ON HYPERSURFACE OF THE FINSLER SPACE OBTAINED BY CONFORMAL \( \beta \)– CHANGE

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Abstract

The conformal \( \beta \)– change of Finsler metric \( L(x,y) \) is given by \( L'(x,y) = e^{\sigma(x)} f(L(x,y)) \), \( \beta(x,y) \), where \( \sigma(x) \) is a function of \( x \), \( \beta(x,y) = b_i(x)y^i \) is a one-form on the underlying manifold \( M^n \), and \( f(L(x,y),\beta(x,y)) \) is a homogeneous function of degree one in \( L \) and \( \beta \). Let \( F^n \) and \( F'^n \) be Finsler spaces with metric functions \( L \) and \( L' \) respectively. In this paper we study the hypersurface of \( F'^n \) and find condition under which this hypersurface becomes a hyperplane of first kind, a hyperplane of second kind and a hyperplane of third kind. In this endeavour we connect quantities of \( F'^n \) with those of \( F^n \). When the hypersurface of \( F'^n \) is a hyperplane of first kind, we investigate the conditions under which it becomes a Landsberg space, a Berwald space, or a locally Minkowskian space.

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1 Introduction

Let \( F^n = (M^n, L) \) be an \( n \)--dimensional Finsler space on the differentiable manifold \( M^n \) equipped with the fundamental function \( L(x,y) \). B. N. Prasad and Bindu Kumari [10] have studied the general \( \beta \)--change, i.e., \( L'(x,y) = f(L,\beta) \), where \( f \) is positively homogeneous function of degree one in \( L \) and \( \beta \), \( \beta(x,y) = b_i(x)y^i \) is a one-form on \( M^n \).

The conformal theory of Finsler space has been dealt by M. Hashiguchi [3], H. Izumi [4, 5], M. Kitayama [6], S. H. Abed [1, 2]. The conformal change is given by \( L''(x,y) = e^{\sigma(x)} L(x,y) \), where \( \sigma(x) \) is a function of position only. In 2009 and 2010, Nabil L. Youssef, S. H. Abed and S. G. Elgendi [14, 15] introduced the transformation \( L''(x,y) = f(e^{\sigma} L,\beta) \), which is general \( \beta \)--change of conformally changed Finsler metric \( L \).

H. S. Shukla and Neelam Mishra [11] have changed the order of combination of the above two changes as

\begin{equation}
L'(x,y) = e^{\sigma(x)} f(L(x,y),\beta(x,y)),
\end{equation}

where \( \sigma(x) \) is a function of \( x \) only and \( \beta(x,y) = b_i(x)y^i \) is a one-form on \( M^n \). They have called this change as conformal \( \beta \)--change of Finsler metric and have studied its geometrical properties in [11] and [12]. When \( \sigma = 0 \), the change (1.1) reduces to general \( \beta \)--change. When \( \sigma = \text{constant}, \) it
becomes a homothetic $\beta$–change. Some properties of homothetic $\beta$–change with $b_i(x)$ as Cartan-parallel have been studied by H. S. Shukla and Neelam Mishra in [13].

In 1985 M. Matsumoto [7] studied the theory of Finsler hypersurfaces. In 2011, S. K. Narasimhamurthy et al. [9] have considered hypersurface of Finsler space with metric $L'''(x, y) = f(e^\alpha L, \beta)$ and studied its geometric properties.

In this paper we shall study the hypersurface of $F^n = (M^n, L^n)$.

**2 Hypersurface of $F^n$**

The metric tensor $g_{ij}(x, y)$ and Cartan’s $C$–tensor $C_{ijk}(x, y)$ of $F^n$ are given by

$$g_{ij} = \frac{1}{2} \frac{\partial^2 L^2}{\partial y^i \partial y^j}, \quad C_{ijk} = \frac{1}{2} \frac{\partial g_{ij}}{\partial y^k}$$

respectively. Let $CT = (F^i_j, G^i_j, C^i_{jk})$ denote the Cartan’s connection of $F^n$.

Let $(M^{n-1}, \overline{L})$ be a hypersurface of $(M^n, L)$ given by

$$x^j = x^j(u^\alpha).$$

Let us suppose that the functions (2.1) are at least of class $C^3$ in $u^\alpha$ and the projection factor $B^i_\beta = \frac{\partial x^i}{\partial u^\alpha}$ are such that their matrix has maximum rank $(n - 1)$. We shall use the following notations:

$$B^i_{\alpha\beta} = \frac{\partial^2 x^i}{\partial u^\alpha \partial u^\beta}, \quad B^i_{0\beta} = v^\alpha B^i_{\alpha\beta}, \quad B^j_{\alpha\beta} = \delta^j_{\alpha} B^i_{\beta\alpha},$$

where $v^\alpha$ is the element of support for the hypersurface satisfying the relation $y^i = B^i_\alpha(u)v^\alpha$. The fundamental metric function of the hypersurface is given by

$$\overline{L}(u^\alpha, v^\alpha) = L(x^j(u^\alpha), B^i_\alpha v^\alpha).$$

At each point $(u^\alpha)$ of $F^{n-1}$ the unit normal vector $N^i(u, v)$ is defined by

$$g_{ij}B^j_i N^j = 0, \quad g_{ij}N^i N^j = 1.$$  

If $(B^i_\alpha, N^i)$ is the inverse matrix of $(B^i_\alpha, N^i)$, we have

$$B^i_\alpha B^j_\beta = \delta^i_\beta, \quad B^i_\alpha N_i = 0, \quad N^i N_i = 1 \quad \text{and} \quad B^j_\alpha B^i_j + N^i N_j = \delta^i_\beta.$$  

Making use of the inverse matrix $(g_{\alpha\beta})$ of $(g_{\alpha\beta})$, we get

$$B^i_\alpha = g_{\alpha\beta} g_{ij} B^j_\beta.$$  

For the induced Cartan’s connection $ICT = (F^\alpha_{\beta\gamma}, G^\alpha_{\beta\gamma}, C^\alpha_{\beta\gamma})$ of $F^{n-1}$ induced from the Cartan’s connection $CT = (F^i_j, G^i_j, C^i_{jk})$ of $F^n$, the second fundamental $h$–tensor $H_{\alpha\beta}$ and the normal curvature vector $H_{\beta}$ are respectively given by [8]:

$$H_{\alpha\beta} = N_i (B^i_{\alpha\beta} + F^i_{jk} B^j_\alpha B^k_\beta) + M_\alpha H_\beta, \quad H_{\beta} = N_i (B^i_{0\beta} + F^i_{0j} B^j_\beta),$$

where $M_\alpha = C_{ijk} B^k_\gamma N^j N^k$.

Contracting $H_{\alpha\beta}$ by $v^\alpha$, we get $H_{0\beta} = H_{\alpha\beta} v^\alpha = H_\beta$. The second fundamental $v$–tensor $M_{\alpha\beta}$ is given by [8]:

$$M_{\alpha\beta} = C_{ijk} B^i_\alpha B^j_\beta N^k.$$  

The Gauss characteristic equation with respect to $ICT$ is written as

$$R_{\alpha\beta\gamma\delta} = R_{ijkh} B^i_{\alpha\gamma\delta} + P_{ijkh} (B^i_{\gamma\delta} H_\gamma - B^i_{\delta\gamma} H_\delta) B^j_{\alpha\beta} + H_{\alpha\gamma} H_{\beta\delta} - H_{\alpha\delta} H_{\beta\gamma}. $$

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3 Fundamental quantities of $F^{**n}$

We shall denote the quantities corresponding to $F^{**n}$ by putting star on the top of them. Differentiating equation (1.1) with respect to $y^i$, we have

$$(3.1) \quad l^i_0 = e^{\sigma^*}(f_1l_i + f_2b_i),$$

where the subscripts 1 and 2 denote the partial derivatives with respect to $L$ and $\beta$ respectively.

Differentiating (3.1) with respect to $y^j$, we get

$$(3.2) \quad h^*_{ij} = e^{2\sigma^*}(ph_{ij} + q_0m_im_j),$$

where $m_i = b_i - \frac{\partial}{\partial L}L_i$, $p = \frac{f_0}{f_1}$, $q_0 = fL^2w$, $w = \frac{f_1}{f_2} = \frac{f_0}{f_2} = \frac{f_1}{f_2}.$

From (3.1) and (3.2), we get the following relation between metric tensors of $F^n$ and $F^{**n}$:

$$(3.3) \quad g^*_{ij} = h^*_{ij} + l^i_0l^j_0 = e^{2\sigma^*}(pg_{ij} + p_0b_ib_j + p_1(l_ib_j + l_jb_i) + p_2l_il_j),$$

where

$$(3.4) \quad p_0 = q_0 + f_2^2, \quad p_1 = ff_1 - fL\beta w, \quad p_2 = \frac{(f_1f_2 - fL\beta w)\beta}{L}.$$  

From (3.3), we get the following relations between Cartan’s $C$–tensors of $F^n$ and $F^{**n}$:

$$(3.5) \quad C^*_i{}^j_k = e^{2\sigma^*}(pC_{i}{}^j_k + \frac{p}{2L}(h_{i}^j m_k + h_{j}^i m_l + h_{l}^i m_j)) + \frac{qL^2}{2}m_im_jm_k,$$

where $q = 3f_2w + f_1w_2$;

$$(3.6) \quad C^*_{ij} = C_{ij} + \frac{p}{2f_1}(h_{i}^j m_k + h_{j}^i m_l + h_{l}^i m_j) + \frac{qL^3}{2f_1}m_im_jm_k,$$

$$(3.7) \quad L \frac{L}{f_1}C_{ij} = \frac{pL\Delta}{2f_1}h_{i}^j m_k - \frac{(2pL + qL^4\Delta)}{2f_1}m_im_jm_k,$$

where $C_{jk} = C_{ijk}b^i, n^h = fL^2wb^i + p^h, h^i_j = g^i_jh_{ij}$.

The spray coefficient of $F^{**n}$ is given by [13]:

$$(3.8) \quad G^i{}^j = \frac{1}{2}g^i{}^j_ky^k = G^i + D^i,$$

where the vector $D^i$ is given by

$$(3.9) \quad D^i = \frac{f_2L}{f_1}S_0 - \frac{L}{f_1f_1}((f_1r_{00} - 2Lf_2s_{00}b')(py^i - L^2wfb^i) + \sigma_0y^i - \frac{1}{2}f^2\sigma^i),$$

with

$$(3.10) \quad 2r_{ij} = b_{ij} + b_{ji}, \quad 2s_{ij} = b_{ij} - b_{ji}, \quad s^i_j = g^{ij}s_{rj}y^r, \quad \sigma_k = \frac{\partial \sigma}{\partial x^k}, \quad \sigma^i = g^{ij}\sigma_j,$$

and ‘0’ standing for contradiction with $y^k$, viz., $\sigma_0 = \sigma_k^k, s^i_0 = s^k_k, \sigma_0^k = s^k_k$, etc.

The Cartan’s non-linear connection with $y^k$, viz., $\sigma_0 = \sigma_k^k, s^i_0 = s^k_k, \sigma_0^k = s^k_k$, etc.

The Cartan’s non-linear connection of $F^{**n}$ is given by [13]:

$$(3.11) \quad G^i{}^j = G^i + D^i,$$

where the tensor $D^i = \partial_iD^i$ is given by

$$(3.12) \quad D^i = \frac{L}{f_1}e^{2\sigma^*}A^i_j + Q^iA^i_jb^* + \frac{pLf_2}{f_2f_1^2t}b_{0j}(-Lf_1b^i + (f^2 - L^2\Delta f_2)y^i)$$

$$ + \sigma_0y^i - f\sigma^i(f_1f_j + f_2b_j),$$

in which

$$(3.13) \quad A^i_j = \frac{1}{2}r_{00}b_{ij} + e^{2\sigma^*}f_2s_{ij} + s_{00}Q_j - \frac{e^{2\sigma^*}f_1}{L}C_{ij} + V_{ijm}D^m,$$
\[ A_j^i = g^{ir}A_{rij}, \quad V_{ijm} = g_{sj}V_{im}^s, \quad Q_t = e^{2\sigma}(p + fL^2w)y_t + f_2^2b_t, \]
\[ B_{jk} = \frac{1}{2}e^{2\sigma}(ph_{jk} + \frac{1}{2}qL^2m_jm_k), \quad \hat{\partial}_kQ_j = \frac{1}{2}B_{jk}. \]

The Berwald’s connection coefficient of \( F^n \) is given by [13]:
\[ (3.11a) \quad G_{jk} = G_{jk}^i + B^i_{jk}, \]
where \( B^i_{jk} = \partial_kD_j^i \). The Cartan’s connection coefficient of \( F^n \) is given by [13]:
\[ (3.12) \quad F^i_{jk} = F^i_{jk} + D^i_{jk}, \]
where
\[ (3.13) \quad D^i_{jk} = \left[ \frac{e^{-2\sigma}}{f^3f_1}g^{ijs} - Q^i_s h^s + e^{2\sigma}\frac{pL}{f^3f_1}(Lfb^j + (f\beta - \Delta L^2 f^2)y^j) \right] \]
\[ (B_{sj}b_{0ik} + B_{sk}b_{0ij} - B_{kj}b_{0ks} + s_{sj}Q_k + s_{sk}Q_j + r_{kj}Q_s) \]
\[ + \frac{ff_1e^{2\sigma}}{L}C_{jkr}D^r_s + V_{jkr}D^r_s - \frac{ff_1e^{2\sigma}}{L}C_{skm}D^m_j - V_{skm}D^m_k \]
\[ - \frac{ff_1e^{2\sigma}}{L}C_{sjm}D^m_k - V_{skm}D^m_j) - e^{2\sigma}w^j g^i_j. \]

The tensor \( D^i_{jk} \), called the difference tensor, has the following properties:
\[ (3.14) \quad D^i_{j0} = B^i_{j0} = D^i_j, \quad D^i_{00} = 2D^i. \]

The \((v)\)–torsion tensor of \( F^n \) is defined as [8]:
\[ (3.15) \quad R^i_{jk} = \delta_kG^i_j - \delta_jG^i_k, \quad \delta_k = \partial_k - G^i_k\hat{\partial}_r, \]
the \((h)\)–curvature tensor of \( F^n \) is defined as [8]:
\[ (3.16) \quad R^i_{hjk} = \theta(j, k)\delta_kF^i_{hj} - F^m_{hk}F^i_{mk} + C^i_{hm}R^m_{jk}, \]
and the \((v)hv\)–torsion tensor of \( F^n \) is defined as [8]:
\[ (3.17) \quad P^i_{jk} = \partial_kG^i_j - F^i_{jk}. \]

### 4 Hypersurface of \( F^n \)

Let us consider a hypersurface \( F^{n-1} = (M^{n-1}, \hat{L}(u, v)) \) of \( F^n \) and another hypersurface \( F^{n-1} = (M^{n-1}, \hat{L}(u, v)) \) of \( F^n \), both hypersurfaces being represented by the same equation (2.1). Let \( N^i \) be the unit normal vector at each point of \( F^{n-1} \), which is invariant under the conformal \( \beta \)–change. The unit normal vector \( N^{\ast i}(u, v) \) of \( F^{n-1} \) is uniquely determined by
\[ (4.1) \quad g_{ij}B^i_aN^{\ast i} = 0, \quad g_{ij}N^{\ast i}N^{\ast j} = 1. \]

Transvecting the first equation of (2.2) by \( \nu^n \), we get
\[ (4.2) \quad \nu^iN^i = 0. \]

Contracting (3.3) by \( N^iN^j \) and using (2.2) and (4.2), we have
\[ (4.3) \quad g_{ij}N^iN^j = e^{2\sigma}(p + p_0(b_iN^i)^2). \]

This gives
\[ g_{ij}^\ast \pm \frac{N^i}{e^{\sigma}\sqrt{(p + p_0(b_iN^i)^2)}} \pm \frac{N^j}{e^{\sigma}\sqrt{(p + p_0(b_iN^i)^2)}} = 1. \]
Hence, we can put

\[(4.4) \quad N^i = \frac{N^j}{e^{\sigma} \sqrt{[p + p_0(b_i N^i)^2]}}.\]

where we have chosen the positive sign of the radical in order to fix the orientation. Using equations (2.2) and (4.2), the first condition of (4.1) gives

\[(4.5) \quad (p_0 b_i B^i_a + p_1 l_i B^i_a) \frac{b_j N^j}{e^{\sigma} \sqrt{[p + p_0(b_i N^i)^2]}} = 0.\]

Suppose that \(p_0 b_i B^i_a + p_1 l_i B^i_a = 0\). Then transvecting it by \(v^\alpha\), we get \(p_0 \beta + p_1 L = 0\). But this is impossible as \(L\) and \(\beta\) are independent. Hence

\[(4.6) \quad b_j N^j = 0.\]

Therefore (4.4) is rewritten as

\[(4.7) \quad N^i = \frac{1}{e^{\sigma} \sqrt{p}} N^i.\]

Thus, we have

\[\text{Proposition 4.1.} \quad \text{For a field of linear frame } (B^i_a, N^i) \text{ of } F^n, \text{ there exists a field of linear frame } (B^i_a, N^*_i) \text{ of } F^{n-1} \text{ such that the conditions (4.1) are satisfied along } F^{n-1} \text{ and } b_i \text{ is tangential to both the hypersurfaces } F^{n-1} \text{ and } F^{n-1}.\]

The quantities \(B^i_a\) are uniquely defined along \(F^{n-1}\) by

\[B^i_a = g^{a\beta} g_{ij} B^j_\beta,\]

where \((g^{a\beta})\) is the inverse matrix of \((g_{a\beta})\). Let \((B^i_a, N^*_i)\) be the inverse matrix of \((B^i_a, N^i)\). Then, we have

\[B^i_a B^j_\beta = \delta^i_\beta, \quad B^i_a N^*_i = 0, \quad N^i N^*_i = 1, \quad B^i_a N^*_i = 0.\]

Also, \(B^i_a B^j_\alpha + N^i N^*_j = \delta^i_j\), where

\[(4.8) \quad N^*_i = e^{\sigma} \sqrt{p} N_i.\]

From (4.8) and (2.5), we get

\[(4.9) \quad H^*_a = e^{\sigma} \sqrt{p} N_i (B^i_{0\beta} + F^i_{0\beta} B^i_\beta).\]

If each path of a hypersurface \(F^{n-1}\) with respect to the induced connection is also a path of the enveloping space \(F^n\), then \(F^{n-1}\) is called a hyperplane of the first kind. A hyperplane of the first kind is characterized by \(H_a = 0\).

We shall use the following theorem which has been proved in [13]:

\[\text{Theorem 4.1.} \quad \text{Under the conformal } \beta-\text{change (1.1) consider the following two assertions:}\]

1. The covariant vector field \(b_i(x)\) is Cartan-parallel.
2. The difference tensor \(D^i_{jk}\) vanishes identically.

Then we have:

(a) If (1) and (2) hold, then \(\sigma\) is homothetic.
(b) If \(\sigma\) is homothetic, then (1) and (2) are equivalent.
Let \( \sigma \) be homothetic and \( b_1(x) \) be Cartan-parallel in \( F^n \). Then from (4.9), (2.5) and Theorem 4.1, we get

(4.10) \[
H^*_\alpha = e^\sigma \sqrt{p} H_\alpha.
\]

From (4.10) we find that \( H^*_\alpha = 0 \) iff \( H_\alpha = 0 \). Thus we have the theorem:

**Theorem 4.2.** Let \( \sigma \) be homothetic and \( b_1(x) \) be Cartan-parallel in \( F^n \). Then the hypersurface \( F^{n-1} \) is a hyperplane of the first kind iff the hypersurface \( F^{n-1} \) is a hyperplane of the first kind.

Next, contracting (3.5) by \( B^i_\alpha N^{i,j}N^{j,k} \), making use of (4.7), \( M_\alpha = C_{ijk}B^i_\alpha N^j N^k \), \( m_i N^i = 0 \), \( b_i N^i = 0 \), \( h_{jk} N^j N^k = 1 \) and \( h_{ij} B^i_\alpha N^j = 0 \), we get

(4.11) \[
M^*_\alpha = M_\alpha.
\]

To compute \( H^*_\alpha \beta \) we use (2.5), (4.7), (4.10), (4.11) and Theorem 4.1 to get

(4.12) \[
H^*_\alpha \beta = e^\sigma \sqrt{p} H_{\alpha \beta}.
\]

If each \( h \)-path of a hypersurface \( F^{n-1} \) with respect to the induced connection is also an \( h \)-path of the enveloping space \( F^n \), then \( F^{n-1} \) is called a hyperplane of the second kind. A hyperplane of the second kind is characterized by \( H_\alpha = 0 \), \( H_{\alpha \beta} = 0 \). From (4.12) we find that \( H^*_\alpha \beta = 0 \) iff \( H_{\alpha \beta} = 0 \). Therefore from (4.10) and (4.12), we have the theorem:

**Theorem 4.3.** Let \( \sigma \) be homothetic and \( b_1(x) \) be Cartan-parallel in \( F^n \). Then the hypersurface \( F^{n-1} \) is a hyperplane of the second kind iff the hypersurface \( F^{n-1} \) is a hyperplane of the second kind.

Finally, contracting (3.5) by \( B^i_\alpha B^j_\beta N^{i,j} \) and making use of (2.6), (4.7), \( m_i N^i = 0 \), \( h_{ij} B^i_\alpha N^j = 0 \) and Theorem 4.1, we have

(4.13) \[
M^*_\alpha \beta = e^\sigma \sqrt{p} M_{\alpha \beta}.
\]

If the unit normal vector of \( F^{n-1} \) is parallel along each curve of \( F^{n-1} \), then \( F^{n-1} \) is called a hyperplane of the third kind. A hyperplane of the third kind is characterized by \( H_\alpha = 0 \), \( H_{\alpha \beta} = 0 \) and \( M_{\alpha \beta} = 0 \). Hence from (4.10), (4.12) and (4.13), we have the theorem:

**Theorem 4.4.** Let \( \sigma \) be homothetic and \( b_1(x) \) be Cartan-parallel in \( F^n \). Then the hypersurface \( F^{n-1} \) is a hyperplane of the third kind iff the hypersurface \( F^{n-1} \) is a hyperplane of the third kind.

For hyperplane of the first kind, the \((v)hv\)-torsion tensor is given by [7]:

(4.14) \[
P^*_\beta \gamma = B^i_\gamma K^i_\beta \gamma,
\]

where

\[
K^i_\beta \gamma = P^i_j B^j_\beta \gamma.
\]

Using (4.14) and the last relation of (2.3), we get

(4.15) \[
K^i_\beta \gamma = B^i_\delta P^\delta_\beta \gamma + N^i N^h K^h_\beta \gamma.
\]

Under homothetic \( \beta \)-change with \( b_1(x) \) as Cartan-parallel it has been proved in [13] that

(i) a Landsberg space remains a Landsberg space,
(ii) a Berwald space remains a Berwald space,
(iii) a locally Minkowskian space remains a locally Minkowskian space.
When $\sigma$ is homothetic and $b_{i}(x)$ is Cartan-parallel, we have $K_{\beta\gamma}^{i} = K_{\beta\gamma}^{i}$, and then it follows that

\begin{equation}
(4.16) \quad P_{\beta\gamma}^{\alpha} = B_{i}^{\alpha} K_{\beta\gamma}^{i}.
\end{equation}

On substituting (4.15) in (4.16) and using (2.3), we get

\begin{equation}
(4.17) \quad P_{\beta\gamma}^{\alpha} = P_{\beta\gamma}^{\alpha}.
\end{equation}

Thus we have the theorem:

**Theorem 4.5.** Let $\sigma$ be homothetic and $b_{i}(x)$ be Cartan-parallel in $F^{n}$. Then the hyperplane $F^{n-1}$ of the first kind is a Landsberg space iff the hyperplane $F^{n-1}$ of the first kind is a Landsberg space.

For hyperplane of the first kind, the Berwald connection coefficients $G_{\beta\gamma}^{a}$ are given by [7]:

\begin{equation}
(4.18) \quad G_{\beta\gamma}^{a} = B_{i}^{a} A_{\beta\gamma}^{i},
\end{equation}

where

\begin{equation}
A_{\beta\gamma}^{i} = G_{jk}^{i} B_{\beta\gamma}^{j} + B_{\beta\gamma}^{i}.
\end{equation}

Using (4.18) and the last relation of (2.3), we get

\begin{equation}
(4.19) \quad A_{\beta\gamma}^{i} = B_{i}^{a} G_{\beta\gamma}^{a} + N_{i}^{a} A_{\beta\gamma}^{i}.
\end{equation}

When $\sigma$ is homothetic and $b_{i}(x)$ is Cartan-parallel, we have $G_{jk}^{i} = G_{jk}^{i}$. Then $A_{\beta\gamma}^{i} = A_{\beta\gamma}^{i}$ and it follows that

\begin{equation}
(4.20) \quad G_{\beta\gamma}^{a} = B_{i}^{a} A_{\beta\gamma}^{i}.
\end{equation}

On substituting (4.19) in (4.20) and using (2.3), we get

\begin{equation}
(4.21) \quad G_{\beta\gamma}^{a} = G_{\beta\gamma}^{a}.
\end{equation}

Then we have the theorem:

**Theorem 4.6.** Let $\sigma$ be homothetic and $b_{i}(x)$ be Cartan-parallel in $F^{n}$. Then the hyperplane $F^{n-1}$ of the first kind is a Berwald space iff the hyperplane $F^{n-1}$ of the first kind is a Berwald space.

From (2.7) the Gauss characteristic equation of hyperplane $F^{n-1}$ of the first kind is written as

\begin{equation}
(4.22) \quad R_{\alpha\beta\gamma\delta} = R_{ijkl} B_{\alpha\beta\gamma\delta}^{ijkl} + H_{\alpha\gamma} H_{\beta\delta} - H_{\alpha\delta} H_{\beta\gamma}.
\end{equation}

The Gauss characteristic equation of hyperplane $F^{n-1}$ of the first kind is similarly written as

\begin{equation}
(4.23) \quad R_{\alpha\beta\gamma\delta}^{*} = R_{ijkl}^{*} B_{\alpha\beta\gamma\delta}^{ijkl} + H_{\alpha\gamma}^{*} H_{\beta\delta}^{*} - H_{\alpha\delta}^{*} H_{\beta\gamma}^{*}.
\end{equation}

Making use of the equation (4.12), the above equation becomes

\begin{equation}
(4.24) \quad R_{\alpha\beta\gamma\delta}^{*} = R_{ijkl}^{*} B_{\alpha\beta\gamma\delta}^{ijkl} + e^{2\sigma} p(H_{\alpha\gamma} H_{\beta\delta} - H_{\alpha\delta} H_{\beta\gamma}).
\end{equation}

Equations (4.24) and (4.22) together give

\begin{equation}
(4.25) \quad R_{\alpha\beta\gamma\delta}^{*} = e^{2\sigma} p R_{\alpha\beta\gamma\delta} + (R_{ijkl}^{*} - e^{2\sigma} p R_{ijkl}) B_{\alpha\beta\gamma\delta}^{ijkl}.
\end{equation}

We know that when $\sigma$ is homothetic and $b_{i}(x)$ is Cartan-parallel, then if $F^{n}$ is locally Minkowskian, so is $F^{n}$; i.e. $R_{ijkl}^{*} = 0$ iff $R_{ijkl} = 0$. Under these conditions the equation (4.25) reduces to

\begin{equation}
R_{\alpha\beta\gamma\delta}^{*} = e^{2\sigma} p R_{\alpha\beta\gamma\delta}.
\end{equation}

Thus, in view of the **Theorem 4.6**, we have the theorem:

**Theorem 4.7.** Let $\sigma$ be homothetic, $b_{i}(x)$ be Cartan-parallel and $F^{n}$ be a locally Minkowskian space. Then the hyperplane $F^{n-1}$ of the first kind is a locally Minkowskian space iff the hyperplane $F^{n-1}$ of the first kind is a locally Minkowskian space.

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References