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(Dedicated to Honor Professor H.M. Srivastava on His 80th Birth Anniversary Celebrations)

ITERATION PRINCIPLE FOR IVPS OF NONLINEAR FIRST ORDER IMPULSIVE DIFFERENTIAL EQUATIONS

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Abstract

In this paper we prove the existence and approximation theorems for the initial value problems of first order nonlinear impulsive differential equations under certain mixed partial Lipschitz and partial compactness type conditions. Our results are based on the Dhage monotone iteration principle embodied in a hybrid fixed point theorem of Dhage involving the sum of two monotone order preserving operators in a partially ordered Banach space. The novelty of the present approach lies the fact that we obtain an algorithm for the solution. Our abstract main result is also illustrated by indicating a numerical example.

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1 Introduction

It is well-known that many of the dynamical systems in the universe involve the jumps or discontinuities due to to impulses at finite number of places in a given period of time and impulsive differential equations are the mathematical models to describe such phenomena precisely. The existence and uniqueness theory for nonlinear impulsive differential equations have received much attention during the last decade, however the theory of approximation of the solutions to such equations is relatively rare in the literature. The dynamical systems, which involve the jumps or discontinuities at finite number of points are modeled on the nonlinear impulsive differential equations. The sudden changes in the dynamic systems for a short period of time can betterly be discussed with he help of impulsive differential equations. The importance of such impulsive differential equations in the dynamic systems as well as exhaustive account of various topics related to this problem may be found in the research monographs of Samoilenko and Perestyuk [21], Lakshmikantam et al [20] and the references therein. The existence theorems so far discussed in the literature for such impulsive differential equations involve either the use of usual Lipschitz or compactness type condition on the nonlinearities and which are considered to be very strong conditions in the subject of nonlinear analysis. Here in the present set up of new Dhage monotone iteration method, we do not need usual Lipschitz and compactness type conditions but require only partial Lipschitz and partial compactness type conditions of the nonlinearity and the existence as well as approximation of the solutions is obtained under certain monotonic conditions. We claim that the results of this paper are new to the literature on impulsive differential equations.

Let $\mathbb{R}$ be the real line and let $J = [0, T]$ be a closed and bounded interval in $\mathbb{R}$. Let $t_0, \dots t_{p+1}$ be the points in J such that $0 = t_0 < t_1 < \cdots, < t_p < t_{p+1} = T$ and let $J' = J \setminus \{t_1, \dots, t_p\}$. Denote

$J_j = (t_j, t_{j+1}) \subset J$ for $j = 1, 2, \ldots, p$. By $X = C(J, \mathbb{R})$ and $L^1(J, \mathbb{R})$ we denote respectively the spaces of continuous and Lebesgue integrable real-valued functions defined on J.

Now, given a function $h \in L^1(J, \mathbb{R}_+)$, consider the initial value problem (in short IVP) for the first order impulsive differential equation (in short IDE)

$$\left.\begin{aligned} x'(t) + h(t)x(t) &= f(t, x(t)), \quad t \in J \setminus \{t_1, \ldots, t_p\}, \\ x(t_j^+) - x(t_j^-) &= \mathcal{I}_j(x(t_j)), \\ x(0) &= x_0 \in \mathbb{R}, \end{aligned}\right\} \tag{1.1}$$

where, the limits $x(t_j^+)$ and $x(t_j^-)$ are respectively the right and left limit of x at $t = t_j$ such that $x(t_j) = x(t_j^-)$, $\mathcal{I}_j \in C(\mathbb{R}, \mathbb{R})$, $\mathcal{I}_j(x(t_j))$ are the impulsive effects at the points $t = t_j$, $j = 1, \ldots, p$ and $f : J \times \mathbb{R} \to \mathbb{R}$ is such that f is continuous on $J' = J - \{t_1, ..., t_p\}$, and there exist the limits

$$\lim_{t \to t_j^-} f(t, u) = f(t_j, u) \quad \text{and} \quad \lim_{t \to t_{j+}} f(t, u), \; u \in \mathbb{R},$$

for each $j = 1, \ldots, p$.

By a *impulsive solution* of the IDE (1.1) we mean a function $x \in PC^1(J, \mathbb{R})$ that satisfies the differential equation and the conditions in (1.1), where $PC^1(J, \mathbb{R})$ is the space of piecewise continuously differentiable real-valued functions defined on J.

The IDE (1.1) has already been discussed in the literature under continuity and compactness type conditions of the function f for various aspects of the solutions. The existence and unqueness theorems for the IDE (1.1) may be proved using the classical hybrid fixed point theorems Schauder and Banach given in Dhage [8] and references therein. Here in the present study, we discuss the IDE (1.1) for existence and approximate impulsive solution under partial Lipschit and partial compactness type conditions via Dhage iteration method based on a hybrid fixed point theorems of Dhage [3, 4].

2 Auxiliary Results

Throughout this paper, unless otherwise mentioned, let $(E, \preceq |\cdot\|)$ denote a partially ordered normed linear space. Two elements x and y in E are said to be **comparable** if either the relation $x \preceq y$ or $y \preceq x$ holds. A non-empty subset C of E is called a **chain** or **totally ordered** if all the elements of C are comparable. It is known that E is **regular** if $\{x_n\}$ is a nondecreasing (resp. nonincreasing) sequence in E such that $x_n \to x^*$ as $n \to \infty$, then $x_n \preceq x^*$ (resp. $x_n \succeq x^*$) for all $n \in \mathbb{N}$. The conditions guaranteeing the regularity of E may be found in Heikkilä and Lakshmikantham [19] and the references therein.

We need the following definitions (see Dhage [2, 3, 4] and the references therein) in what follows.

A mapping $\mathcal{T} : E \to E$ is called **isotone** or **monotone nondecreasing** if it preserves the order relation $\preceq$, that is, if $x \preceq y$ implies $\mathcal{T}x \preceq \mathcal{T}y$ for all $x, y \in E$. Similarly, $\mathcal{T}$ is called **monotone nonincreasing** if $x \preceq y$ implies $\mathcal{T}x \succeq \mathcal{T}y$ for all $x, y \in E$. Finally, $\mathcal{T}$ is called **monotonic** or simply **monotone** if it is either monotone nondecreasing or monotone nonincreasing on E. A mapping $\mathcal{T} : E \to E$ is called **partially continuous** at a point $a \in E$ if for given $\epsilon > 0$ there exists a $\delta > 0$ such that $\|\mathcal{T}x - \mathcal{T}a\| < \epsilon$ whenever x is comparable to a and $\|x - a\| < \delta$. $\mathcal{T}$ is called partially continuous on E if it is partially continuous at every point of it. It is clear that if $\mathcal{T}$ is partially continuous on E, then it is continuous on every chain C contained in E and vice-versa. A non-empty subset S of the partially ordered metric space E is called **partially bounded** if every chain C in S is bounded. A mapping $\mathcal{T}$ on a partially ordered metric space E into itself is called **partially bounded** if $\mathcal{T}(E)$ is a partially bounded subset of E. $\mathcal{T}$ is called **uniformly partially**

bounded if all chains C in $\mathcal{T}(E)$ are bounded by a unique constant. A non-empty subset S of the partially ordered metric space E is called **partially compact** if every chain C in S is a compact subset of E. A mapping $\mathcal{T} : E \to E$ is called **partially compact** if every chain C in $\mathcal{T}(E)$ is a relatively compact subset of E. $\mathcal{T}$ is called **uniformly partially compact** if $\mathcal{T}$ is a uniformly partially bounded and partially compact operator on E. $\mathcal{T}$ is called **partially totally bounded** if for any bounded subset S of E, $\mathcal{T}(S)$ is a partially totally bounded subset of E. If $\mathcal{T}$ is partially continuous and partially totally bounded, then it is called **partially completely continuous** on E.

***Remark* 2.1.** *Suppose that $\mathcal{T}$ is a monotone operator on E into itself. Then $\mathcal{T}$ is a partially bounded or partially compact on E if $\mathcal{T}(C)$ is a bounded or compact subset of E for each chain C in E.*

***Definition* 2.1** (Dhage [5, 6], Dhage and Dhage [15])**.** *The order relation $\preceq$ and the metric d on a non-empty set E are said to be $\mathcal{D}$-**compatible** if $\{x_n\}$ is a monotone sequence, that is, monotone nondecreasing or monotone nonincreasing sequence in E and if a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges to x^* implies that the original sequence $\{x_n\}$ converges to x^*. Similarly, given a partially ordered normed linear space $(E, \preceq, \|\cdot\|)$, the order relation $\preceq$ and the norm $\|\cdot\|$ are said to be $\mathcal{D}$-compatible if $\preceq$ and the metric d defined through the norm $\|\cdot\|$ are $\mathcal{D}$-compatible. A subset S of E is called **Janhavi set** if the order relation $\preceq$ and the metric d or the norm $\|\cdot\|$ are $\mathcal{D}$-compatible in S. In particular, if $S = E$, then E is called a **Janhavi metric space** or **Janhavi Banach space**.*

Clearly, the set $\mathbb{R}$ of real numbers with usual order relation $\leq$ and the norm defined by the absolute value function $|\cdot|$ has this property. Similarly, the finite dimensional Euclidean space $\mathbb{R}^n$ with usual componentwise order relation and the standard norm possesses the compatibility property and so is a **Janhavi Banach space**.

***Definition* 2.2.** *An upper semi-continuous and monotone nondecreasing function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is called a $\mathcal{D}$-function provided $\psi(0) = 0$. A monotone operator $\mathcal{T} : E \to E$ is called nonlinear partial $\mathcal{D}$-contraction if there exists a $\mathcal{D}$-function $\psi_{\mathcal{T}}$ such that*

$$\|\mathcal{T}x - \mathcal{T}y\| \leq \psi_{\mathcal{T}}(\|x - y\|) \tag{2.1}$$

for all comparable elements $x, y \in E$, where $0 < \psi_{\mathcal{T}}(r) < r$ for $r > 0$.

In particular, if $\psi_{\mathcal{T}}(r) = k\,r$, $k > 0$, $\mathcal{T}$ is called a partial Lipschitz operator with a Lipschitz constant k and moreover, if $0 < k < 1$, $\mathcal{T}$ is called a linear partial contraction on E with the contraction constant k.

The **Dhage monotone iteration principle** or **Dhage monotone iteration method** embodied in the following applicable hybrid fixed point theorems of Dhage [3] in a partially ordered normed linear space is used as a key tool for our work contained in this paper. The details of the Dhage monotone iteration principle or method are given in Dhage [5, 6, 7], Dhage *et al.* [13, 14], Dhage and Otrocol [17] and the references therein.

***Theorem* 2.1** (Dhage [3])**.** *Let $(E, \preceq, \|\cdot\|)$ be a partially ordered Banach space and let $\mathcal{T} : E \to E$ be a monotone nondecreasing and nonlinear partial $\mathcal{D}$-contraction. Suppose that there exists an element $x_0 \in E$ such that $x_0 \preceq \mathcal{T}x_0$ or $x_0 \succeq \mathcal{T}x_0$. If $\mathcal{T}$ is continuous or E is regular, then $\mathcal{T}$ has a unique comparable fixed point x^* and the sequence $\{\mathcal{T}^n x_0\}$ of successive iterations converges monotonically to x^*. Moreover, the fixed point x^* is unique if every pair of elements in E has a lower bound or an upper bound.*

***Theorem* 2.2** (Dhage [3])**.** *Let $(E, \preceq, \|\cdot\|)$ be a regular partially ordered complete normed linear space and let every compact chain C in E be Janhavi set. Let $\mathcal{A}, \mathcal{B} : E \to E$ be two monotone nondecreasing operators such that*

(a) $\mathcal{A}$ is partially bounded and nonlinear partial $\mathcal{D}$-contraction,
(b) $\mathcal{B}$ is partially continuous and partially compact, and
(c) there exists an element $x_0 \in E$ such that $x_0 \preceq \mathcal{A}x_0 + \mathcal{B}x_0$ or $x_0 \succeq \mathcal{A}x_0 + \mathcal{B}x_0$.

Then the hybrid operator equation $\mathcal{A}x + \mathcal{B}x = x$ has a solution x^* in E and the sequence $\{x_n\}$ of successive iterations defined by $x_{n+1} = \mathcal{A}x_n + \mathcal{B}x_n$, n=0,1,…, converges monotonically to x^*.

***Remark* 2.2.** *The condition that every compact chain of E is Janhavi set holds if every partially compact subset of E possesses the compatibility property with respect to the order relation $\preceq$ and the norm $\|\cdot\|$ in E.*

***Remark* 2.3.** *1 We remark that hypothesis (a) of Theorem 2.2 implies that the operator $\mathcal{A}$ is partially continuous and consequently both the operators $\mathcal{A}$ and $\mathcal{B}$ in the theorem are partially continuous on E. The regularity of E in above Theorems 2.1 and 2.2 may be replaced with a stronger continuity condition respectively of the operators $\mathcal{T}$ and $\mathcal{A}$ and $\mathcal{B}$ on E which are the results proved in Dhage [2, 3].*

3 Existence and Approximation Theorem

Let $X_j = C(J_j, \mathbb{R})$ denote the class of continuous real-valued functions on the interval $J_j = (t_j, t_{j+1})$. Denote by $PC(J, \mathbb{R})$ the space of piecewise continuous real-valued functions on J defined by

$$PC(J,\mathbb{R}) = \Big\{x \in X_j \mid x(t_j^-) \text{ and } x(t_j^+) \text{ exists for } j = 1,\dots,p; \quad \text{and } x(t_j^-) = x(t_j)\Big\}. \tag{3.1}$$

Define a supremum norm $\|\cdot\|$ in $PC(J, \mathbb{R})$ by

$$\|x\|_{PC} = \sup_{t\in J} |x(t)| \tag{3.2}$$

and define the order cone K in $PC(J, \mathbb{R})$ by

$$K = \{x \in PC(J,\mathbb{R}) \mid x(t) \geq 0 \ \text{ for all } t \in J\}, \tag{3.3}$$

which is obviously a normal cone in $PC(J, \mathbb{R})$. Now, define the order relation $\preceq$ in $PC(J, \mathbb{R})$ by

$$x \preceq y \iff y - x \in K \tag{3.4}$$

which is equivalent to

$$x \preceq y \iff x(t) \leq y(t) \ \text{ for all } \ t \in J.$$

Clearly, $(PC(J, \mathbb{R}), K)$ becomes a regular ordered Banach space with respect to the above norm and order relation in $PC(J, \mathbb{R})$ and every compact chain C in $PC(J, \mathbb{R})$ is Janhavi set in view of the following lemmas proved in Dhage [6, 7].

***Lemma* 3.1** (Dhage [6, 7])**.** *Every ordered Banach space (E, K) is regular.*

***Lemma* 3.2** (Dhage [6, 7])**.** *Every partially compact subset S of an ordered Banach space (E, K) is a Janhavi set in E.*

We need the following definition in what follows.

***Definition* 3.1.** *A function $u \in PC^1(J,\mathbb{R})$ is said to be a lower impulsive solution of the IDE (1.1) if it satisfies*

$$\left.\begin{aligned} u'(t) + h(t)u(t) \le f(t,u(t)),\ \ t \in J \setminus \{t_1,\dots,t_p\},\\ u(t_j^+) - u(t_j^-) \le \mathcal{I}_j(u(t_j)),\\ u(0) \le x_0 \in \mathbb{R}, \end{aligned}\right\}$$

for $j = 1, 2, ..., p$. Similarly, a function $v \in PC^1(J,\mathbb{R})$ is called an upper impulsive solution of the IDE (1.1) if the above inequality is satisfied with reverse sign.

We consider the following set of assumptions in what follows:

(H_1) The impulsive functions $\mathcal{I}_j \in C(\mathbb{R},\mathbb{R})$ are bounded on X with bounds $M_{\mathcal{I}_j}$ for each $j = 1,\dots,p.$,

(H_2) There exists a constants $L_{\mathcal{I}_j} > 0$ such that

$$0 \le \mathcal{I}_j x - \mathcal{I}_j y \le L_{\mathcal{I}_j}(x-y)$$

for all $x, y \in \mathbb{R}$, $x \ge y$, where $j = 1,\dots,p$.

(H_3) The function f is bounded on $J \times \mathbb{R}$ with bound M_f.

(H_4) $f(t,x)$ is nondecreasing in x for each $t \in J$.

(H_5) There exists a constant $L_f > 0$ such that

$$0 \le f(t,x) - f(t,y) \le L_f(x-y)$$

for all $t \in J$ and $x, y \in \mathbb{R}$, $x \ge y$.

(H_6) The IDE (1.1) has a lower impulsive solution $u \in PC^1(J,\mathbb{R})$.

Below we prove some useful results in what follows.

***Lemma* 3.3.** *Given $\sigma \in L^1(J,\mathbb{R})$, a function $x \in PC(J,\mathbb{R})$ is a impulsive solution to the IDE*

$$\left.\begin{aligned} x'(t) + h(t)x(t) = \sigma(t),\ \ t \in J \setminus \{t_1,\dots,t_p\},\\ x(t_j^+) - x(t_j^-) = \mathcal{I}_j(x(t_j)),\\ x(0) = x_0, \end{aligned}\right\} \tag{3.5}$$

if and only if it is an impulsive solution of the impulsive integral equation

$$x(t) = x_0\, e^{-H(t)} + \sum_{0<t_j<t} k(t,t_j)\mathcal{I}_j(x(t_j)) + \int_0^t k(t,s)\sigma(s)\,ds,\ \ t \in J, \tag{3.6}$$

where the kernel function k is given by

$$k(t,s) = e^{-H(t)+H(s)} \quad \text{and} \quad H(t) = \int_0^t h(s)\,ds. \tag{3.7}$$

Proof. First note that the integral in $H(t)$ is a continuous and nonnegative real-valued function on J. Therefore, we have $H(t) > 0$ on J provided h is not an identically zero function. Otherwise $H(t) \equiv 0$ on J. Moreover, we have $H(t^-) = H(t) = H(t^+)$ for all $t \in J$.

First suppose that x is an impulsive solution of the IDE (3.5) on J. Then, we have

$$\left.\begin{aligned} \left(e^{H(t)}x(t)\right)' = e^{H(t)}\sigma(t),\ \ t \in J \setminus \{t_1,\dots,t_p\},\\ x(t_j^+) - x(t_j^-) = \mathcal{I}_j(x(t_j)),\\ x(0) = x_0 \in \mathbb{R}, \end{aligned}\right\} \tag{3.8}$$

for $j = 1, 2, \ldots, p$.

From the theory of integral calculus, it follows that

$$\begin{aligned} e^{H(t_1^-)}x(t_1^-) - e^{H(0)}x(0) &= \int_0^{t_1} \left(e^{H(s)}x(s)\right)' \, ds \\ e^{H(t_2^-)}x(t_2^-) - e^{H(t_1^+)}x(t_1^+) &= \int_{t_1}^{t_2} \left(e^{H(s)}x(s)\right)' \, ds \\ &\vdots \\ e^{H(t)}x(t) - e^{H(t_p^+)}x(t_p^+) &= \int_{t_p}^{t} \left(e^{H(s)}x(s)\right)' \, ds. \end{aligned}$$

Summing up the above equations,

$$e^{H(t)}x(t) - \sum_{0<t_j<t} e^{H(t_j)}\mathcal{I}_j(x(t_j)) = x_0 + \int_0^t e^{H(s)}h(s)\,ds,$$

or

$$x(t) = x_0e^{-H(t)} + \sum_{0<t_j<t} k(t,t_j)\mathcal{I}_j(x(t_j)) + \int_0^t k(t,s)\sigma(s)\,ds.$$

for $t \in J$.

Conversely, suppose that x is an impulsive solution of the impulsive integral equation (3.6). Obviously x satisfies the initial and jump conditions given in (3.5). By the definition of the kernel function k, we obtain

$$e^{H(t)}x(t) = x_0 + \sum_{0<t_j<t} e^{H(t_j)}\mathcal{I}_j(x(t_j)) + \int_0^t e^{H(s)}\sigma(s)\,ds \tag{3.9}$$

for all $t \in J$. Since $\sigma \in L^1(J,\mathbb{R})$, one has $\int_0^t e^{H(s)}\sigma(s)\,ds \in AC(J,\mathbb{R})$. So, a direct differentiation of (3.8) yields,

$$\left(e^{H(t)}x(t)\right)' = e^{H(t)}\sigma(t),$$

or

$$x'(t) + h(t)x(t) = \sigma(t),$$

for $t \in J$ satisfying $x(0) = x_0$ and (3.3). The proof of the lemma is complete.

***Remark* 3.1.** *We note that the kernel function $k(t,s)$ is continuous and nonnegative real-valued function on $J \times J$. Moreover,* $\sup_{t>s} k(t,s) \le 1$.

***Lemma* 3.4.** *Given $\sigma \in L^1(J,\mathbb{R})$, if there is a function $u \in PC(J,\mathbb{R})$ satisfying the impulsive differential inequality*

$$\left.\begin{aligned} u'(t) + h(t)u(t) \le \sigma(t), \quad t \in J \setminus \{t_1,\ldots,t_p\}, \\ u(t_j^+) - u(t_j^-) \le \mathcal{I}_j(u(t_j)), \\ u(0) \le x_0, \end{aligned}\right\} \tag{3.10}$$

then it satisfies the impulsive integral inequality

$$u(t) \le x_0e^{-H(t)} + \sum_{0<t_j<t} k(t,t_j)\mathcal{I}_j(u(t_j)) + \int_0^t k(t,s)\sigma(s)\,ds, \ \ t \in J, \tag{3.11}$$

where the kernel function k is defined by the expression (3.7) *on $J \times J$.*

Proof. Proceeding as in the proof of Lemma 3.3, we obtain

$$\left.\begin{aligned}\left(e^{H(t)}u(t)\right)' \le e^{H(t)}\sigma(t),\ \ t \in J \setminus \{t_1,\dots,t_p\},\\ u(t_j^+) - u(t_j^-) \le \mathcal{I}_j(u(t_j)),\\ u(0) \le x_0,\end{aligned}\right\}$$

for $j = 1, 2, \dots, p$.

From the theory of integral calculus, it follows that

$$\begin{aligned}
e^{H(t_1^-)}u(t_1^-) - e^{H(0)}u(0) &= \int_0^{t_1} \left(e^{H(s)}u(s)\right)' ds\\
e^{H(t_2^-)}u(t_2^-) - e^{H(t_1^+)}u(t_1^+) &= \int_{t_1}^{t_2} \left(e^{H(s)}u(s)\right)' ds\\
&\ \vdots\\
e^{H(t)}u(t) - e^{H(t_p^+)}u(t_p^+) &= \int_{t_p}^{t} \left(e^{H(s)}u(s)\right)' ds.
\end{aligned}$$

Summing up the above equations,

$$e^{H(t)}u(t) - \sum_{0<t_j<t} e^{H(t_j)}\mathcal{I}_j(u(t_j)) \le u_0 + \int_0^t e^{H(s)}h(s)\,ds,$$

or

$$u(t) \le x_0 e^{-H(t)} + \sum_{0<t_j<t} k(t,t_j)\mathcal{I}_j(u(t_j)) + \int_0^t k(t,s)\sigma(s)\,ds$$

for $t \in J$ and the proof of the lemma is complete.

Similarly, we have the following useful result concerning the impulsive differential inequality with reverse sign.

***Lemma* 3.5.** *Given $\sigma \in L^1(J,\mathbb{R})$, if there is a function $v \in PC(J,\mathbb{R})$ satisfying the impulsive differential inequality*

$$\left.\begin{aligned}v'(t) + h(t)v(t) \ge \sigma(t),\ \ t \in J \setminus \{t_1,\dots,t_p\},\\ v(t_j^+) - v(t_j^-) \ge \mathcal{I}_j(v(t_j)),\\ v(0) \ge x_0,\end{aligned}\right\} \tag{3.12}$$

then it satisfies the impulsive integral inequality

$$v(t) \ge x_0 e^{-H(t)} + \sum_{0<t_j<t} k(t,t_j)\mathcal{I}_j(v(t_j)) + \int_0^t k(t,s)\sigma(s)\,ds,\ \ t \in J, \tag{3.13}$$

where the kernel function k is defined by the expression (3.7) *on $J \times J$.*

***Theorem* 3.1.** *Suppose that the hypotheses (H_1) through (H_4) and (H_6) hold. Furthermore, if $\sum_{j=1}^p L_{\mathcal{I}_j} < 1$, then the IDE (1.1) has a impulsive solution x^* defined on J and the sequence $\{x_n\}$ of successive approximations defined by*

$$\begin{aligned}
x_0(t) &= u(t),\\
x_{n+1}(t) &= x_0 e^{-H(t)} + \sum_{0<t_j<t} k(t,t_j)\mathcal{I}_j(x_n(t_j))\\
&\quad + \int_0^t k(t,s) f(s,x_n(s))\,ds
\end{aligned} \tag{3.14}$$

for all $t \in J$, converges monotonically to x^*.

Proof. Set $E = PC(J, \mathbb{R})$. Then, by Lemma 3.2, every compact chain C in E possesses the compatibility property with respect to the norm $\|\cdot\|_{PC}$ and the order relation $\preceq$ so that every compact chain C is a Janhavi set in E.

Now, by Lemma 3.3, the IDE (1.1) is equivalent to the nonlinear impulsive integral equation

$$x(t) = x_0 e^{-H(t)} + \sum_{0<t_j<t} k(t, t_j)\mathcal{I}_j(x(t_j)) + \int_0^t k(t, s) f(s, x(s))\, ds \tag{3.15}$$

for all $t \in J$.

Define two operators $\mathcal{A}$ and $\mathcal{B}$ on E by

$$\mathcal{A}x(t) = \sum_{0<t_j<t} k(t, t_j)\mathcal{I}_j(x(t_j)),\ t \in J, \tag{3.16}$$

and

$$\mathcal{B}x(t) = x_0 e^{-H(t)} + \int_0^t k(t, s) f(s, x(s))\, ds,\ t \in J. \tag{3.17}$$

From the continuity of the integral, it follows that $\mathcal{A}$ and $\mathcal{B}$ define the operators $\mathcal{A}, \mathcal{B} : E \to E$ and the impulsive integral equation (3.15) is transformed into the operator equation as

$$\mathcal{A}x(t) + \mathcal{B}x(t) = x(t),\quad t \in J. \tag{3.18}$$

Now, the problem of finding the impulsive solution of the IDE (1.1) is just reduced to finding impulsive solution of the operator equation (3.18) on J. We show that the operators $\mathcal{A}$ and $\mathcal{B}$ satisfy all the conditions of Theorem 2.2 in a series of following steps.

Step I: *$\mathcal{A}$ and $\mathcal{B}$ are nondecreasing on E.*

Let $x, y \in E$ be such that $x \succeq y$. Then, by hypothesis (H_2), we get

$$\mathcal{A}x(t) = \sum_{0<t_j<t} k(t, t_j)\mathcal{I}_j(x(t_j)) \geq \sum_{0<t_j<t} k(t, t_j)\mathcal{I}_j(y(t_j)) = \mathcal{A}y(t),$$

for all $t \in J$. By definition of the order relation in E, we obtain $\mathcal{A}x \succeq \mathcal{A}y$ and a fortiori, $\mathcal{A}$ is a nondecreasing operator on E. Similarly, using hypothesis (H_4),

$$\begin{aligned}\mathcal{B}x(t) &= x_0 e^{-H(t)} + \int_0^t k(t, s) f(s, x(s))\, ds \\ &\geq x_0 e^{-H(t)} + \int_0^t k(t, s) f(s, x(s))\, ds \\ &= \mathcal{B}y(t),\end{aligned}$$

for all $t \in J$. Therefore, the operator $\mathcal{B}$ is also nondecreasing on E into itself.

Step II: *$\mathcal{A}$ is partially bounded and partially contraction on E.*

Let $x \in E$ be arbitrary. Then by (H_1) we have

$$|\mathcal{A}x(t)| \leq \left|\sum_{0<t_j<t} k(t, t_j)\mathcal{I}_j(x(t_j))\right| \leq \sum_{0<t_j<t} \left|k(t, t_j)\right| \left|\mathcal{I}_j(y(t_j))\right| \leq \sum_{j=1}^{p} M_{\mathcal{I}_j}$$

for all $t \in J$. Taking the supremum over t, we obtain $\|\mathcal{A}x\| \leq \sum_{j=1}^{p} M_{\mathcal{I}_j}$ for all $x \in E$, so $\mathcal{A}$ is a bounded operator on E. This further implies that $\mathcal{A}$ is partially bounded on E.

Next, let $x, y \in E$ be such that $x \succeq y$. Then by (H_2), we have

$$|\mathcal{A}x(t) - \mathcal{A}y(t)| \leq \left|\sum_{0<t_j<t} k(t, t_j)\mathcal{I}_j(x(t_j)) - \sum_{0<t_j<t} k(t, t_j)\mathcal{I}_j(y(t_j))\right|$$

$$\leq \left| \sum_{0<t_j<t} k(t,t_j)[\mathcal{I}_j(x(t_j)) - \mathcal{I}_j(x(t_j))] \right|$$
$$\leq \sum_{0<t_j<t} k(t,t_j)\, L_{\mathcal{I}_j}[x(t_j) - x(t_j)]$$
$$\leq L\,\|x-y\|_{PC}\,,$$

for all $t \in J$, where $L = \sum_{j=1}^{p} L_{\mathcal{I}_j} < 1$. Taking the supremum over t, we obtain

$$\|\mathcal{A}x - \mathcal{A}y\|_{PC} \leq L\,\|x-y\|_{PC}$$

for all $x, y \in E$ with $xx \succeq y$. Hence $\mathcal{A}$ is a partially contraction on E which also implies that $\mathcal{A}$ is partially continuous on E.

Step III: *$\mathcal{B}$ is partially continuous on E.*

Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in a chain C such that $x_n \to x$, for all $n \in \mathbb{N}$. Then

$$\begin{aligned}
\lim_{n\to\infty} \mathcal{B}x_n(t) &= \lim_{n\to\infty} \left[x_0 e^{-H(t)} + \int_0^t k(t,s) f(s, x_n(s))\, ds \right] \\
&= x_0 e^{-H(t)} + \int_0^t k(t,s) \left[\lim_{n\to\infty} f(s, x_n(s)) \right] ds \\
&= x_0 e^{-H(t)} + \int_0^t k(t,s) f(s, x(s))\, ds \\
&= \mathcal{B}x(t),
\end{aligned}$$

for all $t \in J$. This shows that $\mathcal{B}x_n$ converges to $\mathcal{B}x$ pointwise on J.

Now, we show that $\{\mathcal{B}x_n\}_{n\in\mathbb{N}}$ is a quasi-equicontinuous sequence of functions in E. Let $\tau_1, \tau_2 \in (t_j, t_{j+1}] \cap J$, $j = 1, \ldots, p$. Then, we have that

$$\begin{aligned}
&\left| \mathcal{B}x_n(\tau_2) - \mathcal{B}x_n(\tau_1) \right| \\
&= \Big| x_0 e^{-H(\tau_1)} + \int_0^{\tau_1} k(\tau_1, s) f(s, x_n(s))\, ds \\
&\qquad - x_0 e^{-H(\tau_2)} - \int_0^{\tau_2} k(\tau_2, s) f(s, x_n(s))\, ds \Big| \\
&\leq \left| \int_0^{\tau_1} k(\tau_1, s) f(s, x_n(s))\, ds - \int_0^{\tau_2} k(\tau_2, s) f(s, x_n(s))\, ds \right| \\
&\qquad + \left| x_0 e^{-H(\tau_1)} - x_0 e^{-H(\tau_2)} \right| \\
&\leq \left| \int_0^{\tau_1} k(\tau_1, s) f(s, x_n(s))\, ds - \int_0^{\tau_1} k(\tau_2, s) f(s, x_n(s))\, ds \right| \\
&\qquad + \left| \int_0^{\tau_1} k(\tau_2, s) f(s, x_n(s))\, ds - \int_0^{\tau_2} k(\tau_2, s) f(s, x_n(s))\, ds \right| \\
&\qquad + \left| x_0 e^{-H(\tau_1)} - x_0 e^{-H(\tau_2)} \right| \\
&\leq |x_0| \left| e^{-H(\tau_1)} - e^{-H(\tau_2)} \right| \\
&\qquad + \int_0^T \left| k(\tau_1, s) - k(\tau_2, s) \right| \left| f(s, x_n(s)) \right| ds
\end{aligned}$$

$$
\begin{aligned}
&\qquad + \left| \int_{\tau_2}^{\tau_1} |k(\tau_2, s)|\, |f(s, x_n(s))|\, ds \right| \\
&\le |x_0| \left| e^{-H(\tau_1)} - e^{-H(\tau_2)} \right| \\
&\qquad + M_f \int_0^T \left| k(\tau_1, s) - k(\tau_2, s) \right| ds \\
&\qquad + M_f |\tau_1 - \tau_2| \\
&\to 0 \quad \text{as} \quad \tau_2 \to \tau_1,
\end{aligned}
$$

uniformly for all $n \in \mathbb{N}$. This shows that the sequence $\{\mathcal{B}x_n\}$ of functions is quasi- equicontinuous and so convergence $\mathcal{B}x_n \to \mathcal{B}x$ is uniform in view of the arguments given in Samoilenko and Perestyuk [21], Lakshmikantam *et.al* [20]. Hence $\mathcal{B}$ is partially continuous operator on E into itself.

Step IV: *$\mathcal{B}$ is partially compact operator on E.*

Let C be an arbitrary chain in E. We show that $\mathcal{B}(C)$ is uniformly bounded and quasi-equicontinuous set in E. First we show that $\mathcal{B}(C)$ is uniformly bounded. Let $y \in \mathcal{B}(C)$ be any element. Then there is an element $x \in C$ such that $y = \mathcal{B}x$. By hypothesis (H_3)

$$
\begin{aligned}
|y(t)| &= |\mathcal{B}x(t)| \\
&= \left| x_0 e^{-H(t)} + \int_0^t k(t, s) f(s, x(s))\, ds \right| \\
&\le \left| x_0 e^{-H(t)} \right| + \int_0^T |k(t, s)|\, |f(s, x(s))|\, ds \\
&\le \left| x_0 e^{-H(t)} \right| + M_f \int_0^T k(t, s)\, ds \\
&\le |x_0| + M_f T \\
&= r,
\end{aligned}
$$

for all $t \in J$. Taking the supremum over t we obtain $\|y\|_{PC} \le \|\mathcal{B}x\|_{PC} \le r$, for all $y \in \mathcal{B}(C)$. Hence $\mathcal{B}(C)$ is uniformly bounded subset of functions E. Next we show that $\mathcal{B}(C)$ is an quasi-equicontinuous set in E. Let $\tau_1, \tau_2 \in (t_j, t_{j+1}] \cap J$, $j = 1, \ldots, p$. Then proceeding with the arguments as in Step II, it can be shown that $\mathcal{B}(C)$ is an quasi-equicontinuous subset of functions in E. So $\mathcal{B}(C)$ is a uniformly bounded and quasi-equicontinuous set of functions in E and hence it is compact in view of Arzelá-Ascoli theorem (see Samoilenko and Perestyuk [21], Lakshmikantam *et al.* [20]). Consequently $\mathcal{B} : E \to E$ is a partially compact operator of E into itself.

Step V: *u is a lower impulsive solution of the operator equation $x = \mathcal{A}x + \mathcal{B}x$.*

By hypothesis (H_4), the IDE (1.1) has a lower impulsive solution u defined on J. Then, we have

$$
\left.
\begin{aligned}
u'(t) + h(t)u(t) \le f(t, u(t)), \quad t \in J \setminus \{t_1, \ldots, t_p\}, \\
u(t_j^+) - u(t_j^-) \le \mathcal{I}_j(u(t_j)), \\
u(0) \le x_0.
\end{aligned}
\right\}. \tag{3.19}
$$

Now, by a direct application of the impulsive differential inequality established in Lemma 3.4 yields that

$$
u(t) \le u_0\, e^{-H(t)} + \sum_{0 < t_j < t} k(t, t_j) \mathcal{I}_j(u(t_j)) + \int_0^t k(t, s) f(s, u(s))\, ds \tag{3.20}
$$

for $t \in J$. Furthermore, from definitions of the operators $\mathcal{A}$ and $\mathcal{B}$ it follows that $u(t) \leq \mathcal{A}u(t) + \mathcal{B}u(t)$ for all $t \in J$. Hence $u \leq \mathcal{A}u + \mathcal{B}u$. Thus the operators $\mathcal{A}$ and $\mathcal{B}$ satisfy all the conditions of Theorem 2.2 and so the operator equation $\mathcal{A}x + \mathcal{B}x = x$ has a impulsive solution. Consequently the integral equation and a fortiori, the IDE (1.1) has a impulsive solution x^* defined on J. Furthermore, the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations defined by (3.14) converges monotonically to x^*. This completes the proof.

Next, we prove the uniqueness theorem for the IDE on the interval J.

***Theorem* 3.2.** *Suppose that the hypotheses (H_1)-(H_2) and (H_5)-(H_6) hold. Furthermore, if $\sum_{j=1}^{p} L_{\mathcal{I}_j} + L_f < 1$, then the IDE (1.1) has a unique impulsive solution solution x^* defined on J and the sequence $\{x_n\}$ of successive approximations defined by* (3.14) *converges monotonically to x^*.*

Proof. Set $E = PC(J, \mathbb{R})$. Then, every pair of elements in $PC(J, \mathbb{R})$ has a lower bound as well as an upper bound so it is a lattice with respect to the order relation $\leq$ in E.

Now, by Lemma 3.3, the IDE (1.1) is equivalent to the nonlinear impulsive integral equation (3.15). Define two operators $\mathcal{A}$ and $\mathcal{B}$ on E by (3.16) and (3.17). Now, consider the mapping $\mathcal{T} : E \to E$ defined by

$$\mathcal{T}x(t) = \mathcal{A}x(t) + \mathcal{B}x(t), \;\; t \in J. \tag{3.21}$$

Then the impulsive integral equation (3.6) is reduced to the operator equation as

$$\mathcal{T}x(t) = x(t), \;\; t \in J. \tag{3.22}$$

Now, proceeding with the arguments as in the proof of Theorem 3.1 it can shown that the operator $\mathcal{A}$ is a partial Lipschitzian with Lipschitz constant $L_{\mathcal{A}} = \sum_{j=1}^{p} L_{\mathcal{I}_j}$. Similarly, we show that $\mathcal{B}$ is also a Lipschitzian on E into itself. Let $x, y \in E$ be such that $x \geq y$. Then, by hypothesis (H_5), one has

$$\begin{aligned}
\left|\mathcal{B}x(t) - \mathcal{B}y(t)\right| &= \left|\int_0^t k(t,s)f(s,x(s))\,ds - \int_0^t k(t,s)f(s,y(s))\,ds\right| \\
&\leq \int_0^t |k(t,s)| \left|f(s,x(s)) - f(s,y(s))\right| ds \\
&\leq L_f \int_0^t \left|x(t) - y(t)\right| ds \\
&\leq L_f\, T\, \|x - y\|_{PC}
\end{aligned}$$

for all $t \in J$ and $x, y \in E$. Taking the supremum over t in the above inequality, we obtain

$$\|\mathcal{B}x - \mathcal{B}y\|_{PC} \leq L_{\mathcal{B}}\, \|x - y\|_{PC}$$

for all $x, y \in E$, $x \geq y$, where $L_{\mathcal{B}} = L_f\, T$. This shows that $\mathcal{B}$ is again a partial Lipschitzian operator on E into itself with a Lipschitz constant $L_{\mathcal{B}}$. Next, by definition of the operator $\mathcal{T}$, one has

$$\|\mathcal{T}x - \mathcal{T}y\|_{PC} \leq \|\mathcal{A}x - \mathcal{A}y\|_{PC} + \|\mathcal{B}x - \mathcal{B}y\|_{PC} \leq (L_{\mathcal{A}} + L_{\mathcal{B}})\, \|x - y\|_{PC}$$

for all $x, y \in E$, $x \geq y$, where $L_{\mathcal{A}} + L_{\mathcal{B}} = \sum_{j=1}^{p} L_{\mathcal{I}_j} + L_f T < 1$. Hence $\mathcal{T}$ is a partial contraction operator on E into itself. Since the hypothesis (H_6) holds, it is proved as in the step V of the proof of Theorem 3.1 that the operator equation (3.22) has a lower solution u in E. Then, by an application of Theorem 2.1, we obtain that the operator equation (3.22) and consequently the IDE (1.1) has a unique impulsive solution x^* and the sequence $\{x_n\}$ of successive approximations defined by (3.15) converges monotonically to x^*. This competes the proof.

***Remark* 3.2.** *The conclusion of Theorems 3.1 and 3.2 also remains true if we replace the hypothesis (H_6) with the following one.*

(H_7) The IDE (1.1) has an upper impulsive solution $v \in PC(J, \mathbb{R})$.

The proofs of the existence theorems under this new hypothesis are obtained using the similar arguments with appropriate modifications.In this case we invoke the use of Lemma 3.5 in the proofs.

***Example* 3.1.** *Given the interval $J = [0, 1]$ of the real line $\mathbb{R}$ and given the points $t_1 = \frac{1}{5}$, $t_2 = \frac{2}{5}$, $t_3 = \frac{3}{5}$ and $t_4 = \frac{4}{5}$ in $[0, 1]$, consider the initial value problem (in short IVP) for the first order impulsive differential equations (in short IDE)*

$$\left.\begin{aligned} x'(t) + x(t) &= \tanh x(t), \quad t \in [0,1] \setminus \{t_1, t_2, t_3, t_4\}, \\ x(t_j^+) - x(t_j^-) &= \mathcal{I}_j(x(t_j)), \\ x(0) &= 1, \end{aligned}\right\} \tag{3.23}$$

for $t_j \in \{\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}\}$; where $x(t_j^-)$ and $x(t_j^-)$ are respectively, the right and left limit of x at $t = t_j$ such that $x(t_j) = x(t_j^-)$ and $\mathcal{I}_j(x(t_j))$ are the impulsive effects at the points $t = t_j$, $j = 1, \ldots, 4$ given by

$$\mathcal{I}_j(x) = \begin{cases} \dfrac{1}{2^j} \cdot \dfrac{x}{1+x} + 2, & \text{if } x > 0, \\ 2, & \text{if } x \le 0, \end{cases}$$

for all $t \in [0, 1]$. Here $f(t, x) = \tanh x$, so it is continuous and bounded on $[0, 1] \times \mathbb{R}$ with bound $M_f = 2$. Again, the map $x \mapsto f(t, x)$ is nondecreasing for each $t \in [0, 1]$. Next, the impulsive function $\mathcal{I}_j$ are continuous and bounded on $\mathbb{R}$ with bound $M_{\mathcal{I}_j} = 3$ for each $j = 1, \ldots, 4$. It is easy to verify that the impulsive operators $\mathcal{I}_j$ satisfy the hypothesis (H_2) with Lipschitz constants $L_{\mathcal{I}_j} = \dfrac{1}{2^j}$ for $j = 1, \ldots, 4$. Moreover, $\sum_{j=1}^{4} L_{\mathcal{I}_j} = \sum_{j=1}^{4} \dfrac{1}{2^j} < 1$. Finally, the functions $u(t) = e^{-t} - 1$ and $v(t) = 15e^{-t} + 1$ are respectively the lower and upper impulsive solutions of the IDE (1.1) *defined on* $[0, 1]$. *Thus, all the conditions of Theorem 3.1 are satisfied and so the IDE* (3.23) *has a impulsive solution ξ^* and the sequence $\{x_n\}$ of successive approximations defined by*

$$\begin{aligned} x_0(t) &= e^{-t} - 1, \\ x_{n+1}(t) &= e^{-t} + \sum_{0<t_j<t} k(t, t_j)\mathcal{I}_j(x_n(t_j)) \\ &\quad + \int_0^t k(t, s) \tanh x_n(s))\, ds \end{aligned}$$

for all $t \in J$, converges monotonically to x^*. Similarly, the sequence $\{y_n\}$ of successive approximations defined by

$$\begin{aligned} y_0(t) &= 15e^{-t} + 1, \\ y_{n+1}(t) &= e^{-t} + \sum_{0<t_j<t} k(t, t_j)\mathcal{I}_j(y_n(t_j)) \\ &\quad + \int_0^t k(t, s) \tanh y_n(s))\, ds \end{aligned}$$

for all $t \in J$, also converges monotonically to the impulsive solution y^* of the IDE (3.23) *in view of Remark 3.2.*

***Remark* 3.3.** *We note that if the IDE* (1.1) *has a lower impulsive solution u as well as an upper impulsive solution v such that $u \leq v$, then under the given conditions of Theorem 3.1 it has corresponding impulsive solutions x_* and y^* and these impulsive solutions satisfy the inequality*

$$u = x_0 \leq x_1 \leq \cdots \leq x_n \leq x_* \leq y^* \leq y_n \leq \cdots \leq y_1 \leq y_0 = v.$$

Hence x_* and y^* are respectively the minimal and maximal impulsive solutions of the IDE (1.1) *in the vector segment* $[u, v]$ *of the Banach space $E = PC(J, \mathbb{R})$, where the vector segment* $[u, v]$ *is a set of elements in $PC(J, \mathbb{R})$ defined by*

$$[u, v] = \{x \in PC(J, \mathbb{R}) \mid u \leq x \leq v\}.$$

This is because of the order cone K defined by (3.3) *is a closed set in $PC(J, \mathbb{R})$. A few details concerning the order relation by the order cones and the Janhavi sets in an ordered Banach space are given in Dhage [9, 10].*

***Remark* 3.4.** *In this paper we considered a very simple nonlinear first order impulsive differential equation for the existence and approximation theorem via monotone iteration principle or method, however the same method may be extended to other complex nonlinear impulsive differential equations of different orders with appropriate modifications for obtaining the algorithms for approximate solution (see Dhage [1] and references therein).*

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