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(Dedicated to Honor Professor H.M. Srivastava on His 80th Birth Anniversary Celebrations)

CONTOUR INTEGRAL REPRESENTATIONS OF TWO VARIABLE GENERALIZED HYPERGEOMETRIC FUNCTION OF SRIVASTAVA AND DAOUST WITH THEIR APPLICATIONS TO INITIAL VALUE PROBLEMS OF ARBITRARY ORDER

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Abstract

In this paper, we establish two contour integral representations involving Mittag - Leffler functions (i) for a two variable generalized hypergeometric function of Srivastava and Daoust and (ii) a sum of the Kummer's confluent hypergeometric functions. Then, we make their appeal to obtain the contour integrals for many generating functions and bilateral generating relations. Further, in development and extensions of fractional calculus, we obtain various relations of contour integrals with fractional derivatives and integral operators to use them in solving of any order initial value problems.

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1 Introduction

Recently, Pathan and Kumar [27] studied and solved the generalized Cauchy problem by representing the multi-parameter Mittag - Leffler functions ([13], [14]), in terms of two variable generalized hypergeometric function due to Srivastava and Daoust ([24], [28], [29], [30])

$$S\,{}^{A\,:\,B;\,B}_{C\,:\,D;\,D'}\left(\begin{matrix}[(a):\theta,\vartheta]:[(b):\psi];[(b'):\psi'];\\ [(c):\delta,\kappa]:[(d):\varphi];[(d'):\varphi'];\end{matrix} z,w\right)=\sum_{m,n=0}^{\infty}\mathscr{H}\,{}^{A\,:\,B;\,B'}_{C\,:\,D;\,D'}(m,n)\frac{z^m}{m!}\frac{w^n}{n!}, \tag{1.1}$$

where, $\mathscr{H}\,{}^{A\,:\,B;\,B'}_{C\,:\,D;\,D'}(m,n)=\dfrac{\prod_{j=1}^{A}\Gamma(a_j+m\theta_j+n\vartheta_j)}{\prod_{j=1}^{C}\Gamma(c_j+m\delta_j+n\kappa_j)}\dfrac{\prod_{j=1}^{B}\Gamma(b_j+m\psi_j)}{\prod_{j=1}^{D}\Gamma(d_j+m\varphi_j)}\dfrac{\prod_{j=1}^{B'}\Gamma(b'_j+n\psi'_j)}{\prod_{j=1}^{D'}\Gamma(d'_j+n\varphi'_j)}$.

The series (1.1) is convergent under the conditions

$$\sum_{j=1}^{C}\delta_j+\sum_{j=1}^{D}\varphi_j-\sum_{j=1}^{A}\theta_j-\sum_{j=1}^{B}\psi_j+1>0;\sum_{j=1}^{C}\kappa_j+\sum_{j=1}^{D'}\varphi_j{}'-\sum_{j=1}^{A}\vartheta_j-\sum_{j=1}^{B'}\psi'_j+1>0.$$

The Srivastava and Daoust function (1.1) is the generalization of the Kampé de Fériet function [33] including the Appell's functions, Horn's functions and Humbert's confluent hypergeometric functions of two variables (see for instance, ([1], [5], [6], [31], [32]).

Applications and detailed analysis of Euler type and Hankel's contour type integral representations of the Kampé de Fériet function and Appell's functions are studied in various fields of science and technology due to (Exton [5], Srivstava and Karlsson [31]).The Appell's functions are transformed into product of Gaussian and Kummer's confluent hypergeometric functions by many authors ([2], [31], [32]).

In this connection, presently, Fejzullahu [7] established a contour integral representation of the Kummer's confluent hypergeometric function [4] in the form

$$M(\alpha,\beta+\nu,z) = \frac{z^{1-\beta}}{2\pi i}\int_{-\infty}^{(0^+,1^+)} e^{zt}t^{-\beta}\left(1-\frac{1}{t}\right)^{-\alpha}\gamma^*(\nu,zt)dt,\ \alpha,\beta,\nu\in\mathbb{C}, |\arg(z)| < \frac{\pi}{2}, \tag{1.2}$$

where, throughout the paper $i = \sqrt{(-1)}$, $\mathbb{C}$ being the set of complex numbers, $\gamma^*(\nu,z) = \frac{z^{-\nu}}{\Gamma(\nu)}\gamma(\nu,z)$, while

$$\gamma(\nu,z) = \int_0^z u^{\nu-1}e^{-u}du, \Re(\nu) > 0, \tag{1.3}$$

$$\frac{\Gamma(\nu+k)}{\Gamma(\nu)} = (\nu)_k = \begin{cases} 1, k = 0; \\ \nu(\nu+1)\dots(\nu+k-1), \forall k\in\mathbb{N}; \end{cases}$$

$\mathbb{N}$ be the set of natural numbers, and

$$M(\alpha,\beta,z) = \frac{z^{1-\beta}\Gamma(\beta)}{2\pi i}\int_{-\infty}^{(0^+,1^+)} e^{zt}t^{-\beta}\left(1-\frac{1}{t}\right)^{-\alpha}dt, \alpha,\beta\in\mathbb{C},\ |\arg(z)| < \frac{\pi}{2}. \tag{1.4}$$

The function $M(\alpha,\beta,z)$ is a Kummer's function ([4], [7], [32, p.36]) defined by

$$M(\alpha,\beta,z) = \sum_{k=0}^{\infty}\frac{(\alpha)_k}{(\beta)_k}\frac{z^k}{k!}. \tag{1.5}$$

For further innovation and extensions of the results (1.2) - (1.5), we establish two contour integral representations involving

(i) the generalized Mittag - Leffler function, $E_{\nu,\rho}(z)$, of order $\frac{1}{\Re(\nu)}$, $\nu\in\mathbb{C}$, $\Re(\nu) > 0$, and

(ii) the Mittag - Leffler function, $E_{\frac{1}{Q}}(z)$ of order $Q \in \mathbb{N}^* = \{2,3,4,\dots\}$, for two variable Srivastava - Daoust function (1.1) and sum up of Kummer's functions (1.5), respectively.

Then, we make their applications to obtain various contour integral representations for generating functions and many bilateral generating relations.

The generalized Mittag - Leffler function, $E_{\nu,\rho}(z)$ ([10], [34]) is defined by

$$E_{\nu,\rho}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\nu k+\rho)}, \forall z,\nu,\rho\in\mathbb{C}, \Re(\nu) > 0, \Re(\rho) > 0, \tag{1.6}$$

which for $\rho = 1$, reduces to the Mittag - Leffler function, $E_\nu(z)$ ([25], [34]) such that $E_\nu(z) = E_{\nu,1}(z)$. We also have

$$E_0(z) = \frac{1}{1-z}; E_1(z) = e^z; E_1(1) = e; E_2(z) = \cosh(\sqrt{z}); E_2(-z^2) = \cos z;\ \text{and} \tag{1.7}$$

$$E_{\frac{1}{2}}(z^{\frac{1}{2}}) = e^z erf(-\sqrt{z}),\ \text{where,}\ erf(z) = \int_z^{\infty} e^{-u^2}du.$$

The Mittag-Leffler function arises naturally in the solution of fractional order integral equations or fractional order differential equations, and especially in the investigations of the fractional generalization of the kinetic equation, random walks, Lévy flights, super-diffusive transport and in the study of complex systems. (See for instance the literature of Diethelm [3], Hilfer [9], Kilbas et al. [11], Kiryakova [12], Oldham and Spanier [26]). To make our study more applicable in various diffusion and wave problems ([8], [15] - [22]), we transform the contour integrals into various fractional derivative and integral operators and use them in solving of any order initial value problems.

2 Contour integral representations of hypergeometric functions of one and two variables and bilateral generating relations

In this section, we establish two contour integrals involving Mittag - Leffler functions and make their applications to obtain the contour integral representations of generalized hypergeometric functions of two variables of Srivastava and Daoust (1.1) and to sum up of Kummer's functions defined in (1.5). Then, we establish some theorems involving integral representations for some bilateral generating functions:

***Theorem* 2.1.** *If for all $\alpha,\beta,\lambda,\rho,z,w\in\mathbb{C}, |\arg(w)|<\frac{\pi}{2}, \lambda\neq 0, \Re(\rho)>0, \nu\in\mathbb{R}^+, \mathbb{R}^+$ is the set of positive real numbers and then, there exists an integral representation*

$$\frac{1}{2\pi i}\int_{-\infty}^{(0^+,1^+)} E_{\nu,\rho}(\lambda z^\nu t^{-\nu})e^{wt}t^{-\beta}\left(1-\frac{1}{t}\right)^{-\alpha}dt,$$

such that it holds a relation for the Srivastava and Daoust function (1.1) *in the form*

$$\frac{1}{2\pi i}\int_{-\infty}^{(0^+,1^+)} E_{\nu,\rho}(\lambda z^\nu t^{-\nu})e^{wt}t^{-\beta}\left(1-\frac{1}{t}\right)^{-\alpha}dt = \frac{w^{\beta-1}}{\Gamma(\alpha)}S^{0:1;1}_{1:1;0}\left(\begin{matrix}[-:-,-]:[1:1];[\alpha:1];\\ [\beta:\nu,1]:[\rho:\nu];[-:-];\end{matrix}\lambda w^\nu z^\nu, w\right). \tag{2.1}$$

Proof. Making an application of (1.6) in left hand side of (2.1) and changing the order of integration and summation, we get

$$\frac{1}{2\pi i}\int_{-\infty}^{(0^+,1^+)} E_{\nu,\rho}(\lambda z^\nu t^{-\nu})e^{wt}t^{-\beta}\left(1-\frac{1}{t}\right)^{-\alpha}dt = \sum_{k=0}^{\infty}\frac{\lambda^k z^{\nu k}w^{\nu k+\beta-1}}{\Gamma(\nu k+\rho)\Gamma(\nu k+\beta)}\left\{\frac{w^{1-\nu k-\beta}\Gamma(\nu k+\beta)}{2\pi i}\int_{-\infty}^{(0^+,1^+)} e^{wt}t^{-\nu k-\beta}\left(1-\frac{1}{t}\right)^{-\alpha}dt\right\}. \tag{2.2}$$

Making an appeal to (1.5) and (1.6), the (2.2) gives

$$\frac{1}{2\pi i}\int_{-\infty}^{(0^+,1^+)} E_{\nu,\rho}(\lambda z^\nu t^{-\nu})e^{wt}t^{-\beta}\left(1-\frac{1}{t}\right)^{-\alpha}dt = \frac{w^{\beta-1}}{\Gamma(\alpha)}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{1}{\Gamma(\beta+\nu k+m)}\frac{\Gamma(1+k)\Gamma(\alpha+m)}{\Gamma(\rho+\nu k)}\frac{(\lambda w^\nu z^\nu)^k}{k!}\frac{w^m}{m!}, \tag{2.3}$$

which by an appeal to (1.1) gives the result (2.1).

***Theorem* 2.2.** *If all $\alpha,\beta,z,w\in\mathbb{C}, |arg(w+z)|<\frac{\pi}{2}$, then, there exists a new integral*

$$\frac{1}{2\pi i}\int_{-\infty}^{(0^+,1^+)} E_{\frac{1}{Q}}\left((zt)^{\frac{1}{Q}}\right)e^{wt}t^{-\beta}\left(1-\frac{1}{t}\right)^{-\alpha}dt, \forall Q\in N^*,$$

which gives an integral representation for the sum of Kummer's functions as

$$(2.4)\quad \frac{1}{2\pi i}\int_{-\infty}^{(0^+,1^+)} E_{\frac{1}{Q}}\left((zt)^{\frac{1}{Q}}\right)e^{wt}t^{-\beta}\left(1-\frac{1}{t}\right)^{-\alpha}dt$$
$$=\sum_{r=1}^{Q} X_r(w+z)^{\beta-1}\left(\frac{z}{(w+z)}\right)^{1-\frac{r}{Q}} M(\alpha,\beta+\frac{r}{Q}-1,w+z)\forall Q=2,3,4,\dots,$$

where, $X_r=\begin{cases}1, 1\le r\le Q-1;\\ \frac{1}{\Gamma(\beta)}, r=Q.\end{cases}$

Proof. An appeal to the formula due to Mathai and Haubold [23, p.84]

$$E_{\frac{1}{Q}}((z)^{\frac{1}{Q}})=e^{z}\left[1+\sum_{r=1}^{Q-1}\frac{\gamma(1-\frac{r}{Q},z)}{\Gamma(1-\frac{r}{Q})}\right],\forall Q\in N^{*},$$

in the left hand side of (2.4) gives

$$(2.5)\quad \frac{1}{2\pi i}\int_{-\infty}^{(0^+,1^+)} E_{\frac{1}{Q}}\left((zt)^{\frac{1}{Q}}\right)e^{wt}t^{-\beta}\left(1-\frac{1}{t}\right)^{-\alpha}dt=\frac{1}{2\pi i}\int_{-\infty}^{(0^+,1^+)} e^{(w+z)t}t^{-\beta}\left(1-\frac{1}{t}\right)^{-\alpha}dt$$
$$+\sum_{r=1}^{Q-1}\frac{(z)^{1-\frac{r}{Q}}}{2\pi i}\int_{-\infty}^{(0^+,1^+)} e^{(w+z)t}t^{-(\beta+\frac{r}{Q}-1)}\left(1-\frac{1}{t}\right)^{-\alpha}\gamma^{*}(1-\frac{r}{Q},zt)dt$$
$$\Rightarrow \frac{1}{2\pi i}\int_{-\infty}^{(0^+,1^+)} E_{\frac{1}{Q}}\left((zt)^{\frac{1}{Q}}\right)e^{wt}t^{-\beta}\left(1-\frac{1}{t}\right)^{-\alpha}dt=\frac{(w+z)^{1-\beta}\Gamma(\beta)}{(w+z)^{1-\beta}\Gamma(\beta)2\pi i}\int_{-\infty}^{(0^+,1^+)} e^{(w+z)t}t^{-\beta}\left(1-\frac{1}{t}\right)^{-\alpha}dt$$
$$+\sum_{r=1}^{Q-1}\frac{(z)^{1-\frac{r}{Q}}}{2\pi i}\frac{(w+z)^{2-\beta-\frac{r}{Q}}}{(w+z)^{2-\beta-\frac{r}{Q}}}\int_{-\infty}^{(0^+,1^+)} e^{(w+z)t}t^{-(\beta+\frac{r}{Q}-1)}\left(1-\frac{1}{t}\right)^{-\alpha}\gamma^{*}(1-\frac{r}{Q},zt)dt.$$

Finally, making an appeal to (1.1) and (1.4) and (2.5), we obtain

$$(2.6)\quad \frac{1}{2\pi i}\int_{-\infty}^{(0^+,1^+)} E_{\frac{1}{Q}}\left((zt)^{\frac{1}{Q}}\right)e^{wt}t^{-\beta}\left(1-\frac{1}{t}\right)^{-\alpha}dt$$
$$=\frac{(w+z)^{\beta-1}}{\Gamma(\beta)}M(\alpha,\beta,w+z)+\sum_{r=1}^{Q-1}(w+z)^{\beta-1}\left(\frac{z}{(w+z)}\right)^{1-\frac{r}{Q}}M(\alpha,\beta+\frac{r}{Q}-1,w+z),\ \forall Q=2,3,4,\dots.$$

By Eqn. (2.6), we easily obtain the result (2.4).

***Corollary* 2.1.** *If all conditions of the* ***Theorem* 2.1** *are satisfied along with set* $\nu=1,\rho=1$, *then, following result holds*

$$(2.7)\quad \frac{1}{2\pi i}\int_{-\infty}^{(0^+,1^+)} E_{1,1}(\lambda z t^{-1})e^{wt}t^{-\beta}\left(1-\frac{1}{t}\right)^{-\alpha}dt=\frac{w^{\beta-1}}{\Gamma(\beta)}\sum_{k=0}^{\infty}\frac{(\lambda wz)^k}{(\beta)_k}M(\alpha,\beta+k,w).$$

***Corollary* 2.2.** *If all conditions of the* ***Theorem* 2.1** *are satisfied along with set* $\nu=1,\rho=1$, $\lambda=-\frac{1}{2}$, *and replace* z *by* w', w *by* $z+\frac{w'}{2}$, *then, by* (2.7) *following result holds*

$$(2.8)\quad \frac{1}{2\pi i}\int_{-\infty}^{(0^+,1^+)} E_{1,1}\left(-\frac{1}{2}w't^{-1}\right)e^{(z+\frac{w'}{2})t}t^{-\beta}\left(1-\frac{1}{t}\right)^{-\alpha}dt$$
$$=\frac{(z+\frac{w'}{2})^{\beta-1}}{\Gamma(\beta)}\sum_{k=0}^{\infty}\frac{(-\frac{1}{2}(zw'+\frac{(w')^2}{2}))^k}{(\beta)_k}M(\alpha,\beta+k,w').$$

***Corollary* 2.3.** *If all conditions of the* ***Theorem* 2.1** *are satisfied along with set* $\nu = 1, \rho = 1$, $\lambda = -\frac{1}{2}$, *and replace* z *by* w', w *by* $z + \frac{w'}{2}$, *then, by* (2.7) *following result holds*

$$\tag{2.9} \frac{1}{2\pi i}\int_{-\infty}^{(0^+,1^+)} \exp\left[\frac{w'}{2}(t-t^{-1})\right] e^{zt} t^{-\beta}\left(1-\frac{1}{t}\right)^{-\alpha} dt = \sum_{n=-\infty}^{\infty} \frac{z^{\beta-n-1}}{\Gamma(\beta-n)} J_n(w') M(\alpha, \beta-n, z).$$

In the left hand side of equation (2.8), *set some rearrangements and apply the generating relation,* $\exp[\frac{x}{2}(t-t^{-1})] = \sum_{n=-\infty}^{\infty} J_n(x)t^n$, *where,* $J_n(x)$ *are the Bessel functions for all* $n \in \{0, \pm1, \pm2, \ldots\}$, *we get the contour integral representation for bilateral generating function* (2.9).

***Corollary* 2.4.** *If all conditions of the* ***Theorem* 2.1** *are satisfied along with set* $\nu = 1, \rho = 1$, *and again, replace in it,* $\lambda z = \eta t \log(1+t) - xt^2$, *then, following identities hold*

$$\tag{2.10} \begin{aligned} &\frac{1}{2\pi i}\int_{-\infty}^{(0^+,1^+)} E_{1,1}(\log(1+t)^{\eta} - xt) e^{wt} t^{-\beta}\left(1-\frac{1}{t}\right)^{-\alpha} dt \\ &= \frac{1}{2\pi i}\int_{-\infty}^{(0^+,1^+)} (1+t)^{\eta} e^{-xt} e^{wt} t^{-\beta}\left(1-\frac{1}{t}\right)^{-\alpha} dt = \frac{1}{2\pi i}\int_{-\infty}^{(0^+,1^+)} (1+t)^{\eta} e^{(w-x)t} t^{-\beta}\left(1-\frac{1}{t}\right)^{-\alpha} dt. \end{aligned}$$

Again on applying the relation $\sum_{n=0}^{\infty} L_n^{(\eta-n)}(x)t^n = (1+t)^{\eta}e^{-xt}$ *where,* $L_n^{(\eta)}(x)$ *are the Laguerre polynomials* $\forall n \in \{0, 1, 2, \ldots\}$ *in second integral of* (2.10), *we get the contour integrals for a bilateral generating relation*

$$\tag{2.11} \begin{aligned} &\frac{1}{2\pi i}\int_{-\infty}^{(0^+,1^+)} E_{1,1}(\log(1+t)^{\eta} - xt) e^{wt} t^{-\beta}\left(1-\frac{1}{t}\right)^{-\alpha} dt \\ &= \frac{1}{2\pi i}\int_{-\infty}^{(0^+,1^+)} (1+t)^{\eta} e^{-xt} e^{wt} t^{-\beta}\left(1-\frac{1}{t}\right)^{-\alpha} dt = \sum_{n=0}^{\infty} \frac{w^{\beta-n-1}}{\Gamma(\beta-n)} L_n^{(\eta-n)}(x) M(\alpha, \beta-n, w). \end{aligned}$$

Further by (2.10), *we find the contour integrals for a generating function*

$$\tag{2.12} \begin{aligned} &\frac{1}{2\pi i}\int_{-\infty}^{(0^+,1^+)} E_{1,1}(\log(1+t)^{\eta} - xt) e^{wt} t^{-\beta}\left(1-\frac{1}{t}\right)^{-\alpha} dt \\ &= \frac{1}{2\pi i}\int_{-\infty}^{(0^+,1^+)} (1+t)^{\eta} e^{(w-x)t} t^{-\beta}\left(1-\frac{1}{t}\right)^{-\alpha} dt = \sum_{n=0}^{\infty} \frac{(-\eta)_n(-1)^n}{n!} \frac{(w-x)^{\beta-n-1}}{\Gamma(\beta-n)} M(\alpha, \beta-n, w-x). \end{aligned}$$

***Corollary* 2.5.** *If all conditions of the* ***Theorem* 2.1** *are satisfied along with set* $\nu = 1, \rho = 1$, *and replace* $\lambda z = 2xt^2 - t^3$, *then, by the generating relation* $\sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n = exp(2xt - t^2)$, *where,* $H_n(x)$ *are the Hermite polynomials* $\forall n \in \{0, 1, 2, \ldots\}$, *following identities hold*

$$\tag{2.13} \begin{aligned} &\frac{1}{2\pi i}\int_{-\infty}^{(0^+,1^+)} E_{1,1}(2xt - t^2) e^{wt} t^{-\beta}\left(1-\frac{1}{t}\right)^{-\alpha} dt \\ &= \frac{1}{2\pi i}\int_{-\infty}^{(0^+,1^+)} exp(2xt - t^2) e^{wt} t^{-\beta}\left(1-\frac{1}{t}\right)^{-\alpha} dt = \sum_{n=0}^{\infty} \frac{w^{\beta-n-1}}{n!\Gamma(\beta-n)} H_n(x) M(\alpha, \beta-n, w). \end{aligned}$$

3 Extended contour integral representations for generalized hypergeometric functions of two variables

In this section, we extend the contour integral given in the ***Section* 2** and then, by properties of Mttag - Leffler functions (1.6) and (1.7), we obtain some more results for generalized hypergeometric functions of two variables.

***Theorem* 3.1.** *If for all $\alpha, \beta, \lambda, \rho, \sigma \in C, |arg(w)| < \frac{\pi}{2}, \lambda \neq 0, \Re(\rho) > 0, \Re(\sigma) > 0$, $\nu \in \mathbb{R}^+$ then, the contour integral $\frac{1}{2\pi i}\int_{-\infty}^{(0^+,1^+)} E_{\nu,\rho+\sigma}(\lambda z^\nu t^{-\nu}) e^{wt} t^{-\beta}\left(1-\frac{1}{t}\right)^{-\alpha} dt$ exists as*

$$(3.1)\quad \frac{1}{2\pi i}\int_{-\infty}^{(0^+,1^+)} E_{\nu,\rho+\sigma}(\lambda z^\nu t^{-\nu}) e^{wt} t^{-\beta}\left(1-\frac{1}{t}\right)^{-\alpha} dt = \frac{w^{\beta-1}}{\Gamma(\alpha)} S\,{}^{0:1;1}_{1:1;0}\left(\begin{matrix}[-:-,-]:[1:1],;[\alpha:1]; \\ [\beta:\nu,1]:[\rho+\sigma:\nu];[-:-];\end{matrix}\lambda w^\nu z^\nu, w\right).$$

Proof. Consider the contour integral

$$\frac{1}{2\pi i}\int_{-\infty}^{(0^+,1^+)} E_{\nu,\rho+\sigma}(\lambda z^\nu t^{-\nu}) e^{wt} t^{-\beta}\left(1-\frac{1}{t}\right)^{-\alpha} dt$$

and apply the result due to Mathai and Haubold [23, p.90],

$$z^{\beta+\mu-1} E_{\alpha,\beta+\mu}(\lambda z^\alpha) = \frac{1}{\Gamma(\mu)}\int_0^z u^{\beta-1}(z-u)^{\mu-1} E_{\alpha,\beta}(\lambda u^\alpha) du,$$

we get

$$\frac{1}{2\pi i}\int_{-\infty}^{(0^+,1^+)} E_{\nu,\rho+\sigma}(\lambda z^\nu t^{-\nu}) e^{wt} t^{-\beta}\left(1-\frac{1}{t}\right)^{-\alpha} dt = \frac{1}{z^{\rho+\sigma-1}\Gamma(\sigma) 2\pi i}\int_0^z u^{\rho-1}(z-u)^{\sigma-1}\int_{-\infty}^{(0^+,1^+)} t^{-\beta}\left(1-\frac{1}{t}\right)^{-\alpha} e^{wt} E_{\nu,\rho}(\lambda t^{-\nu} u^\nu) dt du.$$

In right hand side of above equality, an appeal to the ***Theorem* 2.1** gives

$$(3.2)\quad \frac{1}{2\pi i}\int_{-\infty}^{(0^+,1^+)} E_{\nu,\rho+\sigma}(\lambda z^\nu t^{-\nu}) e^{wt} t^{-\beta}\left(1-\frac{1}{t}\right)^{-\alpha} dt$$
$$= \frac{w^{\beta-1}}{\Gamma(\alpha) z^{\rho+\sigma-1}\Gamma(\sigma)}\int_0^z u^{\rho-1}(z-u)^{\sigma-1} S\,{}^{0:1;1}_{1:1;0}\left(\begin{matrix}[-:-,-]:[1:1];[\alpha:1]; \\ [\beta:\nu,1]:[\rho:\nu];[-:-];\end{matrix}\ \lambda w^\nu u^\nu, w\right) du$$
$$= \frac{w^{\beta-1}}{\Gamma(\alpha)} S\,{}^{0:1;1}_{1:1;0}\left(\begin{matrix}[-:-,-]:[1:1];[\alpha:1]; \\ [\beta:\nu,1]:[\rho+\sigma:\nu];[-:-];\end{matrix}\ \lambda w^\nu z^\nu, w\right).$$

***Corollary* 3.1.** *In the* ***Theorem* 3.1**, *when set $\nu = 1$, there exists a contour integral for Srivastava and Panda's generalized Kampé de Fériet function* [32], *in the form*

(3.3)
$$\frac{1}{2\pi i}\int_{-\infty}^{(0^+,1^+)} E_{1,\rho+\sigma}(\lambda z t^{-1}) e^{wt} t^{-\beta}\left(1-\frac{1}{t}\right)^{-\alpha} dt = \frac{w^{\beta-1}}{\Gamma(\beta)\Gamma(\rho+\sigma)} F\,{}^{0:1;1}_{1:1;0}\left[\begin{matrix}(-):(1);(\alpha); \\ (\beta):(\rho+\sigma);(-);\end{matrix}\lambda wz, w\right].$$

4 Transformations of contour integrals in various fractional derivative and integral operators

In this section, we present some transformations of the contour integrals defined in the ***Sections* 2** and **3** into Riemann - Liouville fractional derivative and integral operators and in Caputo derivative operators. To obtain the transformation formulae, we define following fractional derivative and integral operators:

The Riemann - Liouville fractional derivative $D^\mu_{a^+} Y$ of order $\mu, \mu \in \mathbb{C}, \Re(\mu) > 0$, as (for instance see [3], [11], [23])

$$(4.1)\quad (D^{\mu}_{a^+}Y)(x) := \left(\frac{d}{dx}\right)^n (I^{n-\mu}_{a^+}Y)(x)$$
$$= \frac{1}{\Gamma(n-\mu)}\left(\frac{d}{dx}\right)^n \int_a^x (x-u)^{n-\mu-1}Y(u)du, (n = [\Re(\mu)]+1; x > a).$$

Here, in formula (4.1), $I^{\alpha}_a Y$ is the Riemann - Liouville integral of order $\alpha, \alpha \in \mathbb{C}, \Re(\alpha) > 0$, given by

$$(I^{\alpha}_a Y)(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-\xi)^{\alpha-1}Y(\xi)d\xi, a \le x \le b.$$

The Caputo derivative of the function $Y(t)$, denoted by ${}^{\mathsf{C}}_{t}D^{\alpha}_{0^+}Y(t)$ where, $m-1 < \alpha \le m, \forall m \in \mathbb{N}$, is defined by

$$(4.2)\qquad ({}^{\mathsf{C}}_{t}D^{\alpha}_{a^+}Y)(t) = (I^{m-\alpha}_{a^+}Y^{(m)})(t), Y^{(m)}(t) = D^m_t Y(t) = \frac{d^m Y(t)}{dt^m} = \frac{d}{dt}\left(\frac{d^{m-1}}{dt^{m-1}}\right)Y(t).$$

***Theorem* 4.1.** *If for all $\alpha, \beta, \lambda, \rho, \sigma \in C, |arg(w)| < \frac{\pi}{2}, \lambda \ne 0, \Re(\rho) > 0, \Re(\sigma) > 0, \nu \in \mathbb{R}^+$ then, by the contour integral $\frac{1}{2\pi i}\int_{-\infty}^{(0^+,1^+)} E_{\nu,\rho+\sigma}(\lambda z^{\nu}t^{-\nu})e^{wt}t^{-\beta}\left(1-\frac{1}{t}\right)^{-\alpha}dt$, there exists following equalities among Riemann - Liouville fractional derivative, contour integral and Srivstava and Daoust function as*

$$(4.3)\quad \left(D^{\rho}_{0^+}\left\{\frac{1}{2\pi i}\int_{-\infty}^{(0^+,1^+)} u^{\rho+\sigma-1}E_{\nu,\rho+\sigma}(\lambda u^{\nu}t^{-\nu})e^{wt}t^{-\beta}\left(1-\frac{1}{t}\right)^{-\alpha}dt\right\}\right)(z)$$
$$= \frac{z^{\sigma-1}}{2\pi i}\int_{-\infty}^{(0^+,1^+)} E_{\nu,\sigma}(\lambda z^{\nu}t^{-\nu})e^{wt}t^{-\beta}\left(1-\frac{1}{t}\right)^{-\alpha}dt$$
$$= \frac{w^{\beta-1}z^{\sigma-1}}{\Gamma(\alpha)} S^{0:1;1}_{1:1;0}\left(\begin{matrix}[-:-,-]:[1:1];[\alpha:1];\\ [\beta:\nu,1]:[\rho:\nu];[-:-];\end{matrix}\lambda w^{\nu}z^{\nu}, w\right).$$

Proof. By the properties of the Mittag - Leffler functions $(I^{\alpha}_{0^+}u^{\beta-1}E_{\nu,\beta}(\lambda u^{\nu}))(x) = x^{\alpha+\beta-1}E_{\nu,\alpha+\beta}(\lambda x^{\nu})$ and the formula (4.1) the contour integral, given in the ***Theorem* 4.1**, is written as

$$(4.4)\quad \frac{1}{2\pi i}\int_{-\infty}^{(0^+,1^+)} z^{\rho+\sigma-1}E_{\nu,\rho+\sigma}(\lambda z^{\nu}t^{-\nu})e^{wt}t^{-\beta}\left(1-\frac{1}{t}\right)^{-\alpha}dt$$
$$= \left(I^{\rho}_{0^+}\left\{\frac{1}{2\pi i}\int_{-\infty}^{(0^+,1^+)} u^{\sigma-1}E_{\nu,\sigma}(\lambda u^{\nu}t^{-\nu})e^{wt}t^{-\beta}\left(1-\frac{1}{t}\right)^{-\alpha}dt\right\}\right)(z).$$

Therefore, by the property that $(D^{\rho}_{0^+}I^{\rho}_{0^+}Y)(z) = Y(z), Y(z) \in L_p(a,b), (1 \le p \le \infty), \forall z \in (a,b) \subset \mathbb{R}$, we have

$$\left(D^{\rho}_{0^+}\left\{\frac{1}{2\pi i}\int_{-\infty}^{(0^+,1^+)} u^{\rho+\sigma-1}E_{\nu,\rho+\sigma}(\lambda u^{\nu}t^{-\nu})e^{wt}t^{-\beta}\left(1-\frac{1}{t}\right)^{-\alpha}dt\right\}\right)(z)$$
$$= \left(D^{\rho}_{0^+}I^{\rho}_{0^+}\left\{\frac{1}{2\pi i}\int_{-\infty}^{(0^+,1^+)} u^{\sigma-1}E_{\nu,\sigma}(\lambda u^{\nu}t^{-\nu})e^{wt}t^{-\beta}\left(1-\frac{1}{t}\right)^{-\alpha}dt\right\}\right)(z)$$
$$= \frac{z^{\sigma-1}}{2\pi i}\int_{-\infty}^{(0^+,1^+)} E_{\nu,\sigma}(\lambda z^{\nu}t^{-\nu})e^{wt}t^{-\beta}\left(1-\frac{1}{t}\right)^{-\alpha}dt.$$

So that by (2.1), the equalities are given by

$$\left(D_{0^+}^{\rho}\left\{\frac{1}{2\pi i}\int_{-\infty}^{(0^+,1^+)} u^{\rho+\sigma-1}E_{\nu,\rho+\sigma}(\lambda u^{\nu}t^{-\nu})e^{wt}t^{-\beta}\left(1-\frac{1}{t}\right)^{-\alpha}dt\right\}\right)(z)$$
$$=\frac{z^{\sigma-1}}{2\pi i}\int_{-\infty}^{(0^+,1^+)} E_{\nu,\sigma}(\lambda z^{\nu}t^{-\nu})e^{wt}t^{-\beta}\left(1-\frac{1}{t}\right)^{-\alpha}dt$$
$$=\frac{w^{\beta-1}z^{\sigma-1}}{\Gamma(\alpha)}S^{0\,:\,1;1}_{1\,:\,1;0}\left(\begin{matrix}[-:-,-]:[1:1];[\alpha:1];\\ [\beta:\nu,1]:[\rho:\nu];[-:-];\end{matrix}\ \lambda w^{\nu}z^{\nu},w\right).$$

Hence, the ***Theorem*** **4.1** is followed.

In the similar manner, we obtain

$$\frac{z^{\sigma-1}}{2\pi i}\int_{-\infty}^{(0^+,1^+)} E_{\nu,\sigma}(\lambda z^{\nu}t^{-\nu})e^{wt}t^{-\beta}\left(1-\frac{1}{t}\right)^{-\alpha}dt$$
$$=\frac{1}{2\pi i}\int_{-\infty}^{(0^+,1^+)} z^{1+(\sigma-1)-1}E_{\nu,1+\sigma-1}(\lambda z^{\nu}t^{-\nu})e^{wt}t^{-\beta}\left(1-\frac{1}{t}\right)^{-\alpha}dt$$
$$=\left(I_{0^+}^{\sigma-1}\left\{\frac{1}{2\pi i}\int_{-\infty}^{(0^+,1^+)} E_{\nu}(\lambda u^{\nu}t^{-\nu})e^{wt}t^{-\beta}\left(1-\frac{1}{t}\right)^{-\alpha}dt\right\}\right)(z).$$

Therefore,

$$\text{(4.5)}\quad \left(D_{0^+}^{\sigma-1}\left\{\frac{1}{2\pi i}\int_{-\infty}^{(0^+,1^+)} u^{\sigma-1}E_{\nu,\sigma}(\lambda u^{\nu}t^{-\nu})e^{wt}t^{-\beta}\left(1-\frac{1}{t}\right)^{-\alpha}dt\right\}\right)(z)$$
$$=\frac{1}{2\pi i}\int_{-\infty}^{(0^+,1^+)} E_{\nu}(\lambda z^{\nu}t^{-\nu})e^{wt}t^{-\beta}\left(1-\frac{1}{t}\right)^{-\alpha}dt.$$

Finally, by an appeal to (4.3) and (4.5) we get

$$\text{(4.6)}\quad \left(D_{0^+}^{\sigma-1+\rho}\left\{\frac{1}{2\pi i}\int_{-\infty}^{(0^+,1^+)} u^{\rho+\sigma-1}E_{\nu,\rho+\sigma}(\lambda u^{\nu}t^{-\nu})e^{wt}t^{-\beta}\left(1-\frac{1}{t}\right)^{-\alpha}dt\right\}\right)(z)$$
$$=\frac{1}{2\pi i}\int_{-\infty}^{(0^+,1^+)} E_{\nu}(\lambda z^{\nu}t^{-\nu})e^{wt}t^{-\beta}\left(1-\frac{1}{t}\right)^{-\alpha}dt.$$

Again, by action of Caputo derivative (4.2) on the Mittag - Leffler functions (1.6, for $\rho=1$), we obtain

$$\text{(4.7)}\quad {}_{z}^{C}D_{0^+}^{\nu}\left(D_{0^+}^{\sigma-1+\rho}\left\{\frac{1}{2\pi i}\int_{-\infty}^{(0^+,1^+)} u^{\rho+\sigma-1}E_{\nu,\rho+\sigma}(\lambda u^{\nu}t^{-\nu})e^{wt}t^{-\beta}\left(1-\frac{1}{t}\right)^{-\alpha}dt\right\}(z)\right)$$
$$=\frac{\lambda}{2\pi i}\int_{-\infty}^{(0^+,1^+)} E_{\nu,1}(\lambda z^{\nu}t^{-\nu})e^{wt}t^{-\nu-\beta}\left(1-\frac{1}{t}\right)^{-\alpha}dt.$$

Thus,

$$\text{(4.8)}\quad {}_{z}^{C}D_{0^+}^{n\nu}\left(D_{0^+}^{\sigma-1+\rho}\left\{\frac{1}{2\pi i}\int_{-\infty}^{(0^+,1^+)} u^{\rho+\sigma-1}E_{\nu,\rho+\sigma}(\lambda u^{\nu}t^{-\nu})e^{wt}t^{-\beta}\left(1-\frac{1}{t}\right)^{-\alpha}dt\right\}(z)\right)$$
$$=\frac{\lambda^{n}}{2\pi i}\int_{-\infty}^{(0^+,1^+)} E_{\nu}(\lambda z^{\nu}t^{-\nu})e^{wt}t^{-n\nu-\beta}\left(1-\frac{1}{t}\right)^{-\alpha}dt,\forall n\in\mathbb{N}.$$

5 Numerical Example

Consider the initial value diffusion and wave problem $\forall \rho \in \mathbb{C}$,

$$(5.1) \qquad (D^{\rho}_{0^+} Y)(z) = \frac{z^{\sigma-1}}{2\pi i} \int_{-\infty}^{(0^+,1^+)} E_{\nu,\sigma}(\lambda z^{\nu} t^{-\nu}) e^{wt} t^{-\beta} \left(1 - \frac{1}{t}\right)^{-\alpha} dt,$$

where, the initial conditions are given by

$$\frac{d^{n-j}}{dz^{n-j}} (I^{n-\rho}_{0^+} Y)(z) \Big|_{z=0^+} = 0, \, j = 1, 2, \ldots, n; \; \Re(\rho) > 0, n = [\Re(\rho)] + 1.$$

To solve the problem (5.1), we operate both sides of equation (5.1) by ρ order Riemann - Liouville fractional integral operator $I^{\rho}_{0^+}$ and apply the formula [11, Eqn. (2.1.44), p. 75]

$$(I^{\rho}_{0^+} D^{\rho}_{0^+} Y)(z) = Y(z) - \sum_{j=1}^{n} \frac{\frac{d^{n-j}}{dz^{n-j}} (I^{n-\rho}_{0^+} Y)(z) \Big|_{z=0^+}}{\Gamma(\rho - j + 1)} z^{\rho - j}, \Re(\rho) > 0, n = [\Re(\rho)] + 1,$$

and then, use the initial conditions given in (5.1), we get

$$(5.2) \qquad Y(z) = \left(I^{\rho}_{0^+} \left\{ \frac{u^{\sigma-1}}{2\pi i} \int_{-\infty}^{(0^+,1^+)} E_{\nu,\sigma}(\lambda u^{\nu} t^{-\nu}) e^{wt} t^{-\beta} \left(1 - \frac{1}{t}\right)^{-\alpha} dt \right\} \right)(z).$$

Now, making an appeal to formula (4.4) in right hand side of (5.2), we derive

$$(5.3) \qquad Y(z) = \frac{z^{\rho+\sigma-1}}{2\pi i} \int_{-\infty}^{(0^+,1^+)} E_{\nu,\rho+\sigma}(\lambda z^{\nu} t^{-\nu}) e^{wt} t^{-\beta} \left(1 - \frac{1}{t}\right)^{-\alpha} dt.$$

6 Conclusion

In the ***Sections*** **2** and **3**, we derive various relations and known functions by the contour integrals in form of some special functions, their generating functions and bilateral generating relations involving Bessel functions, Laguerre polynomials and Hermite polynomials. Then, in the ***Section*** **4**, by the action of Riemann - Liouville fractional derivatives and integrals and by the operation of the Caputo derivative on contour integrals, we derive some identities among other contour integrals, and special functions. In the end of our investigation, we present a simple initial value problem to find its solution in terms of contour integrals. The presented work is applicable in various diffusion and wave problems occurring of Mittag - Leffler functions seen in the literature ([3], [8], [9], [11], [12], [23], [26] among others).

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