Abstract

In this article, we establish some fixed point theorems for weak contraction mappings in the setting of complete $S$-metric spaces. Our results extend, generalize and unify several results from the existing literature regarding $S$-metric space.

Keywords and phrases: Fixed point, weak contraction, $S$-metric space.

1 Introduction and Preliminaries

Fixed point theory is one of the famous and traditional theories in mathematics and has a broad set of applications. In this theory, contraction is one of the main tools to prove the existence and uniqueness of a fixed point. Banach’s contraction principle which gives an answer to the existence and uniqueness of a solution of an operator equation $Tx = x$, is the most widely used fixed point theorem in all of analysis. This principle is constructive in nature and is one of the most useful techniques in the study of nonlinear equations. The statement of the Banach contraction principle is as follows.

Theorem 1.1. (Banach Contraction Principle) Let $(X,d)$ be a complete metric space, $c \in [0,1)$ and $f : X \to X$ a mapping such that for each $x,y \in X$,

\[ d(f(x), f(y)) \leq c d(x,y). \]

Then $f$ has a unique fixed point $p \in X$, such that for each $x \in X$, $\lim_{n \to \infty} f^n x = p$. Inequality (1.1) implies the continuity of $f$.

The Banach contraction principle has been generalized in many ways over the years. In some generalizations, the contractive notion of the map is weakened, see [2, 3, 4, 5, 6, 7, 10, 11] and others.

There are many generalizations of the Banach contraction principle for different metric spaces that exist in the literature of metric fixed point theory.

Recently, Sedghi et al. [8] introduced the notion of $S$-metric space which is a generalization of a $G$-metric space and $D^*$-metric space. In [8] the authors proved some properties of $S$-metric spaces. Also, they obtained some fixed point theorems in $S$-metric space for a self-map.

The definition and properties of $S$-metric spaces are as follows (see, [8]).
Definition 1.1. ([8]), A $S$-metric on a non-empty set $X$ is a function $S : X^3 \to \mathbb{R}^+$ satisfying the following conditions:

(SM$_1$) $S(x, y, z) = 0$ if and only if $x = y = z$;
(SM$_2$) $S(x, y, z) \leq S(x, x, t) + S(y, y, t) + S(z, z, t)$;

for all $x, y, z, t \in X$, where $\mathbb{R}^+ = [0, \infty)$. Then the function $S$ is called an $S$-metric on $X$ and the pair $(X, S)$ is called an $S$-metric space or simply SMS.

Example 1.1. ([8]), Let $X = \mathbb{R}^n$ and $||.||$ a norm on $X$, then $S(x, y, z) = ||y + z - 2x|| + ||y - z||$ is an $S$-metric on $X$.

Example 1.2. ([8]), Let $X = \mathbb{R}^n$ and $||.||$ a norm on $X$, then $S(x, y, z) = ||x - z|| + ||y - z||$ is an $S$-metric on $X$.

Example 1.3. ([9]), Let $X = \mathbb{R}$ be the real line. Then $S(x, y, z) = |x - z| + |y - z|$ for all $x, y, z \in \mathbb{R}$ is an $S$-metric on $X$. This $S$-metric on $X$ is called the usual $S$-metric on $X$.

Lemma 1.1. ([8], Lemma 2.5) If $(X, S)$ is an $S$-metric space, then we have $S(x, y, z) = S(y, y, x)$ for all $x, y \in X$.

Lemma 1.2. ([8], Lemma 2.12) Let $(X, S)$ be an $S$-metric space. If $\{x_n\}$ and $\{y_n\}$ are sequences in $X$ converging to $x$ and $y$ respectively, that is, $x_n \to x$ and $y_n \to y$ as $n \to \infty$, then $S(x_n, y_n, z_n) \to S(x, y, z)$ as $n \to \infty$.

Lemma 1.3. ([8], Lemma 2.10) Let $(X, S)$ be an $S$-metric space. If the sequence $\{x_n\}$ in $X$ converges to $x$, then the limit $x$ is unique.

Lemma 1.4. ([8], Lemma 2.11) Let $(X, S)$ be an $S$-metric space. If the sequence $\{x_n\}$ in $X$ converges to $x$, then $\{x_n\}$ is a Cauchy sequence.

Definition 1.2. ([8]) Let $(X, S)$ be an $S$-metric space.

1. A sequence $\{x_n\}$ in $X$ converges to $x \in X$ if $S(x_n, x_n, x) \to 0$ as $n \to \infty$; that is, for each $\varepsilon > 0$, there exists an $m_0 \in \mathbb{N}$ such that for all $n \geq m_0$ we have $S(x_n, x_n, x) < \varepsilon$. We denote this by $\lim_{n \to \infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.

2. A sequence $\{x_n\}$ in $X$ is called a Cauchy sequence if $S(x_n, x_n, x_m) \to 0$ as $n, m \to \infty$; that is, for each $\varepsilon > 0$, there exists an $m_0 \in \mathbb{N}$ such that for all $n, m \geq m_0$ we have $S(x_n, x_n, x_m) < \varepsilon$.

3. The $S$-metric space $(X, S)$ is called complete if every Cauchy sequence in $(X, S)$ is convergent in $(X, S)$.

Definition 1.3. Let $T$ be a self mapping on an $S$-metric space $(X, S)$. Then $T$ is said to be continuous at $x \in X$ if for any sequence $\{x_n\}$ in $X$ with $x_n \to x$ implies that $Tx_n \to Tx$ as $n \to \infty$.

Definition 1.4. ([8]) Let $(X, S)$ be an $S$-metric space. A mapping $T : X \to X$ is said to be a contraction if there exists a constant $0 \leq L < 1$ such that

\[ S(Tx, Tx, Ty) \leq L S(x, y), \]

for all $x, y \in X$. If the $S$-metric space $(X, S)$ is complete then the mapping defined as above has a unique fixed point.
**Definition 1.5.** ([1]) (Weak Contraction Mapping) Let \((X, d)\) be a complete metric space. A mapping \(T : X \to X\) is said to be weakly contractive if

\[
d(T(x), T(y)) \leq d(x, y) - \psi(d(x, y)),
\]

where \(x, y \in X\), \(\psi : [0, \infty) \to [0, \infty)\) is continuous and non-decreasing, \(\psi(x) = 0\) if and only if \(x = 0\) and \(\lim_{x \to \infty} \psi(x) = \infty\).

If we take \(\psi(x) = cx\) where \(0 < c < 1\) then (1.3) reduces to (1.1).

Now, we introduce the notion of weak contraction in \(S\)-metric space as follows.

**Definition 1.6.** Let \((X, S)\) be an \(S\)-metric space. A mapping \(T : X \to X\) is said to be a weak contraction on \(X\) if there exists a function \(\psi : [0, \infty) \to [0, \infty)\) with \(\psi(t) = 0\) if and only if \(t = 0\) and satisfying the following condition

\[
S(Tx, Tx, Ty) \leq S(x, x, y) - \delta \psi(S(x, x, y)),
\]

for all \(x, y \in X\), where \(0 \leq \delta < 1\).

If we take \(\psi(x) = x\) and \(\delta = L\), then (1.4) reduces to (1.2).

**Example 1.4.** Let \(X = \mathbb{R}\) and defined \(S : X^3 \to \mathbb{R}^+\) by

\[
S(x, x, y) = \begin{cases} 
4x^2 + y^2 & \text{if } x \neq y, \\
0 & \text{if } x = y,
\end{cases}
\]

for all \(x, y \in X\). Then \(S\) is an \(S\)-metric on \(X\) and \((X, S)\) is a \(S\)-metric space. Let \(T : X \to X\) defined by \(T(x) = \frac{x}{4}\) and \(\psi(t) = 15t\) for all \(t \geq 0\), where \(\psi : [0, \infty) \to [0, \infty)\) is continuous and non-decreasing function. Then

\[
S(Tx, Tx, Ty) = S\left(\frac{x}{4}, \frac{x}{4}, \frac{y}{4}\right) = \frac{x^2}{4} + \frac{y^2}{16}
\]

\[
= 4x^2 + y^2 - \frac{15}{16}(4x^2 + y^2)
\]

\[
= S(x, x, y) - \frac{1}{16}\psi(S(x, x, y)).
\]

Thus \(T\) is a weak contraction on \(X\).

The purpose of this paper is to prove some fixed point theorems under a weak contraction condition in the setting of \(S\)-metric spaces. Our results extend, generalize and improve several results from the existing literature in \(S\)-metric spaces.

**2 Main Results**

In this section, we shall prove some fixed point theorem in a complete \(S\)-metric space for weak contraction mapping.

**Theorem 2.1.** Let \((X, S)\) be a complete \(S\)-metric space. Let \(T : X \to X\) be a mapping satisfying the condition:

\[
S(Tx, Tx, Ty) \leq \min\{S(x, x, Tx), S(y, y, Ty)\} - h\phi\left(\max\{S(x, x, Ty), S(y, y, Tx)\}\right),
\]

for all \(x, y \in X\) where \(h > 0\) and \(\phi : [0, \infty) \to [0, \infty)\) is a continuous function with \(\phi(t) = 0\) if and only if \(t = 0\). Then \(T\) has a unique fixed point in \(X\).
Proof. Let \( x_0 \in X \) and \( \{x_n\} \) be a sequence defined by \( x_{n+1} = Tx_n \) for \( n = 0, 1, 2, \ldots \). If \( x_n = x_{n+1} = Tx_n \), then \( x_n \) is a fixed point of \( T \). So we assume that \( x_n \neq x_{n+1} \). It follows from (2.1), (SM2) and Lemma 1.1 that

\[
S(x_n, x_{n+1}) = S(Tx_{n-1}, Tx_{n-1}, Tx_n) \\
\leq \min \{S(x_{n-1}, x_{n-1}, Tx_{n-1}), S(x_n, x_n, Tx_n)\} \\
- h \phi \left( \max \{S(x_{n-1}, x_{n-1}, Tx_{n-1}), S(x_n, x_n, Tx_n)\} \right) \\
= \min \{S(x_{n-1}, x_{n-1}, x_n), S(x_n, x_n, x_{n+1})\} \\
- h \phi \left( \max \{S(x_{n-1}, x_{n-1}, x_n), S(x_n, x_n, x_{n+1})\} \right) \\
= \min \{S(x_{n-1}, x_{n-1}, x_n), S(x_n, x_n, x_{n+1})\} - h \phi \left( S(x_{n-1}, x_{n-1}, x_n) \right).
\]

Since \( \phi \geq 0 \), then we obtain from equation (2.2) that

\[
S(x_n, x_n, x_{n+1}) \leq S(x_{n-1}, x_{n-1}, x_n).
\]

Thus we have a non-negative and non-increasing sequence \( \{S(x_n, x_n, x_{n+1})\} \). Therefore, there exists an \( L \geq 0 \) such that

\[
\lim_{n \to \infty} S(x_n, x_n, x_{n+1}) = L.
\]

Since \( \phi \) is continuous on \([0, \infty)\), using (2.2), (SM2), Lemma 1.1 and taking the limit as \( n \to \infty \), we obtain

\[
L \leq L - h \lim_{n \to \infty} \phi \left( 2S(x_{n-1}, x_{n-1}, x_n) + S(x_n, x_n, x_{n+1}) \right) \\
= L - h \phi \left( \lim_{n \to \infty} 2S(x_{n-1}, x_{n-1}, x_n) + \lim_{n \to \infty} S(x_n, x_n, x_{n+1}) \right) \\
= L - h \phi (3L).
\]

Since \( h > 0 \) and \( \phi(3L) \geq 0 \), then equation (2.5) is possible only if \( \phi(3L) = 0 \). Thus, we get \( L = 0 \). Hence, we obtain that

\[
\lim_{n \to \infty} S(x_n, x_n, x_{n+1}) = L = 0.
\]

This proves that \( \{x_n\} \) is a Cauchy sequence in \( X \). Since \( X \) is complete, then there exists an element \( z \in X \) such that \( \lim_{n \to \infty} x_n = z \), that is, \( x_n \to z \) as \( n \to \infty \). To show that \( z \) is a fixed point of \( T \). Using (2.1), we have

\[
S(x_{n+1}, x_{n+1}, Tz) = S(Tx_n, Tx_n, Tz) \\
\leq \min \{S(x_n, x_n, Tx_n), S(z, z, Tz)\} \\
- h \phi \left( \max \{S(x_n, x_n, Tx_n), S(z, z, Tz)\} \right) \\
= \min \{S(x_n, x_n, x_{n+1}), S(z, z, Tz)\} \\
- h \phi \left( \max \{S(x_n, x_n, x_{n+1}), S(z, z, Tz)\} \right).
\]

Since \( \lim_{n \to \infty} x_n = z \) and \( \lim_{n \to \infty} S(x_n, x_n, x_{n+1}) = 0 \), then from equation (2.7) and taking the limit as \( n \to \infty \), we get

\[
S(z, Tz) \leq \min \{S(z, Tz), S(z, z, Tz)\} \\
- h \phi \left( \max \{S(z, Tz), S(z, z, Tz)\} \right) \\
= \min \{0, S(z, Tz)\}
\]

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Lemma

(2.11) \[ T x \]

Let \( \Psi \)

(2.12) \[ \text{for all } x \]

satisfying the condition:

Theorem

(2.10) \[ T \text{point of } S \]

is a fixed point of \( T \). To prove the uniqueness of fixed point of \( T \), assume that \( v \) be another fixed point of \( T \) such that \( v = Tv \) with \( v \neq z \). Using (2.1) and Lemma 1.1, we have

(2.9) \[ S(z, z, v) = S(Tz, Tz, Tv) \]

\[ \leq \min \{ S(z, z, Tz), S(v, v, Tv) \} \]

\[ = \min \{ S(z, z, Tz), S(v, Tz) \} \]

\[ = \min \{ S(z, z, v), S(v, v, z) \} \]

\[ = 0 - h \phi \left( S(z, z, v) \right) \]

Inequality (2.9) is possible only if \( S(z, z, v) = 0 \). Hence \( z = v \). This shows that the fixed point of \( T \) is unique. This completes the proof.

Theorem 2.2. Let \((X, S)\) be a complete \( S \)-metric space. Let \( T : X \to X \) be a continuous mapping satisfying the condition:

(2.10) \[ \Psi \left( S(Tx, Tx, Ty), S(Ty, Ty, Tx) \right) \leq q \Psi \left( S(x, x, y), S(y, y, x) \right), \]

for all \( x, y \in X \), where \( 0 < q < 1 \) and \( \Psi : [0, \infty)^2 \to [0, \infty)^2 \) is a continuous function on \([0, \infty)^2\) with \( \Psi(a, b) = 0 \) if and only if \( a = 0 = b \). Then \( T \) has a unique fixed point in \( X \).

Proof. Let \( x_0 \in X \) and \( \{x_n\} \) be a sequence defined by \( x_{n+1} = Tx_n \) for \( n = 0, 1, 2, \ldots \) If \( x_n = x_{n+1} = Tx_n \), then \( x_n \) is a fixed point of \( T \). So, we assume that \( x_n \neq x_{n+1} \). It follows from (2.10), \((SM_2)\) and Lemma 1.1 that

(2.11) \[ \Psi \left( S(x_n, x_n, x_1), S(x_1, x_1, x_0) \right) \]

\[ = \Psi \left( S(Tx_n, Tx_n, Tx_{n+1}), S(Tx_{n+1}, Tx_{n+1}, Tx_n) \right) \]

\[ \leq q \Psi \left( S(x_n, x_n, x_{n+1}), S(x_{n+1}, x_{n+1}, x_n) \right) \]

\[ \vdots \]

\[ \leq q^n \Psi \left( S(x_0, x_0, x_1), S(x_1, x_1, x_0) \right). \]

Since \( 0 < q < 1 \) and, for \( n \to \infty \), we get

(2.12) \[ \Psi \left( S(x_{n+1}, x_{n+1}, x_{n+2}), S(x_{n+2}, x_{n+2}, x_{n+1}) \right) \to 0. \]

Since \( \Psi \) is a continuous function,

\[ 0 = \lim_{n \to \infty} \Psi \left( S(x_{n+1}, x_{n+1}, x_{n+2}), S(x_{n+2}, x_{n+2}, x_{n+1}) \right) \]

\[ = \Psi \left( \lim_{n \to \infty} S(x_{n+1}, x_{n+1}, x_{n+2}), \lim_{n \to \infty} S(x_{n+2}, x_{n+2}, x_{n+1}) \right). \]

Thus, by the property of \( \Psi \),

\[ \lim_{n \to \infty} S(x_{n+1}, x_{n+1}, x_{n+2}) = \lim_{n \to \infty} S(x_{n+2}, x_{n+2}, x_{n+1}) = 0. \]
Hence by Lemma 1.1 and Definition 1.2(2), \( \{x_n\} \) is a Cauchy sequence in a complete \( S \)-metric space \( X \). Therefore there exists an \( u \in X \) such that \( \lim_{n \to \infty} x_n = u \). Since \( T \) is continuous, we have

\[
Tu = T(\lim_{n \to \infty} x_n) = \lim_{n \to \infty} Tx_n = \lim_{n \to \infty} x_{n+1} = u,
\]

and \( u \) is a fixed point of \( T \). Now, we shall show that the uniqueness \( u \) is unique. For this, suppose that \( v \) is another fixed point of the mapping \( T \) such that \( v = Tv \) with \( v \neq u \). Using equation (2.10), we have

\[
\Psi(S(u, u, v), S(v, v, u)) = \Psi(S(Tu, Tu, Tv), S(Tv, Tv, Tu)) \\
\leq q \Psi(S(u, u, v), S(v, v, u)).
\]

Since \( 0 < q < 1 \), then we get \( \Psi(S(u, u, v), S(v, v, u)) = 0 \), which by the property of \( \Psi \), implies that \( S(u, u, v) = S(v, v, u) = 0 \). Thus we obtain \( u = v \). This shows that the fixed point of \( T \) is unique. This completes the proof.

If we take \( \Psi(x, y) = x + y, q = L \) and using Lemma 1.1 in Theorem 2.2, then we have the following result as corollary.

**Corollary 2.1.** Let \( (X, S) \) be a complete \( S \)-metric space. Suppose that \( T : X \to X \) be a mapping satisfying the condition:

\[
S(Tx, Tx, Ty) \leq LS(x, x, y),
\]

for all \( x, y \in X \) and \( 0 < L < 1 \) is a constant. Then \( T \) has a unique fixed point in \( X \).

**Remark 2.1.** Corollary 2.1 extends the well known Banach contraction principle from complete metric space to that in the setting of a complete \( S \)-metric space considered in this paper.

**Example 2.1.** Let \( X = \mathbb{R} \) be the real line and \( S \) be the usual \( S \)-metric on \( X \) defined as \( S(x, y, z) = |x - z| + |y - z| \) for all \( x, y, z \in \mathbb{R} \). Then \( (X, S) \) is called an \( S \)-metric space. Consider the mapping \( T : X \to X \) defined by \( T(x) = \frac{x}{7} \) for all \( x \in [0, 1] \). Then, we have

\[
S(Tx, Tx, Ty) = S\left(\frac{x}{7}, \frac{x}{7}, \frac{y}{7}\right) \\
= \frac{|x - y|}{7} + \frac{|x - y|}{7} \\
= 2\left|\frac{x - y}{7}\right| \\
= \frac{2}{7}|x - y| \\
\leq |x - y| \\
= \frac{1}{2}(2|x - y|) \\
= LS(x, x, y),
\]

where \( L = \frac{1}{2} < 1 \). Thus \( T \) satisfies all the conditions of Corollary 2.1. Hence, by applying Corollary 2.1, \( T \) has a unique fixed point in \( X \). It is seen that \( 0 \in X \) is the unique fixed point of \( T \).

3 Conclusion

In this article, we have established some unique fixed point theorems under a weak contractive condition in the framework of complete \( S \)-metric spaces. Our results extend, generalize and unify some recent results from the existing literature.
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References