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(Dedicated to Honor Professor H.M. Srivastava on His 80th Birth Anniversary Celebrations)

ON HYPERSURFACE OF THE FINSLER SPACE OBTAINED BY CONFORMAL $\beta-$ CHANGE

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Abstract

The conformal $\beta-$ change of Finsler metric $L(x,y)$ is given by $L^*(x,y) = e^{\sigma(x)} f(L(x,y),\beta(x,y))$, where $\sigma(x)$ is a function of x, $\beta(x,y) = b_i(x)y^i$ is a one-form on the underlying manifold M^n, and $f(L(x,y),\beta(x,y))$ is a homogeneous function of degree one in L and β. Let F^n and F^{*n} be Finsler spaces with metric functions L and L^* respectively. In this paper we study the hypersurface of F^{*n} and find condition under which this hypersurfcae becomes a hyperplane of first kind, a hyperplane of second kind and a hyperplane of third kind. In this endeavour we connect quantities of F^{*n} with those of F^n. When the hypersurface of F^{*n} is a hyperplane of first kind, we investigate the conditions under which it becomes a Landsberg space, a Berwald space, or a locally Minkowskian space.
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1 Introduction

Let $F^n = (M^n, L)$ be an $n-$dimentional Finsler space on the differentiable manifold M^n equipped with the fundamental function $L(x,y)$. B. N. Prasad and Bindu Kumari [10] have studied the general $\beta-$change, i.e., $L'(x,y) = f(L,\beta)$, where f is positively homogeneous function of degree one in L and β, $\beta(x,y) = b_i(x)y^i$ is a one-form on M^n.

The conformal theory of Finsler space has been dealt by M. Hashiguchi [3], H. Izumi [4, 5], M. Kitayama [6], S. H. Abed [1, 2]. The conformal change is given by $L''(x,y) = e^{\sigma(x)}L(x,y)$, where $\sigma(x)$ is a function of position only. In 2009 and 2010, Nabil L. Youssef, S. H. Abed and S. G. Elgendi [14, 15] introduced the transformation $L''(x,y) = f(e^\sigma L,\beta)$, which is general $\beta-$change of conformally changed Finsler metric L.

H. S. Shukla and Neelam Mishra [11] have changed the order of combination of the above two changes as

$$L^*(x,y) = e^{\sigma(x)} f(L(x,y),\beta(x,y)), \tag{1.1}$$

where $\sigma(x)$ is a function of x only and $\beta(x,y) = b_i(x)y^i$ is a one-form on M^n. They have called this change as conformal $\beta-$change of Finsler metric and have studied its geometrical properties in [11] and [12]. When $\sigma = 0$, the change (1.1) reduces to general $\beta-$change. When $\sigma =$ constant, it becomes a homothetic $\beta-$change. Some properties of homothetic $\beta-$change with $b_i(x)$ as Cartan-parallel have been studied by H. S. Shukla and Neelam Mishra in [13].

In 1985 M. Matsumoto [7] studied the theory of Finsler hypersurfaces. In 2011, S. K. Narasimhamurthy et al. [9] have considered hypersurface of Finsler space with metric $L'''(x,y) = f(e^{\sigma}L, \beta)$ and studied its geometric properties.

In this paper we shall study the hypersurface of $F^{*n} = (M^n, L^*)$.

2 Hypersurface of F^n

The metric tensor $g_{ij}(x,y)$ and Cartan's $C-$tensor $C_{ijk}(x,y)$ of F^n are given by

$$g_{ij} = \frac{1}{2}\frac{\partial^2 L^2}{\partial y^i \partial y^j}, \qquad C_{ijk} = \frac{1}{2}\frac{\partial g_{ij}}{\partial y^k}$$

respectively. Let $C\Gamma = (F^i_{jk}, G^i_j, C^i_{jk})$ denote the Cartan's connection of F^n.

Let $(M^{n-1}, \overline{L})$ be a hypersurface of (M^n, L) given by

$$x^i = x^i(u^\alpha). \tag{2.1}$$

Let us suppose that the functions (2.1) are at least of class C^3 in u^α and the projection factor $B^j_\beta = \frac{\partial x^j}{\partial u^\beta}$ are such that their matrix has maximum rank $(n-1)$. We shall use the following notations:

$$B^i_{\alpha\beta} = \frac{\partial^2 x^i}{\partial u^\alpha \partial u^\beta}, \qquad B^i_{0\beta} = v^\alpha B^i_{\alpha\beta}, \qquad B^{ij}_{\alpha\beta} = B^i_\alpha B^j_\beta,$$

where v^α is the element of support for the hypersurface satisfying the relation $y^i = B^i_\alpha(u)v^\alpha$. The fundamental metric function of the hypersurface is given by

$$\overline{L}(u^\alpha, v^\alpha) = L(x^i(u^\alpha), B^i_\alpha v^\alpha).$$

At each point (u^α) of F^{n-1} the unit normal vector $N^i(u,v)$ is defined by

$$g_{ij}B^i_\alpha N^j = 0, \qquad g_{ij}N^i N^j = 1. \tag{2.2}$$

If (B^α_i, N_i) is the inverse matrix of (B^i_α, N^i), we have

$$B^i_\alpha B^\beta_i = \delta^\beta_\alpha, \quad B^i_\alpha N_i = 0, \quad N^i N_i = 1 \text{ and } B^i_\alpha B^\alpha_j + N^i N_j = \delta^i_j. \tag{2.3}$$

Making use of the inverse matrix $(g^{\alpha\beta})$ of $(g_{\alpha\beta})$, we get

$$B^\alpha_i = g^{\alpha\beta}g_{ij}B^j_\beta. \tag{2.4}$$

For the induced Cartan's connection $IC\Gamma = (F^\alpha_{\beta\gamma}, G^\alpha_\beta, C^\alpha_{\beta\gamma})$ of F^{n-1} induced from the Cartan's connection $C\Gamma = (F^i_{jk}, G^i_j, C^i_{jk})$ of F^n, the second fundamental $h-$tensor $H_{\alpha\beta}$ and the normal curvature vector H_β are respectively given by [8]:

$$H_{\alpha\beta} = N_i(B^i_{\alpha\beta} + F^i_{jk}B^j_\alpha B^k_\beta) + M_\alpha H_\beta, \qquad H_\beta = N_i(B^i_{0\beta} + F^i_{0j}B^j_\beta), \tag{2.5}$$

where $M_\alpha = C_{ijk}B^i_\alpha N^j N^k$.

Contracting $H_{\alpha\beta}$ by v^α, we get $H_{0\beta} = H_{\alpha\beta}v^\alpha = H_\beta$. The second fundamental $v-$tensor $M_{\alpha\beta}$ is given by [8]:

$$M_{\alpha\beta} = C_{ijk}B^i_\alpha B^j_\beta N^k. \tag{2.6}$$

The Gauss characteristic equation with respect to $IC\Gamma$ is written as

$$R_{\alpha\beta\gamma\delta} = R_{ijkh}B^{ijkh}_{\alpha\beta\gamma\delta} + P_{ijkh}(B^h_\gamma H_\delta - B^h_\delta H_\gamma)B^{ij}_{\alpha\beta}H^k + H_{\alpha\gamma}H_{\beta\delta} - H_{\alpha\delta}H_{\beta\gamma}. \tag{2.7}$$

3 Fundamental quantities of F^{*n}

We shall denote the quantities corresponding to F^{*n} by putting star on the top of them. Differentiating equation (1.1) with respect to y^i, we have

$$l^*_i = e^{\sigma}(f_1 l_i + f_2 b_i), \tag{3.1}$$

where the subscripts 1 and 2 denote the partial derivatives with respect to L and β respectively.

Differentiating (3.1) with respect to y^j, we get

$$h^*_{ij} = e^{2\sigma}(ph_{ij} + q_0 m_i m_j), \tag{3.2}$$

where $m_i = b_i - \frac{\beta}{L}L_i$, $\quad p = \frac{ff_1}{L}$, $\quad q_0 = fL^2 w$, $\quad w = \frac{f_{11}}{\beta^2} = \frac{-f_{12}}{L\beta} = \frac{f_{22}}{L^2}$.

From (3.1) and (3.2), we get the following relation between metric tensors of F^n and F^{*n}:

$$g^*_{ij} = h^*_{ij} + l^*_i l^*_j = e^{2\sigma}\{pg_{ij} + p_0 b_i b_j + p_1(l_i b_j + l_j b_i) + p_2 l_i l_j\}, \tag{3.3}$$

where

$$p_0 = q_0 + f_2^2, \quad p_1 = ff_1 - fL\beta w, \quad p_2 = \frac{(ff_1 - fL\beta w)\beta}{L}. \tag{3.4}$$

From (3.3), we get the following relations between Cartan's $C-$tensors of F^n and F^{*n}:

$$C^*_{ijk} = e^{2\sigma}\{pC_{ijk} + \frac{p}{2L}(h_{ij}m_k + h_{jk}m_i + h_{ki}m_j) + \frac{qL^2}{2}m_i m_j m_k\}, \tag{3.5}$$

where $q = 3f_2 w + fw_2$;

$$\begin{aligned} C^{*h}_{ij} =& C^h_{ij} + \frac{p}{2ff_1}(h_{ij}m^h + h^h_j m_i + h^h_i m_j) + \frac{qL^3}{2ff_1}m_i m_j m^h \\ &- \frac{L}{ft}C_{.ij}n^h - \frac{pL\triangle}{2f^2 f_1 t}h_{ij}n^h - \frac{(2pL + qL^4\triangle)}{2f^2 f_1 t}m_i m_j n^h, \end{aligned} \tag{3.6}$$

where $C_{.jk} = C_{ijk}b^i$, $n^h = fL^2 wb^h + pl^h$, $h^i_j = g^{il}h_{lj}$.

The spray coefficient of F^{*n} is given by [13]:

$$G^{*i} = \frac{1}{2}\gamma^{*i}_{jk}y^j y^k = G^i + D^i, \tag{3.7}$$

where the vector D^i is given by

$$D^i = \frac{f_2 L}{f_1}s^i_0 - \frac{L}{ff_1 t}(f_1 r_{00} - 2Lf_2 s_{r0}b^r)(py^i - L^2 wfb^i) + \sigma_0 y^i - \frac{1}{2}f^2\sigma^i, \tag{3.8}$$

with

$$2r_{ij} = b_{i|j} + b_{j|i}, \quad 2s_{ij} = b_{i|j} - b_{j|i}, \quad s^i_0 = g^{ir}s_{rj}y^j, \quad \sigma_k = \frac{\partial\sigma}{\partial x^k}, \quad \sigma^i = g^{ij}\sigma_j. \tag{3.9}$$

and '0' standing for contradiction with y^k, viz., $\sigma_0 = \sigma_k y^k$, $s_{r0} = s_{rk}y^k$, etc.

The Cartan's non-linear connection of F^{*n} is given by [13]:

$$G^{*i}_j = G^i_j + D^i_j, \tag{3.10}$$

where the tensor $D^i_j = \dot{\partial}_j D^i$ is given by

$$\begin{aligned} D^i_j =& \frac{Le^{2\sigma}}{ff_1}A^i_j - Q^i A_{rj}b^r + \frac{pLf_2}{f^2 f_1^2 t}b_{0|j}\{-Lf_1 b^i + (f\beta - L^2\triangle f_2)y^i\} \\ &+ \sigma_j y^i - f\sigma^i(f_1 f_j + f_2 b_j), \end{aligned} \tag{3.11}$$

in which

$$A_{ij} = \frac{1}{2}r_{00}B_{ij} + e^{2\sigma}ff_2 s_{ij} + s_{i0}Q_j - (\frac{e^{2\sigma}ff_1}{L}C_{imj} + V_{ijm})D^m,$$

$$A^i_j = g^{ir}A_{rj}, \quad V_{ijm} = g_{sj}V^s_{im}, \quad Q_i = e^{2\sigma}\{(p + fL^2w)y_i + f_2^2 b_i\},$$
$$B_{jk} = \frac{1}{2}e^{2\sigma}(ph_{jk} + \frac{1}{2}qL^2 m_j m_k), \quad \dot{\partial}_k Q_j = \frac{1}{2}B_{jk}.$$

The Berwald's connection coefficient of F^{*n} is given by [13]:

(3.11a) $$G^{*i}_{jk} = G^i_{jk} + B^i_{jk},$$

where $B^i_{jk} = \dot{\partial}_k D^i_j$. The Cartan's connection coefficient of F^{*n} is given by [13]:

(3.12) $$F^{*i}_{jk} = F^i_{jk} + D^i_{jk},$$

where

(3.13) $$\begin{aligned} D^i_{jk} =& [\frac{e^{-2\sigma}L}{ff_1}g^{is} - Q^i b^s + y^s \frac{e^{-2\sigma}pL}{f^3 f_1 t}\{-Lfb^i + (f\beta - \triangle L^2 f_2)y^i\}] \\ & (B_{sj}b_{0|k} + B_{sk}b_{0|j} - B_{kj}b_{0|s} + s_{sj}Q_k + s_{sk}Q_j + r_{kj}Q_s \\ & + \frac{ff_1 e^{2\sigma}}{L}C_{jkr}D^r_s + V_{jkr}D^r_s - \frac{ff_1 e^{2\sigma}}{L}C_{skm}D^m_j - V_{sjm}D^m_k \\ & - \frac{ff_1 e^{2\sigma}}{L}C_{sjm}D^m_k - V_{skm}D^m_j) - e^{-2\sigma}\sigma^i g^*_{jk}. \end{aligned}$$

The tensor D^i_{jk}, called the difference tensor, has the following properties:

(3.14) $$D^i_{j0} = B^i_{j0} = D^i_j, \qquad D^i_{00} = 2D^i.$$

The $(v)h$−torsion tensor of F^n is defined as [8]:

(3.15) $$R^i_{jk} = \delta_k G^i_j - \delta_j G^i_k, \qquad \delta_k = \partial_k - G^r_k \dot{\partial}_r,$$

the h−curvature tensor of F^n is defined as [8]:

(3.16) $$R^i_{hjk} = \theta(j,k)[\delta_k F^i_{hj} - F^m_{hj}F^i_{mk} + C^i_{hm}R^m_{jk}],$$

and the $(v)hv$−torsion tensor of F^n is defined as [8]:

(3.17) $$P^i_{jk} = \dot{\partial}_k G^i_j - F^i_{jk}.$$

4 Hypersurface of F^{*n}

Let us consider a hypersurface $F^{n-1} = (M^{n-1}, \overline{L}(u,v))$ of F^n and another hypersurface $F^{*n-1} = (M^{n-1}, \overline{L}^*(u,v))$ of F^{*n}, both hypersurfaces being represented by the same equation (2.1). Let N^i be the unit normal vector at each point of F^{n-1}, which is invariant under the conformal β−change. The unit normal vector $N^{*i}(u,v)$ of F^{*n-1} is uniquely determined by

(4.1) $$g^*_{ij}B^i_\alpha N^{*i} = 0, \qquad g^*_{ij}N^{*i}N^{*j} = 1.$$

Transvecting the first equation of (2.2) by v^α, we get

(4.2) $$y_i N^i = 0.$$

Contracting (3.3) by $N^i N^j$ and using (2.2) and (4.2), we have

(4.3) $$g^*_{ij}N^i N^j = e^{2\sigma}\{p + p_0(b_i N^i)^2\}.$$

This gives

$$g^*_{ij}[\pm \frac{N^i}{e^\sigma \sqrt{\{p + p_0(b_i N^i)^2\}}}][\pm \frac{N^j}{e^\sigma \sqrt{\{p + p_0(b_i N^i)^2\}}}] = 1.$$

Hence, we can put

$$N^{*i} = \frac{N^i}{e^{\sigma}\sqrt{\{p + p_0(b_iN^i)^2\}}}, \tag{4.4}$$

where we have chosen the positive sign of the radical in order to fix the orientation. Using equations (2.2) and (4.2), the first condition of (4.1) gives

$$(p_0b_iB^i_\alpha + p_1l_iB^i_\alpha)\frac{b_jN^j}{e^{\sigma}\sqrt{\{p + p_0(b_iN^i)^2\}}} = 0. \tag{4.5}$$

Suppose that $p_0b_iB^i_\alpha + p_1l_iB^i_\alpha = 0$. Then transvecting it by v^α, we get $p_0\beta + p_1L = 0$. But this is impossible as L and β are independent. Hence

$$b_jN^j = 0. \tag{4.6}$$

Therefore (4.4) is rewritten as

$$N^{*i} = \frac{1}{e^{\sigma}\sqrt{p}}\,N^i. \tag{4.7}$$

Thus, we have

***Proposition* 4.1.** *For a field of linear frame (B^i_α, N^i) of F^n, there exists a field of linear frame (B^i_α, N^{*i}) of F^{*n} such that the conditions* (4.1) *are satisfied along F^{*n-1} and b_i is tangential to both the hypersurfaces F^{n-1} and F^{*n-1}.*

The quantities $B^{*\alpha}_i$ are uniquely defined along F^{*n-1} by

$$B^{*\alpha}_i = g^{*\alpha\beta}g^*_{ij}B^j_\beta,$$

where $(g^{*\alpha\beta})$ is the inverse matrix of $(g^*_{\alpha\beta})$. Let $(B^{*\alpha}_i, N^*_i)$ be the inverse matrix of (B^i_α, N^{*i}). Then, we have

$$B^i_\alpha B^{*\beta}_i = \delta^\beta_\alpha, \qquad B^i_\alpha N^*_i = 0, \qquad N^{*i}N^*_i = 1, \qquad B^{*\alpha}_i N^{*i} = 0.$$

Also, $B^i_\alpha B^{*\alpha}_j + N^{*i}N^*_j = \delta^i_j$, where

$$N^*_i = e^{\sigma}\sqrt{p}\,N_i. \tag{4.8}$$

From (4.8) *and* (2.5)*, we get*

$$H^*_\alpha = e^{\sigma}\sqrt{p}\,N_i(B^i_{0\beta} + F^{*i}_{0j}B^j_\beta). \tag{4.9}$$

If each path of a hypersurface F^{n-1} with respect to the induced connection is also a path of the enveloping space F^n, then F^{n-1} is called a hyperplane of the first kind. A hyperplane of the first kind is characterized by $H_\alpha = 0$.

We shall use the following theorem which has been proved in [13]*:*

***Theorem* 4.1.** *Under the conformal $\beta-$change* (1.1) *consider the following two assertions:*

(1) *The covariant vector field $b_i(x)$ is Cartan-parallel.*
(2) *The difference tensor D^i_{jk} vanishes identically.*

Then we have:

(a) *If* (1) *and* (2) *hold, then σ is homothetic.*
(b) *If σ is homothetic, then* (1) *and* (2) *are equivalent.*

Let σ be homothetic and $b_i(x)$ be Cartan-parallel in F^n. Then from (4.9), (2.5) *and* ***Theorem* 4.1.**, *we get*

$$H^*_\alpha = e^\sigma \sqrt{p}\, H_\alpha. \tag{4.10}$$

From (4.10) we find that $H^*_\alpha = 0$ iff $H_\alpha = 0$. Thus we have the theorem:

***Theorem* 4.2.** *Let σ be homothetic and $b_i(x)$ be Cartan-parallel in F^n. Then the hypersurface F^{*n-1} is a hyperplane of the first kind iff the hypersurface F^{n-1} is a hyperplane of the first kind.*

Next, contracting (3.5) *by $B^i_\alpha N^{*j} N^{*k}$, making use of* (4.7), *$M_\alpha = C_{ijk} B^i_\alpha N^j N^k$, $m_i N^i = 0$, $b_i N^i = 0$, $h_{jk} N^j N^k = 1$ and $h_{ij} B^i_\alpha N^j = 0$, we get*

$$M^*_\alpha = M_\alpha. \tag{4.11}$$

To compute $H^*_{\alpha\beta}$ we use (2.5), (4.7), (4.10), (4.11) *and* ***Theorem* 4.1** *to get*

$$H^*_{\alpha\beta} = e^\sigma \sqrt{p}\, H_{\alpha\beta}. \tag{4.12}$$

If each $h-$path of a hypersurface F^{n-1} with respect to the induced connection is also an $h-$path of the enveloping space F^n, then F^{n-1} is called a hyperplane of the second kind. A hyperplane of the second kind is characterized by $H_\alpha = 0$, $H_{\alpha\beta} = 0$. From (4.12) we find that $H^*_{\alpha\beta} = 0$ iff $H_{\alpha\beta} = 0$. Therefore from (4.10) and (4.12) we have the theorem:

***Theorem* 4.3.** *Let σ be homothetic and $b_i(x)$ be Cartan-parallel in F^n. Then the hypersurface F^{*n-1} is a hyperplane of the second kind iff the hypersurface F^{n-1} is a hyperplane of the second kind.*

Finally, contracting (3.5) *by $B^i_\alpha B^j_\beta N^{*k}$ and making use of* (2.6), (4.7), *$m_i N^i = 0$, $h_{ij} B^i_\alpha N^j = 0$ and* ***Theorem* 4.1**, *we have*

$$M^*_{\alpha\beta} = e^\sigma \sqrt{p}\, M_{\alpha\beta}. \tag{4.13}$$

If the unit normal vector of F^{n-1} is parallel along each curve of F^{n-1}, then F^{n-1} is called a hyperplane of the third kind. A hyperplane of the third kind is characterized by $H_\alpha = 0$, $H_{\alpha\beta} = 0$ and $M_{\alpha\beta} = 0$. Hence from (4.10), (4.12) and (4.13), we have the theorem:

***Theorem* 4.4.** *Let σ be homothetic and $b_i(x)$ be Cartan-parallel in F^n. Then the hypersurface F^{*n-1} is a hyperplane of the third kind iff the hypersurface F^{n-1} is a hyperplane of the third kind.*

For hyperplane of the first kind, the $(v)hv-$*torsion tensor is given by* [7]*:*

$$P^\alpha_{\beta\gamma} = B^\alpha_i K^i_{\beta\gamma}, \tag{4.14}$$

where

$$K^i_{\beta\gamma} = P^i{}_{jk} B^{jk}_{\beta\gamma}.$$

Using (4.14) *and the last relation of* (2.3), *we get*

$$K^i_{\beta\gamma} = B^i_\delta P^\delta_{\beta\gamma} + N^i N_h K^h_{\beta\gamma}. \tag{4.15}$$

Under homothetic $\beta-$change with $b_i(x)$ as Cartan-parallel it has been proved in [13] *that*

(i) *a Landsberg space remains a Landsberg space,*
(ii) *a Berwald space remains a Berwald space,*
(iii) *a locally Minkowskian space remains a locally Minkowskian space.*

When σ is homothetic and $b_i(x)$ is Cartan-parallel, we have $K^{*i}_{\beta\gamma} = K^i_{\beta\gamma}$, and then it follows that

(4.16) $$P^{*\alpha}_{\beta\gamma} = B^{*\alpha}_i K^i_{\beta\gamma}.$$

On substituting (4.15) *in* (4.16) *and using* (2.3)*, we get*

(4.17) $$P^{*\alpha}_{\beta\gamma} = P^\alpha_{\beta\gamma}.$$

Thus we have the theorem:

***Theorem* 4.5.** *Let σ be homothetic and $b_i(x)$ be Cartan-parallel in F^n. Then the hyperplane F^{*n-1} of the first kind is a Landsberg space iff the hyperplane F^{n-1} of the first kind is a Landsberg space.*

For hyperplane of the first kind, the Berwald connection coefficients $G^\alpha_{\beta\gamma}$ are given by [7]*:*

(4.18) $$G^\alpha_{\beta\gamma} = B^\alpha_i A^i_{\beta\gamma},$$

where

$$A^i_{\beta\gamma} = G^i_{jk} B^{jk}_{\beta\gamma} + B^i_{\beta\gamma}.$$

Using (4.18) *and the last relation of* (2.3)*, we get*

(4.19) $$A^i_{\beta\gamma} = B^i_\delta G^\delta_{\beta\gamma} + N^i N_h A^h_{\beta\gamma}.$$

When σ is homothetic and $b_i(x)$ is Cartan-parallel, we have $G^{*i}_{jk} = G^i_{jk}$. Then $A^{*i}_{\beta\gamma} = A^i_{\beta\gamma}$ and it follows that

(4.20) $$G^{*\alpha}_{\beta\gamma} = B^{*\alpha}_i A^i_{\beta\gamma}.$$

On substituting (4.19) in (4.20) and using (2.3), we get

(4.21) $$G^{*\alpha}_{\beta\gamma} = G^\alpha_{\beta\gamma}.$$

Then we have the theorem:

***Theorem* 4.6.** *Let σ be homothetic and $b_i(x)$ be Cartan-parallel in F^n. Then the hyperplane F^{*n-1} of the first kind is a Berwald space iff the hyperplane F^{n-1} of the first kind is a Berwald space.*

From (2.7) *the Gauss characteristic equation of hyperplane F^{n-1} of the first kind is written as*

(4.22) $$R_{\alpha\beta\gamma\delta} = R_{ijkh} B^{ijkh}_{\alpha\beta\gamma\delta} + H_{\alpha\gamma} H_{\beta\delta} - H_{\alpha\delta} H_{\beta\gamma}.$$

The Gauss characteristic equation of hyperplane F^{*n-1} of the first kind is similarly written as

(4.23) $$R^*_{\alpha\beta\gamma\delta} = R^*_{ijkh} B^{ijkh}_{\alpha\beta\gamma\delta} + H^*_{\alpha\gamma} H^*_{\beta\delta} - H^*_{\alpha\delta} H^*_{\beta\gamma}.$$

Making use of the equation (4.12)*, the above equation becomes*

(4.24) $$R^*_{\alpha\beta\gamma\delta} = R^*_{ijkh} B^{ijkh}_{\alpha\beta\gamma\delta} + e^{2\sigma} p (H_{\alpha\gamma} H_{\beta\delta} - H_{\alpha\delta} H_{\beta\gamma}).$$

Equations (4.24) *and* (4.22) *together give*

(4.25) $$R^*_{\alpha\beta\gamma\delta} = e^{2\sigma} p R_{\alpha\beta\gamma\delta} + (R^*_{ijkh} - e^{2\sigma} p R_{ijkh}) B^{ijkh}_{\alpha\beta\gamma\delta}.$$

We know that when σ is homothetic and $b_i(x)$ is Cartan-parallel, then if F^n is locally Minkowskian, so is F^{*n}; i.e. $R^*_{ijkh} = 0$ iff $R_{ijkh} = 0$. Under these conditions the equation (4.25) *reduces to*

$$R^*_{\alpha\beta\gamma\delta} = e^{2\sigma} p R_{\alpha\beta\gamma\delta}.$$

Thus, in view of the ***Theorem* 4.6**, we have the theorem:

***Theorem* 4.7.** *Let σ be homothetic, $b_i(x)$ be Cartan-parallel and F^n be a locally Minkowskian space. Then the hyperplane F^{*n-1} of the first kind is a locally Minkowskian space iff the hyperplane F^{n-1} of the first kind is a locally Minkowskian space.*

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References

[1] S. H. Abed, Conformal $\beta-$changes in Finsler spaces, *Proc. Math. Phys. Soc., Egypt*, **86** (2008), 79-89.

[2] S. H. Abed, Cartan connections associated with a $\beta-$conformal change in Finsler geometry, *Tensor, N.S.*, **70** (2008), 146-158.

[3] M. Hashiguchi, On conformal transformations of Finsler metrics, *J. Math. Kyoto Univ.*, **16** (1976), 25-50.

[4] H. Izumi, Conformal transformations of Finsler spaces I, *Tensor, N.S.*, **31** (1977), 33-41.

[5] H. Izumi, Conformal transformations of Finsler spaces II, *Tensor, N.S.*, **33** (1980), 337-369.

[6] M. Kitayama, *Geometry of transformations of Finsler metrics*, Hokkaido University of Education, Kushiro Campus, Japan, 2000.

[7] M. Matsumoto, The induced and intrinsic connections of a hypersurface and Finslerian projective geometry, *J. Math. Kyoto Univ.*, **25** (1985), 107-144.

[8] M. Matsumoto, *Foundations of Finsler geometry and special Finsler spaces*, Kaiseisha Press, Saikawa, Otsu, Japan, 1986.

[9] S. K. Narasimhamurthy, G. N. Latha and C. S. Bagewadi, Geometric properties of Finsler hypersurface given by conformal $\beta-$change, *Analeb Universitatii de Vest din Timisoara. Seria Mathematica-Informatica*, **49** (2011), 77-87.

[10] B. N. Prasad and Bindu Kumari, The $\beta-$change of Finsler metric and imbedding classes of their tangent spaces, *Tensor, N.S.*, **74** (2013), 48-59.

[11] H. S. Shukla and Neelam Mishra, Conformal $\beta-$change of Finsler metric, *J. Int. Acad. Phys. Sci.*, **21(1)** (2017), 19-30.

[12] H. S. Shukla and Neelam Mishra, Some intrinsic properties of conformal $\beta-$change, *Int. J. Math. Combin.*, **2** (2018), 13-23.

[13] H. S. Shukla and Neelam Mishra, Homothetic $\beta-$change of Finsler metric, *J. Rajasthan Acad. Phys. Sci.*, **17(3&4)** (2018), 173-180.

[14] N. L. Youssef, S. H. Abed and S. G. Elgendi, Generalized $\beta-$conformal change of Finsler metrics, *Int. J. Geom. Meth. Mod. Phys.*, **7(4)** (2010). ArXiv No.: math. DG/0906.5369.

[15] N. L. Youssef, S. H. Abed and S. G. Elgendi, Generalized $\beta-$conformal change and special Finsler spaces, *Int. J. Geom. Meth. Mod. Phys.*, **8(6)** (2010). ArXiv No.: math. DG/1004.5478v3.