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(Dedicated to Honor Professor H.M. Srivastava on His 80th Birth Anniversary Celebrations)

BOUNDS FOR THE MAXIMUM MODULUS OF POLYNOMIAL NOT VANISHING IN A DISK

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Abstract

Let $p(z)$ be a polynomial of degree n. We have several results for the bounds of maximum modulus of polynomial in terms of coefficients of polynomial and radius of the disk having no zeros in it. In this paper we have proved some results for the bounds of maximum modulus of polynomial not vanishing in a disk of greater or smaller than unity. Our results improve the earlier proved results.

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1 Introduction and Statement of Results

Let $p(z)$ be a polynomial of degree n. Let us define $m = \min_{|z|=k} |P(z)|$ and $M(p, r) = \max_{|z|=r} |P(z)|$. Concerning the estimate for the maximum modulus of a polynomial on the circle $|z| = R$, $R > 0$, in terms of its degree and the maximum modulus on the unit circle, we know that for every $R \geq 1$,

$$\max_{|z|=R} |p(z)| \leq R^n \max_{|z|=1} |p(z)|. \tag{1.1}$$

The result is best possible for the polynomial having all its zeros at origin.

Inequality (1.1) is a simple deduction from the maximum modulus principle (for reference see [8] or [11]).

For the polynomial of degree n and the case $r \leq 1$, we have the following result due to Varga [13] who attributed it to Zerrantonello.

$$\max_{|z|=r} |p(z)| \geq r^n \max_{|z|=1} |p(z)|. \tag{1.2}$$

Again the result is best possible for the polynomial having all its zeros at origin.

For the class of polynomials having no zeros in $|z| < 1$, the inequalities (1.1) and (1.2) are sharpened by Ankeny and Rivlin [1] and Rivlin [12], by proving following inequality (1.3) and inequality (1.4) respectively

$$\max_{|z|=R\geq 1} |p(z)| \leq \frac{R^n + 1}{2} \max_{|z|=1} |p(z)|, \tag{1.3}$$

$$\max_{|z|=r} |p(z)| \geq \left(\frac{1 + r}{2}\right)^n \max_{|z|=1} |p(z)|. \tag{1.4}$$

Aziz and Dawood [3] improved inequality (1.3) under the same hypothesis as

$$\max_{|z|=R\geq 1}|p(z)| \leq \left(\frac{R^n+1}{2}\right)\max_{|z|=1}|p(z)| - \left(\frac{R^n-1}{2}\right)\min_{|z|=1}|p(z)|. \tag{1.5}$$

There are several results concerning the refinement and generalizations of above mentioned inequalities (see [5], [7] and [14]).

For the case $0 < \rho \leq 1$, when polynomial does not vanish in $|z| < k$, $k \geq 1$ we have the following inequality due to Aziz [2].

$$\max_{|z|=\rho}|p(z)| \geq \left(\frac{\rho+k}{1+k}\right)^n \max_{|z|=1}|p(z)|. \tag{1.6}$$

The result is sharp and equality in (1.6) is attained for $p(z) = c(ze^{i\beta}+k)^n$, $c(\neq 0) \in C$ and $\beta \in R$.

Inequality (1.6) was improved by Govil, Qazi and Rahman [6] by introducing coefficients of polynomial under consideration in it as following

***Theorem* 1.1.** *Let* $p(z) = \sum_{j=0}^{n} a_j z^j$ *does not vanish in* $|z| < k$, $k \geq 1$ *and let* $\lambda = \lambda(k) = \frac{ka_1}{na_0}$. *Then for* $0 < \rho \leq 1$,

$$\max_{|z|=\rho}|p(z)| \geq \left(\frac{k^2+2|\lambda|k\rho+\rho^2}{k^2+2|\lambda|k+1}\right)^{\frac{n}{2}} \max_{|z|=1}|p(z)|. \tag{1.7}$$

In the case when n is even equality in (1.7) *is attained for*

$$p(z) = c(z^2e^{i2\beta} + 2kze^{i\beta}\cos\alpha + k^2)^{\frac{n}{2}}, \quad c(\neq 0) \in C \text{ and } \alpha, \beta \in R.$$

The following result is also due to Govil, Qazi and Rahman [6] and is complement to ***Theorem* 1.1**.

***Theorem* 1.2.** *Let* $p(z) = \sum_{j=0}^{n} a_j z^j$ *does not vanish in* $|z| < k$, $k \in (0, 1)$ *and let* $\lambda = \lambda(k) = \frac{ka_1}{na_0}$. *Then for* $0 \leq \rho \leq k^2$

$$\max_{|z|=\rho}|p(z)| \geq \left(\frac{k^2+2|\lambda|k\rho+\rho^2}{k^2+2|\lambda|k+1}\right)^{\frac{n}{2}} \max_{|z|=1}|p(z)|. \tag{1.8}$$

In the case when n is even equality in (1.8) *is attained for*

$$p(z) = c(z^2e^{i2\beta} + 2kze^{i\beta}\cos\alpha + k^2)^{\frac{n}{2}}, \quad c(\neq 0) \in C \text{ and } \alpha, \beta \in R.$$

Recently, Mir et al. [7] proved the following interesting result and generalized a result due to Govil and Nwaeze [5] and many other results improving the ***Theorem*** of T. J. Rivlin [12].

***Theorem* 1.3.** *Let* $p(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$, $1 \leq \mu < n$ *be a polynomial of degree n that does not vanish in* $|z| < k$, $k \geq 1$.*Then for* $0 < r < R \leq 1$,

$$M(p,r) \geq \frac{(1+r^\mu)^{\frac{n}{\mu}}}{(1+r^\mu)^{\frac{n}{\mu}} + (R^\mu+r^\mu)^{\frac{n}{\mu}} - (k^\mu+r^\mu)^{\frac{n}{\mu}}}\left\{M(p,R) + m\ln\left(\frac{(R^\mu+k^\mu)}{(r^\mu+k^\mu)}\right)^{\frac{n}{\mu}}\right\}, \tag{1.9}$$

where $m = \min_{|z|=k}|p(z)|$ *and* $M(p,r) = \max_{|z|=r}|p(z)|$ *etc.*

2 Main Theorems

In this paper, firstly we prove the following result for the class of polynomials not vanishing in a prescribed disk, which improves upon the bound obtained by ***Theorem* 1.1**.

***Theorem* 2.1.** *Let* $p(z) = \sum_{j=0}^{n} a_j z^j \neq 0$ *for* $|z| < k$, $k \geq 1$ *and let* $\lambda = \lambda(k) = \frac{ka_1}{na_0}$. *Then for* $0 < \rho \leq 1$, *we have*

$$\max_{|z|=\rho} |p(z)| \geq \left(\frac{\rho^2 + 2|\lambda|k\rho + k^2}{1 + 2|\lambda|k + k^2} \right)^{\frac{n}{2}} \max_{|z|=1} |p(z)| + \frac{1}{k^n} \left\{ \left(\frac{\rho^2 + 2|\lambda|k\rho + k^2}{1 + 2|\lambda|k + k^2} \right)^{\frac{n}{2}} - \rho^n \right\} \min_{|z|=k} |p(z)|. \tag{2.1}$$

In the case where n is even, equality in (1.9) *is attained for*

$$p(z) = c(z^2 e^{i2\beta} + 2kze^{i\beta} \cos\alpha + k^2)^{\frac{n}{2}}, \quad c(\neq 0) \in C \text{ and } \alpha, \beta \in R.$$

The above inequality (2.1) always gives better bounds than inequality (1.7) except in the case when $\min_{|z|=k} |p(z)| = 0$.

Next we prove the following result, which is complement to ***Theorem* 2.1**, for the class of polynomials not vanishing in a disk of radius less than (or equal) unity and also improves upon ***Theorem* 1.2**.

***Theorem* 2.2.** *Let* $p(z) = \sum_{j=0}^{n} a_j z^j$ *does not vanish in* $|z| < k$, $k \in (0, 1)$ *and let* $\lambda = \lambda(k) = \frac{ka_1}{na_0}$. *Then for* $0 \leq \rho \leq k^2$

$$\max_{|z|=\rho} |p(z)| \geq \left(\frac{k^2 + 2|\lambda|k\rho + \rho^2}{k^2 + 2|\lambda|k + 1} \right)^{\frac{n}{2}} \max_{|z|=1} |p(z)| + \frac{1}{k^n} \left\{ \left(\frac{k^2 + 2|\lambda|k\rho + \rho^2}{k^2 + 2|\lambda|k + 1} \right)^{\frac{n}{2}} - \rho^n \right\} \min_{|z|=k} |p(z)|. \tag{2.2}$$

In the case when n is even equality in (2.2) *is attained for*

$$p(z) = c(z^2 e^{i2\beta} + 2kze^{i\beta} \cos\alpha + k^2)^{\frac{n}{2}}, \quad c(\neq 0) \in C \text{ and } \alpha, \beta \in R.$$

Finally, we prove the following interesting result, which improves upon ***Theorem* 1.3** by Mir et al. [7] and hence also generalizes and improves upon all those results which are claimed to be improved by ***Theorem* 1.3** as well.

***Theorem* 2.3.** *Let* $p(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$, $1 \leq \mu < n$ *be a polynomial of degree n that does not vanish in* $|z| < k$, $k \geq 1$. *Then for* $0 < r < R \leq 1$,

$$M(p, r) \geq \frac{(1 + r^\mu)^{\frac{n}{\mu}}}{(1 + r^\mu)^{\frac{n}{\mu}} + (R^\mu + k^\mu)^{\frac{n}{\mu}} - (r^\mu + k^\mu)^{\frac{n}{\mu}}} \left[M(p, R) + \left[\frac{n}{k^n} I_n - \frac{r^n}{k^n} \left\{ \left(\frac{(R^\mu + k^\mu)}{(r^\mu + k^\mu)} \right)^{\frac{n}{\mu}} - 1 \right\} + \ln \left(\frac{(R^\mu + k^\mu)}{(r^\mu + k^\mu)} \right)^{\frac{n}{\mu}} \right] \min_{|z|=k} |p(z)| \right]. \tag{2.3}$$

Here, the integral I_n *is defined as*

$$I_n = \int_r^R \frac{t^{n+\mu-1}}{t^\mu + k^\mu} dt. \tag{2.4}$$

Here, the integrand being a rational algebraic function can be evaluated by reduction formulae for a given value of n. For example, $I_0 = \frac{1}{2} \ln\left(\frac{R}{k}\right)$ *and* $I_1 = (R - r) - k \ln\left(\frac{R+k}{r+k}\right)$.

***Remark* 2.1.** *The integrand in* (2.4) *is increasing function of t, so the least approximate value of* I_n *can be taken as* $\left(\dfrac{n(R - r)r^{n+\mu-1}}{k^n(r^\mu + k^\mu)} \right)$.

3 Lemmas

For the proof of the ***Theorems***, we need the following lemmas.

***Lemma* 3.1.** *Let* $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \le \mu \le n$ *is a polynomial of degree n having no zeros in* $|z| < k$, $k \ge 1$. *Then*

$$\max_{|z|=1} |p'(z)| \le \frac{n}{1+k^\mu} \left\{ \max_{|z|=1} |p(z)| - \min_{|z|=1} |p(z)| \right\}. \tag{3.1}$$

The result is sharp and equality holds for the polynomial $p(z) = (z^\mu + k^\mu)^{\frac{n}{\mu}}$.

The above ***Lemma* 3.1** is due to Pukhta [9].

The next ***lemma*** is due to Bidkham and Dewan [4].

***Lemma* 3.2.** *Let* $p(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$, $1 \le \mu \le n$ *is a polynomial of degree n, having no zeros in* $|z| < k$, $k \ge 1$. *Then for* $0 < r < R \le 1$,

$$\max_{|z|=r} |p(z)| \ge \left(\frac{k^\mu + r^\mu}{k^\mu + R^\mu} \right)^{\frac{n}{\mu}} \max_{|z|=R} |p(z)|. \tag{3.2}$$

We improve the above ***Lemma* 3.2** as follows.

***Lemma* 3.3.** *Let* $p(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$ *is a polynomial of degree n, having no zeros in* $|z| < k$, $k \ge 1$. *Then for* $0 < r < R \le 1$,

$$\max_{|z|=r} |p(z)| \ge \left(\frac{k^\mu + r^\mu}{k^\mu + R^\mu} \right)^{\frac{n}{\mu}} \max_{|z|=R} |p(z)| + \left\{ \frac{R^n}{k^n} \left(\frac{k^\mu + r^\mu}{k^\mu + R^\mu} \right)^{\frac{n}{\mu}} - \frac{r^n}{k^n} \right\} \min_{|z|=k} |p(z)|. \tag{3.3}$$

Proof. **[Proof of Lemma 3.3]** Since $p(z)$ does not vanish in $|z| < k$, $k \ge 1$ and $|p(z)| \ge m = \min_{|z|=k} |p(z)|$, therefore by Rouche's theorem, the polynomial $F(z) = p(z) + \lambda \frac{z^n}{k^n} m$, $|\lambda| < 1$, also does not vanish in $|z| < k$, $k \ge 1$. Therefore on applying inequality (3.2) to the polynomial $F(z) = p(z) + \lambda \frac{z^n}{k^n} m$, we have

$$\max_{|z|=r} \left| p(z) + \lambda \frac{z^n}{k^n} m \right| \ge \left(\frac{k^\mu + r^\mu}{k^\mu + R^\mu} \right)^{\frac{n}{\mu}} \max_{|z|=R} \left| p(z) + \lambda \frac{z^n}{k^n} m \right|,$$

or

$$\max_{|z|=r} |p(z)| + |\lambda| \frac{r^n}{k^n} m \ge \left(\frac{k^\mu + r^\mu}{k^\mu + R^\mu} \right)^{\frac{n}{\mu}} \left(\max_{|z|=R} \left| p(z) + \lambda \frac{z^n}{k^n} m \right| \right). \tag{3.4}$$

Now suitably choosing the argument of λ such that R.H.S. of inequality (3.4) becomes

$$\max_{|z|=R} \left| p(z) + \lambda \frac{z^n}{k^n} m \right| = \max_{|z|=R} |p(z)| + |\lambda| \frac{R^n}{k^n} m. \tag{3.5}$$

Now combining inequalities (3.4) and (3.5), we get

$$\max_{|z|=r} |p(z)| + |\lambda| \frac{r^n}{k^n} m \ge \left(\frac{k^\mu + r^\mu}{k^\mu + R^\mu} \right)^{\frac{n}{\mu}} \left(\max_{|z|=R} |p(z)| + |\lambda| \frac{R^n}{k^n} m \right).$$

Or equivalently,

$$\max_{|z|=r} |p(z)| \ge \left(\frac{k^\mu + r^\mu}{k^\mu + R^\mu} \right)^{\frac{n}{\mu}} \max_{|z|=R} |p(z)| + |\lambda| \left\{ \frac{R^n}{r^n} \left(\frac{k^\mu + r^\mu}{k^\mu + R^\mu} \right)^{\frac{n}{\mu}} - \frac{r^n}{k^n} \right\} \min_{|z|=k} |p(z)|.$$

Finally letting $|\lambda| \to 1$, we get the desired result.

4 Proof of the Main Theorems

Proof. **[Proof of *Theorem* 2.1]** Since $p(z)$ does not vanish in $|z| < k$, $k \geq 1$ and $|p(z)| \geq m = \min\limits_{|z|=k} |p(z)|$, therefore by Rouche's theorem, the polynomial $F(z) = p(z) + \mu \frac{z^n}{k^n} m$, $|\mu| < 1$, also does not vanish in $|z| < k$, $k \geq 1$. Therefore on applying inequality (1.7) to the polynomial $F(z) = p(z) + \mu \frac{z^n}{k^n} m$, we have

$$\max_{|z|=\rho} |F(z)| \geq \left(\frac{k^2 + 2|\lambda| k\rho + \rho^2}{k^2 + 2|\lambda| k + 1} \right)^{\frac{n}{2}} \max_{|z|=1} |F(z)|$$

or

$$\max_{|z|=\rho} \left| p(z) + \mu m \frac{z^n}{k^n} \right| \geq \left(\frac{k^2 + 2|\lambda| k\rho + \rho^2}{k^2 + 2|\lambda| k + 1} \right)^{\frac{n}{2}} \max_{|z|=1} \left| p(z) + \mu m \frac{z^n}{k^n} \right|$$

or

$$\max_{|z|=\rho} |p(z)| + |\mu|\, m \frac{\rho^n}{k^n} \geq \left(\frac{k^2 + 2|\lambda| k\rho + \rho^2}{k^2 + 2|\lambda| k + 1} \right)^{\frac{n}{2}} \max_{|z|=1} \left| p(z) + \mu m \frac{z^n}{k^n} \right| . \tag{4.1}$$

Now suitably choosing argument of μ on R.H.S. of (4.1), we have

$$\max_{|z|=1} \left| p(z) + \mu m \frac{z^n}{k^n} \right| = \max_{|z|=1} |p(z)| + |\mu| \frac{m}{k^n} . \tag{4.2}$$

Combining (4.1) and (4.2) we get

$$\max_{|z|=\rho} |p(z)| + |\mu|\, m \frac{\rho^n}{k^n} \geq \left(\frac{k^2 + 2|\lambda| k\rho + \rho^2}{k^2 + 2|\lambda| k + 1} \right)^{\frac{n}{2}} \left\{ \max_{|z|=1} |p(z)| + |\mu| \frac{m}{k^n} \right\}$$

or

$$\max_{|z|=\rho} |p(z)| \geq \left(\frac{\rho^2 + 2|\lambda| k\rho + k^2}{1 + 2|\lambda| k + k^2} \right)^{\frac{n}{2}} \max_{|z|=1} |p(z)| + \frac{|\mu|}{k^n} \left\{ \left(\frac{\rho^2 + 2|\lambda| k\rho + k^2}{1 + 2|\lambda| k + k^2} \right)^{\frac{n}{2}} - \rho^n \right\} \min_{|z|=k} |p(z)| .$$

Finally, on letting $|\mu| \to 1$ the proof of ***Theorem* 2.1** is completed.

Proof. **[Proof of *Theorem* 2.2]** Proof of ***Theorem* 2.2** follows on the same lines as that of proof of ***Theorem* 2.1**. Here we use inequality (1.8) instead of (1.7). Hence we omit the details.

Proof. **[Proof of *Theorem* 2.3]** Let $0 < t \leq k$. Since $p(z)$ does not vanish in $|z| < k$, $k \geq 1$, the polynomial $F(z) = p(tz)$ does not vanish in $|z| < \frac{k}{t}$, $\frac{k}{t} \geq 1$, therefore applying ***Lemma* 3.1** to $F(z)$, we have

$$\max_{|z|=1} |F'(z)| \leq \frac{n}{1 + \left(\frac{k}{t}\right)^{\mu}} \left\{ \max_{|z|=1} |F(z)| - \min_{|z|=\frac{k}{t}} |F(z)| \right\},$$

which is equivalent to

$$\max_{|z|=t} |p'(z)| \leq \frac{n\, t^{\mu-1}}{t^{\mu} + k^{\mu}} \left\{ \max_{|z|=t} |p(z)| - \min_{|z|=k} |p(z)| \right\}. \tag{4.3}$$

We have, now for $0 \leq \theta < 2\pi$ and $0 < r < R \leq 1$,

$$\left| p(Re^{i\theta}) - p(re^{i\theta}) \right| \leq \int_r^R \left| p'(te^{i\theta}) \right| dt.$$

$$\left| p\left(Re^{i\theta}\right) - p\left(re^{i\theta}\right) \right| \leq \int_r^R \frac{n\, t^{\mu-1}}{t^{\mu} + k^{\mu}} \left\{ \max_{|z|=t} |p(z)| - \min_{|z|=k} |p(z)| \right\} dt \quad \text{by using (4.3)} \tag{4.4}$$

Now applying inequality (3.3) of ***Lemma* 3.3** to inequality (4.4), we get

$$\left| p\left(Re^{i\theta}\right) - p\left(re^{i\theta}\right) \right|$$

$$\leq \int_r^R \frac{nt^{\mu-1}}{t^\mu + k^\mu}\left[\left\{\left(\frac{t^\mu + k^\mu}{r^\mu + k^\mu}\right)^{\frac{n}{\mu}} \max_{|z|=r}|p(z)| - \left(\frac{t^n}{k^n} - \frac{r^n}{k^n}\left(\frac{t^\mu + k^\mu}{r^\mu + k^\mu}\right)^{\frac{n}{\mu}}\right)\min_{|z|=k}|p(z)|\right\} - \min_{|z|=k}|p(z)|\right]dt$$

$$= \int_r^R \frac{n\, t^{\mu-1}}{t^\mu + k^\mu}\left(\frac{t^\mu + k^\mu}{r^\mu + k^\mu}\right)^{\frac{n}{\mu}} \max_{|z|=r}|p(z)|dt - n\int_r^R \frac{t^{\mu-1}}{t^\mu + k^\mu}\left[\frac{t^n}{k^n} - \frac{r^n}{k^n}\left(\frac{t^\mu + k^\mu}{r^\mu + k^\mu}\right)^{\frac{n}{\mu}} + 1\right]\min_{|z|=k}|p(z)|dt.$$

The above expression is equivalent to

$$\begin{aligned}
&\left|p\left(Re^{i\theta}\right) - p\left(re^{i\theta}\right)\right| \\
&\leq \int_r^R \frac{nt^{\mu-1}}{(r^\mu + k^\mu)^{\frac{n}{\mu}}}(t^\mu + k^\mu)^{\frac{n}{\mu}-1}\max_{|z|=r}|p(z)|dt \\
&\quad - n\min_{|z|=k}|p(z)|\int_r^R\left[\frac{t^{n+\mu-1}}{(t^\mu + k^\mu)k^n} - \frac{r^n}{k^n}\frac{t^{\mu-1}(t^\mu + k^\mu)^{\frac{n}{\mu}-1}}{(r^\mu + k^\mu)^{\frac{n}{\mu}}} + \frac{t^{\mu-1}}{(t^\mu + k^\mu)}\right]dt \\
&\leq \frac{n}{(r^\mu + k^\mu)^{\frac{n}{\mu}}}M(p,r)\int_r^R t^{\mu-1}(t^\mu + k^\mu)^{\frac{n}{\mu}-1}dt - n\min_{|z|=k}|p(z)| \\
&\quad \int_r^R\left[\frac{t^{n+\mu-1}}{(t^\mu + k^\mu)k^n} - \frac{r^n}{k^n}\frac{t^{\mu-1}(t^\mu + k^\mu)^{\frac{n}{\mu}-1}}{(r^\mu + k^\mu)^{\frac{n}{\mu}}} + \frac{t^{\mu-1}}{(t^\mu + k^\mu)}\right]dt, \\
&\leq \frac{n}{(r^\mu + 1)^{\frac{n}{\mu}}}M(p,r)\int_r^R t^{\mu-1}(t^\mu + k^\mu)^{\frac{n}{\mu}-1}dt - \min_{|z|=k}|p(z)| \\
&\quad \left[\frac{n}{k^n}I_n - \frac{nr^n}{k^n(r^\mu + k^\mu)^{\frac{n}{\mu}}}\int_r^R t^{\mu-1}(t^\mu + k^\mu)^{\frac{n}{\mu}-1} + \int_r^R \frac{t^{\mu-1}}{(t^\mu + k^\mu)}dt\right] \\
&= \frac{(R^\mu + k^\mu)^{\frac{n}{\mu}} - (r^\mu + k^\mu)^{\frac{n}{\mu}}}{(1 + r^\mu)^{\frac{n}{\mu}}}M(p,r) \\
&\quad - \left[\frac{n}{k^n}I_n - \frac{r^n}{k^n}\frac{(R^\mu + k^\mu)^{\frac{n}{\mu}} - (r^\mu + k^\mu)^{\frac{n}{\mu}}}{(k^\mu + r^\mu)^{\frac{n}{\mu}}} + \frac{n}{\mu}\ \ln\left(\frac{(R^\mu + k^\mu)}{(r^\mu + k^\mu)}\right)\right]\min_{|z|=k}|p(z)| \\
&= \frac{(R^\mu + k^\mu)^{\frac{n}{\mu}} - (r^\mu + k^\mu)^{\frac{n}{\mu}}}{(1 + r^\mu)^{\frac{n}{\mu}}}M(p,r) \\
&\quad - \left[\frac{n}{k^n}I_n - \frac{r^n}{k^n}\left\{\left(\frac{R^\mu + k^\mu}{r^\mu + k^\mu}\right)^{\frac{n}{\mu}} - 1\right\} + \ln\left(\frac{(R^\mu + k^\mu)}{(r^\mu + k^\mu)}\right)^{\frac{n}{\mu}}\right]\min_{|z|=k}|p(z)|
\end{aligned}$$

where the integral I_n is as defined in (2.4).

Thus we have shown that for $0 \leq \theta < 2\pi$ and $0 < r < R \leq 1$,

$$\begin{aligned}
\left|p\left(Re^{i\theta}\right) - p\left(re^{i\theta}\right)\right| \leq\ & \frac{(R^\mu + k^\mu)^{\frac{n}{\mu}} - (r^\mu + k^\mu)^{\frac{n}{\mu}}}{(1 + r^\mu)^{\frac{n}{\mu}}}M(p,r) \\
& - \left[\frac{n}{k^n}\,I_n - \frac{r^n}{k^n}\left\{\left(\frac{R^\mu + k^\mu}{r^\mu + k^\mu}\right)^{\frac{n}{\mu}} - 1\right\} + \ln\left(\frac{(R^\mu + k^\mu)}{(r^\mu + k^\mu)}\right)^{\frac{n}{\mu}}\right]\min_{|z|=k}|p(z)|
\end{aligned}$$

Therefore, finally, we have the equivalent result

$$\begin{aligned}
M(p,R) \leq M(p,r) + & \frac{(R^\mu + k^\mu)^{\frac{n}{\mu}} - (r^\mu + k^\mu)^{\frac{n}{\mu}}}{(1 + r^\mu)^{\frac{n}{\mu}}}M(p,r) \\
& - \left[\frac{n}{k^n}I_n - \frac{r^n}{k^n}\left\{\left(\frac{(R^\mu + k^\mu)}{(r^\mu + k^\mu)}\right)^{\frac{n}{\mu}} - 1\right\} + \ln\left(\frac{(R^\mu + k^\mu)}{(r^\mu + k^\mu)}\right)^{\frac{n}{\mu}}\right]\min_{|z|=k}|p(z)|
\end{aligned}$$

or

$$M(p,R) \le \frac{(1+r^{\mu})^{\frac{n}{\mu}} + (R^{\mu}+k^{\mu})^{\frac{n}{\mu}} - (r^{\mu}+k^{\mu})^{\frac{n}{\mu}}}{(1+r^{\mu})^{\frac{n}{\mu}}} M(p,r)$$
$$- \left[\frac{n}{k^n} I_n - \frac{r^n}{k^n}\left\{\left(\frac{(R^{\mu}+k^{\mu})}{(r^{\mu}+k^{\mu})}\right)^{\frac{n}{\mu}} - 1\right\} + \ln\left(\frac{(R^{\mu}+k^{\mu})}{(r^{\mu}+k^{\mu})}\right)^{\frac{n}{\mu}}\right] \min_{|z|=k} |p(z)|$$

or equivalently

$$M(p,r) \ge \frac{(1+r^{\mu})^{\frac{n}{\mu}}}{(1+r^{\mu})^{\frac{n}{\mu}} + (R^{\mu}+k^{\mu})^{\frac{n}{\mu}} - (r^{\mu}+k^{\mu})^{\frac{n}{\mu}}}$$
$$\left[M(p,R) + \left[\frac{n}{k^n} I_n - \frac{r^n}{k^n}\left\{\left(\frac{(R^{\mu}+k^{\mu})}{(r^{\mu}+k^{\mu})}\right)^{\frac{n}{\mu}} - 1\right\} + \ln\left(\frac{(R^{\mu}+k^{\mu})}{(r^{\mu}+k^{\mu})}\right)^{\frac{n}{\mu}}\right] \min_{|z|=k} |p(z)|\right].$$

Thus the desired result is proved.

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