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(Dedicated to Honor Professor H.M. Srivastava on His 80th Birth Anniversary Celebrations)

A GENERALIZED SUBCLASS OF ALPHA CONVEX BI-UNIVALENT FUNCTIONS OF COMPLEX ORDER

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Abstract

In this present investigation a subclass of alpha convex bi-univalent functions of complex order in the open unit disc $U = \{z : |z| < 1\}$, defined by Salagean operator and quasi-subordination is discussed. The estimates on the initial coefficients $|a_2|$ and $|a_3|$ for the functions in this subclass are studied. The results obtained in this paper would generalise those already proved by various authors.

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1 Introduction and Preliminaries

Let A be the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic in the open unit disc $U = \{z : |z| < 1\}$. By S, we denote the class of functions $f(z) \in A$ and univalent in U.

Let us denote by B, the class of bounded or Schwarz functions $w(z)$ satisfying $w(0) = 0$ and $|w(z)| \leq 1$ which are analytic in the open unit disc U and of the form

$$w(z) = \sum_{n=1}^{\infty} c_n z^n, z \in U.$$

A function $f \in S$ is said to be starlike if it satisfies the inequality

$$Re\left(\frac{zf'(z)}{f(z)}\right) > 0 (z \in U).$$

The class of starlike functions is denoted by S^*.
A function $f \in S$ is said to be convex if it satisfies the inequality

$$Re\left(\frac{(zf'(z))'}{f'(z)}\right) > 0 (z \in U).$$

The class of convex functions is denoted by K.

A function $f \in S$ is said to be α-convex if it satisfies the inequality

$$Re\left((1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\frac{(zf'(z))'}{f'(z)}\right) > 0 (0 \leq \alpha \leq 1, z \in U).$$

The class of α-convex functions is denoted by $M(\alpha)$ and was introduced by Mocanu [8]. In particular $M(0) \equiv S^*$ and $M(1) \equiv K$.

For $f \in A$, Salagean [14] introduced the following operator:

$$D^0 f(z) = f(z), D^1 f(z) = zf'(z),$$

and in general,

$$D^n f(z) = D(D^{n-1} f(z)), n \in N$$

or equivalent to

$$D^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k, n \in N_0 = N \cup (0).$$

The inverse functions of the functions in the class S may not be defined on the entire unit disc U although the functions in the class S are invertible. However using Koebe-one quarter theorem [4] it is obvious that the image of U under every function $f \in S$ contains a disc of radius $\frac{1}{4}$. Hence every univalent function f has an inverse f^{-1}, defined by

$$f^{-1}(f(z)) = z (z \in U)$$

and

$$f(f^{-1}(w)) = w\left(| w |< r_0(f) : r_0(f) \geq \frac{1}{4}\right),$$

where

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \dots . \tag{1.2}$$

A function $f \in A$ is said to be bi-univalent in U if both f and f^{-1} are univalent in U.

By Σ, we denote the class of bi-univalent functions in U defined by (1.1).

Consider two functions f and g analytic in U. We say that f is subordinate to g (symbolically $f \prec g$) if there exists a bounded function $u(z) \in B$ for which $f(z) = g(u(z))$. This result is known as principle of subordination.

Robertson [13] introduced the concept of quasi-subordination in 1970. If f and ϕ are analytic functions, then we say that f is quasi-subordinate to ϕ (symbolically $f \prec_q \phi$) if there exists analytic functions k and ω with $|k(z)| \leq 1$, $\omega(0) = 0$ and $|\omega(z)| < 1$ such that

$$\frac{f(z)}{k(z)} \prec \phi(z),$$

or it is equivalent to

$$f(z) = k(z)\phi(\omega(z)). \tag{1.3}$$

In particular for $k(z) = 1$, $f(z) = \phi(\omega(z))$, so that $f(z) \prec \phi(z)$ in U. It is obvious to see that the quasi-subordination is a generalization of the usual subordination. The work on quasi-subordination is quite extensive which finds interesting dimensions in some recent investigations [1,5,7,12].

Lewin [6] discussed the class Σ of bi-univalent functions and obtained the bound for the second coefficient. Brannan and Taha [2] investigated certain subclasses of bi-univalent functions, similar to the familiar subclasses of univalent functions consisting of strongly starlike, starlike and convex functions. They introduced bi-starlike functions and bi-convex functions and obtained estimates on the initial coefficients. Also the subclasses of bi-univalent functions defined by Salagean operator were studied by various authors [3,9,11,15].

The earlier work on bi-univalent functions defined by quasi-subordination and Salagean operator motivate us to define the following subclass:

Also we assume that $\phi(z)$ is analytic in U with $\phi(0) = 1$ and let

$$\phi(z) = 1 + B_1 z + B_2 z^2 + ...(B_1 \in R^+) \tag{1.4}$$

and

$$k(z) = A_0 + A_1 z + A_2 z^2 + ...(|k(z)| \leq 1, z \in U). \tag{1.5}$$

To avoid repetition, throughout the paper we assume that $0 \leq \alpha \leq 1$ and $z \in U$.

***Definition* 1.1.** *A function $f \in \Sigma$ given by* (1.1) *is said to be in the class $M_\Sigma(n, \alpha, \gamma, \phi)$ if it satisfy the following conditions:*

$$\frac{1}{\gamma}\left[(1-\alpha)\frac{z(D^{n-1}f(z))'}{D^{n-1}f(z)} + \alpha\frac{z(D^n f(z))'}{D^n f(z)} - 1\right] \prec_q (\phi(z) - 1) \tag{1.6}$$

and

$$\frac{1}{\gamma}\left[(1-\alpha)\frac{w(D^{n-1}g(w))'}{D^{n-1}g(w)} + \alpha\frac{w(D^n g(w))'}{D^n g(w)} - 1\right] \prec_q (\phi(w) - 1), \tag{1.7}$$

where $g = f^{-1}$ and $z, w \in U$.

The folowing observations are obvious:

(i) $M_\Sigma(n, \alpha, 1, \phi) \equiv M_\Sigma(n, \alpha, \phi)$.

(ii) $M_\Sigma(1, \alpha, 1, \phi) \equiv M_\Sigma(\alpha, \phi)$.

(iii) $M_\Sigma(1, 0, 1, \phi) \equiv S^*_\Sigma(\phi)$*, the class of bi-starlike functions defined with quasi subordination.*

(iv) $M_\Sigma(1, 1, 1, \phi) \equiv K_\Sigma(\phi)$*, the class of bi-conves functions defined with quasi subordination.*

For deriving our main results, we need the following lemma:

***Lemma* 1.1.** [10] *If $p \in P$ be family of all functions p analytic in U for which $Re[p(z)] > 0$ and have the form $p(z) = 1 + p_1 z + p_2 z^2 + ...$ for $z \in U$, then $|p_n| \leq 2$ for each n.*

2 Coefficient bounds for the function class $M_\Sigma(n, \alpha, \gamma, \phi)$

***Theorem* 2.1.** *If $f \in M_\Sigma(n, \alpha, \gamma, \phi)$, then*

$$|a_2| \leq min.\left[\frac{|A_0\gamma|B_1}{(n+\alpha+1)}, \sqrt{\frac{|A_0\gamma|(B_1 + |B_2 - B_1|)}{(n+\alpha+1)}}\right] \tag{2.1}$$

and

$$|a_3| \leq min.\left[\frac{|A_0\gamma|(B_1 + |B_2 - B_1|)}{(n+\alpha+1)} + \frac{|A_1\gamma|B_1 + |A_0\gamma|B_1}{(n+2)(n+2\alpha+1)}\right. \tag{2.2}$$

$$\left.\frac{|\gamma|}{(n+2)(n+2\alpha+1)}\left[|\gamma|\left[\frac{(n+1)^2 + \alpha(2n+3)}{(n+\alpha+1)^2}\right]B_1^2|A_0|^2 + (B_1 + |B_2 - B_1|)|A_0| + |A_1|B_1\right]\right].$$

Proof. As $f \in M_\Sigma(n,\alpha,\gamma,\phi)$, so by ***Definition* 1.1**, using the concept of quasi-subordination, there exists Schwarz functions $r(z)$ and $s(z)$ and analytic function $k(z)$ such that

$$\frac{1}{\gamma}\left[(1-\alpha)\frac{z(D^{n-1}f(z))'}{D^{n-1}f(z)}+\alpha\frac{z(D^{n}f(z))'}{D^{n}f(z)}-1\right]=k(z)(\phi(r(z))-1) \tag{2.3}$$

and

$$\frac{1}{\gamma}\left[(1-\alpha)\frac{w(D^{n-1}g(w))'}{D^{n-1}g(w)}+\alpha\frac{w(D^{n}g(w))'}{D^{n}g(w)}-1\right]=k(w)(\phi(s(w))-1), \tag{2.4}$$

where $r(z)=1+r_1z+r_2z^2+...$ and $s(w)=1+s_1w+s_2w^2+....$

Define the functions $p(z)$ and $q(z)$ by

$$r(z)=\frac{p(z)-1}{p(z)+1}=\frac{1}{2}\left[c_1z+(c_2-\frac{c_1^2}{2})z^2+...\right] \tag{2.5}$$

and

$$s(z)=\frac{q(z)-1}{q(z)+1}=\frac{1}{2}\left[d_1z+(d_2-\frac{d_1^2}{2})z^2+...\right]. \tag{2.6}$$

Using (2.5) and (2.6) in (2.3) and (2.4) respectively, it yields

$$\frac{1}{\gamma}\left[(1-\alpha)\frac{z(D^{n-1}f(z))'}{D^{n-1}f(z)}+\alpha\frac{z(D^{n}f(z))'}{D^{n}f(z)}-1\right]=k(z)\left[\phi\left(\frac{p(z)-1}{p(z)+1}\right)-1\right] \tag{2.7}$$

and

$$\frac{1}{\gamma}\left[(1-\alpha)\frac{w(D^{n-1}g(w))'}{D^{n-1}g(w)}+\alpha\frac{w(D^{n}g(w))'}{D^{n}g(w)}-1\right]=k(w)\left[\phi\left(\frac{q(w)-1}{q(w)+1}\right)-1\right]. \tag{2.8}$$

But

$$\frac{1}{\gamma}\left[(1-\alpha)\frac{z(D^{n-1}f(z))'}{D^{n-1}f(z)}+\alpha\frac{z(D^{n}f(z))'}{D^{n}f(z)}-1\right]$$

$$=\frac{1}{\gamma}\left[(n+\alpha+1)a_2z+[(n+2)(n+2\alpha+1)a_3-((n+1)^2+\alpha(2n+3))a_2^2]z^2+...\right] \tag{2.9}$$

and

$$\frac{1}{\gamma}\left[(1-\alpha)\frac{w(D^{n-1}g(w))'}{D^{n-1}g(w)}+\alpha\frac{w(D^{n}g(w))'}{D^{n}g(w)}-1\right] \tag{2.10}$$

$$=\frac{1}{\gamma}\left[-(n+\alpha+1)a_2w+[(n+2)(n+2\alpha+1)(2a_2^2-a_3)-((n+1)^2+\alpha(2n+3))a_2^2]w^2+...\right].$$

Again using (1.4) and (1.5) in (2.5) and (2.6) respectively, we get

$$k(z)\left[\phi\left(\frac{p(z)-1}{p(z)+1}\right)-1\right]=\frac{1}{2}A_0B_1c_1z+\left[\frac{1}{2}A_1B_1c_1+\frac{1}{2}A_0B_1\left(c_2-\frac{c_1^2}{2}\right)+\frac{A_0B_2c_1^2}{4}\right]z^2+... \tag{2.11}$$

and

$$k(w)\left[\phi\left(\frac{q(w)-1}{q(w)+1}\right)-1\right]=\frac{1}{2}A_0B_1d_1w+\left[\frac{1}{2}A_1B_1d_1+\frac{1}{2}A_0B_1\left(d_2-\frac{d_1^2}{2}\right)+\frac{A_0B_2d_1^2}{4}\right]w^2+... \tag{2.12}$$

Using (2.9) and (2.11) in (2.7) and equating the coefficients of z and z^2, we get

$$\frac{(n+\alpha+1)}{\gamma}a_2=\frac{1}{2}A_0B_1c_1 \tag{2.13}$$

and

$$(2.14)\quad \frac{(n+2)(n+2\alpha+1)a_3-((n+1)^2+\alpha(2n+3))a_2^2}{\gamma} = \frac{1}{2}A_1B_1c_1+\frac{1}{2}A_0B_1\left(c_2-\frac{c_1^2}{2}\right)+\frac{A_0B_2}{4}c_1^2.$$

Again using (2.10) and (2.12) in (2.8) and equating the coefficients of w and w^2, we get

$$(2.15)\quad -\frac{(n+\alpha+1)}{\gamma}a_2 = \frac{1}{2}A_0B_1d_1$$

and

$$(2.16)\quad \frac{(n+2)(n+2\alpha+1)(2a_2^2-a_3)-((n+1)^2+\alpha(2n+3))a_2^2}{\gamma}$$
$$= \frac{1}{2}A_1B_1d_1+\frac{1}{2}A_0B_1\left(d_2-\frac{d_1^2}{2}\right)+\frac{A_0B_2}{4}d_1^2.$$

From (2.13) and (2.15), it is clear that

$$(2.17)\quad c_1 = -d_1$$

and

$$(2.18)\quad a_2 = \frac{A_0B_1c_1\gamma}{2(n+\alpha+1)} = -\frac{A_0B_1d_1\gamma}{2(n+\alpha+1)}.$$

Therefore on applying triangle inequality and using ***Lemma* 1.1**, (2.18) yields

$$(2.19)\quad |a_2| \leq \frac{|A_0\gamma|B_1}{(n+\alpha+1)}.$$

Adding (2.14) and (2.16), it yields

$$(2.20)\quad \frac{2[(n+2)(n+2\alpha+1)-(n+1)^2-\alpha(2n+3)]}{\gamma}a_2^2 = \frac{1}{2}A_0B_1(c_2+d_2)+\frac{A_0(B_2-B_1)}{4}(c_1^2+d_1^2).$$

Using ***Lemma* 1.1** and on applying triangle inequality in (2.20), we obtain

$$(2.21)\quad |a_2|^2 \leq \frac{|A_0\gamma|(B_1+|B_2-B_1|)}{(n+\alpha+1)}.$$

So, the result (2.1) can be easily obtained from (2.19) and (2.21).

Now subtracting (2.16) from (2.14), we obtain

$$(2.22)\quad a_3 = a_2^2+\frac{A_1B_1(c_1-d_1)+A_0B_1(c_2-d_2)}{4(n+2)(n+2\alpha+1)}\gamma.$$

Applying triangle inequality and using ***Lemma* 1.1** and (2.21) in (2.22), it yields

$$(2.23)\quad |a_3| \leq \frac{|A_0|(B_1+|B_2-B_1|)}{(n+\alpha+1)}+\frac{|A_1\gamma|B_1+|A_0\gamma|B_1}{(n+2)(n+2\alpha+1)}.$$

From (2.13) and (2.14), we have

(2.24)

$$|a_3| \leq \frac{|\gamma|}{(n+2)(n+2\alpha+1)}\left[|\gamma|\left[\frac{(n+1)^2+\alpha(2n+3)}{(n+\alpha+1)^2}\right]B_1^2|A_0|^2+(B_1+|B_2-B_1|)|A_0|+|A_1|B_1\right].$$

Again from (2.15) and (2.17), it gives

(2.25)

$$|a_3| \leq \frac{|\gamma|}{(n+2)(n+2\alpha+1)}\left[|\gamma|\left[\frac{n^2+2n\alpha+5\alpha+3}{(n+\alpha+1)^2}\right]B_1^2|A_0|^2+(B_1+|B_2-B_1|)|A_0|+|A_1|B_1\right].$$

Since R.H.S. of (2.25) is greater than that of (2.24), so result (2.2) is obvious.

For $\gamma = 1$, ***Theorem* 2.1** gives the following result:

***Corollary* 2.1.** *If $M_{\Sigma}(n,\alpha,\phi)$, then*

$$|a_2| \leq min.\left[\frac{|A_0|B_1}{(n+\alpha+1)}, \sqrt{\frac{|A_0|(B_1+|B_2-B_1|)}{(n+\alpha+1)}}\right]$$

and

$$|a_3| \leq min.\left[\frac{|A_0|(B_1+|B_2-B_1|)}{(n+\alpha+1)} + \frac{|A_1|B_1+|A_0|B_1}{(n+2)(n+2\alpha+1)},\right.$$
$$\left.\frac{1}{(n+2)(n+2\alpha+1)}\left[\left[\frac{(n+1)^2+\alpha(2n+3)}{(n+\alpha+1)^2}\right]B_1^2|A_0|^2+(B_1+|B_2-B_1|)|A_0|+|A_1|B_1\right]\right].$$

For $\gamma = 1$ and $n = 1$, the following result is obvious from ***Theorem* 2.1**:

***Corollary* 2.2.** *If $f(z) \in M_{\Sigma}(\alpha,\phi)$, then*

$$|a_2| \leq min.\left[\frac{|A_0|B_1}{(2+\alpha)}, \sqrt{\frac{|A_0|(B_1+|B_2-B_1|)}{(2+\alpha)}}\right]$$

and

$$|a_3| \leq min.\left[\frac{|A_0|(B_1+|B_2-B_1|)}{(2+\alpha)} + \frac{|A_1|B_1+|A_0|B_1}{6(1+\alpha)},\right.$$
$$\left.\frac{1}{6(1+\alpha)}\left[\left[\frac{4+5\alpha}{(2+\alpha)^2}\right]B_1^2|A_0|^2+(B_1+|B_2-B_1|)|A_0|+|A_1|B_1\right]\right].$$

For $\gamma = 1, \alpha = 0$ and $n = 1$, ***Theorem* 2.1** gives the following result:

***Corollary* 2.3.** *If $f(z) \in S^*_{\Sigma}(\phi)$, then*

$$|a_2| \leq min.\left[\frac{|A_0|B_1}{2}, \sqrt{\frac{|A_0|(B_1+|B_2-B_1|)}{2}}\right]$$

and

$$|a_3| \leq min.\left[\frac{|A_0|(B_1+|B_2-B_1|)}{2} + \frac{|A_1|B_1+|A_0|B_1}{6},\right.$$
$$\left.\frac{1}{6}\left[B_1^2|A_0|^2+(B_1+|B_2-B_1|)|A_0|+|A_1|B_1\right]\right].$$

For $\gamma = 1, \alpha = 1$ and $n = 1$, the following result is obvious from *Theorem* 2.1:

***Corollary* 2.4.** *If $f(z) \in K_{\Sigma}(\phi)$, then*

$$|a_2| \leq min.\left[\frac{|A_0|B_1}{3}, \sqrt{\frac{|A_0|(B_1+|B_2-B_1|)}{3}}\right]$$

and

$$|a_3| \leq min.\left[\frac{|A_0|(B_1+|B_2-B_1|)}{3} + \frac{|A_1|B_1+|A_0|B_1}{12},\right.$$
$$\left.\frac{1}{12}\left[B_1^2|A_0|^2+(B_1+|B_2-B_1|)|A_0|+|A_1|B_1\right]\right].$$

3. Conclusion

This paper is concerened with a very generalized subclass of alpha convex bi-univalent functions of complex order in the open unit disc. The class is associated with Salagean operator and is defined by means of quasi-subordination. We have studied the estimates of the initial coefficients $|a_2|$ and $|a_3|$ for the functions in this class. By giving the particular values to the various paprameters like α, γ, n and q, the results already proved by earlier researchers can be easily obtained. So this paper will work as a milestone to the future researchers in this field.

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