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(Dedicated to Honor Professor H.M. Srivastava on His 80th Birth Anniversary Celebrations)

ADJOINTNESS FOR SHEFFER POLYNOMIALS

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Abstract

In recent papers, new sets of Sheffer and Brenke polynomials based on higher order Bell numbers, and several integer sequences related to them have been studied. In this article new sets of Sheffer polynomials are introduced defining a sort of adjointness property. As an application, we show the adjoint set of Actuarial polynomials and derive their main characteristics.
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1 Introduction

In recent articles [24, 8], new sets of Sheffer and Brenke polynomials [7] based on higher order Bell numbers, have been studied. Furthermore, several integer sequences associated with the considered polynomials sets both of exponential and logarithmic type have been introduced.

In this article adjoint sets of Sheffer polynomials are considered and a particular case is analyzed.

2 Sheffer polynomials

The Sheffer polynomials $\{s_n(x)\}$ are introduced [26] by means of the exponential generating function [28] of the type:

$$A(t)\exp(xH(t)) = \sum_{n=0}^{\infty} s_n(x)\frac{t^n}{n!}\,, \tag{2.1}$$

where

$$A(t) = \sum_{n=0}^{\infty} a_n\frac{t^n}{n!}\,, \qquad (a_0 \neq 0), \tag{2.2}$$

$$H(t) = \sum_{n=0}^{\infty} h_n\frac{t^n}{n!}\,, \qquad (h_0 = 0).$$

According to a different characterization (see [25, p. 18]), the same polynomial sequence can be defined by means of the pair $(g(t), f(t))$, where $g(t)$ is an invertible series and $f(t)$ is a delta series:

$$g(t) = \sum_{n=0}^{\infty} g_n\frac{t^n}{n!}\,, \qquad (g_0 \neq 0), \tag{2.3}$$

$$f(t) = \sum_{n=0}^{\infty} f_n \frac{t^n}{n!}\,, \qquad (f_0 = 0, f_1 \neq 0).$$

Denoting by $f^{-1}(t)$ the compositional inverse of $f(t)$, i.e. such that

$$f\left(f^{-1}(t)\right) = f^{-1}\left(f(t)\right) = t,$$

the exponential generating function of the sequence $\{s_n(x)\}$ is given by

$$\frac{1}{g[f^{-1}(t)]} \exp\left(x f^{-1}(t)\right) = \sum_{n=0}^{\infty} s_n(x) \frac{t^n}{n!}\,, \tag{2.4}$$

so that

$$A(t) = \frac{1}{g[f^{-1}(t)]}, \qquad H(t) = f^{-1}(t). \tag{2.5}$$

When $g(t) \equiv 1$, the Sheffer sequence corresponding to the pair $(1, f(t))$ is called the associated Sheffer sequence $\{\sigma_n(x)\}$ for $f(t)$, and its exponential generating function is given by

$$\exp\left(x f^{-1}(t)\right) = \sum_{n=0}^{\infty} \sigma_n(x) \frac{t^n}{n!}\,. \tag{2.6}$$

A list of known Sheffer polynomial sequences and their associated ones can be found in [5, 6].

3 Adjointness for Sheffer polynomial sequences

According to the above considerations, Sheffer polynomials are characterized both by the ordered couples $(A(t), H(t))$, or by $(g(t), f(t))$.

***Definition* 3.1.** *Adjoint Sheffer polynomials are defined by interchanging the ordered couple $(A(t), H(t))$ with $(g(t), f(t))$, when writing the generating function.*

Here and in the following the tilde *"$\sim$", above the symbol of a polynomial set stands for the adjective* "adjoint"*.*

Examples of adjoint polynomial sets are listed here.

Adjoint-Hermite polynomials

$$A(t) = \exp(t^2/4), \qquad H(t) = t/2, \tag{3.1}$$

$$G(t, x) = \exp\left[\frac{t(t+2x)}{4}\right] = \sum_{n=0}^{\infty} \tilde{H}e_n(x) \frac{t^n}{n!}.$$

Adjoint-generalized Hermite polynomials

$$A(t) = \exp\left[(t/\nu)^m\right], \qquad H(t) = t/\nu, \tag{3.2}$$

$$G(t, x) = \exp\left[\left(\frac{t}{\nu}\right)^m + \frac{xt}{\nu}\right] = \sum_{n=0}^{\infty} \tilde{H}_n^{(m)}(x) \frac{t^n}{n!}.$$

Adjoint modified Pidduck polynomials

$$A(t) = \frac{2}{e^t + 1}, \qquad H(t) = \frac{e^t - 1}{e^t + 1}, \tag{3.3}$$

$$G(t, x) = \frac{2}{e^t + 1} \exp\left[x\left(\frac{e^t - 1}{e^t + 1}\right)\right] = \sum_{n=0}^{\infty} \tilde{\mathcal{P}}_n(x) \frac{t^n}{n!}.$$

Adjoint Actuarial polynomials

$$A(t) = (1-t)^{-\beta}, \qquad H(t) = \log(1-t), \tag{3.4}$$

$$G(t,x) = (1-t)^{x-\beta} = \sum_{n=0}^{\infty} \tilde{\alpha}_n^{(\beta)}(x) \frac{t^n}{n!}.$$

Adjoint Poisson-Charlier polynomials

(3.5)
$$A(t) = \exp\left(a(e^t-1)\right), \qquad H(t) = a\left(e^t - 1\right),$$
$$G(t,x) = \exp\left(a(1+x)(e^t-1)\right) = \sum_{n=0}^{\infty} \tilde{c}_n(x;a) \frac{t^n}{n!}.$$

Adjoint Peters polynomials

(3.6)
$$A(t) = \left(1+e^{\lambda t}\right)^{\mu}, \qquad H(t) = e^t - 1,$$
$$G(t,x) = \left(1+e^{\lambda t}\right)^{\mu} \exp\left(x(e^t-1)\right) = \sum_{n=0}^{\infty} \tilde{s}_n(x;\lambda,\mu) \frac{t^n}{n!}.$$

Adjoint Bernoulli polynomials of the second kind

(3.7)
$$A(t) = \frac{t}{e^t-1}, \qquad H(t) = e^t - 1,$$
$$G(t,x) = \frac{t}{e^t-1} \exp\left(x\left(e^t-1\right)\right) = \sum_{n=0}^{\infty} \tilde{b}_n(x) \frac{t^n}{n!}.$$

Adjoint Related polynomials

(3.8)
$$A(t) = \frac{1+e^t}{2}, \qquad H(t) = e^t - 1,$$
$$G(t,x) = \frac{1+e^t}{2} \exp\left[x\left(e^t-1\right)\right] = \sum_{n=0}^{\infty} \tilde{r}_n(x) \frac{t^n}{n!}.$$

Adjoint Hahn polynomials

(3.9)
$$A(t) = \sec t, \qquad H(t) = \tan t,$$
$$G(t,x) = \sec t \exp(x \tan t) = \sum_{n=0}^{\infty} \tilde{R}_n(x) \frac{t^n}{n!}.$$

***Remark* 3.1.** *In the above list we have not considered the Laguerre polynomials $L_n(x)$ and their generalized form $L_n^{\alpha}(x)$, since they are self-adjoint, in the sense that they coincide respectively with $\tilde{L}_n(x)$ and $\tilde{L}_n^{\alpha}(x)$.*

Furthermore, recalling the Hermite polynomials $H_n(x)$ and their generalized form $H_n^m(x)$, introduced by M. Lahiri [19] (*see also the article by R.C.S. Chandel* [11]), *it is worth to note that their respective Adjoint form $\tilde{H}_n(x)$ and $\tilde{H}_n^{(m)}(x)$ reduce to a particular case of the Appell-Kampé de Fériet polynomials in two variables* [1], *so that they will not be considered in the following.*

***Remark* 3.2.** *It is also worth to note that generalized Laguerre-type polynomial families have been considered by G. Dattoli* [13], *and further extensions have been recently obtained by using umbral methods* [2, 20].

4 Adjoint Actuarial polynomials

Actuarial polynomials have been considered in the book by R.P. Boas and R.C. Buck [6]. They were previously introduced by J.F. Steffensen [29] and also sudied by L. Toscano [31].

Here we consider the adjoint Actuarial polynomials, defined through their generating function, i.e. by putting

(4.1)
$$A(t) = (1-t)^{-\beta}, \qquad H(t) = \log(1-t),$$
$$G(t,x) = \exp\left[(x-\beta)\log(1-t)\right] = \sum_{k=0}^{\infty} \tilde{\alpha}_k^{(\beta)}(x) \frac{t^k}{k!}.$$

4.1 Recurrence relation

***Theorem* 4.1.** *For any $k \geq 0$, the polynomials $\tilde{\alpha}_k^{(\beta)}(x)$ satisfy the following recurrence relation:*

$$\tilde{\alpha}_{k+1}^{(\beta)}(x) = \textstyle\sum_{h=0}^{k} \frac{k!}{(k-h)!}(\beta - x)\tilde{\alpha}_{k-h}^{(\beta)}(x). \tag{4.2}$$

Proof. Differentiating $G(t,x)$ with respect to t, we have

$$\frac{\partial G(t,x)}{\partial t} = \left(\frac{x-\beta}{t-1}\right)\exp\left[(\beta - x)\log(1-t)\right], \tag{4.3}$$

$$\frac{\partial G(t,x)}{\partial t} = G(t,x)(\beta - x)\frac{1}{1-t} = (\beta - x)\sum_{k=0}^{\infty}\tilde{\alpha}_k^{(\beta)}(x)\frac{t^k}{k!}\sum_{k=0}^{\infty}k!\frac{t^k}{k!} = \tag{4.4}$$

$$= \sum_{k=1}^{\infty}\tilde{\alpha}_k^{(\beta)}(x)\frac{t^{k-1}}{(k-1)!}.$$

i.e.

$$(\beta - x)\sum_{k=0}^{\infty}\left[\sum_{h=0}^{k}\binom{k}{h}\tilde{\alpha}_{k-h}^{(\beta)}(x)h!\right]\frac{t^k}{k!} = \sum_{k=0}^{\infty}\tilde{\alpha}_{k+1}^{(\beta)}(x)\frac{t^k}{k!}.$$

so that the recurrence relation (4.2) follows.

4.2 Generating function's PDE

***Theorem* 4.2.** *The generating function* (4.1) *satisfies the homogeneous linear PDE*

$$(x-\beta)\frac{\partial G}{\partial x} + (1-t)\log(1-t)\frac{\partial G}{\partial t} = 0. \tag{4.5}$$

Proof. Differentiating $G(t,x)$ with respect to x, we have

$$\frac{\partial G(t,x)}{\partial x} = \log(1-t)\exp\left[(x-\beta)\log(1-t)\right]. \tag{4.6}$$

From equation (4.6) we find

$$\exp\left[(x-\beta)\log(1-t)\right] = \frac{1}{\log(1-t)}\frac{\partial G}{\partial x}.$$

Eliminating the exponential function in equation (4.3), by using the above equation, we find

$$\frac{\partial G(t,x)}{\partial t} = \left(\frac{x-\beta}{t-1}\right)\frac{1}{\log(1-t)}\frac{\partial G}{\partial x},$$

so that our result is proved.

4.3 Shift operators

We recall that a polynomial set $\{p_n(x)\}$ is called quasi-monomial if and only if there exist two operators $\hat{P}$ and $\hat{M}$ such that

$$\hat{P}(p_n(x)) = np_{n-1}(x), \qquad \hat{M}(p_n(x)) = p_{n+1}(x), \qquad (n = 1, 2, \dots). \tag{4.7}$$

$\hat{P}$ is called the *derivative* operator and $\hat{M}$ the *multiplication* operator, as they act in the same way of classical operators on monomials.

This definition traces back to a paper by J.F. Steffensen [30], recently improved by G. Dattoli [12] and widely used in several applications [14, 17].

Y. Ben Cheikh [3] proved that every polynomial set is quasi-monomial under the action of suitable derivative and multiplication operators (see also the article by G. Dattoli et al. [16]). In particular, in the same article (***Corollary* 3.2**), the following result is proved

***Theorem* 4.3.** *Let $(p_n(x))$ denote a Boas-Buck polynomial set, i.e. a set defined by the generating function*

$$A(t)\psi(xH(t)) = \sum_{n=0}^{\infty} p_n(x)\frac{t^n}{n!}\,, \tag{4.8}$$

where

$$A(t) = \sum_{n=0}^{\infty} \tilde{a}_n t^n\,, \qquad (\tilde{a}_0 \neq 0), \tag{4.9}$$

$$\psi(t) = \sum_{n=0}^{\infty} \tilde{\gamma}_n t^n\,, (\tilde{\gamma}_n \neq 0 \quad \text{for all } n),$$

with $\psi(t)$ not a polynomial, and lastly

$$H(t) = \sum_{n=0}^{\infty} \tilde{h}_n t^{n+1}\,, \qquad (\tilde{h}_0 \neq 0). \tag{4.10}$$

Let $\sigma \in \Lambda^{(-)}$ the lowering operator defined by

$$\sigma(1) = 0, \qquad \sigma(x^n) = \frac{\tilde{\gamma}_{n-1}}{\tilde{\gamma}_n}\, x^{n-1}, \qquad (n = 1, 2, \dots). \tag{4.11}$$

Put

$$\sigma^{-1}(x^n) = \frac{\tilde{\gamma}_{n+1}}{\tilde{\gamma}_n}\, x^{n+1} \quad (n = 0, 1, 2, \dots). \tag{4.12}$$

Denoting, as before, by $f(t)$ the compositional inverse of $H(t)$, the Boas-Buck polynomial set $\{p_n(x)\}$ is quasi-monomial under the action of the operators

$$\hat{P} = f(\sigma), \qquad \hat{M} = \frac{A'[f(\sigma)]}{A[f(\sigma)]} + xD_x H'[f(\sigma)]\sigma^{-1}, \tag{4.13}$$

where prime *denotes the ordinary derivatives with respect to t.*

Note that in our case we are dealing with a Sheffer polynomial set, so that since we have $\psi(t) = e^t$, the operator σ defined by equation (4.11) simply reduces to the derivative operator D_x. Furthermore, we have:

$$f(t) = H^{-1} \qquad (t) = 1 - e^t,$$
$$A(t) = (1-t)^{-\beta}, \qquad H(t) = \log(1-t),$$
$$\frac{A'(t)}{A(t)} = \beta(1-t)^{-1}, \qquad H'(t) = \frac{1}{t-1},$$

and consequently

$$f(\sigma) = 1 - e^{D_x},$$
$$\frac{A'[f(\sigma)]}{A[f(\sigma)]} = \beta[1 - f(\sigma)]^{-1} = \beta e^{-D_x},$$
$$H'[f(\sigma)] = -e^{-D_x}.$$

Comparing the last three equations with equation (4.13), the following result follows:

***Theorem* 4.4.** *The adjoint Actuarial polynomial set $\{\tilde{\alpha}_k^{(\beta)}(x)\}$ is quasi-monomial under the action of the operators*

$$\hat{P} = 1 - e^{D_x} = -\sum_{k=0}^{\infty} \frac{D_x^{k+1}}{(k+1)!}, \tag{4.14}$$

$$\hat{M} = (\beta - x)e^{-D_x} = (\beta - x)\sum_{k=0}^{\infty} \frac{(-1)^k D_x^k}{k!}.$$

There is no problem about the convergence of the above series, since they reduce to finite sums when applied to polynomials.

4.4 Differential equation

According to the results of monomiality principle [12, 17], the quasi-monomial polynomials $\{p_n(x)\}$ satisfy the differential equation

$$\hat{M}\hat{P}p_n(x) = np_n(x). \tag{4.15}$$

In the present case, recalling equations (4.14), we have

***Theorem* 4.5.** *The adjoint Actuarial polynomials* $\{\tilde{\alpha}_k^{(\beta)}(x)\}$ *satisfy the differential equation*

$$(\beta - x)\sum_{k=0}^{n-1}\frac{(-1)^{k+1}}{(k+1)!}D_x^{k+1}\tilde{\alpha}_n^{(\beta)}(x) = n\tilde{\alpha}_n^{(\beta)}(x). \tag{4.16}$$

Proof. Equation (4.15), by using equations (4.14), becomes

$$\begin{aligned}\hat{M}\hat{P}\tilde{\alpha}_n^{(\beta)}(x) &= (\beta - x)e^{-D_x}\left[1 - e^{D_x}\right]\tilde{\alpha}_n^{(\beta)}(x) = \\ &= (\beta - x)\left[e^{-D_x} - 1\right]\tilde{\alpha}_n^{(\beta)}(x) = n\tilde{\alpha}_n^{(\beta)}(x),\end{aligned}$$

i.e.

$$(\beta - x)\sum_{k=0}^{\infty}\frac{(-1)^{k+1}}{(k+1)!}D_x^{k+1}\tilde{\alpha}_n^{(\beta)}(x) = n\tilde{\alpha}_n^{(\beta)}(x),$$

and furthermore, for any fixed n, the last series expansion reduces to a finite sum, with upper limit $n-1$, when it is applied to a polynomial of degree n, because the last not vanishing term (for $k = n-1$) contains the derivative of order n.

4.5 First few values

Here we show the first few values for the adjoint Actuarial polynomials, defined by the generating function (4.1)

$$\begin{aligned}
\tilde{\alpha}_0^{(\beta)}(x) &= 1,\\
\tilde{\alpha}_1^{(\beta)}(x) &= (\beta - x),\\
\tilde{\alpha}_2^{(\beta)}(x) &= (\beta - x)^2 + (\beta - x),\\
\tilde{\alpha}_3^{(\beta)}(x) &= (\beta - x)^3 + 3(\beta - x)^2 + 2(\beta - x),\\
\tilde{\alpha}_4^{(\beta)}(x) &= (\beta - x)^4 + 6(\beta - x)^3 + 11(\beta - x)^2 + 6(\beta - x),\\
\tilde{\alpha}_5^{(\beta)}(x) &= (\beta - x)^5 + 10(\beta - x)^4 + 35(\beta - x)^3 + 50(\beta - x)^2 + 24(\beta - x),\\
\tilde{\alpha}_6^{(\beta)}(x) &= (\beta - x)^6 + 15(\beta - x)^5 + 85(\beta - x)^4 + 225(\beta - x)^3 + 274(\beta - x)^2 +\\
&\quad +120(\beta - x),\\
\tilde{\alpha}_7^{(\beta)}(x) &= (\beta - x)^7 + 21(\beta - x)^6 + 175(\beta - x)^5 + 735(\beta - x)^4 + 1624(\beta - x)^3 +\\
&\quad +1764(\beta - x)^2 + 720(\beta - x),\\
\tilde{\alpha}_8^{(\beta)}(x) &= (\beta - x)^8 + 28(\beta - x)^7 + 322(\beta - x)^6 + 1960(\beta - x)^5 + 6769(\beta - x)^4 +\\
&\quad +13132(\beta - x)^3 + 13068(\beta - x)^2 + 5040(\beta - x).
\end{aligned}$$

Note that the coefficients of the considered adjoint Actuarial polynomials appear in the Encyclopedia of Integer Sequences [27], under A094638: *Triangle read by rows*: $T(n,k) = |s(n, n+1-k)|$, *where* $s(n,k)$ *are the signed Stirling numbers of the first kind* ($1 \le k \le n$; in other words, the unsigned Stirling numbers of the first kind in reverse order).

In several articles [9, 10, 21, 22, 23], further adjoint sets of Sheffer polynomials have been examined.

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