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(Dedicated to Honor Professor H.M. Srivastava on His 80th Birth Anniversary Celebrations)

MULTIPLE FRACTIONAL DIFFUSIONS VIA MULTIVARIABLE H- FUNCTION

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Abstract

In this paper, we introduce a diffusion and wave equation consisting of multidimensional space Riesz-Feller fractional operators and Caputo time fractional derivative. Imposing certain boundary values, we obtain its solution in terms of multivariable H-function and finally making an appeal to our results, we evaluate various multiple fractional diffusions.

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1 Introduction, formulae and definitions

Since 1976, various researchers and authors studied the multivariable H-function ([19], [20]) as considering as an initial function to solve many physical and scientific problems, for example (see [1], [2]). This multivariable H-function is widely used by many researchers in the derivation of the results involving fractional derivatives and fractional integrals ([12], [15], [16], [18]). Also, the Fourier series representations are studied for the multivariable H-function (see [3]). In 2005, Mainardi et al. [10] employed Fox's H-function in fractional diffusion problems, formerly this H-function was introduced by [5] and contour integrals for H-function were appeared in [17].

To explore new ideas for enlarging to the field of fractional diffusions in multidimensional space, in the present investigation, we obtain a solution of multidimensional in space fractional, time fractional diffusion and wave problem in terms of the multivariable H-function involving a multiple contour integral of Mellin - Barnes type [9], defined by ([19], [20]) as

$$(1.1)\quad H[z_1,...,z_r] = H^{0,n:m_1,n_1;...;m_r,n_r}_{p,q:p_1,q_1;...;p_r,q_r}\left[\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix}\left| \begin{matrix} (a_j:\alpha_j^{(1)},\dots,\alpha_j^{(r)})_{1,p} : (c_j^{(1)}:\sigma_j^{(1)})_{1,p_1};\dots;(c_j^{(r)}:\sigma_j^{(r)})_{1,p_r} \\ (b_j:\beta_j^{(1)},\dots,\beta_j^{(r)})_{1,q} : (d_j^{(1)}:\rho_j^{(1)})_{1,q_1};\dots;(d_j^{(r)}:\rho_j^{(r)})_{1,q_r} \end{matrix}\right.\right]$$

$$= \frac{1}{(2\pi\omega)^r}\int_{L_1}\dots\int_{L_r}\Psi(\varrho_1,\dots,\varrho_r)\left\{\prod_{i=1}^{r}\Phi_i(\varrho_i)(z_i)^{\varrho_i}\right\}d\varrho_1\dots d\varrho_r.$$

Here in Eqn. (1.1), $\omega = \sqrt{(-1)}$ throughout this paper, and also

$$(1.2)\quad \Psi(\varrho_1,\dots,\varrho_r) = \frac{\prod_{j=1}^{n}\Gamma(1-a_j+\sum_{i=1}^{r}\alpha_j^{(i)}\varrho_i)}{[\prod_{j=n+1}^{p}\Gamma(a_j-\sum_{i=1}^{r}\alpha_j^{(i)}\varrho_i)][\prod_{j=1}^{q}\Gamma(1-b_j+\sum_{i=1}^{r}\beta_j^{(i)}\varrho_i)]},$$

$$\Phi_i(\varrho_i) = \frac{[\prod_{j=1}^{m_i}\Gamma(d_j^{(i)}-\rho_j^{(i)}\varrho_i)][\prod_{j=1}^{n_i}\Gamma(1-c_j^{(i)}+\sigma_j^{(i)}\varrho_i)]}{[\prod_{j=n_i+1}^{p_i}\Gamma(c_j^{(i)}-\sigma_j^{(i)}\varrho_i)][\prod_{j=m_i+1}^{q_i}\Gamma(1-d_j^{(i)}+\rho_j^{(i)}\varrho_i)]},\ \forall i=1,2,\dots,r.$$

Again, $L_i = L_{\gamma_i \omega\infty}$ represents the contours which start at the point $\gamma_i - \omega\infty$ and goes to the point $\gamma_i + \omega\infty$ with $\gamma_i \in \mathbb{R} = (-\infty, \infty), \forall i = 1, \dots, r$ such that all the poles of $\Gamma(d_j^{(i)} - \rho_j^{(i)} \varrho_i), \forall j = 1, \dots, m_i; i = 1, \dots, r$ are separated from those of $\Gamma(1 - c_j^{(i)} + \sigma_j^{(i)} \varrho_i)$ $\forall j = 1, \dots, n_i; i = 1, \dots, r$ and $\Gamma(1 - a_j + \sum_{i=1}^{r} \alpha_j^{(i)} \varrho_i) \forall j = 1, \dots, n$. Here, the integers n, p, q, m_i, n_i, p_i and q_i satisfy the inequalities $0 \leq n \leq p$; $q \geq 0; 1 \leq m_i \leq q_i$ and $1 \leq n_i \leq p_i, i = 1, \dots, r$. The parameters $a_j, \forall j = 1, \dots, p; c_j^{(i)}, \forall j = 1, \dots, p_i; i = 1, \dots, r; b_j, \forall j = 1, \dots, q; d_j^{(i)}, \forall j = 1, \dots, q_i; i = 1, \dots, r$, are complex numbers and the associated coefficients $\alpha_j^{(i)}, \forall j = 1, \dots, p; i = 1, \dots, r$; $\sigma_j^{(i)}, \forall j = 1, \dots, p_i; i = 1, \dots, r$; $\beta_j^{(i)}, \forall j = 1, \dots, q; i = 1, \dots, r; \rho_j^{(i)}, \forall j = 1, \dots, q_i; i = 1, \dots, r$, are positive real numbers, such that, $\forall i = 1, \dots, r$

$$\Delta_i = \sum_{j=1}^{p} \alpha_j^{(i)} + \sum_{j=1}^{p_i} \sigma_j^{(i)} - \sum_{j=1}^{q} \beta_j^{(i)} - \sum_{j=1}^{q_i} \rho_j^{(i)} \leq 0; \tag{1.3}$$

and

$$\Omega_i = \sum_{j=1}^{n} \alpha_j^{(i)} - \sum_{j=n+1}^{p} \alpha_j^{(i)} - \sum_{j=1}^{q} \beta_j^{(i)} + \sum_{j=1}^{n_i} \sigma_j^{(i)} - \sum_{j=n_i+1}^{p_i} \sigma_j^{(i)} + \sum_{j=1}^{m_i} \rho_j^{(i)} - \sum_{j=m_i+1}^{q_i} \rho_j^{(i)} > 0. \tag{1.4}$$

In the integrand of (1.1), the poles are supposed to be simple. The integral (1.1) converges absolutely for $|arg\,(z_i)| < \frac{\pi}{2}\Omega_i, i = 1, \dots, r$. The points $z_i = 0, i = 1, 2, \dots, r$ and various exceptional parameter values being tacitly excluded.

Particularly, for $r = 1$, the Eqns. (1.1) - (1.4) reduce to Fox's H- function ([5], [10], [12, p.2], [17]).

By Eqns. (1.1) - (1.4) an approximation formula is given by

$$H[z_1, \dots, z_r] = O(|z_1|^{\vartheta_1} \dots |z_r|^{\vartheta_r}), \max_{1 \leq j \leq r} \|z_j\| \to 0; \tag{1.5}$$

where $\vartheta_i = \min_{1 \leq j \leq m_i} \left(\frac{\Re(d_j^{(i)})}{\rho_j^{(i)}} \right), i = 1, \dots, r.$

Although for $n = 0$, another approximation formula holds

$$H\,[z_1, \dots, z_r] = O\,(|z_1|^{\varsigma_1} \dots |z_r|^{\varsigma_r})\,, \min_{1 \leq j \leq r} \|z_j\| \to \infty; \tag{1.6}$$

where

$$\varsigma_i = \min_{1 \leq j \leq n_i} \left(\frac{\Re(c_j^{(i)}) - 1}{\sigma_j^{(i)}} \right), i = 1, \dots, r.$$

Again, for $n = p = q = 0$, the multivariable H-function (1.1) - (1.4) consists a relation with Fox's H-function ([5], [10], [17]) in the form (see [12, p.207])

$$H_{0,0:p_1,q_1;\dots;p_r,q_r}^{0,0:m_1,n_1;\dots;m_r,n_r} \left[\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix} \middle| \begin{matrix} - : (c_j^{(1)} : \sigma_j^{(1)})_{1,p_1}; \dots; (c_j^{(r)} : \sigma_j^{(r)})_{1,p_r} \\ - : (d_j^{(1)} : \rho_j^{(1)})_{1,q_1}; \dots; (d_j^{(r)} : \rho_j^{(r)})_{1,q_r} \end{matrix} \right] = \prod_{i=1}^{r} H_{p_i,q_i}^{m_i,n_i} \left[z_i \middle| \begin{matrix} (c_j^{(i)} : \sigma_j^{(i)})_{1,p_i} \\ (d_j^{(i)} : \rho_j^{(i)})_{1,q_i} \end{matrix} \right], \tag{1.7}$$

provided that $\Delta_i' = \sum_{j=1}^{p_i} \sigma_j^{(i)} - \sum_{j=1}^{q_i} \rho_j^{(i)} \leq 0$; and

$$\Omega_i' = \sum_{j=1}^{n_i} \sigma_j^{(i)} - \sum_{j=n_i+1}^{p_i} \sigma_j^{(i)} + \sum_{j=1}^{m_i} \rho_j^{(i)} - \sum_{j=m_i+1}^{q_i} \rho_j^{(i)} > 0 \text{ with } |\arg(z_i)| < \frac{\pi}{2}\Omega_i', i = 1, \dots, r.$$

Recently, series and analytic solutions of fractional in time and space - fractional anomalous diffusion problems are obtained and studied by many authors ([6], [7], [8], [13]). In [10] the solution of fractional diffusion is found in terms of Fox's H-function (see also [5], [12, p.2], [17]). Now, in the present paper, we point out a generalization of a diffusion equation introduced and studied by various researchers ([8], [10], [12, p.199 Eqn. (6.186)]) as

$$(1.8)\quad {}_t^{\mathrm{C}}D_{0^+}^{\alpha} u(x,t) = {}_xD_{\theta}^{\beta} u(x,t), -\infty < x < +\infty, t \geq 0; 0 < \alpha \leq 2; |\theta| \leq \min\{\beta, 2-\beta\},$$

where, $0 < \beta \leq 2$.

In Eqn. (1.8), the *Caputo derivative* of the function $f(t)$, denoted by ${}_t^{\mathrm{C}}D_{0^+}^{\alpha} f(t)$ where, $m-1 < \alpha \leq m, \forall m \in \mathbb{N}$, is defined by [4, p.49]

$$(1.9)\qquad ({}_t^{\mathrm{C}}D_{0^+}^{\alpha} f)(t) = (I^{m-\alpha} f^{(m)})(t),$$

where, $f^{(m)}(t) = D_t^m f(t), \{D_t^m \equiv \frac{d^m}{dt^m} = \frac{d}{dt}(\frac{d^{m-1}}{dt^{m-1}})\}, I^{m-\alpha}$ being the Riemann - Liouville fractional integral given by

$$(I^{m-\alpha} f)(t) = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} f(\tau) d\tau, t > 0, m-1 < \alpha \leq m, \\ f(t), \alpha = m, \forall m \in \mathbb{N}. \end{cases}$$

The Laplace transformation of a sufficiently well behaved function $v(t)$ is denoted and defined by $L\{v(t); s\} = V(s) = \int_0^\infty e^{-st} v(t) dt, s > 0$. In our researches, we use this Laplace transformation of the Caputo time fractional derivative (1.9) in the form [4, p.134]

$$(1.10)\qquad L\{({}_t^{\mathrm{C}}D_{0^+}^{\alpha} v)(t); s\} = s^{\alpha} V(s) - \sum_{k=0}^{m-1} s^{\alpha-1-k} v^{(k)}(0^+)\ \forall m-1 < \alpha \leq m.$$

Further in the x- domain, we are needful of an operation by *an integro - differential Riesz-Feller operator* ${}_xD_{\theta}^{\beta}$, on $f(x)$, denoted by ${}_xD_{\theta}^{\beta} f(x)$ where $|\theta| \leq \min\{\beta, 2-\beta\}, 0 < \beta \leq 2$, is given as [8], [12, p.192, Eqn. (6.150)]

$$(1.11)\quad {}_xD_{\theta}^{\beta} f(x) = \frac{\Gamma(1+\beta)}{\pi} \left[\sin \frac{(\beta+\theta)\pi}{2} \int_0^\infty \frac{f(x+\varrho) - f(x)}{\varrho^{\beta+1}} d\varrho + \sin \frac{(\beta-\theta)\pi}{2} \int_0^\infty \frac{f(x-\varrho) - f(x)}{\varrho^{\beta+1}} d\varrho \right].$$

The Fourier transformation of a sufficiently well behaved function $f(x)$, denoted by $\bar{f}(\varrho) = \mathcal{F}\{f(x); \varrho\} = \int_{-\infty}^{+\infty} e^{+\omega\varrho x} f(x) dx, \varrho \in \mathbb{R}$, then, the Fourier transform of the integro - differential *Riesz-Feller* operator (1.11) is obtained as ([8], [12, p.191])

$$(1.12)\qquad \mathcal{F}\{{}_xD_{\theta}^{\beta} f(x); \varrho\} = -\psi_{\beta}^{\theta}(\varrho) \bar{f}(\varrho).$$

Here in (1.12), we have

$$(1.13)\qquad \psi_{\beta}^{\theta}(\varrho) = |\varrho|^{\beta} \exp[\omega(sign\ \varrho)\frac{\theta\pi}{2}],\ \text{where, } |\theta| \leq \min\{\beta, 2-\beta\}, 0 < \beta \leq 2.$$

To make extensions in the area of fractional calculus and its applications in various scientific fields, we refer the books ([4], [12], [13], [16]). Again, for enriching in the present investigation, we introduce a diffusion and wave equation consisting of Riesz - Feller fractional operators in multidimensional space and Caputo time - fractional operator. We solve this diffusion and wave equation on imposing certain boundary conditions and obtain its solution in terms of multivariable H-function given in Eqns. (1.1) - (1.4). Its estimations are found by the Eqns. (1.5) and (1.6). The action of Riesz - Feller operator on multiple space - variable function $U(x_1, \ldots, x_r)$ is given by

$$(1.14)\quad {}_{x_1}D^{\beta_1}_{\theta_1}U(x_1,...,x_r)=\frac{\Gamma(1+\beta_1)}{\pi}[\sin\frac{(\beta_1+\theta_1)\pi}{2}\int_0^\infty\frac{U(x_1+\xi,\ldots,x_r)-U(x_1,\ldots,x_r)}{\xi^{\beta_1+1}}d\xi$$
$$+\sin\frac{(\beta_1-\theta_1)\pi}{2}\int_0^\infty\frac{U(x_1-\xi,\ldots,x_r)-U(x_1,\ldots,x_r)}{\xi^{\beta_1+1}}d\xi]$$
$$\vdots$$
$${}_{x_r}D^{\beta_r}_{\theta_r}U(x_1,...,x_r)=\frac{\Gamma(1+\beta_r)}{\pi}[\sin\frac{(\beta_r+\theta_r)\pi}{2}\int_0^\infty\frac{U(x_1,\ldots,x_r+\xi)-U(x_1,\ldots,x_r)}{\xi^{\beta_r+1}}d\xi$$
$$+\sin\frac{(\beta_r-\theta_r)\pi}{2}\int_0^\infty\frac{U(x_1,\ldots,x_r-\xi)-U(x_1,\ldots,x_r)}{\xi^{\beta_r+1}}d\xi].$$

Motivated by above work, in this paper, we introduce to a diffusion and wave equation consisting of multidimensional in space Riesz-Feller fractional operators defined by (1.11) and Caputo time fractional derivative by (1.9). Then, we shall dissipate to the multiple function $u(x_1,\ldots,x_r,t)$ into the product of the functions involving of separate space and time variables as given in Eqn. (2.2) and on imposing certain boundary conditions (2.3) - (2.5), obtain a solution and some results in terms of multivariable H-function and thus on applying our results, we evaluate various multiple fractional diffusions.

2 A multiple generalized diffusion and wave equation and its degenerations into several equations of different variables

In this section, we make a generalization of the Eqn. (1.8) on introducing many space fractional Riesz - Feller derivatives as defined in Eqn. (1.11) and the time fractional derivative by Eqn. (1.9) and thus find a multiple generalized diffusion and wave equation in the form

$$(2.1)\quad {}^{\mathrm{C}}_{t}D^{\alpha}_{0^+}u(x_1,\ldots,x_r,t)={}_{x_1}D^{\beta_1}_{\theta_1}u(x_1,\ldots,x_r,t)+\ldots+{}_{x_r}D^{\beta_r}_{\theta_r}u(x_1,\ldots,x_r,t),t>0,0<\alpha\le 2;$$
$$|\theta_i|\le\min\{\beta_i,2-\beta_i\},0<\beta_i\le 2;-\infty<x_i<+\infty;\forall i=1,2,\ldots,r.$$

Now, we equip following initial and boundary conditions to degenerate of the equation (2.1) into several equations consisting of fractional in time and space fractional variables

$$(2.2)\qquad u(x_1,\ldots,x_r,t)=\prod_{i=1}^{r}v(x_i,t)\ \forall x_i\in\mathbb{R},t\in\mathbb{R}^+,\ i=1,2,\ldots,r.$$

For all $x_i\in\mathbb{R},i=1,2,\ldots,r$; the initial conditions are followed by

$$(2.3)\quad u(x_1,\ldots,x_r,t)=\sum_{i=1}^{r}\left\{\frac{\prod_{j=1,j\neq i}^{r}(t-\tau_j)v(x_j,t)}{\prod_{j=1,j\neq i}^{r}(\tau_i-\tau_j)}\right\}\varphi(x_i),\ \text{at } t=\tau_i,\tau_i\to 0^+\forall i=1,2,3,\ldots,r;$$

and

$$(2.4)\qquad u_t(x_1,\ldots,x_r,0^+)=\lim_{t\to 0^+}\frac{\partial}{\partial t}u(x_1,\ldots,x_r,t)=0.$$

For $t>0$, the boundary conditions are given by

$$(2.5)\qquad \lim_{x_1,\ldots,x_r\to\pm\infty}u(x_1,\ldots,x_r,t)=0.$$

***Theorem* 2.1.** *If the relation* (2.2) *is applied on both sides of Eqn.* (2.1), *then the equation* (2.1) *degenerates into r-generalized diffusion and wave equations as*

$$(2.6)\qquad {}^{\mathrm{C}}_{t}D^{\alpha}_{0^+}v(x_i,t)={}_{x_i}D^{\beta_i}_{\theta_i}v(x_i,t),\ \textit{where } 0<\alpha\le 2;\ \textit{and } |\theta_i|\le\min\{\beta_i,2-\beta_i\},0<\beta_i\le 2;$$
$-\infty<x_i<+\infty,\ \forall i=1,2,...,r.$

Along with the initial and boundary conditions

$$(2.7)\qquad v(x_i,\tau_i)=\varphi(x_i)\ \textit{at}\ \tau_i\to 0^+,v_t(x_i,0^+)=0\ \forall i=1,2,\ldots,r;$$

and

$$\lim_{x_i\to\pm\infty}v(x_i,t)=0\ \forall i=1,2,\ldots,r.$$

Proof. Under the conditions $0 < \alpha \leq 2, |\theta_i| \leq \min\{\beta_i, 2-\beta_i\}, 0 < \beta_i \leq 2; -\infty \leq x_i \leq \infty$, use the formula (2.2) in the Eqn. (2.1), we find that

$$\sum_{i=1}^{r}\prod_{j=1,j\neq i}^{r} v(x_j,t)\,{}_t^C D_{0^+}^{\alpha} v(x_i,t) = \sum_{i=1}^{r}\prod_{j=1,j\neq i}^{r} v(x_j,t)\,{}_{x_i}D_{\theta_i}^{\beta_i} v(x_i,t). \tag{2.8}$$

Then, equating the i^{th} element in both sides of Eqn. (2.8), we obtain r-equations given in the Eqn. (2.6).

Again, expand the right hand side of Eqn. (2.2) by Lagrange's interpolation formula in regard of t, we get the identities

$$u(x_1,\ldots,x_r,t) = \prod_{i=1}^{r} v(x_i,t) = \sum_{i=1}^{r}\left(\frac{\prod_{j=1,j\neq i}^{r}(t-\tau_j)}{\prod_{j=1,j\neq i}^{r}(\tau_i-\tau_j)}\right) v(x_i,\tau_i)\prod_{j=1,j\neq i}^{r} v(x_j,t)$$
$$= \sum_{i=1}^{r}\left(\frac{\prod_{j=1,j\neq i}^{r}(t-\tau_j)v(x_j,t)}{\prod_{j=1,j\neq i}^{r}(\tau_i-\tau_j)}\right) v(x_i,\tau_i). \tag{2.9}$$

Thus, use the Eqns. (2.3) and (2.9), we obtain the first condition given in Eqn. (2.7) as $v(x_i,\tau_i) = \varphi(x_i)$ at $\tau_i \to 0^+$. In the similar manner, other conditions of (2.7) are found as $v_t(x_i,0^+) = 0\ \forall i = 1,2,\ldots,r$; and $\lim_{x_i\to\pm\infty} v(x_i,t) = 0\ \forall i = 1,2,\ldots,r$.

3 Solution of the problem (2.1)-(2.5)

Before finding out the solution of the problem (2.1)-(2.5), we first prove following Lemmas:

***Lemma* 3.1.** *If* $\Delta_i' = \sum_{j=1}^{p_i}\sigma_j^{(i)} - \sum_{j=1}^{q_i}\rho_j^{(i)} \leq 0$; *and* $\Omega_i' = \sum_{j=1}^{n_i}\sigma_j^{(i)} - \sum_{j=n_i+1}^{p_i}\sigma_j^{(i)} + \sum_{j=1}^{m_i}\rho_j^{(i)} - \sum_{j=m_i+1}^{q_i}\rho_j^{(i)} > 0$ *with* $|arg(z_i)| < \frac{\pi}{2}\Omega_i', i = 1,\ldots,r$, *then, there exists an equality of multivariable H-function as*

$$H_{0,0:p_1,q_1;\ldots;p_r,q_r}^{0,0:m_1,n_1;\ldots;m_r,n_r}\left[\begin{matrix} z_1 \\ \cdot \\ \cdot \\ \cdot \\ z_r\end{matrix}\left|\begin{matrix} - : \left(c_j^{(1)} : \sigma_j^{(1)}\right)_{1,p_1};\ldots;\left(c_j^{(r)} : \sigma_j^{(r)}\right)_{1,p_r} \\ - : \left(d_j^{(1)} : \rho_j^{(1)}\right)_{1,q_1};\ldots;\left(d_j^{(r)} : \rho_j^{(r)}\right)_{1,q_r}\end{matrix}\right.\right]$$
$$= H_{0,0:q_1,p_1;\ldots;q_r,p_r}^{0,0:n_1,m_1;\ldots;n_r,m_r}\left[\begin{matrix} (z_1)^{-1} \\ \cdot \\ \cdot \\ \cdot \\ (z_r)^{-1}\end{matrix}\left|\begin{matrix} - : \left(1-d_j^{(1)} : \rho_j^{(1)}\right)_{1,q_1};\ldots;\left(1-d_j^{(r)} : \rho_j^{(r)}\right)_{1,q_r} \\ - : \left(1-c_j^{(1)} : \sigma_j^{(1)}\right)_{1,p_1};\ldots;\left(1-c_j^{(r)} : \sigma_j^{(r)}\right)_{1,p_r}\end{matrix}\right.\right]. \tag{3.1}$$

Proof. Consider the property of Fox's H-function given in ([10], [12, p.11])

$$H_{p,q}^{m,n}\left[z\left|\begin{matrix}\left(a_j : A_j\right)_{1,p} \\ \left(b_j : B_j\right)_{1,q}\end{matrix}\right.\right] = H_{q,p}^{n,m}\left[\frac{1}{z}\left|\begin{matrix}\left(1-b_j : B_j\right)_{1,q} \\ \left(1-a_j : A_j\right)_{1,p}\end{matrix}\right.\right]. \tag{3.2}$$

Then, use the result (3.2) in right hand side of the result (1.7), we obtain the equality (3.1).

***Lemma* 3.2.** *If theory and conditions of the* ***Theorem* 2.1** *are followed, then the fundamental solution of the problem* (2.1) - (2.5) *exists as*

$$u(x_1,\ldots,x_r,t) = \prod_{i=1}^{r} G(\alpha,\beta_i,\theta_i;t,x_i). \tag{3.3}$$

Here in Eqn. (3.3), $\forall i = 1,2,\ldots,r$, *the functions* $G(\alpha,\beta_i,\theta_i;t,x_i)$ *satisfy the integral equations* $G(\alpha,\beta_i,\theta_i;t,x_i) = \int_{-\infty}^{\infty} G(\alpha,\beta_i,\theta_i;t,\zeta_i)\delta(\zeta_i - x_i)d\zeta_i, \delta(x)$ *is a well known Dirac - delta function.*

Proof. We make an appeal to the ***Theorem* 2.1**, to find r-generalized diffusion and wave equations with initial and boundary values given in the Eqns. (2.6) - (2.7). Now, take Laplace and Fourier transformation of the equations given in (2.6) - (2.7) and find the relations in following transformed form

$$\bar{V}(\varrho_i, s) = \frac{s^{\alpha-1}}{s^\alpha + \psi_{\beta_i}^{\theta_i}(\varrho_i)}\bar{\varphi}(\varrho_i)\forall i = 1, 2, \ldots, r. \tag{3.4}$$

Again, by Eqn. (3.4) consider that a function $\bar{G}(\alpha, \beta_i, \theta_i; s, \varrho_i) = \frac{s^{\alpha-1}}{s^\alpha+\psi_{\beta_i}^{\theta_i}(\varrho_i)}$, which by taking inverse Laplace transformation as on applying the formulae of [12, p.49, Eqns. (2.12) and (2.14)], and by Eqn. (1.13), we obtain

$$\bar{G}(\alpha, \beta_i, \theta_i; t, \varrho_i) = E_\alpha(-\psi_{\beta_i}^{\theta_i}(\varrho_i)t^\alpha) = E_\alpha(-(\varrho_i)^{\beta_i} \exp[\omega\frac{\theta_i\pi}{2}]t^\alpha)\forall i = 1, 2, \ldots, r. \tag{3.5}$$

Now, use Eqn. (3.5) in the Eqn. (3.4) and thus apply Fourier convolution, we obtain the integral equations

$$V(x_i, t) = \int_{-\infty}^{\infty} G(\alpha, \beta_i, \theta_i; t, \zeta_i)\varphi(\zeta_i - x_i)d\zeta_i. \tag{3.6}$$

Further, in Eqn. (3.6) if we take $\varphi(x_i) = \delta(x_i)\forall i = 1, 2, \ldots, r, (\delta(x)$ is well known Dirac delta function), then, $V(x_i, t) = G(\alpha, \beta_i, \theta_i; t, x_i)$, and then, we find another integral equations

$$G(\alpha, \beta_i, \theta_i; t, x_i) = \int_{-\infty}^{\infty} G(\alpha, \beta_i, \theta_i; t, \zeta_i)\delta(\zeta_i - x_i)d\zeta_i. \tag{3.7}$$

Since $\forall i = 1, 2, \ldots, r$, the functions $G(\alpha, \beta_i, \theta_i; t, x_i)$ satisfy all Eqns. and conditions given in ***Theorem* 2.1**, hence $G(\alpha, \beta_i, \theta_i; t, x_i)$ are general Green functions. Finally, by our assumption (2.2) and Eqns. (3.6) and (3.7), we obtain the fundamental solution (3.3).

***Lemma* 3.3.** *If* $\forall i = 1, 2, \ldots, r, |\theta_i| \le \min\{\beta_i, 2 - \beta_i\}, 0 < \beta_i \le 2$. *Then, for* $t > 0, x_i > 0$, *the Green functions* $G(\alpha, \beta_i, \theta_i; t, x_i), i = 1, 2, \ldots, r$, *defined in **Lemma 3.2**, are expressed by several Fox's H -functions as*

$$G(\alpha, \beta_i, \theta_i; t, x_i) = \begin{cases} \frac{1}{\beta_i x_i} H_{3,3}^{1,2}\left[(x_i)^{-1}(t)^{\frac{\alpha}{\beta_i}} \middle| \begin{matrix} (0, \frac{1}{\beta_i}), (0, 1), (0, \frac{\beta_i-\theta_i}{2\beta_i}) \\ (0, \frac{1}{\beta_i}), (0, \frac{\alpha}{\beta_i}), (0, \frac{\beta_i-\theta_i}{2\beta_i}) \end{matrix}\right], \\ \text{when } \beta_i < \alpha, \text{also, singular at } x_i = 0; \\ \frac{1}{\beta_i x_i} H_{3,3}^{2,1}\left[(x_i)(t)^{\frac{-\alpha}{\beta_i}} \middle| \begin{matrix} (1, \frac{1}{\beta_i}), (1, \frac{\alpha}{\beta_i}), (1, \frac{\beta_i-\theta_i}{2\beta_i}) \\ (1, \frac{1}{\beta_i}), (1, 1), (1, \frac{\beta_i-\theta_i}{2\beta_i}) \end{matrix}\right], \\ \text{when, } \beta_i > \alpha, \text{ also singular at } x_i \to \infty. \end{cases} \tag{3.8}$$

Also, when, $\alpha = \beta_i, G(\alpha, \beta_i, \theta_i; t, x_i)$ *is singular at* $x_i = t\forall i = 1, \ldots, r$.

Proof. In Eqn. (3.5), by the relation $\psi_{\beta_i}^{\theta_i}(\varrho_i) = \psi_{\beta_i}^{-\theta_i}(-\varrho_i)$, there is a symmetric relation $\psi_{\beta_i}^{\theta_i}(-x_i) = \psi_{\beta_i}^{-\theta_i}(x_i)\ \forall x_i > 0,\ i = 1, 2, \ldots, r$. Thus, on taking inverse Fourier transformation of the function $\bar{G}(\alpha, \beta_i, \theta_i; t, \varrho_i)$ of (3.5), we find (see also [11])

$$G(\alpha, \beta_i, \theta_i; t, x_i) = \begin{cases} \frac{1}{\pi}\int_0^\infty E_\alpha\left(-(\varrho_i)^{\beta_i} \exp\left[\omega\frac{\theta_i\pi}{2}\right]t^\alpha\right)\cos\varrho_i x_i d\varrho_i, \\ \frac{1}{\pi}\int_0^\infty E_\alpha\left(-(\varrho_i)^{\beta_i} \exp\left[\omega\frac{\theta_i\pi}{2}\right]t^\alpha\right)\sin\varrho_i x_i d\varrho_i. \end{cases} \tag{3.9}$$

Now, to achieve the result of above ***Lemma* 3.3**, we have to define the Mellin transformation of a sufficiently well behaved function $f(\varrho)$ (see [14]) as

$$\mathfrak{M}\{f(\varrho); s\} = f^*(s) = \int_0^{+\infty} f(\varrho)\varrho^{s-1}d\varrho, \gamma_1 < \Re(s) = \gamma < \gamma_2, \tag{3.10}$$

and the inverse Mellin transformation as

$$\mathfrak{M}^{-1}\{f^*(s);\varrho\} = f(\varrho) = \frac{1}{2\pi\omega}\int_{\gamma-\omega\infty}^{\gamma+\omega\infty} f^*(s)\varrho^{-s}ds, \varrho > 0. \tag{3.11}$$

Also, the properties of juxtaposition $\overset{\mathfrak{M}}{\leftrightarrow}$ of a function $f(\varrho)$ with its Mellin transform $f^*(s)$ are

$$f(a\varrho) \overset{\mathfrak{M}}{\leftrightarrow} a^{-s}f^*(s), a > 0, \tag{3.12}$$

$$(\varrho^p) \overset{\mathfrak{M}}{\leftrightarrow} \frac{1}{|p|}f^*(\frac{s}{p}), p \neq 0. \tag{3.13}$$

The Parseval's formula is given by

$$\int_0^{+\infty} f(\varrho)g(\varrho)d\varrho = \frac{1}{2\pi\omega}\int_{\gamma-\omega\infty}^{\gamma+\omega\infty} f^*(s)g^*(1-s)ds. \tag{3.14}$$

Then by (3.10), (3.11) and (3.12), for $x > 0$ there exist the formulae for trigonometric functions

$$\mathfrak{M}\{\sin(x\varrho);s\} = x^{-s}\Gamma(s)\sin(\frac{\pi s}{2}), -1 < \Re(s) < 1] \tag{3.15}$$

and

$$\mathfrak{M}\{\cos(x\varrho);s\} = x^{-s}\Gamma(s)\cos(\frac{\pi s}{2}),\ 0 < \Re(s) < 1. \tag{3.16}$$

By above formula (3.10) for $\varrho_i > 0,\ t > 0$, the Mellin transform of Eqn. (3.5) is written by

$$G^*(s_i) = \int_0^{+\infty} E_\alpha(-(\varrho_i)^{\beta_i}\exp[\omega\frac{\theta_i\pi}{2}]t^\alpha)(\varrho_i)^{s_i-1}d\varrho_i, \tag{3.17}$$

for $\gamma_{i,1} < \Re(s_i) < \gamma_{i,2}\ \forall i = 1,2,\ldots,r$.

Then, for $\gamma_{i,1} < \Re(s_i) < \gamma_{i,2}\ \forall i = 1,2,\ldots,r$, by the Mellin transforms of trigonometrical functions (3.15), (3.16) in the Eqn. (3.9), and using the Eqn. (3.17) and thus on applying the above Parseval's formula (3.14), there exists a relation

$$G(\alpha,\beta_i,\theta_i;t,x_i) = \begin{cases} \frac{1}{\pi x_i}\frac{1}{2\pi\omega}\int_{\gamma_i-\omega\infty}^{\gamma_i+\omega\infty} G^*(s_i)\Gamma(1-s_i)\sin(\frac{\pi s_i}{2})(x_i)^{s_i}ds_i, \\ \qquad x_i > 0, 0 < \gamma_i < 1; \\ \frac{1}{\pi x_i}\frac{1}{2\pi\omega}\int_{\gamma_i-\omega\infty}^{\gamma_i+\omega\infty} G^*(s_i)\Gamma(1-s_i)\cos(\frac{\pi s_i}{2})(x_i)^{s_i}ds_i, \\ \qquad x_i > 0, 0 < \gamma_i < 2. \end{cases} \tag{3.18}$$

Again, use above juxtaposition (3.13) in Eqn. (3.5), and apply the techniques of [11], we write

$$G^*(s_i) = \frac{1}{\beta_i}\frac{\Gamma(\frac{s_i}{\beta_i})\Gamma(1-\frac{s_i}{\beta_i})}{\Gamma(1-\frac{\alpha s_i}{\beta_i})}\exp[-\omega\frac{\pi s_i\theta_i}{2\beta_i}](t)^{\frac{-\alpha s_i}{\beta_i}}, \tag{3.19}$$

for $t > 0, |\theta_i| \leq \{2-\alpha\}, 0 < \Re(s_i) < \beta_i \forall i = 1,2,\ldots,\ r$. (See also [10]).

Therefore on applying (3.19) in the Eqns. of (3.18), we find a result for $x_i > 0, t > 0, |\theta_i| \leq \{2-\alpha\}, 0 < \Re(s_i) < \beta_i \forall i = 1,\ldots,r$, in the form

$$\begin{aligned} G(\alpha,\beta_i,\theta_i;t,x_i) = \\ \frac{1}{\pi\beta_i x_i}\frac{1}{2\pi\omega}\int_{\gamma_i-\omega\infty}^{\gamma_i+\omega\infty} \frac{\Gamma\left(\frac{s_i}{\beta_i}\right)\Gamma\left(1-\frac{s_i}{\beta_i}\right)}{\Gamma\left(1-\frac{\alpha s_i}{\beta_i}\right)}\Gamma(1-s_i)\sin\left(\frac{\pi s_i}{2}\right)\cos\left(\frac{\pi\theta_i s_i}{2\beta_i}\right)\left(x_i\,(t)^{\frac{-\alpha}{\beta_i}}\right)^{s_i}ds_i \\ -\frac{1}{\pi\beta_i x_i}\frac{1}{2\pi\omega}\int_{\gamma_i-\omega\infty}^{\gamma_i+\omega\infty} \frac{\Gamma\left(\frac{s_i}{\beta_i}\right)\Gamma\left(1-\frac{s_i}{\beta_i}\right)}{\Gamma\left(1-\frac{\alpha s_i}{\beta_i}\right)}\Gamma(1-s_i)\cos\left(\frac{\pi s_i}{2}\right)\sin\left(\frac{\pi\theta_i s_i}{2\beta_i}\right)\left(x_i\,(t)^{\frac{-\alpha}{\beta_i}}\right)^{s_i}ds_i. \end{aligned} \tag{3.20}$$

The Eqn. (3.20) gives us the formula, for $x_i > 0, t > 0, |\theta_i| \leq \{2-\alpha\}, 0 < \Re(s_i) < \beta_i \forall i = 1, \ldots, r,$ as

$$(3.21)\quad G(\alpha, \beta_i, \theta_i; t, x_i) = \frac{1}{\pi\beta_i x_i}\frac{1}{2\pi\omega}\int_{\gamma_i-\omega\infty}^{\gamma_i+\omega\infty}\frac{\Gamma(\frac{s_i}{\beta_i})\Gamma(1-\frac{s_i}{\beta_i})}{\Gamma(1-\frac{\alpha s_i}{\beta_i})}\Gamma(1-s_i)\sin\left(\left\{\frac{(\beta_i-\theta_i)\pi s_i}{2\beta_i}\right\}\right)(x_i(t)^{\frac{-\alpha}{\beta_i}})^{s_i}ds_i.$$

Now in Eqn. (3.21), set $\left\{\frac{(\beta_i-\theta_i)}{2\beta_i}\right\} = \lambda_i$, and then, use the property of Gamma function that $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$, we find the Green function for $x_i > 0, t > 0, |\theta_i| \leq \{2-\alpha\}, 0 < \Re(s_i) < \beta_i$, in the Mellin - Barnes contour integrals $\forall i = 1, \ldots, r$, as

$$(3.22)\qquad G(\alpha, \beta_i, \theta_i; t, x_i) = \frac{1}{\beta_i x_i}\frac{1}{2\pi\omega}\int_{\gamma_i-\omega\infty}^{\gamma_i+\omega\infty}\frac{\Gamma\left(\frac{s_i}{\beta_i}\right)\Gamma\left(1-\frac{s_i}{\beta_i}\right)\Gamma(1-s_i)}{\Gamma\left(1-\frac{\alpha s_i}{\beta_i}\right)\Gamma(\lambda_i s_i)\Gamma(1-\lambda_i s_i)}\left(x_i(t)^{\frac{-\alpha}{\beta_i}}\right)^{s_i}ds_i.$$

Finally, in right hand side of Eqn. (3.22) make an application of the definitions (1.1) - (1.4) (for $r = 1$), and again use the relation (3.2), we obtain all results of Eqn. (3.8).

$\forall i = 1, 2, \ldots, r$, in the complex s_i- planes, the right hand side of Eqn. (3.22) shows that there is an extension of probability distribution in the ranges $\{0 < \beta_i \leq 2\} \cap \{0 < \alpha \leq 1\}$ and $\{1 < \alpha \leq \beta_i \leq 2\}$, thus we present following Main Theorem:

Theorem 3.1. *If $|\theta_i| \leq min\{\beta_i, 2-\beta_i\}, 0 < \beta_i \leq 2; -\infty < x_i < +\infty; \forall i = 1, 2, \ldots, r$, such that the ranges are $\{0 < \beta_i \leq 2\} \cap \{0 < \alpha \leq 1\}$ and $\{1 < \alpha \leq \beta_i \leq 2\}$, then the Eqn.* (2.1) *on imposing the conditions* (2.2)-(2.5)*, have following three solutions in terms of a multivariable Green function, defined by $u(x_1, \ldots, x_r, t) = G^{\theta_1,\ldots,\theta_r}_{\alpha,\beta_1,\ldots,\beta_r}(x_1, \ldots, x_r, t) = \prod_{i=1}^r G(\alpha, \beta_i, \theta_i; t, x_i)$, as*

Case 1 *If $\beta_i < \alpha$ and $x_i > 0\ \forall i = 1, 2, \ldots, r$, then*

$$(3.23)\quad G^{\theta_1,\ldots,\theta_r}_{\alpha,\beta_1,\ldots,\beta_r}(x_1, \ldots, x_r, t) = \left\{\prod_{i=1}^r \frac{1}{\beta_i x_i}\right\}$$
$$\times H^{0,0:1,2;\ldots;1,2}_{0,0:3,3;\ldots;3,3}\left[\begin{matrix}(x_1)^{-1}(t)^{\frac{\alpha}{\beta_1}}\\ \cdot\\ \cdot\\ \cdot\\ (x_r)^{-1}(t)^{\frac{\alpha}{\beta_r}}\end{matrix}\left|\begin{matrix}-:\left(0,\frac{1}{\beta_1}\right),(0,1),\left(0,\frac{\beta_1-\theta_1}{2\beta_1}\right);\ldots;\left(0,\frac{1}{\beta_r}\right),(0,1),\left(0,\frac{\beta_r-\theta_r}{2\beta_r}\right)\\ -:\left(0,\frac{1}{\beta_1}\right),\left(0,\frac{\alpha}{\beta_1}\right),\left(0,\frac{\beta_1-\theta_1}{2\beta_1}\right);\ldots;\left(0,\frac{1}{\beta_r}\right),\left(0,\frac{\alpha}{\beta_r}\right),\left(0,\frac{\beta_r-\theta_r}{2\beta_r}\right)\end{matrix}\right.\right],$$

also, singular at $x_i = 0$.

Case 2 *If $\beta_i > \alpha$ and $x_i > 0\ \forall i = 1, 2, \ldots, r$, then*

$$(3.24)\quad G^{\theta_1,\ldots,\theta_r}_{\alpha,\beta_1,\ldots,\beta_r}(x_1, \ldots, x_r, t) = \left\{\prod_{i=1}^r \frac{1}{\beta_i x_i}\right\}$$
$$\times H^{0,0:2,1;\ldots;2,1}_{0,0:3,3;\ldots;3,3}\left[\begin{matrix}x_1(t)^{\frac{-\alpha}{\beta_1}}\\ \cdot\\ \cdot\\ \cdot\\ x_r(t)^{\frac{-\alpha}{\beta_r}}\end{matrix}\left|\begin{matrix}-:\left(1,\frac{1}{\beta_1}\right),\left(1,\frac{\alpha}{\beta_1}\right),\left(1,\frac{\beta_1-\theta_1}{2\beta_1}\right);\ldots;\left(1,\frac{1}{\beta_r}\right),\left(1,\frac{\alpha}{\beta_r}\right),\left(1,\frac{\beta_r-\theta_r}{2\beta_r}\right)\\ -:\left(1,\frac{1}{\beta_1}\right),(1,1),\left(1,\frac{\beta_1-\theta_1}{2\beta_1}\right);\ldots;\left(1,\frac{1}{\beta_r}\right),(1,1),\left(1,\frac{\beta_r-\theta_r}{2\beta_r}\right)\end{matrix}\right.\right],$$

also singular at $x_i \to \infty$.

***Case* 3** *If $\alpha = \beta_i\ \forall i = 1, \ldots, r$,*

$$G^{\theta_1,\ldots,\theta_r}_{\alpha,\beta_1,\ldots,\beta_r}(x_1, \ldots, x_r, t) \text{ is singular at } x_i = t, \quad \forall\ i = 1, 2, ..., r. \tag{3.25}$$

Proof. Consider the Eqn. (2.2) in the Eqns. (2.1), (2.3), (2.4) and (2.5) and thus use the theory and results obtained in the ***Lemmas*** **3.2** and **3.3**, we obtain the Green functions $G(\alpha, \beta_i, \theta_i; t, x_i)\ \forall i = 1, 2, \ldots, r$. Then, make an application of Eqn. (1.7) and ***Lemma*** **3.1** in the statement of the ***Theorem*** **3.1**, we find the multivariable Green functions in three cases given in the Eqns. (3.23), (3.24) and (3.25).

4 Various multiple diffusions

In Eqn. (2.1), if we set $\theta_i = 0 \forall i = 1, \ldots, r$, *the integro - differential Riesz-Feller operators*, ${}_{x_i}D^{\beta_i}_{\theta_i}$, $\forall i = 1, \ldots, r$, in the domain $(x_1, \ldots, x_r)$, become symmetric operators with respect to the variables $x_1, \ldots, x_r$, as

$${}_{x_i}D^{\beta_i}_{0} = -\left(-\frac{d^2}{dx_i^2}\right)^{\frac{\beta_i}{2}} \forall i = 1, \ldots, r, \tag{4.1}$$

and thus in Eqn. (4.1), we interpret that

$$-|\varrho|^{\beta_i} = -(\varrho^2)^{\frac{\beta_i}{2}} \forall i = 1, \ldots, r.$$

Now, on specializing the values of the parameters α, β_i and $\theta_i \forall i = 1, \ldots, r$, by above results of the ***Theorem*** **3.1**, we discuss following multiple diffusions:

(I) In ***Theorem*** **3.1**, if we set $\alpha = 1, \beta_i = 2$ and $\theta_i = 0$, $\forall i = 1,\ldots, r$, then there exists, a standard diffusion equation $\frac{\partial u}{\partial t} = \sum_{i=1}^{r} \frac{\partial^2 u}{\partial x_i^2}$, $u = u(x_1, x_2, ..., x_r, t)$, for $t > 0$, $x_i > 0$, $\forall i = 1, \ldots, r$, and thus in ***Case*** **2** of this ***Theorem*** **3.1**, the multivariable Green function $G^{\theta_1,\ldots,\theta_r}_{\alpha,\beta_1,\ldots,\beta_r}(x_1, \ldots, x_r, t)$ becomes as

$$G^{0,\ldots,0}_{1,2,\ldots,2}(x_1, \ldots, x_r, t) = \left\{\frac{(t)^{\frac{-1}{2}}}{2}\right\}^r H^{0,0:1,0;\ldots;1,0}_{0,0:1,1;\ldots;1,1}\left[\begin{matrix} x_1(t)^{\frac{-1}{2}} \\ \cdot \\ \cdot \\ \cdot \\ x_r(t)^{\frac{-1}{2}} \end{matrix} \left| \begin{matrix} - : \left(\frac{1}{2}, \frac{1}{2}\right); \ldots; \left(\frac{1}{2}, \frac{1}{2}\right) \\ - : (0, 1); \ldots; (0, 1) \end{matrix}\right.\right]. \tag{4.2}$$

Again as we are familiar with the results

$$H^{1,0}_{1,1}\left[(x(t)^{\frac{-1}{2}} \left| \begin{matrix} \left(\frac{1}{2}, \frac{1}{2}\right) \\ (0, 1) \end{matrix}\right.\right] = \frac{1}{\sqrt{\pi}} e^{-\frac{\left(x(t)^{\frac{-1}{2}}\right)^2}{4}} = \frac{1}{\sqrt{\pi}} e^{-\frac{x^2}{4t}}. \tag{4.3}$$

Hence, use the results (1.7), (4.2) and (4.3), we find a multivariable normal distribution as

$$G^{0,\ldots,0}_{1,2,\ldots,2}(x_1, \ldots, x_r, t) = \left\{\frac{1}{2\sqrt{\pi t}}\right\}^r \exp\left[-\left\{\frac{(x_1)^2}{4t} + \ldots + \frac{(x_r)^2}{4t}\right\}\right]. \tag{4.4}$$

(II) When, $0 < \beta_i < 2, |\theta_i| \le \min\{\beta_i, 2 - \beta_i\}\ \forall i = 1, \ldots, r$; and $\alpha = 1$ then this case is called space *fractional diffusion*, in which two situations are arised for $x_i > 0, \forall i = 1, \ldots, r$,

(a) $0 < \beta_i < 1, |\theta_i| \le \beta_i \forall i = 1, \ldots, r$, $\alpha = 1$, so that by ***Theorem*** **3.1**, ***Case*** **1**, we get

$$G^{\theta_1,\ldots,\theta_r}_{1,\beta_1,\ldots,\beta_r}(x_1, \ldots, x_r, t) = \left\{\prod_{i=1}^{r} \frac{(t)^{\frac{-1}{\beta_i}}}{\beta_i}\right\} \tag{4.5}$$

$$\times H^{0,0:1,1;\ldots;1,1}_{0,0:2,2;\ldots;2,2}\left[\begin{matrix}(x_1)^{-1}(t)^{\frac{1}{\beta_1}}\\ \cdot\\ \cdot\\ \cdot\\ (x_r)^{-1}(t)^{\frac{1}{\beta_r}}\end{matrix}\left|\begin{matrix}-:(1,1),\left(\frac{\beta_1-\theta_1}{2\beta_1},\frac{\beta_1-\theta_1}{2\beta_1}\right);\ldots;(1,1),\left(\frac{\beta_r-\theta_r}{2\beta_r},\frac{\beta_r-\theta_r}{2\beta_r}\right)\\ -:\left(\frac{1}{\beta_1},\frac{1}{\beta_1}\right),\left(\frac{\beta_1-\theta_1}{2\beta_1},\frac{\beta_1-\theta_1}{2\beta_1}\right);\ldots;\left(\frac{1}{\beta_r},\frac{1}{\beta_r}\right),\left(\frac{\beta_r-\theta_r}{2\beta_r},\frac{\beta_r-\theta_r}{2\beta_r}\right)\end{matrix}\right.\right].$$

(b) $1 < \beta_i < 2, |\theta_i| \leq \{2-\beta_i\ \forall i = 1,\ldots,r, \alpha = 1$, so that by ***Theorem* 3.1**, ***Case* 2**), we get

$$\text{(4.6)}\quad G^{\theta_1,\ldots,\theta_r}_{1,\beta_1,\ldots,\beta_r}(x_1,\ldots,x_r,t) = \left\{\prod_{i=1}^{r}\frac{(t)^{-\frac{1}{\beta_i}}}{\beta_i}\right\}$$

$$\times H^{0,0:1,1;\ldots;1,1}_{0,0:2,2;\ldots;2,2}\left[\begin{matrix}x_1(t)^{\frac{-1}{\beta_1}}\\ \cdot\\ \cdot\\ \cdot\\ x_r(t)^{\frac{-1}{\beta_r}}\end{matrix}\left|\begin{matrix}-:\left(\frac{\beta_1-1}{\beta_1},\frac{1}{\beta_1}\right),\left(\frac{\beta_1+\theta_1}{2\beta_1},\frac{\beta_1-\theta_1}{2\beta_1}\right);\ldots;\left(\frac{\beta_r-1}{\beta_r},\frac{1}{\beta_r}\right),\left(\frac{\beta_r+\theta_r}{2\beta_r},\frac{\beta_r-\theta_r}{2\beta_r}\right)\\ -:(0,1),\left(\frac{\beta_1+\theta_1}{2\beta_1},\frac{\beta_1-\theta_1}{2\beta_1}\right);\ldots;(0,1),\left(\frac{\beta_r+\theta_r}{2\beta_r},\frac{\beta_r-\theta_r}{2\beta_r}\right)\end{matrix}\right.\right].$$

(III) *When,* $\beta_i = 2, \theta_i = 0, \forall i = 1,\ldots,r$; *and* $0 < \alpha < 2$, *then, this case is called time fractional diffusion,* in which for $x_i > 0, \forall i = 1,\ldots,r$, ***Theorem* 3.1**, ***Case* 2**) arises and hence, we find
(4.7)

$$G^{0,\ldots,0}_{\alpha,2,\ldots,2}(x_1,\ldots,x_r,t) = \left\{\frac{(t)^{-\frac{\alpha}{2}}}{2}\right\}^r H^{0,0:1,0;\ldots;1,0}_{0,0:1,1;\ldots;1,1}\left[\begin{matrix}x_1(t)^{\frac{-\alpha}{2}}\\ \cdot\\ \cdot\\ \cdot\\ x_r(t)^{\frac{-\alpha}{2}}\end{matrix}\left|\begin{matrix}-:\left(\frac{2-\alpha}{2},\frac{\alpha}{2}\right);\ldots;\left(\frac{2-\alpha}{2},\frac{\alpha}{2}\right)\\ -:(0,1);\ldots;(0,1)\end{matrix}\right.\right].$$

(IV) As the case discussed in Eqn. (4.7), where put $r = 2$ and $0 < \alpha < 1$, then this becomes generalized anomalous diffusion of Kumar, Pathan and Yadav [7] of which another solution in the form of Green function is found by

$$\text{(4.8)}\qquad G^{0,0}_{\alpha,2,2}(x_1,x_2,t) = \left\{\frac{(t)^{-\alpha}}{4}\right\} H^{0,0:1,0;1,0}_{0,0:1,1;1,1}\left[\begin{matrix}x_1(t)^{\frac{-\alpha}{2}}\\ x_2(t)^{\frac{-\alpha}{2}}\end{matrix}\left|\begin{matrix}-:\left(\frac{2-\alpha}{2},\frac{\alpha}{2}\right);\left(\frac{2-\alpha}{2},\frac{\alpha}{2}\right)\\ -:(0,1);(0,1)\end{matrix}\right.\right].$$

In the similar manner, by ***Theorem* 3.1**, we also obtain the Green function solution of the Eqn. (2.1) of the case for $r = 2$, $0 < \alpha < 1, 0 < \beta_1 < 1, 1 < \beta_2 < 2$; of the anomalous diffusion problem due to Kumar, Pathan and Srivastava [6]. For further directions of the researches in this field, we omit them.

5 Special cases.

In this section, we specialize the values of the parameters involving in the results (4.2) to (4.7) of ***Section* 4** (where set $r = 1$, then take $\beta_1 = \beta$ and $\theta_1 = \theta$) and then, we obtain various diffusions as studied and derived by many authors to them (see [4], [8], [10] and [12]) given in followings:

(i) when $\theta = 0, \beta = 2, \alpha = 1$, by (4.2) to (4.4), there exists a normal diffusion.

(ii) when $\theta = 0, 0 < \beta < 2, \alpha = 1$, by (4.5) and (4.6), there exists a space fractional diffusion.

(iii) when $\theta = 0, 0 < \alpha < 2, \beta = 2$, by (4.7), there exists a time fractional diffusion.

(iv) when $\theta \leq min\{\beta, 2-\beta\}, 0 < \alpha = \beta < 2$, there exists a neutral fractional diffusion (see [10], Eqn. (4.3)).

6 Conclusions

A solution of multidimensional in space fractional and time fractional diffusion and wave problem, in terms of the multivariable H-function involving a multiple contour integral of Mellin - Barnes type [9], defined by ([19], [20]), is obtained by imposing certain conditions and the relations given in Eqns. (2.2) - (2.5) in the Eqn. (2.1). The obtained solution is converted into a classical multivariable Green function by which various multiple diffusions as particular cases are discussed in section 4 on specializing of the parameters involving in multidimensional space fractional operators with Caputo time fractional derivative, in which ***Case*** I) represents the standard diffusion, ***Case*** II) represents space fractional diffusion problem in which two cases are raised and in ***Case*** III) time diffusion problem is analyzed. In ***Case*** IV), the fundamental solution of anomalous diffusion problem is obtained. On putting $r = 1$, the special cases are checked by the results in one dimensional in space-time fractional derivatives of previous work of many researchers in the literature for example ([4], [6], [7], [8], [10], [12]).

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