ t \in J, \tag{3.16}$$

and

$$\mathcal{B}x(t) = x_0 e^{-H(t)} + \int_0^t k(t,s)f(s,x(s))\,ds,\ t \in J. \tag{3.17}$$

From the continuity of the integral, it follows that $\mathcal{A}$ and $\mathcal{B}$ define the operators $\mathcal{A},\mathcal{B} : E \to E$ and the impulsive integral equation (3.15) is transformed into the operator equation as

$$\mathcal{A}x(t) + \mathcal{B}x(t) = x(t),\quad t \in J. \tag{3.18}$$

Now, the problem of finding the impulsive solution of the IDE (1.1) is just reduced to finding impulsive solution of the operator equation (3.18) on J. We show that the operators $\mathcal{A}$ and $\mathcal{B}$ satisfy all the conditions of Theorem 2.2 in a series of following steps.

Step I: *$\mathcal{A}$ and $\mathcal{B}$ are nondecreasing on E.*

Let $x, y \in E$ be such that $x \succeq y$. Then, by hypothesis (H_2), we get

$$\mathcal{A}x(t) = \sum_{0<t_j<t} k(t,t_j)\mathcal{I}_j(x(t_j)) \geq \sum_{0<t_j<t} k(t,t_j)\mathcal{I}_j(y(t_j)) = \mathcal{A}y(t),$$

for all $t \in J$. By definition of the order relation in E, we obtain $\mathcal{A}x \succeq \mathcal{A}y$ and a fortiori, $\mathcal{A}$ is a nondecreasing operator on E. Similarly, using hypothesis (H_4),

$$\begin{aligned}\mathcal{B}x(t) &= x_0 e^{-H(t)} + \int_0^t k(t,s)f(s,x(s))\,ds \\ &\geq x_0 e^{-H(t)} + \int_0^t k(t,s)f(s,x(s))\,ds \\ &= \mathcal{B}y(t),\end{aligned}$$

for all $t \in J$. Therefore, the operator $\mathcal{B}$ is also nondecreasing on E into itself.

Step II: *$\mathcal{A}$ is partially bounded and partially contraction on E.*

Let $x \in E$ be arbitrary. Then by (H_1) we have

$$|\mathcal{A}x(t)| \leq \left|\sum_{0<t_j<t} k(t,t_j)\mathcal{I}_j(x(t_j))\right| \leq \sum_{0<t_j<t} \left|k(t,t_j)\right| \left|\mathcal{I}_j(y(t_j))\right| \leq \sum_{j=1}^{p} M_{\mathcal{I}_j}$$

for all $t \in J$. Taking the supremum over t, we obtain $\|\mathcal{A}x\| \leq \sum_{j=1}^{p} M_{\mathcal{I}_j}$ for all $x \in E$, so $\mathcal{A}$ is a bounded operator on E. This further implies that $\mathcal{A}$ is partially bounded on E.

Next, let $x, y \in E$ be such that $x \succeq y$. Then by (H_2), we have

$$|\mathcal{A}x(t) - \mathcal{A}y(t)| \leq \left|\sum_{0<t_j<t} k(t,t_j)\mathcal{I}_j(x(t_j)) - \sum_{0<t_j<t} k(t,t_j)\mathcal{I}_j(y(t_j))\right|$$

$$
\begin{aligned}
&\leq \left| \sum_{0<t_j<t} k(t,t_j)[\mathcal{I}_j(x(t_j)) - \mathcal{I}_j(x(t_j))] \right| \\
&\leq \sum_{0<t_j<t} k(t,t_j)\, L_{\mathcal{I}_j}[x(t_j) - x(t_j)] \\
&\leq L\,\|x-y\|_{PC}\,,
\end{aligned}
$$

for all $t \in J$, where $L = \sum_{j=1}^{p} L_{\mathcal{I}_j} < 1$. Taking the supremum over t, we obtain

$$
\|\mathcal{A}x - \mathcal{A}y\|_{PC} \leq L\,\|x-y\|_{PC}
$$

for all $x, y \in E$ with $xx \succeq y$. Hence $\mathcal{A}$ is a partially contraction on E which also implies that $\mathcal{A}$ is partially continuous on E.

Step III: *$\mathcal{B}$ is partially continuous on E.*

Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in a chain C such that $x_n \to x$, for all $n \in \mathbb{N}$. Then

$$
\begin{aligned}
\lim_{n\to\infty} \mathcal{B}x_n(t) &= \lim_{n\to\infty} \left[x_0 e^{-H(t)} + \int_0^t k(t,s) f(s, x_n(s))\, ds \right] \\
&= x_0 e^{-H(t)} + \int_0^t k(t,s) \left[\lim_{n\to\infty} f(s, x_n(s)) \right] ds \\
&= x_0 e^{-H(t)} + \int_0^t k(t,s) f(s, x(s))\, ds \\
&= \mathcal{B}x(t),
\end{aligned}
$$

for all $t \in J$. This shows that $\mathcal{B}x_n$ converges to $\mathcal{B}x$ pointwise on J.

Now, we show that $\{\mathcal{B}x_n\}_{n\in\mathbb{N}}$ is a quasi-equicontinuous sequence of functions in E. Let $\tau_1, \tau_2 \in (t_j, t_{j+1}] \cap J$, $j = 1, \ldots, p$. Then, we have that

$$
\begin{aligned}
&\left| \mathcal{B}x_n(\tau_2) - \mathcal{B}x_n(\tau_1) \right| \\
&= \Big| x_0 e^{-H(\tau_1)} + \int_0^{\tau_1} k(\tau_1, s) f(s, x_n(s))\, ds \\
&\qquad - x_0 e^{-H(\tau_2)} - \int_0^{\tau_2} k(\tau_2, s) f(s, x_n(s))\, ds \Big| \\
&\leq \left| \int_0^{\tau_1} k(\tau_1, s) f(s, x_n(s))\, ds - \int_0^{\tau_2} k(\tau_2, s) f(s, x_n(s))\, ds \right| \\
&\qquad + \left| x_0 e^{-H(\tau_1)} - x_0 e^{-H(\tau_2)} \right| \\
&\leq \left| \int_0^{\tau_1} k(\tau_1, s) f(s, x_n(s))\, ds - \int_0^{\tau_1} k(\tau_2, s) f(s, x_n(s))\, ds \right| \\
&\qquad + \left| \int_0^{\tau_1} k(\tau_2, s) f(s, x_n(s))\, ds - \int_0^{\tau_2} k(\tau_2, s) f(s, x_n(s))\, ds \right| \\
&\qquad + \left| x_0 e^{-H(\tau_1)} - x_0 e^{-H(\tau_2)} \right| \\
&\leq |x_0| \left| e^{-H(\tau_1)} - e^{-H(\tau_2)} \right| \\
&\qquad + \int_0^T \left| k(\tau_1, s) - k(\tau_2, s) \right| \left| f(s, x_n(s)) \right| ds
\end{aligned}
$$

$$
\begin{aligned}
&\quad + \left| \int_{\tau_2}^{\tau_1} |k(\tau_2, s)|\, |f(s, x_n(s))|\, ds \right| \\
&\le |x_0| \left| e^{-H(\tau_1)} - e^{-H(\tau_2)} \right| \\
&\quad + M_f \int_0^T \left| k(\tau_1, s) - k(\tau_2, s) \right| ds \\
&\quad + M_f |\tau_1 - \tau_2| \\
&\to 0 \quad \text{as} \quad \tau_2 \to \tau_1,
\end{aligned}
$$

uniformly for all $n \in \mathbb{N}$. This shows that the sequence $\{\mathcal{B}x_n\}$ of functions is quasi- equicontinuous and so convergence $\mathcal{B}x_n \to \mathcal{B}x$ is uniform in view of the arguments given in Samoilenko and Perestyuk [21], Lakshmikantam *et.al* [20]. Hence $\mathcal{B}$ is partially continuous operator on E into itself.

Step IV: *$\mathcal{B}$ is partially compact operator on E.*

Let C be an arbitrary chain in E. We show that $\mathcal{B}(C)$ is uniformly bounded and quasi-equicontinuous set in E. First we show that $\mathcal{B}(C)$ is uniformly bounded. Let $y \in \mathcal{B}(C)$ be any element. Then there is an element $x \in C$ such that $y = \mathcal{B}x$. By hypothesis (H_3)

$$
\begin{aligned}
|y(t)| &= |\mathcal{B}x(t)| \\
&= \left| x_0 e^{-H(t)} + \int_0^t k(t, s) f(s, x(s))\, ds \right| \\
&\le \left| x_0 e^{-H(t)} \right| + \int_0^T |k(t, s)|\, |f(s, x(s))|\, ds \\
&\le \left| x_0 e^{-H(t)} \right| + M_f \int_0^T k(t, s)\, ds \\
&\le |x_0| + M_f T \\
&= r,
\end{aligned}
$$

for all $t \in J$. Taking the supremum over t we obtain $\|y\|_{PC} \le \|\mathcal{B}x\|_{PC} \le r$, for all $y \in \mathcal{B}(C)$. Hence $\mathcal{B}(C)$ is uniformly bounded subset of functions E. Next we show that $\mathcal{B}(C)$ is an quasi-equicontinuous set in E. Let $\tau_1, \tau_2 \in (t_j, t_{j+1}] \cap J$, $j = 1, \ldots, p$. Then proceeding with the arguments as in Step II, it can be shown that $\mathcal{B}(C)$ is an quasi-equicontinuous subset of functions in E. So $\mathcal{B}(C)$ is a uniformly bounded and quasi-equicontinuous set of functions in E and hence it is compact in view of Arzelá-Ascoli theorem (see Samoilenko and Perestyuk [21], Lakshmikantam *et al.* [20]). Consequently $\mathcal{B} : E \to E$ is a partially compact operator of E into itself.

Step V: *u is a lower impulsive solution of the operator equation $x = \mathcal{A}x + \mathcal{B}x$.*

By hypothesis (H_4), the IDE (1.1) has a lower impulsive solution u defined on J. Then, we have

$$
\left.
\begin{aligned}
u'(t) + h(t)u(t) \le f(t, u(t)), \quad t \in J \setminus \{t_1, \ldots, t_p\}, \\
u(t_j^+) - u(t_j^-) \le \mathcal{I}_j(u(t_j)), \\
u(0) \le x_0.
\end{aligned}
\right\} . \tag{3.19}
$$

Now, by a direct application of the impulsive differential inequality established in Lemma 3.4 yields that

$$
u(t) \le u_0\, e^{-H(t)} + \sum_{0 < t_j < t} k(t, t_j) \mathcal{I}_j(u(t_j)) + \int_0^t k(t, s) f(s, u(s))\, ds \tag{3.20}
$$

for $t \in J$. Furthermore, from definitions of the operators $\mathcal{A}$ and $\mathcal{B}$ it follows that $u(t) \leq \mathcal{A}u(t) + \mathcal{B}u(t)$ for all $t \in J$. Hence $u \leq \mathcal{A}u + \mathcal{B}u$. Thus the operators $\mathcal{A}$ and $\mathcal{B}$ satisfy all the conditions of Theorem 2.2 and so the operator equation $\mathcal{A}x + \mathcal{B}x = x$ has a impulsive solution. Consequently the integral equation and a fortiori, the IDE (1.1) has a impulsive solution x^* defined on J. Furthermore, the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations defined by (3.14) converges monotonically to x^*. This completes the proof.

Next, we prove the uniqueness theorem for the IDE on the interval J.

***Theorem* 3.2.** *Suppose that the hypotheses (H_1)-(H_2) and (H_5)-(H_6) hold. Furthermore, if $\sum_{j=1}^{p} L_{\mathcal{I}_j} + L_f < 1$, then the IDE (1.1) has a unique impulsive solution solution x^* defined on J and the sequence $\{x_n\}$ of successive approximations defined by* (3.14) *converges monotonically to x^*.*

Proof. Set $E = PC(J, \mathbb{R})$. Then, every pair of elements in $PC(J, \mathbb{R})$ has a lower bound as well as an upper bound so it is a lattice with respect to the order relation $\leq$ in E.

Now, by Lemma 3.3, the IDE (1.1) is equivalent to the nonlinear impulsive integral equation (3.15). Define two operators $\mathcal{A}$ and $\mathcal{B}$ on E by (3.16) and (3.17). Now, consider the mapping $\mathcal{T} : E \to E$ defined by

$$\mathcal{T}x(t) = \mathcal{A}x(t) + \mathcal{B}x(t), \ t \in J. \tag{3.21}$$

Then the impulsive integral equation (3.6) is reduced to the operator equation as

$$\mathcal{T}x(t) = x(t), \ t \in J. \tag{3.22}$$

Now, proceeding with the arguments as in the proof of Theorem 3.1 it can shown that the operator $\mathcal{A}$ is a partial Lipschitzian with Lipschitz constant $L_{\mathcal{A}} = \sum_{j=1}^{p} L_{\mathcal{I}_j}$. Similarly, we show that $\mathcal{B}$ is also a Lipschitzian on E into itself. Let $x, y \in E$ be such that $x \geq y$. Then, by hypothesis (H_5), one has

$$\begin{aligned} \left|\mathcal{B}x(t) - \mathcal{B}y(t)\right| &= \left|\int_0^t k(t,s)f(s,x(s))\,ds - \int_0^t k(t,s)f(s,y(s))\,ds\right| \\ &\leq \int_0^t |k(t,s)| \left|f(s,x(s)) - f(s,y(s))\right| ds \\ &\leq L_f \int_0^t \left|x(t) - y(t)\right| ds \\ &\leq L_f\, T\, \|x-y\|_{PC} \end{aligned}$$

for all $t \in J$ and $x, y \in E$. Taking the supremum over t in the above inequality, we obtain

$$\|\mathcal{B}x - \mathcal{B}y\|_{PC} \leq L_{\mathcal{B}} \|x-y\|_{PC}$$

for all $x, y \in E$, $x \geq y$, where $L_{\mathcal{B}} = L_f\, T$. This shows that $\mathcal{B}$ is again a partial Lipschitzian operator on E into itself with a Lipschitz constant $L_{\mathcal{B}}$. Next, by definition of the operator $\mathcal{T}$, one has

$$\|\mathcal{T}x - \mathcal{T}y\|_{PC} \leq \|\mathcal{A}x - \mathcal{A}y\|_{PC} + \|\mathcal{B}x - \mathcal{B}y\|_{PC} \leq (L_{\mathcal{A}} + L_{\mathcal{B}}) \|x-y\|_{PC}$$

for all $x, y \in E$, $x \geq y$, where $L_{\mathcal{A}} + L_{\mathcal{B}} = \sum_{j=1}^{p} L_{\mathcal{I}_j} + L_f T < 1$. Hence $\mathcal{T}$ is a partial contraction operator on E into itself. Since the hypothesis (H_6) holds, it is proved as in the step V of the proof of Theorem 3.1 that the operator equation (3.22) has a lower solution u in E. Then, by an application of Theorem 2.1, we obtain that the operator equation (3.22) and consequently the IDE (1.1) has a unique impulsive solution x^* and the sequence $\{x_n\}$ of successive approximations defined by (3.15) converges monotonically to x^*. This competes the proof.

***Remark* 3.2.** *The conclusion of Theorems 3.1 and 3.2 also remains true if we replace the hypothesis (H_6) with the following one.*

(H_7) The IDE (1.1) has an upper impulsive solution $v \in PC(J, \mathbb{R})$.

The proofs of the existence theorems under this new hypothesis are obtained using the similar arguments with appropriate modifications.In this case we invoke the use of Lemma 3.5 in the proofs.

***Example* 3.1.** *Given the interval $J = [0, 1]$ of the real line $\mathbb{R}$ and given the points $t_1 = \frac{1}{5}$, $t_2 = \frac{2}{5}$, $t_3 = \frac{3}{5}$ and $t_4 = \frac{4}{5}$ in $[0, 1]$, consider the initial value problem (in short IVP) for the first order impulsive differential equations (in short IDE)*

$$\left.\begin{aligned} x'(t) + x(t) &= \tanh x(t), \quad t \in [0,1] \setminus \{t_1, t_2, t_3, t_4\}, \\ x(t_j^+) - x(t_j^-) &= \mathcal{I}_j(x(t_j)), \\ x(0) &= 1, \end{aligned}\right\} \tag{3.23}$$

for $t_j \in \{\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}\}$; where $x(t_j^-)$ and $x(t_j^-)$ are respectively, the right and left limit of x at $t = t_j$ such that $x(t_j) = x(t_j^-)$ and $\mathcal{I}_j(x(t_j))$ are the impulsive effects at the points $t = t_j$, $j = 1, \ldots, 4$ given by

$$\mathcal{I}_j(x) = \begin{cases} \dfrac{1}{2^j} \cdot \dfrac{x}{1+x} + 2, & \text{if } x > 0, \\ 2, & \text{if } x \le 0, \end{cases}$$

for all $t \in [0, 1]$. Here $f(t, x) = \tanh x$, so it is continuous and bounded on $[0, 1] \times \mathbb{R}$ with bound $M_f = 2$. Again, the map $x \mapsto f(t, x)$ is nondecreasing for each $t \in [0, 1]$. Next, the impulsive function $\mathcal{I}_j$ are continuous and bounded on $\mathbb{R}$ with bound $M_{\mathcal{I}_j} = 3$ for each $j = 1, \ldots, 4$. It is easy to verify that the impulsive operators $\mathcal{I}_j$ satisfy the hypothesis (H_2) with Lipschitz constants $L_{\mathcal{I}_j} = \dfrac{1}{2^j}$ for $j = 1, \ldots, 4$. Moreover, $\sum_{j=1}^{4} L_{\mathcal{I}_j} = \sum_{j=1}^{4} \dfrac{1}{2^j} < 1$. Finally, the functions $u(t) = e^{-t} - 1$ and $v(t) = 15e^{-t} + 1$ are respectively the lower and upper impulsive solutions of the IDE (1.1) *defined on* $[0, 1]$. *Thus, all the conditions of Theorem 3.1 are satisfied and so the IDE* (3.23) *has a impulsive solution ξ^* and the sequence $\{x_n\}$ of successive approximations defined by*

$$\begin{aligned} x_0(t) &= e^{-t} - 1, \\ x_{n+1}(t) &= e^{-t} + \sum_{0<t_j<t} k(t, t_j) \mathcal{I}_j(x_n(t_j)) \\ &\quad + \int_0^t k(t, s) \tanh x_n(s))\, ds \end{aligned}$$

for all $t \in J$, converges monotonically to x^*. Similarly, the sequence $\{y_n\}$ of successive approximations defined by

$$\begin{aligned} y_0(t) &= 15e^{-t} + 1, \\ y_{n+1}(t) &= e^{-t} + \sum_{0<t_j<t} k(t, t_j) \mathcal{I}_j(y_n(t_j)) \\ &\quad + \int_0^t k(t, s) \tanh y_n(s))\, ds \end{aligned}$$

for all $t \in J$, also converges monotonically to the impulsive solution y^* of the IDE (3.23) *in view of Remark 3.2.*

***Remark* 3.3.** *We note that if the IDE* (1.1) *has a lower impulsive solution u as well as an upper impulsive solution v such that $u \leq v$, then under the given conditions of Theorem 3.1 it has corresponding impulsive solutions x_* and y^* and these impulsive solutions satisfy the inequality*

$$u = x_0 \leq x_1 \leq \cdots \leq x_n \leq x_* \leq y^* \leq y_n \leq \cdots \leq y_1 \leq y_0 = v.$$

Hence x_* and y^* are respectively the minimal and maximal impulsive solutions of the IDE (1.1) *in the vector segment $[u, v]$ of the Banach space $E = PC(J, \mathbb{R})$, where the vector segment $[u, v]$ is a set of elements in $PC(J, \mathbb{R})$ defined by*

$$[u, v] = \{x \in PC(J, \mathbb{R}) \mid u \leq x \leq v\}.$$

This is because of the order cone K defined by (3.3) *is a closed set in $PC(J, \mathbb{R})$. A few details concerning the order relation by the order cones and the Janhavi sets in an ordered Banach space are given in Dhage [9, 10].*

***Remark* 3.4.** *In this paper we considered a very simple nonlinear first order impulsive differential equation for the existence and approximation theorem via monotone iteration principle or method, however the same method may be extended to other complex nonlinear impulsive differential equations of different orders with appropriate modifications for obtaining the algorithms for approximate solution (see Dhage [1] and references therein).*

References

[1] B.C. Dhage, *Quadratic perturbations of periodic boundary value problems of second order ordinary differential equations*, Differ. Equ. Appl. **2** (2010), 65–486.

[2] B.C. Dhage, Hybrid fixed point theory in partially ordered normed linear spaces and applications to fractional integral equations, *Differ. Equ. Appl.* **5** (2013), 155-184.

[3] B.C. Dhage, Partially condensing mappings in ordered normed linear spaces and applications to functional integral equations, *Tamkang J. Math.* **45** (2014), 397-426.

[4] B.C. Dhage, Nonlinear $\mathcal{D}$-set-contraction mappings in partially ordered normed linear spaces and applications to functional hybrid integral equations, *Malaya J. Matematik* **3** (2015), 62-86.

[5] B.C. Dhage, Some generalizations of a hybrid fixed point theorem in a partially ordered metric space and nonlinear functional integral equations, *Differ. Equ. Appl.* **8** (2016), 77-97.

[6] B.C. Dhage, Dhage iteration method in the theory of ordinary nonlinear PBVPs of first order functional differential equations, *Commun. Optim. Theory* **2017** (2017), Article ID 32, pp. 22.

[7] B.C. Dhage, A coupled hybrid fixed point theorem involving the sum of two coupled operators in a partially ordered Banach space with applications, *Differ. Equ. Appl.* **9**(4) (2017), 453-477.

[8] B.C. Dhage, Some variants of two basic hybrid fixed point theorem of Krasnoselskii and Dhage with applications, *Nonlinear Studies* **25** (3) (2018), 559-573.

[9] B.C. Dhage, A coupled hybrid fixed point theorem for sum of two mixed monotone coupled operators in a partially ordered Banach space with applications, *Tamkang J. Math.* **50**(1) (2019), 1-36.

[10] B.C. Dhage, Coupled and mixed coupled hybrid fixed point principles in a partially ordered Banach algebra and PBVPs of nonlinear coupled quadratic differential equations, *Differ. Equ. Appl.* **11**(1) (2019), 1-85.

[11] B.C. Dhage, Dhage monotone iteration method for for PBVPs of nonlinear first order impulsive differential equations, *Electronic J. Math. Anal. Appl.* **9**(1) (2021), 37-51.
[12] S.B. Dhage, B.C. Dhage, Dhage iteration method for Approximating positive solutions of PBVPs of nonlinear hybrid differential equations with maxima, *Intern. Jour. Anal. Appl.* **10**(2) (2016), 101-111.
[13] S.B. Dhage, B.C. Dhage, J.B. Graef, Dhage iteration method for initial value problem for nonlinear first order integrodifferential equations, *J. Fixed Point Theory Appl.* **18** (2016), 309-325.
[14] S.B. Dhage, B.C. Dhage, J.B. Graef, Dhage iteration method for approximating the positive solutions of IVPs for nonlinear first order quadratic neutral functional differential equations with delay and maxima, *Intern. J. Appl. Math.* **31**(6) (2018), 1-21.
[15] B.C. Dhage, S.B. Dhage, Approximating solutions of nonlinear first order ordinary differential equations, *GJMS Special Issue for Recent Advances in Mathematical Sciences and Applications-13*, GJMS Vol. **2**, No. 2, (2014), 25-35.
[16] B.C. Dhage, S.B. Dhage, Approximating positive solutions of PBVPs of nonlinear first order ordinary hybrid differential equations, *Appl. Math. Lett.* **46** (2015), 133-142.
[17] B.C. Dhage, D. Otrocol, Dhage iteration method for approximating solutions of nonlinear differential equations with maxima, *Fixed Point Theory* **19** (2) (2018), 545-556.
[18] D. Guo, V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, Academic Press, New York, London 1988.
[19] S. Heikkilä, V. Lakshmikantham, *Monotone Iterative Techniques for Discontinuous Nonlinear Differential Equations*, Marcel Dekker inc., New York 1994.
[20] V. Lakshmikantham, D.D. Bainov, and P.S. Simeonov, *Theory of Impulsive Differential Equations*, World Scientific Pub. Co., Singapore, 1989.
[21] A.N. Samoilenko and N.A. Perestyuk, *Impulsive Differential Equations*, World Scientific, Singapore, 1995.

Jñānābha, Vol. 50(1) (2020), 279-290
(Dedicated to Honor Professor H.M. Srivastava on His 80th Birth Anniversary Celebrations)

NUMERICAL SOLUTION OF MAGNETOHYDRODYNAMIC *(MHD)* RADIATIVE BOUNDARY LAYER FLOW AND HEAT TRANSFER ALONG A WEDGE IN THE PRESENCE OF SUCTION/INJECTION

By
Kuldeep S Rathore* and Mamta Goyal
Department of Mathematics
University of Rajasthan, Jaipur-302004, India
Email:*rathorekuldeep384@gmail.com and mamtagoyal1245@gmail.com

(*Received : May* 13, 2020 *; Revised: June* 27, 2020)
DOI: https://doi.org/10.58250/jnanabha.2020.50126

Abstract

An approximate numerical solution for the two dimensional laminar *MHD* radiative boundary layer flow along a wall of wedge with appropriate suction and injection in the presence of viscous dissipation and porous medium is considered in the present article. The fluid is considered to be viscous and incompressible. By applying appropriate similarity transformation, the governing flow equations have been transformed into corresponding higher order ordinary differential equations. The flow model is shown to be controlled by a number of flow parameters, viz. the radiation parameter, magnetic parameter, the permeability parameter, Prandtl number, Eckert number, the Hartree pressure gradient parameter and suction-injection parameter. The system of governing differential equations is solved numerically by shooting method and numerical calculations are carried out for different values of above mentioned dimensionless parameters. An analysis of the drawn results predicts that the velocity boundary layer and thermal boundary layer are influenced appreciably by the suction-injection, radiation and viscous dissipation at the wall of wedge.

2010 Mathematics Subject Classifications: 76D10, 76E25, 76S05, 76W05, 80A20.
Keywords and phrases: *MHD*, Boundary layer, Heat transfer, Suction-Injection, radiation, porous media.

1 Introduction

The flow problems combined with heat transfer over a wedge shaped configuration is encountered in numerous thermal engineering applications such as geothermal systems, heat exchangers, crude oil extraction, the flow in the desert cooler, nuclear waste management and thermal insulation. Historically in such types of flow problems, the popular Falkner-Skan transformation is obeyed to convert the boundary layer partial differential equations into the ordinary differential equations. In fact, a model of such fluid flow over a wedge shaped bodies was first formulated by Falkner and Skan[12] to illustrate the applications of Prandtls boundary layer phenomena. In the later years Hartree [13] studied the similar problem and gave the numerical results for shear stress with different wedge angles. The other pioneer was Eckert[10], who investigated Falkner-Skan flow past an isothermal wedge and gave initial heat transfer values. In *MHD*, we consider the electrically conducting fluid flow with magnetic characteristics. *MHD* plays significant role in plasma studies, *MHD* power generators, construction of heat exchangers etc. There is plenty of literature available in which fluid flows have been studied with or without *MHD* under different fluid properties using the prominent Falkner-Skan transformations. The following studies give the clear insight of the fundamental problem of the wedge flow and associated applications with different fluid properties.

Thermal characteristics in boundary layer wedge flow under different fluid parameters were extensively investigated by Chen [9] and Watanabe [27]. Kafoussias and Nanousis [15] studied the laminar boundary layer wedge flow with *MHD* and suction injection effects. Yih [28] reported the uniform blowing effect on forced convection flow along a wedge with heat flux. Further Anjalidevi et al. [3] and Kandasamy et al. [16] investigated the thermal stratification and chemical reaction effects with heat source and concentration in the presence of suction/injection. Martin and Boyd [18] reported the Falkner-Skan flow with slip conditions and derived the significant effects on flow. On the similar lines Sattar [24] used unsteady fluid flow past a wedge and draw important conclusions. Ashwani et al. [4], Abbasbandy et al. [2] and Khan et al. [17] also provided the significant contribution to the *MHD* wedge flow with different governing fluid parameters. Arthur et al. [5] studied the stagnation point flow over a porous surface with heat transfer effects and viscous dissipation. El-Dabe et al. [11] and Srinivasharya et al. [23] considered non-Newtonian fluid including Casson fluid and nanofluid in their respective studies and gave substantial contribution to the wedge flow literature. Stretching wedge and convective heat transfer on the boundaries have been studied by Nagendramma et al. [20]. Increasing technical importance and variety of applications of wedge flow leads many young minds of globe such as Majety et al. [19], Ullah et al. [26], Alam et al. [1] and Ramesh et al. [22]. Recently Ibrahim and Tulu [14] reported the *MHD* boundary layer flow and heat transfer considering nanofluid and viscous dissipation, in porous media.

In the present analysis we consider the numerical study of *MHD* boundary layer flow along a wedge with radiation and suction-injection effects. Suction and injection of a fluid through surfaces can give the desired heat transfer rates. Injection or blowing through porous bounding heated or cooled surfaces is widely used in boundary layer control. The effect of various flow parameters on velocity and temperature fields are derived and analyzed with tabular and graphical mode of representation.

2 Mathematical formulation

In the present analysis, we propose two dimensional *MHD* boundary layer flow along a wedge with radiation and suction-injection effects. As proposed in ***Fig.* 2.1**[9], x-coordinate is considered parallel to the wedge and y-coordinate is taken along the free stream. T_w is temperature at wall of the wedge and T_∞ is ambient temperature. The fluid regime is considered to have constant fluid properties. A constant magnetic field of strength B_0 is assumed in the normal direction to the wall of wedge. The induced magnetic field is not taken into consideration as it is too small to compare with the applied magnetic field as suggested by Ullah et al. [26].

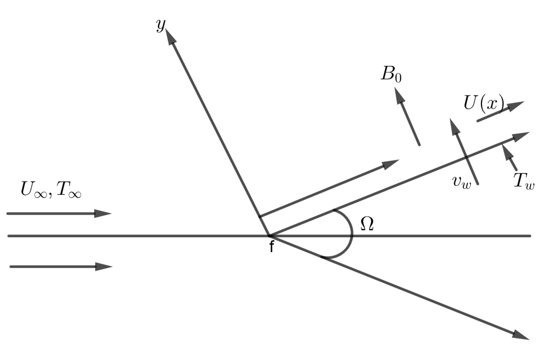


***Figure* 2.1:** Flow analysis along the wall of wedge

Following the above assumptions together with boundary layer approximation, the governing flow equations i.e. continuity, momentum and energy equations can be expressed as (Srinivasacharya et al. [22] and Alam et al. [26]) following:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{2.1}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho_f}\frac{\partial p}{\partial x}+\upsilon_f\frac{\partial^2 u}{\partial y^2}-\left(\frac{\sigma B_o^2}{\rho_f}+\frac{\upsilon_f}{K_o}\right)u, \tag{2.2}$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha_f\frac{\partial^2 T}{\partial y^2}+\frac{\upsilon_f}{C_p}\left(\frac{\partial u}{\partial y}\right)^2-\frac{\partial q_r}{\partial y}. \tag{2.3}$$

The appropriate boundary conditions as per formulation are prescribed as:

$$\begin{aligned} &u=0 \qquad v=-v_w(x) \qquad T=T_w \qquad at \quad y=0\\ &u=U(x)=U_\infty x^m \qquad T\to T_\infty \qquad as \quad y\to\infty. \end{aligned} \tag{2.4}$$

Here $v_w(x)>0$ is the velocity of suction and $v_w(x)<0$ is the velocity of injection. $v_w(x)=v_0x^{\frac{m-1}{2}}$ is a prescribed velocity considered at the wall of the wedge and v_0 is the initial strength of suction. Again u and v are velocities in x and y directions respectively and ρ_f is fluid density, υ_f is kinematic viscosity, α_f is thermal diffusivity and C_p is specific heat of the fluid.

Equation (2.2) shows that pressure p in the boundary layer must be equal to that of free stream for any prescribed value of x . As velocity does not change in free stream, so there is no vorticity involved. In such a case simple Bernoullis equation can be used as suggested by Falkner and Skan [1]. Fluid velocity outside the boundary layer is taken as $U(x)=U_\infty x^m$. For a uniform stream, the equation (2.2) can be expressed as (Falkner and Skan [12])

$$-\frac{1}{\rho_f}\frac{\partial p}{\partial x}=U\frac{dU}{dx}+\left(\frac{\sigma B_o^2}{\rho_f}+\frac{\upsilon_f}{K_o}\right)U. \tag{2.5}$$

On using equation (2.5) into equation (2.2), the momentum equation becomes

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=U\frac{dU}{dx}+\upsilon_f\frac{\partial^2 u}{\partial y^2}+\left(\frac{\sigma B_o^2}{\rho_f}+\frac{\upsilon_f}{K_o}\right)(U-u). \tag{2.6}$$

In the above equations, x is measured from tip of the wedge, x is the Falkner-Skan power law parameter, $\beta=\frac{2m}{1+m}$ is the Hartree pressure gradient parameter corresponding to $\beta=\frac{\Omega}{\pi}$ for the total angle Ω of the wedge as mentioned in ***Fig.* 2.1**. Positive Falkner-Skan power law parameter m represents favorable pressure gradient while negative value of m represents adverse pressure gradient. (Nagendramma et al. [20]).

The radiative heat flux q_r mentioned in equation (2.3) is modelled as per Rosseland approximation [21] by the following

$$q_r=-\frac{4\alpha}{3\beta}\frac{\partial T^4}{\partial y}, \tag{2.7}$$

In the above equation, α represents Stefan-Boltzmann constant and β represents the mean absorption constant. The above approximation holds good at points optically far from the boundary surface and fair for intensive absorption. Now we assume that the temperature difference with in the fluid flow varies as a linear function of temperature so that expanding the term T^4 by the well known Taylor series about T_∞ and omitting the higher-order terms

$$T^4\cong 4T_\infty^3T-3T_\infty^4. \tag{2.8}$$

Using (2.8) into the equation (2.7), we get

$$q_r=-\frac{16\alpha T_\infty^3}{3\beta}\frac{\delta T}{\delta y}. \tag{2.9}$$

Now substituting the value of q_r from (2.9) in (2.3), we get

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\left(\alpha_f+\frac{16\alpha T_\infty^3}{3\beta}\right)\frac{\partial^2 T}{\partial y^2}+\frac{\upsilon_f}{C_p}\left(\frac{\partial u}{\partial y}\right)^2. \tag{2.10}$$

3 Similarity Analysis

In order to solve above system of equations, we apply suitable similarity transformation to convert above partial differential equations into higher order ordinary differential equations. We choose stream function ψ in such a manner that continuity equation (2.1) is identically satisfied, for this

$$u = \frac{\partial \psi}{\partial y} \quad and \quad v = -\frac{\partial \psi}{\partial x}. \tag{3.1}$$

Following the transformation suggested by Bansal [6], the corresponding momentum equation (2.6) and energy equation (2.10) can be converted into ordinary differential equation as:

$$\psi(x,\eta) = \sqrt{\frac{2}{1+m}\upsilon_f U_\infty} x^{\frac{m-1}{2}} f(\eta), \tag{3.2}$$
$$\eta = y\sqrt{\frac{1+m}{2}\frac{U_\infty}{\upsilon_f}} x^{\frac{m-1}{2}},$$
$$\theta(\eta) = \frac{T - T_\infty}{T_w - T_\infty}.$$

In the above transformation equations, η is dimensionless similarity variable, $f(\eta)$nondimensional stream function, $f'(\eta)$is nondimensional velocity and $\theta(\eta)$ is nondimensional temperature. Now using the above transformations and approximation suggested by Kafoussias et al. [15], we get the following transformed equations together with boundary conditions,

$$\frac{d^3 f}{d\eta^3} + f\frac{d^2 f}{d\eta^2} + \beta\left[1 - \left(\frac{df}{d\eta}\right)^2\right] + \frac{1}{1+m}(M+K)\left(1 - \frac{df}{d\eta}\right) = 0, \tag{3.3}$$

$$\left[1 + \frac{4N}{3}\right]\frac{d^2\theta}{d\eta^2} + Pr\left[f\frac{d\theta}{d\eta} + Ec\left(\frac{d^2\theta}{d\eta^2}\right)^2\right] = 0, \tag{3.4}$$

with associated boundary conditions:

$$f = \frac{2}{1+m}S, \quad f' = 0 \quad and \quad \theta = 1 \quad at \quad \eta = 0, \tag{3.5}$$
$$f' \to 1 \quad and \quad \theta \to 0 \quad as \quad \eta \to \infty.$$

The non-dimensional parameters involved in the present study can be summarized as follows:

Radiation parameter $N = \frac{4\alpha T_\infty^3}{\beta\kappa}$, Magnetic parameter $M = \frac{2\sigma B_o^2 x^{1-m}}{\rho_f U_\infty}$,
Eckert number $Ec = \frac{U^2}{C_p(T_w - T_\infty)}$, Prandtl number $Pr = \frac{\upsilon_f}{\alpha_f}$,
Permeability parameter $K = \frac{2\upsilon_f x^{1-m}}{K_o U_\infty}$, Suction parameter $S = v_o\sqrt{\frac{2}{\upsilon_f U_\infty (m+1)}} > 0$,
and $S > 0$ for injection parameter.

In the present problem, the physical quantities of technical importance are the skin- friction coefficient C_f and local Nusselt number Nu_x, which may be described as follows:
$C_f = \frac{2\tau_w}{\rho U^2(x)}$, where τ_w is the surface shear stress which may be expressed as:
$\tau_w = \mu_f\left(\frac{\partial u}{\partial y}\right)_{y=0}$.

So the nondimensional skin friction coefficient C_f is

$$C_f\sqrt{Re_x} = 2\sqrt{\frac{2}{m+1}}f''(0), \text{ where } Re_x \text{ is reynold number.} \tag{3.6}$$

Similarly local Nusselt number can be expressed as:
$Nu_x = \frac{xq_w}{k(T_w - T_\infty)}$, where q_w is surface heat flux that can be defined as
$q_w = -k_f\left(\frac{\delta T}{\delta y}\right)_{y=0}$.

Hence the nondimensional local Nusselt number is

$$\frac{Nu_x}{\sqrt{Re_x}} = -\sqrt{\frac{m+1}{2}}\theta'(0). \tag{3.7}$$

4 Numerical solution

The system of differential equations (3.3) and (3.4) with boundary conditions (3.5) are solved by shooting method. For this, the above equations are converted into the system of first order differential equations as mentioned below:

$$f_1 = f, \quad f_2 = f', \quad f_3 = f'', \quad f_4 = \theta, \quad f_5 = \theta'. \tag{4.1}$$

Using above notations the first order differential equations are as follows

$$f_3' = f_1 f_3 - \beta\left(1 - f_2^2\right) - \frac{1}{1+m}(M+K)(1-f_2), \tag{4.2}$$

$$f_5' = -\frac{3Pr}{3+4N}\left(f_1 f_5 + Ec f_3^2\right), \tag{4.3}$$

with boundary conditions

$$f_1(0) = \frac{2}{1+m}S, \quad f_2(0) = 1, \quad f_4(0) = 1, \tag{4.4}$$

$$f_2(\infty) = 0 \text{ and } f_4(\infty) = 0.$$

For the solution of above equations, we require $f_3(0)$ and $f_5(0)$, but no such values are available at the boundary. So according to shooting technique, we apply initial guess for $f_3(0)$ and $f_5(0)$. Then we compare the numerical values for f_2 and f_4 at η_∞ with the boundary conditions and adjust the values of f_3 and f_5. Secant method is applied for better approximation. The above procedure is carried out until the proper accuracy is achieved.

5 Result and Discussion

Numerical solutions for the velocity and temperature profiles across the boundary layer for different values of flow parameters have been obtained.

***Table* 5.1** describes the influence of various governing flow parameters on skin friction coefficient and local Nusselt number. It is observed that the skin friction coefficient enhances with increasing Falkner-Skan power law parameter, Magnetic parameter, Permeability parameter and suction parameter while reduces with increasing injection parameter. Also the local Nusselt number enhances with increasing Prandtl number and suction parameter and shows opposite behavior while increasing Falkner-Skan power law parameter, Magnetic parameter, Permeability parameter, Eckert number, Radiation parameter and injection parameter.

***Table* 5.1:** Numerical computation of skin friction coefficient $-f''(0)$ and local Nusselt number $-\theta'(0)$ for different physical parameters

m	M	K	Pr	Ec	N	S	$-f''(0)$	$-\theta'(0)$
0.0	0.4	.05	0.7	0.5	0.2	0.2	1.066156	0.458095
.14							1.177544	0.432976
.33							1.273992	0.408410
.50							1.332252	0.392289
.14	0.0						0.995269	0.444004
	0.5						1.218687	0.430197
	1.0						1.405526	0.416433
	0.5	0.0					1.198301	0.431582
		0.2					1.277789	0.426039
		0.4					1.352354	0.420533
		0.6					1.422781	0.415079
		0.1	0.2				1.238719	0.358333
			0.7				1.238719	0.433060
			2.0				1.238719	0.586770
			0.7	0.1			1.238719	0.569115
				0.5			1.238719	0.433060
				1.0			1.238719	0.262997
				0.5	0.0		1.238719	0.460142
					0.5		1.238719	0.408067
					2.0		1.238719	0.365363
					0.2	0.5	1.602576	0.602457
						1.0	2.293073	0.919708
						1.5	3.049663	1.261584
						0.0	1.025276	0.333150
						-0.1	0.928828	0.287969
						-0.2	0.839800	0.246321

***Table* 5.2:** Numerical computation of skin friction coefficient $f''(0)$ for various values of Falkner-Skan power law parameter m for the values $M = K = Ec = N = S = 0$ and Pr=0.73.

$f''(0)$				
m	Present Value	Ashwani[16]	Ullah[25]	Wubshet[28]
1429	0.7322	0.7320	0.7320	0.73200
.2000	0.8022	0.8021	0.8021	0.80213
.3333	0.9278	0.9277	0.9277	0.92765
1.000	1.2327	1.2326	1.2326	1.23258

***Table* 5.2** reveals that value of for various different values of Falkner-Skan power law parameter finds excellent agreement with the previous published results such as Ashwani et al.[4], Ullah et al.[26] and Wubshet et al.[14]. The above confirmation shows that present results are also accurate.

***Figure* 5.1** shows the effect of Falkner-Skan power law parameter on velocity profile. It is observed that while increasing the Falkner-Skan power law parameter velocity boundary layer accelerates. The similar behavior is seen in the case of Hartree pressure gradient parameter β.

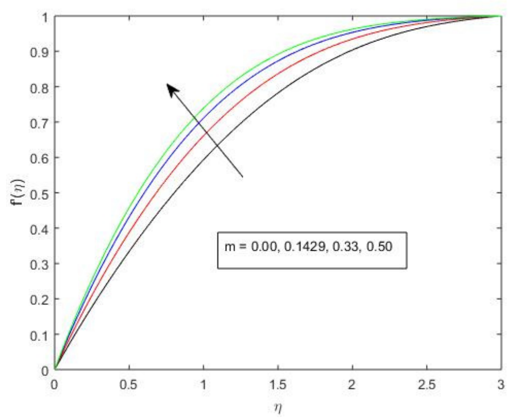


***Figure* 5.1:** Effect of m on velocity profile for $M = 0.4$ and $K = 0.5$

Increment in wedge angle β results into lesser space for fluid flow. Hence velocity boundary layer thickness affects accordingly. ***Figure* 5.2** reports the significant effect of Magnetic parameter$M$ on the velocity field which shows that increasing the value of magnetic parameter results into enhancement on fluid velocity. The effect of permeability parameter on velocity field is described in ***Figure* 5.3**. According to this while increasing the permeability parameter velocity is slightly increased on the porous surface and reduces the boundary layer thickness. The similar effects of m, M, K on velocity have been reported by Wubshet et al. [14].

The effect of Prandtl number on temperature profile is reported in ***Figure* 5.4**. It clearly describes that increasing the value of Prandtl number results into decrease in temperature field. This is because of increasing the Prandtl number tends to reduce the thermal diffusivity of the fluid and causes the weak penetration of heat inside the fluid. The influence of viscous dissipation i.e. Eckert number on the temperature field is shown in ***Figure* 5.5**. The Eckert number describes the conversion the kinetic number into internal energy by work done against the viscous fluid stress. It is observed that increasing the value of Eckert number causes the rise in the temperature. The effect of radiation parameter on temperature field is reported in ***Figure* 5.6**. It is observed from figure that on increasing the value of results into the significant increment of temperature.

The effect of suction and injection parameters on velocity as well as in temperature fields are reported in ***Figures* 5.7-5.10**. Suction and injection are effective tools to control the flow field. Fluid flow can be made laminar using the suction injection. The effect of suction S on velocity field is reported in ***Figure* 5.7** which describes that increasing the amount of suction tends into significant increment in fluid velocity. This is due to creation of more space for fluid particles in which they can move with greater velocity. Similarly the effect of suction parameter S on temperature field is shown in ***Figure* 5.8**. It is observed that increasing the value of S results into decrease in the temperature which can be used to cool the fluid flow. Similarly the effect of injection parameter S on velocity field is reported in ***Figure* 5.9**. It is shown that increasing the value of injection parameter tends to lower the velocity. Also injection causes significant rise in temperature field which is described in ***Figure* 5.10**

6 Conclusion

Numerical solution for the two dimensional laminar *MHD* boundary layer flow along a wall of wedge with uniform suction and injection in the presence of radiation and porous medium has been discussed. Using the suitable similarity transformation, the governing partial differential equations

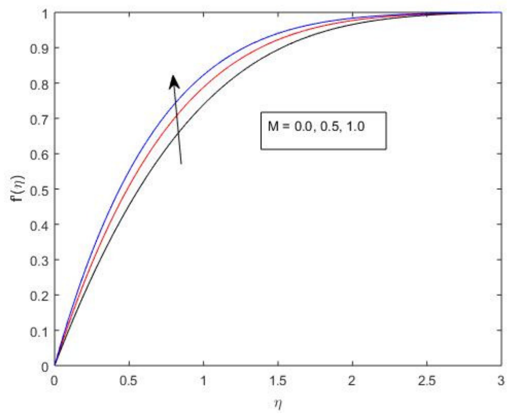


***Figure* 5.2:** Effect of M on velocity profile for $m = .14$ and $K = .05$

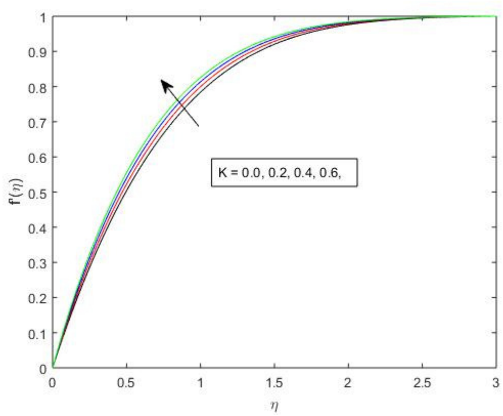


***Figure* 5.3:** Effect of K on velocity profile for $M = 0.4$ and $K = .14$

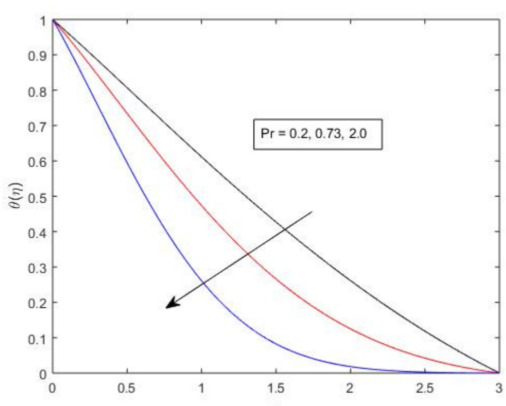


***Figure* 5.4:** Effect of Pr on temperature profile for $Ec = 0.5$ and $N = 0.2$

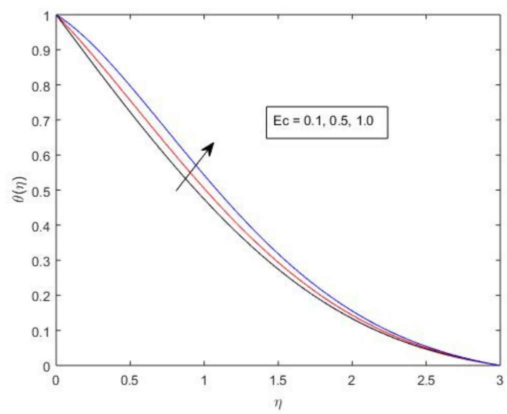


***Figure* 5.5:** Effect of Ec on temperature profile for $Pr = 0.73$ and $N = 0.2$

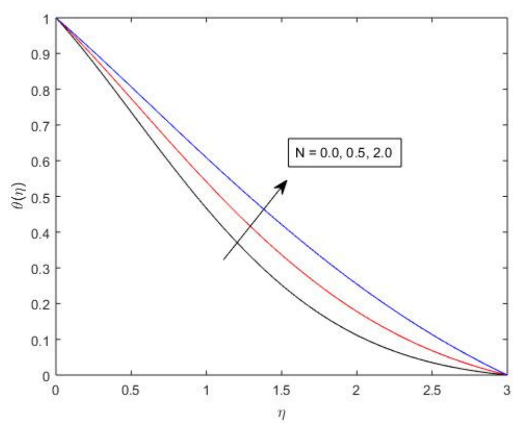


***Figure* 5.6:** Effect of N on temperature profile for $Pr = 0.73$ and $Ec = 0.5$

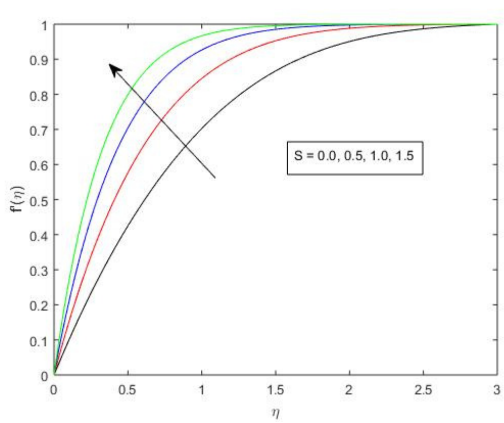


***Figure* 5.7:** Effect of S on velocity profile for $M = 0.4$ and $K = 0.5$

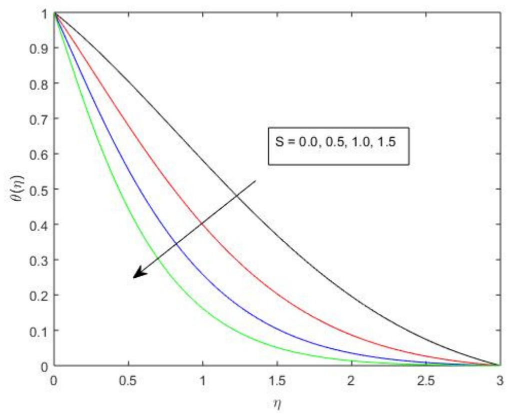


***Figure* 5.8:** Effect of S on temperature profile for $Pr = 0.73$ and $N = 0.2$

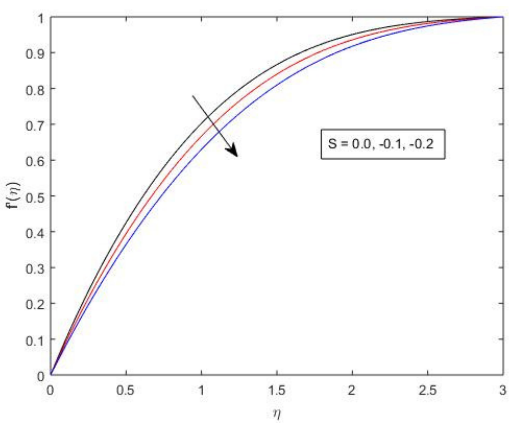


***Figure* 5.9:** Effect of S on velocity profile for $M = 0.4$ and $K = 0.5$

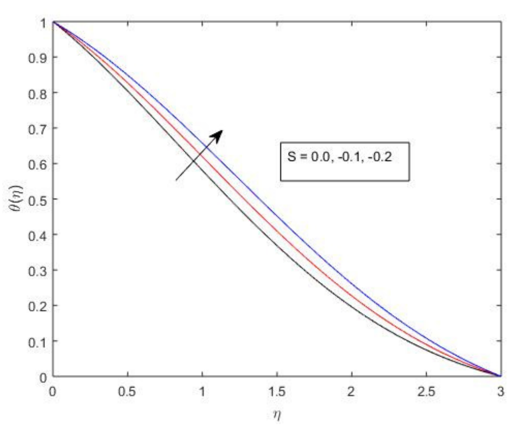


***Figure* 5.10:** Effect of S on temperature profile for $Pr = 0.73$ and $N = 0.2$

are converted into the higher order ordinary differential equations and then solved numerically applying shooting technique. From the aforesaid discussion following conclusions are made:

1. The velocity of fluid enhances with increase in suction parameter, permeability parameter, magnetic parameter and Falkner-Skan power law parameter while reduces with increase in injection parameter.
2. The temperature of fluid enhances with increase in injection parameter, radiation and Eckert number while reduces with increase in Prandtl number and suction parameter.
3. The skin friction coefficient enhances with increasing magnetic parameter, permeability parameter, Falkner-Skan power law parameter and suction parameter.
4. The Nusselt number is a decreasing function for the permeability parameter , Falkner-Skan power law parameter, magnetic parameter, radiation parameter, Eckert number and injection parameter.

Acknowledgement. Authors are very much thankful to the Editor and the referees for their valuable and helpful suggestions which improve the quality of paper.

References

[1] M.S. Alam and M. Ali et al., Solution of Falkner-Skan unstesdy *MHD* boundary layer flow and heat transfer past a moving porous wedge in a nanofluid, *Science Direct Procedia Engineering*, **194** (2017), 414-420.

[2] S. Abbasbandy, R. Naz, T. Hayat and A. Alsaedi, Numerical and analytical solutions for Falkner-Skan flow of *MHD* Maxwell fluid, *Applied Mathematics and Computation*, **242** (2014), 569-575.

[3] S.P. Anjali Devi and R. Kandasamy, Thermal stratification effects on laminar boundary-layer flow over a wedge with suction or injection, *Mech. Res. Commun.*, **28** (2001), 349-354.

[4] G. Ashwani and A.T. Eswara, *MHD* Falkner-Skan boundary layer flow with internal heat generation or absorption, *Int. J. of Mathematical and Computational Sciences*, **6(5)** (2012), 556-559.

[5] E.M. Arthur and I.Y. Seini, Hydromagnetic Stagnation point flow over a porous stretching surface in the presence of radiation and viscous dissipation, *Applied and Compiutational Mathematics*, **3(5)** (2014), 191-196.

[6] J.L. Bansal, *Boundary Layer Theory*, Oxford and IBM Publishing Company, 1986, 169 (Chapter 6).

[7] R. Byron Bird, Warren E. Stewart and Edwin N. Lightfoot, *Transport Phenomena*, John Wiley and Sons, New York, 1992, 605.

[8] T. Cebeci and P. Bradshaw, Physical and Computational Aspects of Convective Heat Transfer, *Springer-Verlag, New York*, **79** (1984), 79-80.

[9] James LS Chen, Thermal response behaviour of laminar boundary layers in wedge flow, *Int. J. Heat and Mass Transfer*, **13** (1970), 1101-1114

[10] E.R.G. Eckert, Die berechnungs/des warmeuberganges in der laminaren grenzschicht um stomter korper, *VDI-Forschungsheft*, **416** (1942), 1-24.

[11] N.T. El-Dabe, A.Y. Ghaly et al., Numerical solution of *MHD* boundary layer flow of non-newtonian casson fluid on a moving wedge with heat and mass transfer and induced magnetic field, *Journal of Applied Mathematics and Physics*, **3(6)** (2015), 649-663.

[12] V.M. Falkner and S.W. Skan, Some approximate solutions of the boundary layer equations, *Philosophical Magazine*, **12** (1931), 865-896.
[13] D.R. Hartree, On an equation occurring in Falkner and Skans approximate treatment of the equations of the boundary layer, *Mathematical Proceeding of the Cambridge Philosophical Society*, **33(2)** (1937), 223-239.
[14] Wubshet Ibrahim and Ayele Tulu, Magnetohydrodynamic *(MHD)* boundary layer flow past a wedge with heat transfer and viscous effects of nanofluid embedded in porous media, *Mathematical Problems in Engineering*, 2019, (2019), Article ID 4507852, 12 pages.
[15] N.G. Kafoussias and N.D. Nanousis, Magnetohydrodynamic laminar boundary- layer flow over a wedge with suction or injection, *Can. J. Phys.*, **75** (1997), 733-745.
[16] R. Kandasamy, K. Periasamy and K.K. Sivaganana Prabhu, Effect of chemical reaction, heat mass transfer along a wedge with heat source and concentration in the presence of suction or injection, *Int. J. of Heat and Mass Transfer*, **48** (2005), 1388-1394.
[17] M.S. Khan, I. Karim, M.S. Islam and M. Wahiduzzaman, *MHD* boundary layer radiative, heat generating and chemical reacting flow past a wedge moving in a nanofluid, *Nano Convergence*, **1** (2014), 20-28.
[18] M.J. Martin and I.D. Boyd, Falkner-Skan flow over a wedge with slip boundary conditions, *Journal of Thermophysics and Heat Transfer*, **24(2)** (2010), 263-270.
[19] S.S. Majety and K. Gangadhar, Viscous dissipation effects on radiative *MHD* boundary layer flow of nano fluid past a wedge porous medium with chemical reaction, *IOSR Journal of Mathematics*, **12(5)** (2016), 71-81.
[20] V. Nagendramma, K. Sreelakshmi and G. Sarojamma, *MHD* heat and mass transfer flow over a stretching wedge with convective boundary condition and thermophoresis, *Science Direct, Procedia Engineering*, **127** (2015), 963-969.
[21] S. Rosseland, *Theoretical Astrophysics*, New York: Oxford University, 1936.
[22] B.K. Ramesh et al., Similarity solution of the *MHD* boundary layer flow past a constant wedge with porous media, *Mathematical Problems in Engineering*, 2017, (2017), Article ID 1428137, 11 pages.
[23] D. Srinivasacharya, U. Mendu et al., *MHD* boundary layer flow of a nanofluid past a wedge, Science Direct, *Procedia Engineering*, **127** (2015), 1064-1070.
[24] M.A. Sattar, A local similarity transformation for the unsteady two-dimensional hydromagnetic boundary layer equation of a flow past a wedge, *Int. J. of Applied Mathematics and Mechanics*, **7(1)** (2011), 15-28.
[25] M. Turkyilmazoglu, Slip flow and heat transfer over a specific wedge: an exactly solvable Falkner-Skan equation, *Journal of Engineering Mathematics*, **92(1)** (2015), 73-81.
[26] I. Ullah, I. Khan and S. Shafie, Hydromagnetic Falkner-Skan flow of Casson fluid past a moving wedge with heat transfer, *Alexandria Engineering Journal*, **55(3)** (2016), 2139-2148.
[27] T. Watanabe, Thermal boundary layer over a wedge with uniform suction and injection in forced flow, *Acta Mechanica*, **83(3-4)** (1990), 119-126.
[28] K.A. Yih, Uniform suction/blowing effect on forced convection about a wedge: Uniform heat flux, *Acta Mech.*,**128** (1998), 173-181.

ISSN 0304-9892 (Print) ISSN 2455-7463 (Online)

Jñānābha

Statement of ownership and particulars about the journal

1.	Place of Publication	D.V. Postgraduate College Orai-285001, U.P., India
2.	Periodicity of Publication	Bi-annual
3.	Printer's Name Nationality Address	Creative Laser Graphics (Iqbal Ahmad) Indian IIT Gate, Kanpur
4.	Publisher's Name	Dr. R.C. Singh Chandel For Vijñāna Parishad of India
	Nationality Address	Indian D.V. Postgraduate College Orai-285001, U.P. India
5.	Editor's Name Nationality Address	Dr. R.C. Singh Chandel Indian D.V. Postgraduate College Orai-285001, U.P. India
6.	Name and Address of the individuals who own the journal and partners of share holders holding more than one percent of the total capital	**Vijñāna Parishad of India** Address: D.V. Postgraduate College Orai-285001, U.P. India

I, *Dr. R. C. Singh Chandel* hereby declare that the particulars given above are true to the best of my knowledge and belief.

RChandel

Dr. R.C. Singh Chandel
Publisher/Editor
rc_chandel@yahoo.com

INSTRUCTIONS TO AUTHORS / REVIEWERS / SUBSCRIBERS

1. 'Jñānābha' is published annually since 1971. Effective with Vol. 47 (2017) it is bi-annual published in (June and December). Since 1971, its content is available in hard copy as well as on its website http://www.vijnanaparishadofindia.org/jnanabha.

2. APC (Article Processing Charge) is no longer requirement for publication in 'Jñānābha'.

3. It is interdisciplinary journal devoted primarily to research articles in all areas of mathematical and physical sciences; it does, however, encourage original mathematical works which are motivated by and relevant to applications in the social, management, mathematical or engineering sciences. Papers intended for publication in this journal should be in typed form or offset-reproduced (not dittoed), A4 size double spaced with generous margin and they may be written in Hindi or English. Manuscripts (soft copy typed in MS Word/La-TeX, mentioning 2010 Mathematics Subject Classification, Keywords and authors postal and E-mail addresses also on front page strictly in the format of 'Jñānābha', may be submitted to either of the Editors). It is mandatory for every author in 'Jñānābha' to be member of 'Vijñāna Parishad of India' in good standing.

The submission of the paper implies the author's assurance that the paper that has not been widely circulated, copyrighted, published or submitted for publication elsewhere.

Authors are advised to submit only neatly (and carefully) type written and thoroughly checked manuscripts employing accepted conventions of references, notations, displays, etc.; all typescripts not in a format suitable to publication in this journal will be returned unrefereed.

4. **La-TeX Template:** La-TeX template is available here to facilitate adherence to publication 'Jñānābha' standards.

5. **Information for Reviewers/Members on Editorial Board.** Members on Editorial Board are responsible to review the papers of their field or to get reviewed by other suitable reviewers of their field that paper is original and publishable in any standard International Journal and is strictly in the format of JÑĀNĀBHA having correct language and proper methodology.

6. Effective Vol. 50 the price per volume is Rs. 1000.00 (or US $ 50.00). Individual members of Vijñāna Parishad of India are entitled to free subscription to the current issues of this journal. Individual membership: Rs. 300.00 (or US $ 40.00) per calendar year; Life membership: Rs. 3000.00 (or US $ 400.00). Back volumes are available at special price.(Payments of dues by cheques must include approximate bank charges).For online payment, visit http://www.vijnanaparishadofindia.org.

[By a reciprocity agreement with the American Mathematical Society, an individual /life member of the Parishad residing outside North American continent may join the Society by submitting an application for membership on the form that may be obtained from the office of the Society **(P.O. Box 6248, Providence, Rhode 02940, USA)** and by paying the society's current dues as a reciprocity member at a considerably reduced rate; the usual requirements that the applicant be endorsed by two members of the Society and that the candidate be elected by the Council are waived, but this reduction in dues to the Society does not apply to those individual/life members of the Parishad who reside , even on temporary basis, in the North American area(i.e. USA and Canada)]

The mathematical content of this journal is reviewed among others by **Zentralblat für Mathematk (Germany).**

It is also indexed in Indian Citation Index (www.indiancitation index.com), Google Scholar (https://scholar.google.com). One may also visit dmr.xml-MathSciNet-American Mathematical Society http://www.mathscinetams.org/dmr/dmrxml. **Jñānābha is included in Serials covered by Zentralblatt Math, Romanian Unit** http://www.zbl.theta.ro/too/zs/j.html

All communications regarding subscriptions, order of back volumes, membership of Vijñāna Parishad of India, change of address, etc. and all books for review, should be addressed to:

The Secretary
Vijñāna Parishad of India
D.V. Postgraduate College, Orai- 285001, UP, India
E-Mail: rc_chandel@yahoo.com
Jñānãbha –Vijñāna Parishad of India

Contents