**FIXED POINT OF SUZUKI-TYPE GENERALIZED MULTIVALUED CONTRACTION MAPPINGS ON WEAK PARTIAL METRIC SPACES**

By

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**Abstract**

Motivated by the recent result of Aydi et al.[4], we establish a fixed point theorem of Suzuki-type generalized multivalued contraction mappings in the framework of weak partial metric space. An example is also given to show the significance of our result.

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1 **Introduction**

The notion of partial metric space was introduced in 1994 by Matthews[7] as a generalization of the metric space. Such spaces are useful for modeling the problems occurring in computer science. He also proved the famous Banach Contraction Principle in partial metric space. Later on many fixed point results in partial metric space have been proved (see, for instance [1], [2], [6] and references therein). Further Aydi et al.[3] obtained an analogue of Nadler’s fixed point theorem in partial metric space using the concept of a partial Hausdorff metric. Recently, Beg and Pathak[5] introduced a weaker form of partial metric called weak partial metric and gave a fixed point theorem. This result extended and generalized by Negi and Gairola[8].

The aim of this paper is to generalize the result of Aydi et al. [4] by introducing generalized Suzuki type multivalued contraction mapping in weak partial metric space. Our result extend various known comparable results in the literature.

2 **Preliminaries**

The following definitions and results are followed by Beg and Pathak[5].

**Definition 2.1.** Let $X$ be a nonempty set. A function $q : X \times X \to \mathbb{R}^+$ is called a weak partial metric on $X$ if for all $x, y, z \in X$, the following conditions hold:

1. \(q(x, x) = q(x, y) \iff x = y\);
2. \(q(x, x) \leq q(x, y)\);
3. \(q(x, y) = q(y, x)\);
4. \(q(x, y) \leq q(x, z) + q(z, y)\).

The pair $(X, q)$ is called a weak partial metric space.

Let $CB^p(X)$ be the family of all non-empty, closed and bounded subsets of weak partial metric space $(X, q)$. For $E, F \in CB^p(X)$ and $x \in X$, define $\delta_q : CB^p(X) \times CB^p(X) \to [0, \infty)$ and
Let \( \phi \neq E \) be a set in \((X, q)\). Then
\( a \in \overline{E} \) if and only if \( q(a, E) = q(a, a) \),
where \( \overline{E} \) denotes the closure of \( E \) with respect to the weak partial metric space.

**Definition 2.2.** A sequence \( \{x_n\} \) in \((X, q)\) converges to a point \( x \in X \) with respect to \( \tau_q \) if and only if
\[ q(x, x_n) = \lim_{n \to \infty} q(x, x_n). \]

**Remark 2.2.** If \( q \) is a weak partial metric on \( X \), the function \( q^* : X \times X \to \mathbb{R}^+ \) given by
\[ q^*(x, y) = q(x, y) - \frac{1}{2}[q(x, x) + q(y, y)] \]
defines a metric on \( X \). Further, a sequence \( \{x_n\} \) converges in \((X, q^*)\) to a point \( x \in X \) if and only if
\[ q(x, x_n) = \lim_{n \to \infty} q(x, x_n) = \lim_{m \to \infty} q(x_n, x_m). \]

**Proposition 2.1.** Let \((X, q)\) be a weak partial metric space. For any \( E, F, H \in CB^q(X) \), the following holds:
(i) \( \delta_q(E, E) = \sup\{q(a, a) : a \in E\} \);
(ii) \( \delta_q(E, E) \leq \delta_q(E, F) \);
(iii) \( \delta_q(E, F) = 0 \Rightarrow E \subseteq F \);
(iv) \( \delta_q(E, F) \leq \delta_q(E, H) + \delta_q(H, F) \).

**Proposition 2.2.** Let \((X, q)\) be a weak partial metric space. For all \( E, F, H \in CB^q(X) \), we have
(wp1) \( H_q^+(E, E) \leq H_q^+(E, F) \);
(wp2) \( H_q^+(E, F) = H_q^+(F, E) \);
(wp3) \( H_q^+(E, F) \leq H_q^+(E, H) + H_q^+(H, F) \).

**Definition 2.3.** Let \((X, q)\) be a weak partial metric space. For \( E, F \in CB^q(X) \), define
\[ H_q^+(E, F) = \frac{1}{2}q(E, F) + \frac{1}{2}q(F, E). \]

The mapping \( H_q^+ : CB^q(X) \times CB^q(X) \to [0, +\infty) \), is called \( H_q^+ \)-type Hausdorff metric induced by \( q \).

**Definition 2.4.** Let \((X, q)\) be a complete weak partial metric space. A multi-valued map \( T : X \to CB^q(X) \) is called \( H_q^+ \)-contraction if for every \( x, y \in X \),
(i) there exists \( r \) in \((0, 1)\) such that
\[ H_q^+(T(x) \setminus \{x\}, T(y) \setminus \{y\}) \leq r \cdot q(x, y), \]
(ii) for every \( x \) in \( X \), \( y \) in \( T(x) \) and \( \epsilon > 0 \), there exists \( z \) in \( T(y) \) such that
\[ q(y, z) \leq H_q^+(T(y), T(x)) + \epsilon. \]

Beg and Pathak[5] gave the following variant of Nadler’s fixed point theorem.
**Theorem 2.1.** [5] Every $H_q^+$ type multivalued contraction on a complete weak partial metric space $(X, q)$ has a fixed point.

Recently Aydi et al.[4] introduced $H_q^+$ type Suzuki multivalued contraction mappings and prove the following theorem.

**Theorem 2.2.** [4] Let $(X, q)$ be a complete weak partial metric space, and let $T : X \to CB^q(X)$ be a multivalued mapping. Let $\psi : [0, 1) \to (0, 1]$ be the non increasing function defined by

$$(2.2) \quad \psi(r) = \begin{cases} 1 & \text{if } 0 \leq r < \frac{1}{2} \\ 1 - r & \text{if } \frac{1}{2} \leq r < 1. \end{cases}$$

Suppose that there exists $0 \leq r < 1$ such that $T$ satisfies the condition $\psi(r)q(x, Tx) \leq q(x, y)$ implies

$$(3.1) \quad H_q^+(Tx \setminus \{x\}, Ty \setminus \{y\}) \leq rq(x, y),$$

for all $x, y \in X$.

Suppose also that, for all $x$ in $X$, $y$ in $Tx$, and $t > 1$, there exists $z$ in $Ty$ such that

$$q(y, z) \leq tH_q^+(Ty, Tx).$$

Then $T$ has a fixed point.

### 3 Main result

Now we state our main result.

**Theorem 3.1.** Let $(X, q)$ be a complete weak partial metric space and let $T : X \to CB^q(X)$ be a multivalued mapping and $\psi : [0, 1) \to (0, 1]$ be the non-increasing function defined by (2.2). If there exists $0 \leq r < 1$ such that $T$ satisfies the condition

$$(3.1) \quad \psi(r)q(x, Tx) \leq q(x, y) \text{ implies } H_q^+(Tx \setminus \{x\}, Ty \setminus \{y\}) \leq rM(x, y),$$

where $M(x, y) = \max(q(x, y), q(x, Tx), q(y, Ty))$ for all $x, y \in X$.

Suppose also that for every $x \in X, y \in Tx$ and $t > 1$, there exists $z \in Ty$ such that

$$(3.2) \quad q(y, z) \leq tH_q^+(Ty, Tx).$$

Then $T$ has a fixed point.

**Proof.** Let $r_1$ be a real number such that $0 \leq r \leq r_1 < 1$ and $w_0 \in X$. Since $Tw_0$ is nonempty, it follows that if $w_0 \in Tw_0$, then proof is finished. Let $w_1 \in Tw_0$ be such that $w_0 \neq w_1$. Similarly $w_1 \in Tw_1$ such that $w_1 \neq w_2$. From (3.2) we have

$$(3.3) \quad q(w_1, w_2) \leq \frac{1}{\sqrt{r_1}}H_q^+(Tw_0, Tw_1).$$

Since $\psi(r) \leq 1$, we have

$$\psi(r)q(w_1, Tw_1) \leq q(w_1, Tw_1) \leq q(w_1, w_2).$$

Using (3.1) in (3.3), we have

$$q(w_1, w_2) \leq \frac{1}{\sqrt{r_1}}H_q^+(Tw_0, Tw_1)$$

$$\leq \frac{1}{\sqrt{r_1}}H_q^+(Tw_0 \setminus \{w_0\}, Tw_1 \setminus \{w_1\})$$
Taking limit in the above inequalities, we get

\[
\lim_{n \to \infty} q(w_n, w_{n+1}) \leq q(w_0, w_1)
\]

If \( q(w_0, w_1) \leq q(w_1, w_2) \) then

\[
q(w_1, w_2) \leq \sqrt{r_1}q(w_0, w_1)
\]

but \( \sqrt{r_1} < 1 \) then we get a contradiction.

Thus, we have

\[
q(w_1, w_2) \leq \sqrt{r_1}q(w_0, w_1).
\]

Continuing this process, we obtain a sequence \( \{w_n\} \) in \( X \) such that

\[
q(w_n, w_{n+1}) \leq (\sqrt{r_1})^nq(w_0, w_1).
\]

Now, we prove that \( \{w_n\} \) is a Cauchy sequence in \((X, q^s)\). For all \( k \in \mathbb{N} \), we have

\[
q^s(w_n, w_{n+k}) \leq q(w_n, w_{n+k}) \\
= q(w_n, w_{n+1}) + q(w_{n+1}, w_{n+2}) + \ldots + q(w_{n+k-1}, w_{n+k}) \\
\leq \left[ (\sqrt{r_1})^n + (\sqrt{r_1})^{n+1} + \ldots + (\sqrt{r_1})^{n+k-1} \right] q(w_0, w_1) \\
\leq \frac{(\sqrt{r_1})^n}{1 - \sqrt{r_1}} q(w_0, w_1)
\]

\[
\to 0 \quad \text{as} \quad n \to \infty.
\]

Hence, \( \lim_{n \to \infty} q^s(w_n, w_{n+k}) = 0 \).

This implies that \( \{w_n\} \) is a Cauchy sequence in \((X, q^s)\). Since \((X, q)\) is complete, therefore \((X, q^s)\) is also complete metric space. It follows that there exists \( u \in X \) such that \( \lim_{n \to \infty} w_n = u \) in \((X, q^s)\).

Therefore, \( \lim_{n \to \infty} q^s(w_n, u) = 0 \).

From (2.1), we have

\[
q(u, u) = \lim_{n \to \infty} q(w_n, u) = \lim_{n \to \infty} q(w_n, w_k) = 0.
\]

Now, from triangle inequality

\[
q(u, Tx) \leq q(u, w_{n+1}) + q(w_{n+1}, Tx)
\]

and

\[
q(w_{n+1}, Tx) \leq q(w_{n+1}, w_n) + q(w_n, u) + q(u, Tx).
\]

Taking limit in the above inequalities, we get

(3.4)

\[
q(u, Tx) = \lim_{n \to \infty} q(w_{n+1}, Tx).
\]

We claim that

\[
q(u, Tx) \leq 2r \max\{q(u, x), q(x, Tx)\}, \quad \forall \ x \in X \setminus \{u\}.
\]

Since, \( \lim_{n \to \infty} q(w_n, u) = 0 \), \( \exists n_0 \in \mathbb{N} \) such that \( q(w_n, u) \leq \frac{1}{3}q(u, x) \), \( \forall n \geq n_0 \).

As \( w_{n+1} \in Tw_n \) then we have

\[
\psi(r)q(w_n, Tw_n) \leq q(w_n, Tw_n) \leq q(w_n, w_{n+1}) \\
\leq q(w_n, u) + q(u, w_{n+1}) \\
\leq \frac{1}{3}q(u, x) + \frac{1}{3}q(u, x) = \frac{2}{3}q(u, x)
\]

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\[ q(u, x) - \frac{1}{3}q(u, x) \leq q(u, x) - q(w_n, u) \leq q(x, w_n). \]

Hence, for any \( n \geq n_0 \) we get
\[ \psi(r)q(w_n, Tw_n) \leq q(x, w_n). \]

From (3.1) we have
\[ H^+_q(Tw_n, Tx) \leq rM(w_n, x). \]

Since
\[ q(w_{n+1}, Tx) \leq \delta q(Tw_n, Tx) \leq 2H^+_q(Tw_n, Tx) \leq 2rM(w_n, x) = 2r \max\{q(w_n, x), q(w_n, Tw_n), q(x, Tx)\} \leq 2r \max\{q(w_n, u) + q(u, x), q(w_n, w_{n+1}), q(x, Tx)\}. \]

Taking limit as \( n \to \infty \), we get
\[ \lim_{n \to \infty} q(w_{n+1}, Tx) \leq 2r \max\{q(u, x), q(x, Tx)\}. \]

From (3.4) we get
\[ q(u, Tx) \leq 2r \max\{q(u, x), q(x, Tx)\}, \quad \forall x \in X \setminus \{u\}. \]

Now, we claim that
\[ H^+_q(Tx, Tu) \leq r \max\{q(x, u), q(x, Tx), q(u, Tu)\} \]
for all \( x \in X \) such that \( x \neq u \).

For each \( n \in \mathbb{N} \), \( \exists y_n \in Tx \) such that
\[ q(u, y_n) \leq q(u, Tx) + \frac{1}{n}q(u, x). \]

Therefore
\[ q(x, Tx) \leq q(x, y_n) \leq q(x, u) + q(u, y_n) \leq q(x, u) + q(u, Tx) + \frac{1}{n}q(u, x) \leq q(x, u) + 2r \max\{q(u, x), q(x, Tx)\} + \frac{1}{n}q(u, x). \]

Suppose \( \max\{q(u, x), q(x, Tx)\} = q(u, x) \) then
\[ q(x, Tx) \leq (1 + 2r + \frac{1}{n})q(u, x), \]
which implies
\[ \frac{1}{1 + 2r + \frac{1}{n}}q(x, Tx) \leq q(u, x). \]

This further implies that
\[ H^+_q(Tu, Tx) \leq rM(x, u), \]
and equation (3.6) holds. Now if \( \max\{q(x, u), q(x, T x)\} = q(x, T x) \) then from (3.7), we get

\[
q(x, T x) \leq q(x, u) + 2rq(x, T x) + \frac{1}{n}q(x, u)
\]

\[
\frac{(1 - 2r)}{(1 + \frac{1}{n})} q(x, T x) \leq q(x, u).
\]

This also implies that \( H_q^+(T x, T u) \leq rM(x, u) \) and equation (3.6) holds.

Finally, let \( b \in T u \) then

(3.8) \[
q(b, T w_n) \leq \delta_q(T u, T w_n).
\]

Also, we know that

\[
q(u, T u) \leq q(u, T w_n) + q(T w_n, T u)
\]

\[
\leq q(u, w_{n+1}) + q(b, T w_n).
\]

From (3.8), we have

\[
q(u, T u) \leq q(u, w_{n+1}) + \delta_q(T u, T w_n).
\]

Taking limit, we get

(3.9) \[
q(u, T u) \leq \lim_{n \to \infty} \delta_q(T u, T w_n).
\]

Also we know that

\[
q(w_{n+1}, T u) \leq \delta_q(T w_n, T u).
\]

Taking limit, we have

(3.10) \[
q(u, T u) \leq \lim_{n \to \infty} \delta_q(T w_n, T u).
\]

From the definition (2.3), we know that

\[
\frac{1}{2}[\delta_q(T w_n, T u) + \delta_q(T u, T w_n)] = H_q^+(T w_n, T u).
\]

Taking limit in the above expression and using (3.9) and (3.10) we get

\[
\frac{1}{2}[q(u, T u) + q(u, T u)] \leq \lim_{n \to \infty} \frac{1}{2}[\delta_q(T w_n, T u) + \delta_q(T u, T w_n)]
\]

\[
= \lim_{n \to \infty} H_q^+(T w_n, T u)
\]

\[
= \lim_{n \to \infty} H_q^+(T w_n \setminus \{w_n\}, T u \setminus \{u\})
\]

\[
\leq \lim_{n \to \infty} r \max\{q(w_n, u), q(w_n, T w_n), q(u, T u)\}
\]

\[
= rq(u, T u)
\]

which implies that

\[
q(u, T u) \leq rq(u, T u).
\]

Since \( r < 1 \) then we get a contradiction.

Hence, we have \( q(u, T u) = 0 = q(u, u) \). Since \( T u \) is closed then \( u \in \overline{T u} = T u \).

Now we give an example to verify our result.
Example 3.1. Let $X = \{0, \frac{1}{3}, 1\}$ and define a weak partial metric $q : X \times X \to [0, \infty)$ as follows:

$q(0, 0) = 0$, $q(\frac{1}{3}, \frac{1}{3}) = \frac{1}{3}$, $q(1, 1) = \frac{1}{3}$, $q(0, 1) = q(1, 0) = \frac{2}{3}$, $q(1, \frac{1}{3}) = q(\frac{1}{3}, 1) = \frac{3}{4}$, $q(0, \frac{1}{3}) = q(\frac{1}{3}, 0) = \frac{1}{4}$.

Define a mapping $T : X \to CB^s(X)$ by

$$
T(x) = \begin{cases} 
0 & \text{if } x = 0 \\
1 & \text{if } x = \frac{1}{3} \\
0, \frac{1}{3} & \text{if } x = 1.
\end{cases}
$$

Clearly $(X, q)$ is a weak partial metric space.

Choose $r = 0.9$. From the definition of $\psi$ we have $\psi(0.9) = 0.1$. To investigate the contraction condition (3.1) holds for all $x, y \in X$, we assume the following cases:

Case (I) When $x = 0$, we have

$$
\psi(r)q(0, T(0)) = 0 \leq q(0, y), \ \forall \ y \in X.
$$

For $y = 0$, we have

$$
H_q^+(T(0) \setminus \{0\}, T(0) \setminus \{0\}) = H_q^+(\phi, \phi) = 0 \leq rM(0, 0).
$$

For $y = \frac{1}{3}$, we have

$$
H_q^+(T(0) \setminus \{0\}, T(\frac{1}{3}) \setminus \{\frac{1}{3}\}) = H_q^+(\phi, \{1\}) = 0 \leq rM(0, \frac{1}{3}).
$$

For $y = 1$, we have

$$
H_q^+(T(0) \setminus \{0\}, T(1) \setminus \{1\}) = H_q^+(\phi, \{0, \frac{1}{3}\}) = 0 \leq rM(0, 1).
$$

Case (II) When $x = \frac{1}{3}$, we have

$$
\psi(r)q(\frac{1}{3}, T(\frac{1}{3})) = (0.1)q(\frac{1}{3}, 1) = 0.06 \leq q(\frac{1}{3}, y), \ \forall \ y \in X.
$$

For $y = 0$, we have

$$
H_q^+(T(\frac{1}{3}) \setminus \{\frac{1}{3}\}, T(0) \setminus \{0\}) = H_q^+(\{1\}, \phi) = 0 \leq rM(\frac{1}{3}, 0).
$$

For $y = \frac{1}{3}$, we have

$$
H_q^+(T(\frac{1}{3}) \setminus \{\frac{1}{3}\}, T(\frac{1}{3}) \setminus \{\frac{1}{3}\}) = H_q^+(\{1\}, \{1\}) = \frac{1}{3} \leq rM(\frac{1}{3}, \frac{1}{3}) = r^3.
$$

For $y = 1$, we have

$$
H_q^+(T(\frac{1}{3}) \setminus \{\frac{1}{3}\}, T(1) \setminus \{1\}) = H_q^+(\{1\}, \{0, \frac{1}{3}\}) = \frac{1}{2} \leq rM(\frac{1}{3}, 1) = r^3.
$$

Case (III) When $x = 1$, we have

$$
\psi(r)q(1, T(1)) = (0.1)q(1, \{0, \frac{1}{3}\}) = 0.04 \leq q(1, y), \ \forall \ y \in X.
$$

For $y = 0$, we have

$$
H_q^+(T(0) \setminus \{0\}, T(1) \setminus \{1\}) = H_q^+(\phi, \{0, \frac{1}{3}\}) = 0 \leq rM(1, 0).
$$
For $y = \frac{1}{3}$, we have
\[ H_q^+(T(0) \setminus \{0\}, T(\frac{1}{3}) \setminus \{\frac{1}{3}\}) = H_q^+(\phi, \{1\}) = 0 \leq rM(0, \frac{1}{3}) = r^3. \]

For $y = 1$, we have
\[ H_q^+(T(1) \setminus \{1\}, T(1) \setminus \{1\}) = H_q^+(\{0, \frac{1}{3}\}, \{0, \frac{1}{3}\}) = \frac{1}{3} \leq rM(1, 1) = r^2. \]

Finally, we will enquire the condition (3.2) with $t = 2$. For this, we discuss the following situations:

(i) If $x = 0$, then $y \in T(0) = \{0\}$, so $\exists z \in T(y) = \{0\}$ such that
\[ 0 = q(y, z) \leq 2H_q^+(T(y), T(x)). \]

(ii) If $x = \frac{1}{3}$, then $y \in T(\frac{1}{3}) = \{1\}$, so $\exists z$ (say $z = 0$) $\in T(1) = \{0, \frac{1}{3}\}$ such that
\[ \frac{2}{5} = q(y, z) \leq 2H_q^+(T(1), T(\frac{1}{3})) = 1. \]

(iii) If $x = 1$, then $y \in T(1) = \{0, \frac{1}{3}\}$. If $y = 0$, then $z = 0$, and condition holds.
Also if $y = \frac{1}{3}$, then $\exists z \in T(\frac{1}{3}) = \{1\}$ such that
\[ \frac{3}{5} = q(y, z) \leq 2H_q^+(\{1\}, \{0, \frac{1}{3}\}) = 1. \]

Hence all the conditions of Theorem 3.1 are satisfied. Here $x = 0$ is fixed point of $T$.

On the other hand the result of Aydi et al. [4] is not applicable. We see that
\[ H_q^+(T(\frac{1}{3}) \setminus \{\frac{1}{3}\}, T(\frac{1}{3}) \setminus \{\frac{1}{3}\}) = H_q^+(\{1\}, \{1\}) = \frac{1}{3} \leq rM(\frac{1}{3}, \frac{1}{3}) = r^\frac{1}{3}. \]

is not satisfied for any $r \in (0, 1)$.

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