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(Dedicated to Honor Professor H.M. Srivastava on His 80th Birth Anniversary Celebrations)

CERTAIN LAURENT TYPE LINEAR AND BILATERAL GENERATING RELATIONS

By
Mohammad Idris Qureshi, Mahvish Ali⋆ and Dilshad Ahamad
Department of Applied Sciences and Humanities
Faculty of Engineering and Technology
Jamia Millia Islamia (A Central University)
New Delhi-110025, India
Email:miqureshi_delhi@yahoo.co.in, mahvishali37@gmail.com,
dlshdhmd4@gmail.com

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Abstract

In this article, certain Laurent type linear and bilateral hypergeometric generating relations are derived by using series rearrangement technique, summation theorems of Pfaff-Saalchütz, Chu-Vandermonde and some reduction formulas.

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1 Introduction and preliminaries

The generalized hypergeometric series is defined as:

$$(1.1)\qquad {}_pF_q\left[\begin{matrix}(\alpha_p);\\ \\ (\beta_q);\end{matrix}\quad z\right] = {}_pF_q\left[\begin{matrix}\alpha_1,\alpha_2,\ldots,\alpha_p;\\ \\ \beta_1,\beta_2,\ldots,\beta_q;\end{matrix}\quad z\right] = \sum_{n=0}^{\infty}\frac{(\alpha_1)_n(\alpha_2)_n\ldots(\alpha_p)_n}{(\beta_1)_n(\beta_2)_n\ldots(\beta_q)_n}\frac{z^n}{n!}.$$

We recall here the definition of a more general double hypergeometric function (than the one defined by Kampé de Fériet) in a slightly modified notation [7, p.423, Eq.(26)]:

$$(1.2)\qquad F^{p:\,q;\,k}_{\ell:\,m;\,n}\left[\begin{matrix}(a_p)\ :\ (b_q)\ ;\ (c_k)\ ;\\ \\ (\alpha_\ell)\ :\ (\beta_m)\ ;\ (\gamma_n)\ ;\end{matrix}\quad x,\ y\right] = \sum_{r,s=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_{r+s}\prod_{j=1}^{q}(b_j)_r\prod_{j=1}^{k}(c_j)_s}{\prod_{j=1}^{\ell}(\alpha_j)_{r+s}\prod_{j=1}^{m}(\beta_j)_r\prod_{j=1}^{n}(\gamma_j)_s}\frac{x^r}{r!}\frac{y^s}{s!},$$

where, for convergence,

$$(1.3)\qquad (i)\quad p+q<\ell+m+1,\quad p+k<\ell+n+1,\quad |x|<\infty,\quad |y|<\infty,\ \text{or}$$

$$(1.4)\qquad (ii)\quad p+q=\ell+m+1,\quad p+k=\ell+n+1\ \text{and}$$

$$(1.5)\qquad \begin{cases}|x|^{1/(p-\ell)}+|y|^{1/(p-\ell)}<1, & \text{if } p>\ell\\ \max\{|x|,|y|\}<1, & \text{if } p\leq\ell.\end{cases}$$

Series rearrangement technique is based upon certain interchanges of the order of a double (or multiple) summation. Several hypergeometric generating relations have been established using series rearrangement technique.

Here, we consider some well known results.

Cauchy's double series identity [6, p.100]:

$$\sum_{m,n=0}^{\infty} \Phi(m,n) = \sum_{m=0}^{\infty}\sum_{n=0}^{m} \Phi(m-n,n), \tag{1.6}$$

provided that the associated double series are absolutely convergent.

Chu-Vandermonde theorem [2, p.69, Q.No. 4]:

$$ {}_2F_1\left[\begin{array}{ccc} -N,\ G & ; & \\ & & 1 \\ H & ; & \end{array}\right] = \frac{(H-G)_N}{(H)_N}; \quad N = 0,1,2,\cdots, \tag{1.7}$$

such that ratio of Pochhammer symbols in r.h.s. is well defined and $H, G \in \mathbb{C}\setminus\mathbb{Z}_0^-$.

Pfaff-Saalschütz theorem [2, p.87, Theorem 29]:

If n is a non-negative integer, then

$$ {}_3F_2\left[\begin{array}{ccc} -n,\ a,\ b & ; & \\ & & 1 \\ c,\ a+b-c-n+1 & ; & \end{array}\right] = \frac{(c-a)_n\,(c-b)_n}{(c)_n\,(c-a-b)_n}, \tag{1.8}$$

such that ratios of Pochhammer symbols in r.h.s. are well defined and $a, b, c, 1+a+b-c-n \in \mathbb{C}\setminus\mathbb{Z}_0^-$.

Srivastava's multiple series identity [5, p.4, Eqn(12)]:

$$\sum_{m=0}^{\infty} f(m)\frac{(x_1+x_2+\cdots+x_n)^m}{m!} = \sum_{m_1,m_2,\cdots,m_n=0}^{\infty} f(m_1+m_2+\cdots+m_n)\frac{{x_1}^{m_1}}{m_1!}\frac{{x_2}^{m_2}}{m_2!}\cdots\frac{{x_n}^{m_n}}{m_n!}, \tag{1.9}$$

provided that the multiple series involved are absolutely convergent.

Motivated by the work on generating functions and generating relations recorded in beautiful monographs of Rainville [2, Chapter 8], Srivastava-Manocha [6] and recent investigations including [3, 1, 4], in this article, we derive certain Laurent type generating relations.

The paper is organized as: In Section 2, some auxiliary results are derived by using series rearrangement technique which are used in our main results. In Section 3, some hypergeometric generating relations are established with the help of the auxiliary results obtained in Section 2.

2 Some reduction formulae

Here, we prove the following auxiliary results:

***Lemma* 2.1.** *The following result holds true:*

$$F^{1:1;0}_{1:1;0}\left[\begin{array}{ccccccc} A & : & C & ; & — & ; & \\ & & & & & & X,\ -X \\ B & : & D & ; & — & ; & \end{array}\right] = {}_2F_2\left[\begin{array}{ccc} A,\ D-C & ; & \\ & & -X \\ B,\ D & ; & \end{array}\right], \tag{2.1}$$

where $B, D \in \mathbb{C}\setminus\mathbb{Z}_0^-$ *and for all finite values of* X.

***Lemma* 2.2.** *The following result holds true:*

$$F^{0:1;2}_{1:0;1}\left[\begin{array}{ccccccc} — & : & G & ; & H,\ J & ; & \\ & & & & & & X,\ X \\ E & : & — & ; & K & ; & \end{array}\right] = \sum_{m=0}^{\infty}\frac{(G)_m X^m}{(E)_m\, m!}\,{}_3F_2\left[\begin{array}{ccc} -m,\ H,\ J & ; & \\ & & 1 \\ K,\ 1-G-m & ; & \end{array}\right], \tag{2.2}$$

where $E, K \in \mathbb{C}\setminus\mathbb{Z}_0^-$ *and for all finite values of* X.

The ${}_3F_2(1)$ in the r.h.s. of equation (2.2) can be summed with the help of hypergeometric summation theorems of Dixon, Whipple, Watson and Pfaff-Saalschütz and other theorems for terminating Clausen series.

***Lemma* 2.3.** *The following result holds true:*

$$F^{0:1;2}_{1:0;1}\left[\begin{array}{ccccc} — & : & -\nu+n & ; & 1+\alpha,\ -p-n\ ; \\ & & & & X,\ X \\ 1+\alpha-\nu+n & : & — & ; & 1-\nu+\alpha-p\ ; \end{array}\right] \tag{2.3}$$

$$= {}_1F_1\left[\begin{array}{cc} -\nu-p & ; \\ & X \\ 1-\nu+\alpha-p & ; \end{array}\right],$$

where $1+\alpha-\nu+n,\ 1-\nu+\alpha-p \in \mathbb{C}\setminus\mathbb{Z}_0^-$ *and for all finite values of* X*.*

Proof of Lemma 2.1:

Suppose the power series form of l.h.s. of equation (2.1) is denoted by Π. Then, we have

$$\Pi = \sum_{m,n=0}^{\infty}\frac{(A)_{m+n}(C)_m X^m(-X)^n}{(B)_{m+n}(D)_m\ m!\ n!}. \tag{2.4}$$

Replacing m by $m-n$ in equation (2.4) and using Cauchy's double series identity (1.6), we get

$$\Pi = \sum_{m=0}^{\infty}\frac{(A)_m(C)_m X^m}{(B)_m(D)_m\ m!}\sum_{n=0}^{m}\frac{(1-D-m)_n(-m)_n}{(1-C-m)_n\ n!} \tag{2.5}$$

$$= \sum_{m=0}^{\infty}\frac{(A)_m(C)_m X^m}{(B)_m(D)_m\ m!}\ {}_2F_1\left[\begin{array}{cc} -m,\ 1-D-m & ; \\ & 1 \\ 1-C-m & ; \end{array}\right].$$

Now, applying Chu-Vandermonde summation ***Theorem* 1.7** in equation (2.5), we get

$$\Pi = \sum_{m=0}^{\infty}\frac{(A)_m(C)_m X^m}{(B)_m(D)_m\ m!}\frac{(D-C)_m}{(1-C-m)_m}.$$

Simplifying above equation, we get equation (2.1).

Proof of Lemma 2.2:

Suppose the power series form of l.h.s. of equation (2.2) is denoted by Φ. Then, we have

$$\Phi = \sum_{m,n=0}^{\infty}\frac{(G)_m(H)_n(J)_n X^{m+n}}{(E)_{m+n}(K)_n\ m!\ n!} \tag{2.6}$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{m}\frac{(G)_{m-n}(H)_n(J)_n X^m(-m)_n}{(E)_m(K)_n\ m!(-1)^n\ n!},\ \text{(on replacing } m \text{ by } m-n)$$

$$= \sum_{m=0}^{\infty}\frac{(G)_m\ X^m}{(E)_m\ m!}\sum_{n=0}^{m}\frac{(G+m)_{-n}(H)_n(J)_n(-m)_n}{(K)_n(-1)^n\ n!}.$$

Simplifying equation (2.6), we get equation (2.2).

Proof of Lemma 2.3:

If we choose $G=-\nu+n$, $E=1+\alpha-\nu+n$, $H=1+\alpha$, $J=-p-n$, $K=1-\nu+\alpha-p$ in ***Lemma* 2.2**, we find

$$F^{0:1;2}_{1:0;1}\left[\begin{array}{ccccc} — & : & -\nu+n & ; & 1+\alpha,\ -p-n\ ; \\ & & & & X,\ X \\ 1+\alpha-\nu+n & : & — & ; & 1-\nu+\alpha-p\ ; \end{array}\right] \tag{2.7}$$

$$= \sum_{m=0}^{\infty}\frac{(-\nu+n)_m X^m}{(1+\alpha-\nu+n)_m\ m!}\ {}_3F_2\left[\begin{array}{cc} -m,\ 1+\alpha,\ -p-n & ; \\ & 1 \\ 1-\nu+\alpha-p,\ 1+\nu-n-m & ; \end{array}\right].$$

On using Pfaff-Saalchütz summation ***Theorem* 1.8** in equation (2.7) and applying some algebraic properties of Pochhammer symbols, we obtain ***Lemma* 2.3**.

3 Laurent type generating relations

In this section, we prove the following Laurent type hypergeometric generating relations:

***Theorem* 3.1.** *The following Laurent type bilateral generating relation for hypergeometric functions* ${}_2F_2(-ab)$ *or* ${}_1F_1(x)$ *holds true for all finite values of a, b, x, t:*

$$\exp(axt)(1-at)^{\alpha-\nu}{}_1F_1\left[\begin{array}{ccc} -\nu & ; & \\ & & x-ab-axt+\frac{b}{t} \\ 1+\alpha-\nu & ; & \end{array}\right] \tag{3.1}$$

$$= \sum_{p=-\infty}^{\infty} \frac{(\nu-\alpha)_p(-\nu)_{p^\star}(-ab)^{p^\star}}{(p+p^\star)!(p^\star)!}\, {}_2F_2\left[\begin{array}{ccc} -\nu+p^\star,\ 1 & ; & \\ & & -ab \\ 1+p+p^\star,\ 1+p^\star & ; & \end{array}\right]\times$$

$$\times{}_1F_1\left[\begin{array}{ccc} -\nu-p & ; & \\ & & x \\ 1-\nu+\alpha-p & ; & \end{array}\right](at)^p, \quad 0<|at|<1; \quad t\neq 0,$$

where

$$p^\star = \max\{0,-p\} = \begin{cases} -p, & when\ \ p=\cdots,-3,-2,-1 \\ 0, & when\ p=0,1,2,\cdots \end{cases} \tag{3.2}$$

and numerator, denominator parameters are neither zero nor negative integers in each hypergeometric function.

***Theorem* 3.2.** *The following Laurent type linear generating relation for the hypergeometric function* ${}_3F_2\left(\frac{a}{b}\right)$ *holds true:*

$$\left(a+\frac{c}{t}\right)^{\alpha-\gamma}(bt+d)^{-\alpha}\, {}_2F_1\left[\begin{array}{ccc} \alpha,\ \lambda & ; & \\ & & \frac{at+c}{bt+d} \\ \mu & ; & \end{array}\right] \tag{3.3}$$

$$= \frac{a^{\alpha-\gamma}}{d^\alpha}\sum_{p=-\infty}^{\infty}(\alpha)_p\left(\frac{-b}{d}\right)^p\sum_{\ell=0}^{\infty}\frac{(\gamma-\alpha)_{\ell+p^\star}(\alpha+p)_{\ell+p^\star}\left(\frac{bc}{ad}\right)^{\ell+p^\star}}{(\ell+p^\star)!\,(\ell+p+p^\star)!}\times$$

$$\times{}_3F_2\left[\begin{array}{ccc} \lambda,\ 1-\gamma+\alpha,\ -\ell-p-p^\star & ; & \\ & & \frac{a}{b} \\ \mu,\ 1-\gamma+\alpha-\ell-p^\star & ; & \end{array}\right]t^p$$

$$\left(0<\left|\frac{c}{at}\right|<1,\quad 0<\left|\frac{bt}{d}\right|<1,\quad 0<\left|\frac{at+c}{bt+d}\right|<1;\quad t\neq 0\right),$$

where $p^\star$ *is defined by equation* (3.2) *and numerator, denominator parameters are neither zero nor negative integers in each hypergeometric function.*

Proof of Theorem 3.1

Suppose the power series form of l.h.s. of equation (3.1) is denoted by Ω. Then, we have

$$\Omega = \exp(axt)\ {}_1F_0\left[\begin{array}{ccc} \nu-\alpha & ; & \\ & & at \\ \text{——} & ; & \end{array}\right]\sum_{N=0}^{\infty}\frac{(-\nu)_N\left(x-ab-axt+\frac{b}{t}\right)^N}{(1+\alpha-\nu)_N\, N!}, \tag{3.4}$$

where $\nu-\alpha, -\nu, 1+\alpha-\nu \in \mathbb{C}\setminus\mathbb{Z}_0^-$.

Using Srivastava's multiple series identity (1.9) in the r.h.s. of equation (3.4), we obtain

$$(3.5)\qquad \Omega = \sum_{\ell=0}^{\infty}\frac{(axt)^{\ell}}{\ell!}\sum_{m=0}^{\infty}\frac{(\nu-\alpha)_m(at)^m}{m!}\sum_{r,s,q,n=0}^{\infty}\frac{(-\nu)_{r+s+q+n}\, x^r(-ab)^s(-axt)^q\left(\frac{b}{t}\right)^n}{(1+\alpha-\nu)_{r+s+q+n}\, r!\, s!\, q!\, n!}$$

$$= \sum_{m=0}^{\infty}\sum_{r,s,n=0}^{\infty}\frac{(\nu-\alpha)_m(at)^m(-\nu)_{r+s+n}\, x^r(-ab)^s\left(\frac{b}{t}\right)^n}{m!(1+\alpha-\nu)_{r+s+n}\, r!\, s!\, n!}\times$$

$$\times\sum_{\ell=0}^{\infty}\sum_{q=0}^{\infty}\frac{(-\nu+r+s+n)_q(-1)^q(axt)^{\ell+q}}{(1+\alpha-\nu+r+s+n)_q\, \ell!\, q!}.$$

On replacing ℓ by $\ell-q$ in equation (3.5), it follows that

$$(3.6)\qquad \Omega = \sum_{m,r,s,n=0}^{\infty}\frac{(\nu-\alpha)_m(-\nu)_{r+s+n}(at)^m\, x^r(-ab)^s\left(\frac{b}{t}\right)^n}{(1+\alpha-\nu)_{r+s+n}\, m!\, r!\, s!\, n!}\sum_{\ell=0}^{\infty}\frac{(axt)^{\ell}}{\ell!}\times$$

$$\times{}_2F_1\left[\begin{array}{ll}-\ell,\ -\nu+r+s+n & ;\\ & \quad 1\\ 1+\alpha-\nu+r+s+n & ;\end{array}\right].$$

Now, applying Chu-Vandermonde summation ***Theorem* 1.7** in equation (3.6), we get

$$(3.7)\qquad \Omega = \sum_{m,r,s,n=0}^{\infty}\frac{(\nu-\alpha)_m(-\nu)_{r+s+n}(at)^m\, x^r(-ab)^s\left(\frac{b}{t}\right)^n}{(1+\alpha-\nu)_{r+s+n}\, m!\, r!\, s!\, n!}$$

$$\times\sum_{\ell=0}^{\infty}\frac{(axt)^{\ell}}{\ell!}\frac{(1+\alpha)_{\ell}}{(1+\alpha-\nu+r+s+n)_{\ell}}$$

$$= \sum_{m,r,s,n,\ell=0}^{\infty}\frac{(\nu-\alpha)_m(-\nu)_{r+s+n}(1+\alpha)_{\ell}}{(1+\alpha-\nu)_{r+s+n+\ell}}\frac{(-1)^s a^{m+s+\ell}b^{s+n}x^{r+\ell}t^{m-n+\ell}}{m!\, r!\, s!\, n!\, l!}.$$

Further, putting $m-n+\ell=p$, equation (3.7) becomes

$$(3.8)\qquad \Omega = \sum_{p=-\infty}^{\infty}\sum_{r,s,n,\ell=0}^{\infty}\frac{(\nu-\alpha)_{p+n-\ell}(-\nu)_{r+s+n}(1+\alpha)_{\ell}}{(1+\alpha-\nu)_{r+s+n+\ell}}\frac{(-1)^s a^{p+n+s}b^{s+n}x^{r+\ell}t^p}{(p+n-\ell)!\, r!\, s!\, n!\, \ell!}$$

$$= \sum_{p=-\infty}^{\infty}\sum_{r,\ell=0}^{\infty}\frac{(\nu-\alpha)_{p-\ell}(-\nu)_r(1+\alpha)_{\ell}\, x^r x^{\ell}}{(1+\alpha-\nu)_{r+\ell}\,(1)_{p-\ell}\, r!\, \ell!}\times$$

$$\times\sum_{s,n=0}^{\infty}\frac{(\nu-\alpha+p-\ell)_n(-\nu+r)_{s+n}(-ab)^s(ab)^n}{(1+\alpha-\nu+r+\ell)_{s+n}(1+p-\ell)_n\, s!\, n!}(at)^p$$

$$= \sum_{p=-\infty}^{\infty}\sum_{r,\ell=0}^{\infty}\frac{(\nu-\alpha)_{p-\ell}(-\nu)_r(1+\alpha)_{\ell}\, x^r x^{\ell}}{(1+\alpha-\nu)_{r+\ell}\,(1)_{p-\ell}\, r!\, \ell!}\times$$

$$\times F^{1:1;0}_{1:1;0}\left[\begin{array}{lllll}-\nu+r & : & \nu-\alpha+p-\ell & ; & — \;;\\ & & & & \quad ab,\ -ab\\ 1+\alpha-\nu+r+\ell & : & 1+p-\ell & ; & — \;;\end{array}\right](at)^p.$$

Now, applying ***Lemma* 2.1** in equation (3.8), we obtain

$$(3.9)\qquad \Omega = \sum_{p=-\infty}^{\infty}\sum_{r,\ell=0}^{\infty}\frac{(\nu-\alpha)_{p-\ell}(-\nu)_r(1+\alpha)_{\ell}\, x^r x^{\ell}}{(1+\alpha-\nu)_{r+\ell}\,(1)_{p-\ell}\, r!\, \ell!}\times$$

$$\times {}_2F_2\left[\begin{array}{ccc} -\nu+r,\ 1-\nu+\alpha & ; & \\ & & -ab \\ 1+\alpha-\nu+r+\ell,\ 1+p-\ell & ; & \end{array}\right](at)^p$$

$$= \sum_{p=-\infty}^{\infty}\sum_{r,\ell,n=0}^{\infty}\frac{(\nu-\alpha)_{p-\ell}(-\nu)_{r+n}(1+\alpha)_\ell\,(1-\nu+\alpha)_n\; x^r x^\ell(-ab)^n}{(1+\alpha-\nu)_{r+\ell+n}\,(1)_{p-\ell+n}\; r!\;\ell!\;n!}(at)^p$$

$$= \sum_{p=-\infty}^{\infty}\sum_{n=0}^{\infty}\frac{(\nu-\alpha)_p(-\nu)_n(-ab)^n(at)^p}{(1)_{p+n}\;n!}\times$$

$$\times\sum_{r,\ell=0}^{\infty}\frac{(-\nu+n)_r(1+\alpha)_\ell\,(-p-n)_\ell\; x^r x^\ell}{(1+\alpha-\nu+n)_{r+\ell}\,(1-\nu+\alpha-p)_\ell\; r!\;\ell!}$$

$$= \sum_{p=-\infty}^{\infty}\sum_{n=0}^{\infty}\frac{(\nu-\alpha)_p(-\nu)_n(-ab)^n(at)^p}{(1)_{p+n}\;n!}\times$$

$$\times F^{0:1;2}_{1:0;1}\left[\begin{array}{ccccccc} \text{——} & : & -\nu+n & ; & 1+\alpha,\ -p-n & ; & \\ & & & & & & x,\ x \\ 1+\alpha-\nu+n & : & \text{——} & ; & 1-\nu+\alpha-p & ; & \end{array}\right].$$

Further, using ***Lemma* 2.3** in equation (3.9), we find

(3.10)
$$\Omega = \sum_{p=-\infty}^{\infty}(\nu-\alpha)_p\sum_{n=0}^{\infty}\frac{(-\nu)_n(-ab)^n}{(n+p)!\;n!}\,{}_1F_1\left[\begin{array}{cc} -\nu-p & ; \\ & x \\ 1-\nu+\alpha-p & ; \end{array}\right](at)^p$$

$$= \sum_{p=-\infty}^{-1}(\nu-\alpha)_p\sum_{n=0}^{\infty}\frac{(-\nu)_n(-ab)^n}{(n+p)!)\;n!}\,{}_1F_1\left[\begin{array}{cc} -\nu-p & ; \\ & x \\ 1-\nu+\alpha-p & ; \end{array}\right](at)^p +$$

$$+\sum_{p=0}^{\infty}(\nu-\alpha)_p\sum_{n=0}^{\infty}\frac{(-\nu)_n(-ab)^n}{(n+p)!\;n!}\,{}_1F_1\left[\begin{array}{cc} -\nu-p & ; \\ & x \\ 1-\nu+\alpha-p & ; \end{array}\right](at)^p$$

$$= \sum_{p=1}^{\infty}(\nu-\alpha)_{-p}\sum_{n=p}^{\infty}\frac{(-\nu)_n(-ab)^n}{(n-p)!\;n!}\,{}_1F_1\left[\begin{array}{cc} -\nu+p & ; \\ & x \\ 1-\nu+\alpha+p & ; \end{array}\right](at)^{-p}$$

$$+\sum_{p=0}^{\infty}(\nu-\alpha)_p\sum_{n=0}^{\infty}\frac{(-\nu)_n(-ab)^n}{(n+p)!\;n!}\,{}_1F_1\left[\begin{array}{cc} -\nu-p & ; \\ & x \\ 1-\nu+\alpha-p & ; \end{array}\right](at)^p$$

$$= \sum_{p=-\infty}^{-1}(\nu-\alpha)_p\sum_{n=-p}^{\infty}\frac{(-\nu)_n(-ab)^n}{(n+p)!\;n!}\,{}_1F_1\left[\begin{array}{cc} -\nu-p & ; \\ & x \\ 1-\nu+\alpha-p & ; \end{array}\right](at)^p$$

$$+\sum_{p=0}^{\infty}(\nu-\alpha)_p\sum_{n=0}^{\infty}\frac{(-\nu)_n(-ab)^n}{(n+p)!\;n!}\,{}_1F_1\left[\begin{array}{cc} -\nu-p & ; \\ & x \\ 1-\nu+\alpha-p & ; \end{array}\right](at)^p$$

$$= \sum_{p=-\infty}^{\infty}(\nu-\alpha)_p\sum_{n=p^\star}^{\infty}\frac{(-\nu)_n(-ab)^n}{(n+p)!\;n!}\,{}_1F_1\left[\begin{array}{cc} -\nu-p & ; \\ & x \\ 1-\nu+\alpha-p & ; \end{array}\right](at)^p,$$

where $p^\star$ is defined by equation (3.2).

Replacing n by $n+p^\star$ in equation (3.10), we get

$$\Omega = \sum_{p=-\infty}^{\infty}(\nu-\alpha)_p \sum_{n=0}^{\infty}\frac{(-\nu)_{n+p^\star}(-ab)^{n+p^\star}}{(n+p+p^\star)!\,(n+p^\star)!}\,{}_1F_1\left[\begin{array}{cc} -\nu-p & ; \\ & x \\ 1-\nu+\alpha-p & ; \end{array}\right](at)^p \tag{3.11}$$

$$= \sum_{p=-\infty}^{\infty}\frac{(\nu-\alpha)_p(-\nu)_{p^\star}(-ab)^{p^\star}}{(p+p^\star)!\,(p^\star)!}\sum_{n=0}^{\infty}\frac{(-\nu+p^\star)_n(1)_n(-ab)^n}{(1+p+p^\star)_n(1+p^\star)_n\,n!}\times$$

$$\times{}_1F_1\left[\begin{array}{cc} -\nu-p & ; \\ & x \\ 1-\nu+\alpha-p & ; \end{array}\right](at)^p.$$

Using definition of ${}_2F_2$ in the r.h.s. of equation (3.11), assertion (3.1) follows.

Proof of Theorem 3.2

Suppose the l.h.s of equation (3.3) is denoted by Δ. Then we have

$$\Delta = \frac{a^{\alpha-\gamma}}{d^\alpha}\sum_{k=0}^{\infty}\frac{(\alpha)_k\,(\lambda)_k\,(at)^k}{(\mu)_k\,d^k\,k!}\left(1+\frac{c}{at}\right)^{k+\alpha-\gamma}\left(1+\frac{bt}{d}\right)^{-(k+\alpha)} \tag{3.12}$$

$$= \frac{a^{\alpha-\gamma}}{d^\alpha}\sum_{k=0}^{\infty}\frac{(\alpha)_k\,(\lambda)_k\,(at)^k}{(\mu)_k\,d^k\,k!}\,{}_1F_0\left[\begin{array}{cc} \gamma-\alpha-k & ; \\ & -\frac{c}{at} \\ \text{—} & ; \end{array}\right]{}_1F_0\left[\begin{array}{cc} \alpha+k & ; \\ & -\frac{bt}{d} \\ \text{—} & ; \end{array}\right]$$

$$= \frac{a^{\alpha-\gamma}}{d^\alpha}\sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty}\sum_{m=0}^{\infty}\frac{(\alpha)_k\,(\lambda)_k\,(at)^k}{(\mu)_k\,d^k\,k!}\frac{(\gamma-\alpha-k)_\ell}{\ell!}\left(\frac{-c}{at}\right)^\ell\frac{(\alpha+k)_m}{m!}\left(\frac{-bt}{d}\right)^m$$

$$= \frac{a^{\alpha-\gamma}}{d^\alpha}\sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty}\sum_{m=0}^{\infty}\frac{(\alpha)_{k+m}(\lambda)_k\left(\frac{a}{d}\right)^k(\gamma-\alpha-k)_\ell\left(\frac{-c}{a}\right)^\ell\left(\frac{-b}{d}\right)^m}{(\mu)_k\,k!\,\ell!\,m!}t^{k-\ell+m}.$$

Replacing m by $m-k$, we get

(3.13)

$$\Delta = \frac{a^{\alpha-\gamma}}{d^\alpha}\sum_{\ell=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{m}\frac{(\alpha)_m(\lambda)_k\left(\frac{a}{d}\right)^k(\gamma-\alpha-k)_\ell\left(\frac{-c}{a}\right)^\ell\left(\frac{-b}{d}\right)^{m-k}}{(\mu)_k\,k!\,\ell!\,(m-k)!}\,t^{m-\ell}$$

$$= \frac{a^{\alpha-\gamma}}{d^\alpha}\sum_{\ell=0}^{\infty}\frac{(\gamma-\alpha)_\ell\left(\frac{-c}{a}\right)^\ell}{\ell!}\sum_{m=0}^{\infty}\frac{(\alpha)_m\left(\frac{-b}{d}\right)^m}{m!}\sum_{k=0}^{m}(\lambda)_k\frac{(\gamma-\alpha+\ell)_{-k}}{(\gamma-\alpha)_{-k}}\frac{\left(\frac{-b}{d}\right)^{-k}\left(\frac{a}{d}\right)^k}{(\mu)_k\,k!\,(1+m)_{-k}}\,t^{m-\ell}$$

$$= \frac{a^{\alpha-\gamma}}{d^\alpha}\sum_{\ell,m=0}^{\infty}\frac{(\gamma-\alpha)_\ell(\alpha)_m\left(\frac{-c}{a}\right)^\ell\left(\frac{-b}{d}\right)^m}{\ell!\,m!}\sum_{k=0}^{m}\frac{(\lambda)_k\,(1-\gamma+\alpha)_k\,(-m)_k}{(\mu)_k\,(1-\gamma+\alpha-\ell)_k}\frac{\left(\frac{a}{b}\right)^k}{k!}t^{m-\ell}$$

$$= \frac{a^{\alpha-\gamma}}{d^\alpha}\sum_{\ell,m=0}^{\infty}\frac{(\gamma-\alpha)_\ell(\alpha)_m\left(\frac{-c}{a}\right)^\ell\left(\frac{-b}{d}\right)^m}{\ell!\,m!}\,{}_3F_2\left[\begin{array}{cc} \lambda,\ 1-\gamma+\alpha,\ -m & ; \\ & \frac{a}{b} \\ \mu,\ 1-\gamma+\alpha-\ell & ; \end{array}\right]t^{m-\ell}.$$

Now, putting $m-\ell=p$ or $m=p+\ell$, we get

(3.14)

$$\Delta = \frac{a^{\alpha-\gamma}}{d^\alpha}\sum_{p=-\infty}^{\infty}\sum_{\ell=0}^{\infty}\frac{(\gamma-\alpha)_\ell(\alpha)_{\ell+p}\left(\frac{-c}{a}\right)^\ell\left(\frac{-b}{d}\right)^{\ell+p}}{\ell!\,(\ell+p)!}\,{}_3F_2\left[\begin{array}{cc} \lambda,\ 1-\gamma+\alpha,\ -\ell-p & ; \\ & \frac{a}{b} \\ \mu,\ 1-\gamma+\alpha-\ell & ; \end{array}\right]t^p$$

$$= \frac{a^{\alpha-\gamma}}{d^{\alpha}} \sum_{p=-\infty}^{\infty} (\alpha)_p \left(\frac{-b}{d}\right)^p \sum_{\ell=0}^{\infty} \frac{(\gamma-\alpha)_\ell(\alpha+p)_\ell \left(\frac{bc}{ad}\right)^\ell}{\ell!\,(\ell+p)!} {}_3F_2\left[\begin{array}{c} \lambda,\ 1-\gamma+\alpha,\ -\ell-p\ ; \\ \mu,\ 1-\gamma+\alpha-\ell\ ; \end{array} \frac{a}{b}\right] t^p$$

$$= \frac{a^{\alpha-\gamma}}{d^{\alpha}} \left\{ \sum_{p=-\infty}^{-1} (\alpha)_p \left(\frac{-b}{d}\right)^p \sum_{\ell=0}^{\infty} \frac{(\gamma-\alpha)_\ell(\alpha+p)_\ell \left(\frac{bc}{ad}\right)^\ell}{\ell!\,(\ell+p)!} {}_3F_2\left[\begin{array}{c} \lambda,\ 1-\gamma+\alpha,\ -\ell-p\ ; \\ \mu,\ 1-\gamma+\alpha-\ell\ ; \end{array} \frac{a}{b}\right] t^p + \right.$$

$$\left. + \sum_{p=0}^{\infty} (\alpha)_p \left(\frac{-b}{d}\right)^p \sum_{\ell=0}^{\infty} \frac{(\gamma-\alpha)_\ell(\alpha+p)_\ell \left(\frac{bc}{ad}\right)^\ell}{\ell!\,(\ell+p)!} {}_3F_2\left[\begin{array}{c} \lambda,\ 1-\gamma+\alpha,\ -\ell-p\ ; \\ \mu,\ 1-\gamma+\alpha-\ell\ ; \end{array} \frac{a}{b}\right] t^p \right\}$$

$$= \frac{a^{\alpha-\gamma}}{d^{\alpha}} \left\{ \sum_{p=1}^{\infty} (\alpha)_{-p} \left(\frac{-b}{d}\right)^{-p} \sum_{\ell=p}^{\infty} \frac{(\gamma-\alpha)_\ell(\alpha-p)_\ell \left(\frac{bc}{ad}\right)^\ell}{\ell!\,(\ell-p)!} {}_3F_2\left[\begin{array}{c} \lambda,\ 1-\gamma+\alpha,\ -\ell+p\ ; \\ \mu,\ 1-\gamma+\alpha-\ell\ ; \end{array} \frac{a}{b}\right] t^{-p} + \right.$$

$$\left. + \sum_{p=0}^{\infty} (\alpha)_p \left(\frac{-b}{d}\right)^p \sum_{\ell=0}^{\infty} \frac{(\gamma-\alpha)_\ell(\alpha+p)_\ell \left(\frac{bc}{ad}\right)^\ell}{\ell!\,(\ell+p)!} {}_3F_2\left[\begin{array}{c} \lambda,\ 1-\gamma+\alpha,\ -\ell-p\ ; \\ \mu,\ 1-\gamma+\alpha-\ell\ ; \end{array} \frac{a}{b}\right] t^p \right\}$$

$$= \frac{a^{\alpha-\gamma}}{d^{\alpha}} \sum_{p=-\infty}^{\infty} (\alpha)_p \left(\frac{-b}{d}\right)^p \sum_{\ell=p^\star}^{\infty} \frac{(\gamma-\alpha)_\ell(\alpha+p)_\ell \left(\frac{bc}{ad}\right)^\ell}{\ell!\,(\ell+p)!} {}_3F_2\left[\begin{array}{c} \lambda,\ 1-\gamma+\alpha,\ -\ell-p\ ; \\ \mu,\ 1-\gamma+\alpha-\ell\ ; \end{array} \frac{a}{b}\right] t^p,$$

where $p^\star = \max\{0, -p\}$.

On replacing ℓ by $\ell + p^\star$ in equation (3.14), we obtain assertion (3.3).

Several other bilinear (multilinear) and bilateral (multilateral) hypergeometric generating relations may also be derived by using series rearrangement technique.

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