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(Dedicated to Honor Professor H.M. Srivastava on His 80th Birth Anniversary Celebrations)

GROWTHS OF COMPOSITE ENTIRE FUNCTIONS DEPENDING ON GENERALIZED RELATIVE LOGARITHMIC ORDER

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Abstract

In this paper we introduce the concept of generalized relative logarithmic order and generalized relative lower logarithmic order of an entire function. We investigate some newly developed results on the growth rates of composite entire functions depending on generalized relative logarithmic orders and generalized relative lower logarithmic orders.
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1 Introduction

The maximum modulus $M_f(r) = \max\{|f(z)| : |z| \leq r\}$ of an entire function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is nondecreasing function of $r > 0$. The order $\rho(f)$ and lower order $\lambda(f)$ of the entire function f are

$$\rho(f) = \limsup_{r\to\infty} \frac{\log\log M_f(r)}{\log r}$$

and

$$\lambda(f) = \liminf_{r\to\infty} \frac{\log\log M_f(r)}{\log r}$$

respectively.

Here we use the following notations:

$$\begin{aligned}\log^{[k]} x &= \log\left(\log^{[k-1]} x\right), \text{ for } k = 1, 2, 3, ...\\ \log^{[0]} x &= x\end{aligned}$$

and

$$\begin{aligned}\exp^{[k]} x &= \exp\left(\exp^{[k-1]} x\right), \text{ for } k = 1, 2, 3, ...\\ \exp^{[0]} x &= x.\end{aligned}$$

***Definition* 1.1.** [3] *Also for a entire function f with order zero, the logarithmic order $\rho_{\log}(f)$ and lower logarithmic order $\lambda_{\log}(f)$ are defined as*

$$\rho_{\log}(f)=\limsup_{r\to\infty}\frac{\log\log M_f(r)}{\log\log r}, \tag{1.1}$$

$$\lambda_{\log}(f)=\liminf_{r\to\infty}\frac{\log\log M_f(r)}{\log\log r}. \tag{1.2}$$

Now the maximum term $\mu_f(r)$ of the function f is defined as

$$\mu_f(r)=\max_{n\geq 0}|a_n|r^n.$$

For $0\leq r<R$, we have [7]

$$\mu_f(r)\leq M_f(r)\leq\frac{R}{R-r}\mu_f(R). \tag{1.3}$$

***Definition* 1.2.** *Using the maximum term we can define $\rho_{\log}(f)$ and $\lambda_{\log}(f)$ as*

$$\rho_{\log}(f)=\limsup_{r\to\infty}\frac{\log\log\mu_f(r)}{\log\log r} \tag{1.4}$$

and

$$\lambda_{\log}(f)=\liminf_{r\to\infty}\frac{\log\log\mu_f(r)}{\log\log r}. \tag{1.5}$$

Since maximum modulus M_f of a nonconstant entire function f is continuous and strictly increasing, there exists

$$M_f^{-1}:(|f(0)|,\infty)\to(0,\infty)$$

such that $\lim_{s\to\infty}M_f^{-1}(s)=\infty$.

Bernel [1] introduced the following concept of relative order of an entire function.

***Definition* 1.3.** *The relative order of f with respect to g is defined as*

$$\begin{aligned}\rho_g(f) &= \inf\left\{\mu>0: M_f(r)<M_g(r^{\mu})\text{ for all } r>r_0(\mu)>0\right\}\\ &= \limsup_{r\to\infty}\frac{\log M_g^{-1}M_f(r)}{\log r}.\end{aligned}$$

and the relative lower order of f with respect to g is defined as

$$\lambda_g(f)=\liminf_{r\to\infty}\frac{\log M_g^{-1}M_f(r)}{\log r}.$$

Lahiri and Banerjee [6] gave us a generalized concept of relative order.

***Definition* 1.4.** *If $p\geq 1$ is a positive integer, then the p-th generalized relative order of f with respect to g, denoted by $\rho_g^{[p]}(f)$ and is defined as*

$$\begin{aligned}\rho_g^{[p]}(f) &= \inf\left\{\mu>0: M_f(r)<M_g(\exp^{[p-1]}r^{\mu})\text{ for all } r>r_0(\mu)>0\right\}\\ &= \limsup_{r\to\infty}\frac{\log^{[p]}M_g^{-1}M_f(r)}{\log r}.\end{aligned}$$

and the generalized relative lower order of f with respect to g, denoted by $\lambda_g^{[p]}(f)$ and is defined as

$$\lambda_g^{[p]}(f)=\liminf_{r\to\infty}\frac{\log^{[p]}M_g^{-1}M_f(r)}{\log r}.$$

Datta and Maji [5] defined relative order of an entire function in terms of its maximum term as:

***Definition* 1.5.** *The relative order and relative lower order of an entire function f with respect to g are defined as*

$$\rho_g(f) = \limsup_{r\to\infty} \frac{\log \mu_g^{-1}\mu_f(r)}{\log r},$$
$$\lambda_g(f) = \liminf_{r\to\infty} \frac{\log \mu_g^{-1}\mu_f(r)}{\log r}.$$

Again in terms of maximum term, ***Definition*** 1.4 can be rewritten as

***Definition* 1.6.** *If $p \geq 1$ is a positive integer, then $\rho_g^{[p]}(f)$ and $\lambda_g^{[p]}(f)$ are defined as:*

$$\rho_g^{[p]}(f) = \limsup_{r\to\infty} \frac{\log^{[p]} \mu_g^{-1}\mu_f(r)}{\log r},$$
$$\lambda_g^{[p]}(f) = \liminf_{r\to\infty} \frac{\log^{[p]} \mu_g^{-1}\mu_f(r)}{\log r}.$$

In view of ***Definitions*** 1.1 and 1.4, here we define the generalized relative logarithmic order as:

***Definition* 1.7.** *If $p \geq 1$ is a positive integer, then the p-th generalized relative logarithmic order of f with respect to g, denoted by $\rho_{\log g}^{[p]}(f)$, is defined by*

$$\rho_{\log g}^{[p]}(f) = \limsup_{r\to\infty} \frac{\log^{[p]} M_g^{-1}M_f(r)}{\log\log r},$$

and the p-th generalized relative lower logarithmic order of f with respect to g, denoted by $\lambda_{\log g}^{[p]}(f)$ and is defined as:

$$\lambda_{\log g}^{[p]}(f) = \liminf_{r\to\infty} \frac{\log^{[p]} M_g^{-1}M_f(r)}{\log\log r}.$$

Also we define the generalized relative logarithmic order and generalized relative lower logarithmic order by using maximum term as:

***Definition* 1.8.** *If $p \geq 1$ is a positive integer, then $\rho_{\log g}^{[p]}(f)$ and $\lambda_{\log g}^{[p]}(f)$ are defined as:*

$$\rho_{\log g}^{[p]}(f) = \limsup_{r\to\infty} \frac{\log^{[p]} \mu_g^{-1}\mu_f(r)}{\log\log r},$$

and

$$\lambda_{\log g}^{[p]}(f) = \liminf_{r\to\infty} \frac{\log^{[p]} \mu_g^{-1}\mu_f(r)}{\log\log r}.$$

Biswas et. al. [2] established some results on the growth rates of composite entire functions depending on their generalized relative orders and generalized relative lower orders. In this paper we will prove some results of [2] on the basis of their generalized relative logarithmic orders and generalized relative lower logarithmic orders.

From Valiron [8] we get the general theory of entire functions and so we do not explain them in details.

2 Lemmas

In this section we present some lemmas which will be needed to prove our results.

***Lemma* 2.1.** [4] *If f and g are two entire functions, then for all sufficiently large values of r,*

$$M_f\left(\frac{1}{8}M_g\left(\frac{r}{2}\right) - |g(0)|\right) \leq M_{f\circ g}(r) \leq M_f\left(M_g(r)\right).$$

***Lemma* 2.2.** [7] *If f and g are any two entire functions. Then for every $\alpha > 1$ and $0 < r < R$,*

$$\mu_{f\circ g}(r) \leq \frac{\alpha}{\alpha - 1}\mu_f\left(\frac{\alpha R}{R - r}\mu_g(R)\right).$$

***Lemma* 2.3.** [7] *If f and g are two entire functions with $g(0) = 0$, then for all sufficiently large values of r,*

$$\mu_{f\circ g}(r) \geq \frac{1}{2}\mu_f\left(\frac{1}{8}\mu_g(\frac{r}{4}) - |g(0)|\right).$$

***Lemma* 2.4.** [1] *If f is an entire function and $\alpha > 1, 0 < \beta < \alpha$. Then for all sufficiently large r,*

$$M_f(\alpha r) \geq \beta M_f(r).$$

***Lemma* 2.5.** [5] *If f is an entire function and $\alpha > 1, 0 < \beta < \alpha$. Then for all sufficiently large r,*

$$\mu_f(\alpha r) \geq \beta \mu_f(r).$$

3 Main results:

In this section we will present the main results of this paper.

***Theorem* 3.1.** *Let f and h be any two entire functions such that $0 < \lambda^{[p]}_{\log h}(f) \leq \rho^{[p]}_{\log h}(f) < \infty$ and g be an entire function with $\lambda^{[q]}_{\log}(g) > 0$ where p, q are any integers with $p > 1$ and $q > 2$. Then for every positive constant δ and every real number α,*

$$\lim_{r\to\infty} \frac{\log^{[p]} M_h^{-1} M_{f\circ g}(r)}{\left[\log^{[p]} M_h^{-1} M_f\left(\{\exp^{[q-3]} r\}^{\delta}\right)\right]^{1+\alpha}} = \infty.$$

Proof. If α is such that $1 + \alpha \leq 0$, then the theorem is trivial. So we suppose that $1 + \alpha > 0$. Since $M_h^{-1}(r)$ is an increasing function of r, it follows from the first part of ***Lemma* 2.1** for all sufficiently large values of r that

$$\begin{aligned} \log^{[p]} M_h^{-1} M_{f\circ g}(r) &\geq \left(\lambda^{[p]}_{\log h}(f) - \varepsilon\right) \log\log\left(\frac{1}{8}M_g(\frac{r}{2}) - |g(0)|\right) \\ &\geq \left(\lambda^{[p]}_{\log h}(f) - \varepsilon\right) \log\log M_g\left(\frac{r}{2}\right) + O(1) \\ &\geq \left(\lambda^{[p]}_{\log h}(f) - \varepsilon\right) \exp^{[q-3]} (\log r)^{\lambda^{[q]}_{\log}(g) - \varepsilon}. \end{aligned} \tag{3.1}$$

Choose ε, $0 < \varepsilon < \min\left(\lambda^{[p]}_{\log h}(f), \lambda^{[q]}_{\log}(g)\right)$.

Again for sufficiently large values of r we have from the definition of $\rho^{[p]}_{\log h}(f)$,

$$\begin{aligned} \left[\log^{[p]} M_h^{-1} M_f\left(\left(\exp^{[q-3]} r\right)^{\delta}\right)\right]^{1+\alpha} &\leq \left[\left(\rho^{[p]}_{\log h}(f) + \varepsilon\right) \log\log\left(\exp^{[q-3]} r\right)^{\delta}\right]^{1+\alpha} \\ &= \left(\rho^{[p]}_{\log h}(f) + \varepsilon\right)^{1+\alpha} \left[\log\left\{\delta \exp^{[q-4]} r\right\}\right]^{1+\alpha} \end{aligned} \tag{3.2}$$

$$\leq \left(\rho^{[p]}_{\log h}(f)+\varepsilon\right)^{1+\alpha}\left[\exp^{[q-5]} r\right]^{1+\alpha}+O(1).$$

From (3.2) and (3.3) ,

$$\frac{\log^{[p]} M_h^{-1} M_{f\circ g}(r)}{\left[\log^{[p]} M_h^{-1} M_f\left((\exp^{[q-2]} r)^{\delta}\right)\right]^{1+\alpha}} \geq \frac{\left(\lambda^{[p]}_{\log h}(f)-\varepsilon\right)\exp^{[q-3]}(\log r)^{\lambda^{[q]}_{\log}(g)-\varepsilon}}{\left(\rho^{[p]}_{\log h}(f)+\varepsilon\right)^{1+\alpha}\left[\exp^{[q-5]} r\right]^{1+\alpha}+O(1)}.$$

Since $\lim\limits_{r\to\infty} \frac{\exp^{[q-3]}(\log r)^{\lambda^{[q]}_{\log}(g)-\varepsilon}}{\left[\exp^{[q-5]} r\right]^{1+\alpha}} \to \infty$, the theorem follows.

In view of ***Theorem* 3.1**, one can easily prove the following theorem.

***Theorem* 3.2.** *Let f, g, h and k be any four entire functions with $\lambda^{[p]}_{\log h}(f) > 0, \lambda^{[q]}_{\log}(g) > 0$ and $\rho^{[m]}_{\log k}(g) < \infty$ where $p(> 1), q(> 2)$ and $m(> 1)$ be any three integers. Then for every $\delta > 0$ and for every real number α,*

$$\lim_{r\to\infty} \frac{\log^{[p]} M_h^{-1} M_{f\circ g}(r)}{\left[\log^{[m]} M_k^{-1} M_g\left(\{\exp^{[q-3]} r\}^{\delta}\right)\right]^{1+\alpha}} = \infty.$$

Using ***Lemma* 2.3** and ***Definition* 1.8**, the following theorems can be proved in view of ***Theorem* 3.1** and ***Theorem* 3.2**,

***Theorem* 3.3.** *Let f and h be any two entire functions such that $0 < \lambda^{[p]}_{\log h}(f) \leq \rho^{[p]}_{\log h}(f) < \infty$ and g be an entire function with $\lambda^{[q]}_{\log}(g) > 0$ where $p(> 1)$ and $q(> 2)$ be any two integers. Then for every positive constant δ and every real number α,*

$$\lim_{r\to\infty} \frac{\log^{[p]} \mu_h^{-1} \mu_{f\circ g}(r)}{\left[\log^{[p]} \mu_h^{-1} \mu_f\left(\{\exp^{[q-3]} r\}^{\delta}\right)\right]^{1+\alpha}} = \infty.$$

***Theorem* 3.4.** *Let f, g, h and k be any four entire functions with $\lambda^{[p]}_{\log h}(f) > 0, \lambda^{[q]}_{\log}(g) > 0$ and $\rho^{[m]}_{\log k}(g) < \infty$ where $p(> 1), q(> 2)$ and $m(> 1)$ be any three integers. Then for every $\delta > 0$ and for every real number α,*

$$\lim_{r\to\infty} \frac{\log^{[p]} \mu_h^{-1} \mu_{f\circ g}(r)}{\left[\log^{[m]} \mu_k^{-1} \mu_g\left(\{\exp^{[q-3]} r\}^{\delta}\right)\right]^{1+\alpha}} = \infty.$$

***Remark* 3.1.** *If we consider $0 < \lambda^{[p]}_{\log h}(f) < \infty$ instead of $0 < \lambda^{[p]}_{\log h}(f) \leq \rho^{[p]}_{\log h}(f) < \infty$ in **Theorems 3.1** and **3.3** and the other conditions remain the same, the conclusion of **Theorems 3.1** and **3.3** remain valid with "limit superior" is replaced by "limit".*

***Remark* 3.2.** *If we consider $0 < \lambda^{[m]}_{\log k}(g) < \infty$ instead of $\rho^{[m]}_{\log k}(g) < \infty$ in **Theorems 3.2** and **3.4** and the other conditions remain the same, the conclusion of **Theorems 3.2** and **3.4** remain valid with "limit superior" is replaced by "limit".*

***Theorem* 3.5.** *Let f, g and h be any three entire functions such that $0 < \lambda^{[p]}_{\log h}(f) \leq \rho^{[p]}_{\log h}(f) < \infty$ and $\rho^{[q]}_{\log}(g) < \infty$ where p, q are integers with $p > 1$ and $q > 2$. Then for every $\delta > 0$ and each $\alpha \in (-\infty, \infty)$,*

$$\lim_{r\to\infty} \frac{\left[\log^{[p]} M_h^{-1} M_{f\circ g}(r)\right]^{1+\alpha}}{\log^{[p]} M_h^{-1} M_f\left(\exp^{[q-1]}(\log r)^{\delta}\right)} = 0,$$

where $\delta > (1+\alpha)\rho^{[q]}_{\log}(g)$.

Proof. For $1+\alpha \leq 0$, there is nothing to proof. Let us consider $1+\alpha > 0$. As $M_h^{-1}(r)$ is an increasing function, then from the last part of ***Lemma* 2.1** we have for large values of r,

$$\begin{aligned}\log^{p} M_h^{-1} M_{f\circ g}(r) &\leq \log^{p} M_h^{-1} M_f\left(M_g(r)\right) \\ &\leq \left(\rho_{\log h}^{[p]}(f)+\varepsilon\right) \log\log\left(M_g(r)\right) \\ &\leq \left(\rho_{\log h}^{[p]}(f)+\varepsilon\right) \exp^{[q-3]}(\log r)^{\rho_{\log}^{[q]}(g)+\varepsilon}. \end{aligned} \tag{3.3}$$

Also for sufficiently large values of r we have from the definition of $\lambda_{\log h}^{[p]}(f)$,

$$\begin{aligned}\log^{[p]} M_h^{-1} M_f\left(\exp^{[q-1]}(\log r)^{\delta}\right) &\geq \left(\lambda_{\log h}^{[p]}(f)-\varepsilon\right)\log\log\left(\exp^{[q-1]}(\log r)^{\delta}\right) \\ &= \left(\lambda_{\log h}^{[p]}(f)-\varepsilon\right)\exp^{[q-3]}(\log r)^{\delta}. \end{aligned} \tag{3.4}$$

For sequence of values of r tending to infinity we have from (3.4) and (3.5) ,

$$\begin{aligned}\frac{\left[\log^{[p]} M_h^{-1} M_{f\circ g}(r)\right]^{1+\alpha}}{\log^{[p]} M_h^{-1} M_f\left(\exp^{[q-1]}(\log r)^{\delta}\right)} &\leq \frac{\left[\left(\rho_{\log h}^{[p]}(f)+\varepsilon\right)\exp^{[q-3]}(\log r)^{\rho_{\log}^{[q]}(g)+\varepsilon}\right]^{1+\alpha}}{\left(\lambda_{\log h}^{[p]}(f)-\varepsilon\right)\exp^{[q-3]}(\log r)^{\delta}} \\ &= \frac{\left(\rho_{\log h}^{[p]}(f)+\varepsilon\right)^{1+\alpha}\exp^{[q-3]}(\log r)^{\left(\rho_{\log}^{[q]}(g)+\varepsilon\right)(1+\alpha)}}{\left(\lambda_{\log h}^{[p]}(f)-\varepsilon\right)\exp^{[q-3]}(\log r)^{\delta}}, \end{aligned} \tag{3.5}$$

where we choose $0<\varepsilon<\min\left\{\lambda_{\log h}^{[p]}(f), \frac{\delta}{1+\alpha}-\rho_{\log}^{[q]}(g)\right\}$. So from (3.6) we get

$$\lim_{r\to\infty}\frac{\exp^{[q-3]}(\log r)^{\left(\rho_{\log}^{[q]}(g)+\varepsilon\right)(1+\alpha)}}{\exp^{[q-3]}(\log r)^{\delta}} = 0.$$

Therefore the theorem is proved.

In the line of ***Theorem* 3.5**, one may state the following Theorem with the similar proof.

***Theorem* 3.6.** *Let f, g, h and k be any four entire functions such that $\rho_{\log h}^{[p]}(f) < \infty$, $\rho_{\log}^{[q]}(g) < \infty$ and $\lambda_{\log k}^{[m]}(g) > 0$ where p, q, m are integers with $p > 1$, $q > 2$ and $m > 1$. Then for every $\delta > 0$ and each $\alpha \in (-\infty, \infty)$,*

$$\lim_{r\to\infty}\frac{\left[\log^{[p]} M_h^{-1} M_{f\circ g}(r)\right]^{1+\alpha}}{\log^{[m]} M_k^{-1} M_g\left(\exp^{[q-1]}(\log r)^{\delta}\right)} = 0,$$

where $\delta > (1+\alpha)\rho_{\log}^{[q]}(g)$.

In view of ***Theorem* 3.5** and ***Theorem* 3.6**, the following two Theorems can be proved by using ***Lemma* 2.2**, ***Lemma* 2.5** and ***Definition* 1.8** and hence their proofs are omitted.

***Theorem* 3.7.** *Let f, g and h be any three entire functions such that $0 < \lambda_{\log h}^{[p]}(f) \leq \rho_{\log h}^{[p]}(f) < \infty$ and $\rho_{\log}^{[q]}(g) < \infty$ where p, q are integers with $p > 1$ and $q > 2$. Then for every $\delta > 0$ and each $\alpha \in (-\infty, \infty)$,*

$$\lim_{r\to\infty}\frac{\left[\log^{[p]} \mu_h^{-1} \mu_{f\circ g}(r)\right]^{1+\alpha}}{\log^{[p]} \mu_h^{-1} \mu_f\left(\exp^{[q-1]}(\log r)^{\delta}\right)} = 0,$$

where $\delta > (1+\alpha)\rho_{\log}^{[q]}(g)$.

***Theorem* 3.8.** *Let f, g, h and k be any four entire functions such that $\rho^{[p]}_{\log h}(f) < \infty$, $\rho^{[q]}_{\log}(g) < \infty$ and $\lambda^{[m]}_{\log k}(g) > 0$ where p, q, m are integers with $p > 1$, $q > 2$ and $m > 1$. Then for every $\delta > 0$ and each $\alpha \in (-\infty, \infty)$,*

$$\lim_{r\to\infty} \frac{\left[\log^{[p]} \mu_h^{-1}\mu_{f\circ g}(r)\right]^{1+\alpha}}{\log^{[m]} \mu_k^{-1}\mu_g\left(\exp^{[q-1]}(\log r)^{\delta}\right)} = 0,$$

where $\delta > (1+\alpha)\rho^{[q]}_{\log}(g)$.

***Remark* 3.3.** *If we consider $0 < \rho^{[p]}_{\log h}(f) < \infty$ instead of $0 < \lambda^{[p]}_{\log h}(f) \leq \rho^{[p]}_{\log h}(f) < \infty$ in **Theorems** **3.5** and **3.7** and the other conditions remain the same, the conclusion of **Theorems** **3.5** and **3.7** remain valid with "limit inferior" is replaced by "limit".*

***Remark* 3.4.** *If we consider $\rho^{[m]}_{\log k}(g) > 0$ instead of $\lambda^{[m]}_{\log k}(g) > 0$ in **Theorems** **3.6** and **3.8** and the other conditions remain the same, the conclusion of **Theorems** **3.6** and **3.8** remain valid with "limit inferior" is replaced by "limit".*

***Theorem* 3.9.** *Let f, g and h be any three entire functions such that $\rho^{[p]}_{\log h}(f) < \infty$ and $\lambda^{[p]}_{\log h}(f\circ g) = \infty$ where p is any integer > 1. Then for every $A(>0)$,*

$$\lim_{r\to\infty} \frac{\log^{[p]} M_h^{-1} M_{f\circ g}(r)}{\log^{[p]} M_h^{-1} M_f\left((\log r)^A\right)} = \infty.$$

Proof. Let us consider the contrarary part, i.e $\lim_{r\to\infty} \frac{\log^{[p]} M_h^{-1} M_{f\circ g}(r)}{\log^{[p]} M_h^{-1} M_f\left((\log r)^A\right)}$ is finite, then there exists a constant B such that for a sequence of values of r tending to infinity we have,

(3.6) $$\log^{[p]} M_h^{-1} M_{f\circ g}(r) \leq B.\log^{[p]} M_h^{-1} M_f\left((\log r)^A\right)$$

Also for sufficiently large values of r we have from the definition of $\rho^{[p]}_{\log h}(f)$,

(3.7) $$\begin{aligned}\log^{[p]} M_h^{-1} M_f\left((\log r)^A\right) &\leq \left(\rho^{[p]}_{\log h}(f) + \varepsilon\right)\log\log(\log r)^A \\ &< \left(\rho^{[p]}_{\log h}(f) + \varepsilon\right)\log(\log r)^A \\ &= \left(\rho^{[p]}_{\log h}(f) + \varepsilon\right)A\log\log r.\end{aligned}$$

For sequence of values of r tending to infinity we have from (3.6) and (3.8),

$$\log^{[p]} M_h^{-1} M_{f\circ g}(r) \leq B.A.\left(\rho^{[p]}_{\log h}(f) + \varepsilon\right)\log\log r,$$

i.e,

$$\lambda^{[p]}_{\log h}(f\circ g) \leq B.A.\left(\rho^{[p]}_{\log h}(f) + \varepsilon\right),$$

which contradicts the fact that $\lambda^{[p]}_{\log h}(f\circ g)$ is infinite. So our assumption is wrong and hence the theorem follows.

One can prove the following theorem in view of ***Theorem* 3.9**,

***Theorem* 3.10.** *Let f, g and h be any three entire functions such that $\rho^{[p]}_{\log h}(f) < \infty$ and $\lambda^{[p]}_{\log h}(f\circ g) = \infty$ where $p(>1)$ is any integer. Then for every $A(>0)$ we have*

$$\lim_{r\to\infty} \frac{\log^{[p]} \mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[p]} \mu_h^{-1}\mu_f\left((\log r)^A\right)} = \infty.$$

***Remark* 3.5.** *If we replace "limit" by "limit superior" and $\lambda^{[p]}_{\log h}(f \circ g) = \infty$ by $\rho^{[p]}_{\log h}(f \circ g) = \infty$ in **Theorem** 3.9 and **Theorem** 3.10 then they are also valid.*

***Corollary* 3.1.** *Under the assumption of **Theorems** 3.9 and 3.10,*

$$\lim_{r\to\infty} \frac{M_h^{-1} M_{f\circ g}(r)}{M_h^{-1} M_f\left((\log r)^A\right)} = \infty$$
$$\text{and } \lim_{r\to\infty} \frac{\mu_h^{-1} \mu_{f\circ g}(r)}{\mu_h^{-1} \mu_f\left((\log r)^A\right)} = \infty$$

hold.

Proof. For all sufficiently large values of r and for $K > 1$ we have from ***Theorem*** **3.9**,

$$\log^{[p]} M_h^{-1} M_{f\circ g}(r) \geq K. \log^{[p]} M_h^{-1} M_f\left((\log r)^A\right)$$
$$\log^{[p-1]} M_h^{-1} M_{f\circ g}(r) \geq \left[\log^{[p-1]} M_h^{-1} M_f\left((\log r)^A\right)\right]^K.$$

Therefore first part is proved.

Similarly from ***Theorem*** **3.10**, we get the second part.

***Corollary* 3.2.** *Under the assumption of **Remark** 3.5, one can prove the following results.*

$$\limsup_{r\to\infty} \frac{M_h^{-1} M_{f\circ g}(r)}{M_h^{-1} M_f\left((\log r)^A\right)} = \infty$$
$$\text{and } \limsup_{r\to\infty} \frac{\mu_h^{-1} \mu_{f\circ g}(r)}{\mu_h^{-1} \mu_f\left((\log r)^A\right)} = \infty.$$

Using ***Theorems*** **3.9** and **3.10, *Remark*** **3.5**, ***Corollaries*** **3.1** and **3.2**, one may also state the following theorems and corollaries without their proofs.

***Theorem* 3.11.** *Let f, g and k be any three entire functions such that $\rho^{[m]}_{\log k}(g) < \infty$ and $\rho^{[m]}_{\log k}(f \circ g) = \infty$ where $m(> 1)$ is any integer. Then for every $B(> 0)$ we have*

$$\limsup_{r\to\infty} \frac{\log^{[m]} M_k^{-1} M_{f\circ g}(r)}{\log^{[m]} M_k^{-1} M_g\left((\log r)^B\right)} = \infty.$$

***Theorem* 3.12.** *Let f, g and k be any three entire functions such that $\rho^{[m]}_{\log k}(g) < \infty$ and $\rho^{[m]}_{\log k}(f \circ g) = \infty$ where $m(> 1)$ is any integer. Then for every $B(> 0)$ we have*

$$\limsup_{r\to\infty} \frac{\log^{[m]} \mu_k^{-1} \mu_{f\circ g}(r)}{\log^{[m]} \mu_k^{-1} \mu_g\left((\log r)^B\right)} = \infty.$$

***Remark* 3.6.** *If we replace "limit superior" by "limit" and $\rho^{[m]}_{\log k}(f \circ g) = \infty$ by $\lambda^{[m]}_{\log k}(f \circ g) = \infty$ then **Theorems** 3.11 and 3.12 also hold.*

***Corollary* 3.3.** *Again under the assumption of **Theorems** 3.11 and 3.12, we get*

$$\limsup_{r\to\infty} \frac{M_k^{-1} M_{f\circ g}(r)}{M_k^{-1} M_g\left((\log r)^B\right)} = \infty,$$
$$\limsup_{r\to\infty} \frac{\mu_k^{-1} \mu_{f\circ g}(r)}{\mu_k^{-1} \mu_g\left((\log r)^B\right)} = \infty.$$

***Corollary* 3.4.** *Also under the assumption of* ***Remark* 3.6** *we have*

$$\lim_{r\to\infty}\frac{M_k^{-1}M_{f\circ g}(r)}{M_k^{-1}M_g\left((\log r)^B\right)} = \infty,$$

$$\lim_{r\to\infty}\frac{\mu_k^{-1}\mu_{f\circ g}(r)}{\mu_k^{-1}\mu_g\left((\log r)^B\right)} = \infty.$$

***Theorem* 3.13.** *Let f and h be any two entire functions such that $0<\lambda^{[p]}_{\log h}(f)\le\rho^{[p]}_{\log h}(f)<\infty$, $p(>1)$ be any integer. Also suppose g be an entire function with $0<\delta<\rho^{[q]}_{\log}(g)\le\infty$, $q(>2)$ be any integer. Then for a sequence of values of r tending to infinity,*

$$M_h^{-1}M_{f\circ g}(r)>M_h^{-1}M_f\left(\exp^{[q-1]}(\log r)^{\delta}\right).$$

Proof. Since $M_h^{-1}(r)$ is an increasing function, then from ***Lemma* 2.1**, for a sequence of values of r tending to infinity,

$$\begin{aligned}\log^{[p]}M_h^{-1}M_{f\circ g}(r) &\ge \left(\lambda^{[p]}_{\log h}(f)-\varepsilon\right)\log\log\left(\frac{1}{8}M_g\left(\frac{r}{2}\right)\right)\\ &\ge \left(\lambda^{[p]}_{\log h}(f)-\varepsilon\right)\exp^{[q-3]}\left(\log\frac{r}{2}\right)^{\rho^{[q]}_{\log}(g)-\varepsilon}+O(1).\end{aligned}\tag{3.8}$$

Also for sufficiently large values of r we have from the definition of $\rho^{[p]}_{\log h}(f)$,

$$\begin{aligned}\log^{[p]}M_h^{-1}M_f\left(\exp^{[q-1]}(\log r)^{\delta}\right) &\le \left(\rho^{[p]}_{\log h}(f)+\varepsilon\right)\log\log\left(\exp^{[q-1]}(\log r)^{\delta}\right)\\ &= \left(\rho^{[p]}_{\log h}(f)+\varepsilon\right)\exp^{[q-3]}(\log r)^{\delta}.\end{aligned}\tag{3.9}$$

For sequence of values of r tending to infinity we have from (3.9) and (3.10) ,

$$\frac{\log^{[p]}M_h^{-1}M_{f\circ g}(r)}{\log^{[p]}M_h^{-1}M_f\left(\exp^{[q-1]}(\log r)^{\delta}\right)}\ge\frac{\left(\lambda^{[p]}_{\log h}(f)-\varepsilon\right)\exp^{[q-3]}\left(\log\frac{r}{2}\right)^{\rho^{[q]}_{\log}(g)-\varepsilon}+O(1)}{\left(\rho^{[p]}_{\log h}(f)+\varepsilon\right)\exp^{[q-3]}(\log r)^{\delta}}.\tag{3.10}$$

As $\delta<\rho^{[q]}_{\log}(g)$, choose $\varepsilon(>0)$ in such that

$$\delta<\rho^{[q]}_{\log}(g)-\varepsilon.\tag{3.11}$$

Using (3.11) in (3.10) we get

$$\limsup_{r\to\infty}\frac{\log^{[p]}M_h^{-1}M_{f\circ g}(r)}{\log^{[p]}M_h^{-1}M_f\left(\exp^{[q-1]}(\log r)^{\delta}\right)}=\infty.\tag{3.12}$$

Therefore we have for $K>1$ and for a sequence of values of r tending to infinity,

$$M_h^{-1}M_{f\circ g}(r)>M_h^{-1}M_f\left(\exp^{[q-1]}(\log r)^{\delta}\right).$$

Hence the theorem is proved.

***Theorem* 3.14.** *Let f and h be any two entire functions with $0<\lambda^{[p]}_{\log h}(f)\le\rho^{[p]}_{\log h}(f)<\infty$, $p(>1)$ be any integer. Also suppose g and k be any two entire functions such that $\rho^{[m]}_{\log k}(g)<\infty$ and $0<\delta<\rho^{[q]}_{\log}(g)$ where $q(>1)$ and $m(>2)$ are two integers. Then for a sequence of values of r tending to infinity,*

$$\log^{[p-1]}M_h^{-1}M_{f\circ g}(r)>\log^{[m-1]}M_k^{-1}M_g\left(\exp^{[q-1]}(\log r)^{\delta}\right).$$

Proof. Let us consider $0 < \delta < \delta_0 < \rho^{[q]}_{\log}(g)$ and for a sequence of values of r tending to infinity we get from (3.9),

$$\log^{[p]} M_h^{-1} M_f\left(\exp^{[q-1]}(\log r)^{\delta}\right) > \left(\lambda^{[p]}_{\log h}(f) - \varepsilon\right)\exp^{[q-3]}(\log r)^{\delta_0}. \quad (3.13)$$

Also for sufficiently large values of r we have from the definition of $\rho^{[m]}_{\log k}(g)$,

$$\begin{aligned}\log^{[m]} M_k^{-1} M_g\left(\exp^{[q-1]}(\log r)^{\delta}\right) &\le \left(\rho^{[m]}_{\log k}(g) + \varepsilon\right)\log\log\left(\exp^{[q-1]}(\log r)^{\delta}\right)\\ &= \left(\rho^{[m]}_{\log k}(g) + \varepsilon\right)\exp^{[q-3]}(\log r)^{\delta}.\end{aligned} \quad (3.14)$$

For sequence of values of r tending to infinity we have from (3.13) and (3.15),

$$\frac{\log^{[p]} M_h^{-1} M_{f\circ g}(r)}{\log^{[m]} M_k^{-1} M_g\left(\exp^{[q-1]}(\log r)^{\delta}\right)} \ge \frac{\left(\lambda^{[p]}_{\log h}(f) - \varepsilon\right)\exp^{[q-3]}(\log r)^{\delta_0}}{\left(\rho^{[m]}_{\log k}(g) + \varepsilon\right)\exp^{[q-3]}(\log r)^{\delta}}. \quad (3.15)$$

Since $\delta < \delta_0$, then from (3.15), we have

$$\limsup_{r\to\infty} \frac{\log^{[p]} M_h^{-1} M_{f\circ g}(r)}{\log^{[m]} M_k^{-1} M_g\left(\exp^{[q-1]}(\log r)^{\delta}\right)} = \infty. \quad (3.16)$$

Hence the theorem is proved.

Again in view of ***Theorem* 3.13** and **3.14**, the following two theorems can be proved by using ***Lemma* 2.3** and ***Definition* 1.8**. Here we state these theorems without their proofs.

***Theorem* 3.15.** *Let f and h be any two entire functions such that $0 < \lambda^{[p]}_{\log h}(f) \le \rho^{[p]}_{\log h}(f) < \infty$, $p(> 1)$ be any integer. Also suppose g be an entire function with $0 < \delta < \rho^{[q]}_{\log}(g) \le \infty$, $q(> 2)$ be any integer. Then for a sequence of values of r tending to infinity,*

$$\mu_h^{-1}\mu_{f\circ g}(r) > \mu_h^{-1}\mu_f\left(\exp^{[q-1]}(\log r)^{\delta}\right).$$

***Theorem* 3.16.** *Let f and h be any two entire functions with $0 < \lambda^{[p]}_{\log h}(f) \le \rho^{[p]}_{\log h}(f) < \infty$, $p(> 1)$ be any integer. Also suppose g and k be any two entire functions such that $\rho^{[m]}_{\log k}(g) < \infty$ and $0 < \delta < \rho^{[q]}_{\log}(g)$ where $q(> 1)$ and $m(> 2)$ are two integers. Then for a sequence of values of r tending to infinity,*

$$\log^{[p-1]}\mu_h^{-1}\mu_{f\circ g}(r) > \log^{[m-1]}\mu_k^{-1}\mu_g\left(\exp^{[q-1]}(\log r)^{\delta}\right).$$

***Theorem* 3.17.** *Let f, g and h be any three entire functions such that $0 < \lambda^{[p]}_{\log h}(f) < \rho^{[p]}_{\log h}(f) < \infty$ and $\lambda^{[q]}_{\log}(g) < \delta < \infty$ where $p(> 1)$ and $q(> 2)$ are any two integers. Then for a sequence of values of r tending to infinity,*

$$M_h^{-1}M_{f\circ g}(r) < M_h^{-1}M_f\left(\exp^{[q-1]}(\log r)^{\delta}\right).$$

Proof. Since $M_h^{-1}(r)$ is an increasing function of r, then for a sequence of values of r tending to infinity we have from the last part of ***Lemma* 2.1**,

$$\begin{aligned}\log^{p} M_h^{-1} M_{f\circ g}(r) &\le \left(\rho^{[p]}_{\log h}(f) + \varepsilon\right)\log\log M_g(r)\\ &\le \left(\rho^{[p]}_{\log h}(f) + \varepsilon\right)\exp^{[q-3]}(\log r)^{\lambda^{[q]}_{\log}(g)+\varepsilon}.\end{aligned} \quad (3.17)$$

Now for sequence of values of r tending to infinity we have from (3.5) and (3.18),

$$\frac{\log^{[p]} M_h^{-1} M_f\left(\exp^{[q-1]}(\log r)^{\delta}\right)}{\log^{p} M_h^{-1} M_{f\circ g}(r)} \ge \frac{\left(\lambda^{[p]}_{\log h}(f) - \varepsilon\right)\exp^{[q-3]}(\log r)^{\delta}}{\left(\rho^{[p]}_{\log h}(f) + \varepsilon\right)\exp^{[q-3]}(\log r)^{\lambda^{[q]}_{\log}(g)+\varepsilon}}. \quad (3.18)$$

Since $\lambda_{\log}^{[q]}(g) < \delta$, we choose $\varepsilon(> 0)$ such that

$$\lambda_{\log}^{[q]}(g) + \varepsilon < \delta < \rho_{\log}^{[q]}(g). \tag{3.19}$$

Then from (3.18) and (3.19) ,we have

$$\limsup_{r\to\infty} \frac{\log^{[p]} M_h^{-1} M_f \left(\exp^{[q-1]} (\log r)^{\delta}\right)}{\log^{p} M_h^{-1} M_{f\circ g}(r)} = \infty. \tag{3.20}$$

Therefore we have for $K > 1$ and for a sequence of values of r tending to infinity,

$$M_h^{-1} M_f \left(\exp^{[q-1]} (\log r)^{\delta}\right) > M_h^{-1} M_{f\circ g}(r).$$

Hence the theorem is proved.

In view of ***Theorem* 3.17**, one can get the following theorem.

***Theorem* 3.18.** *Let f, g, h and k be any four entire functions such that $\lambda_{\log k}^{[m]}(g) > 0$ and $\rho_{\log h}^{[p]}(f) < \infty$ where $p(> 1), m(> 1)$ are two integers. Also $\lambda_{\log}^{[q]}(g) < \delta < \infty$ where $q(> 2)$ be any two integers. Then for a sequence of values of r tending to infinity,*

$$\log^{[p-1]} M_h^{-1} M_{f\circ g}(r) < \log^{[m-1]} M_k^{-1} M_g \left(\exp^{[q-1]} (\log r)^{\delta}\right).$$

In view of ***Theorems* 3.17** and **3.18**, the following two theorems can be proved in similar way,

***Theorem* 3.19.** *Let f, g and h be any three entire functions such that $0 < \lambda_{\log h}^{[p]}(f) < \rho_{\log h}^{[p]}(f) < \infty$ and $\lambda_{\log}^{[q]}(g) < \delta < \infty$ where $p(> 1)$ and $q(> 2)$ are any two integers. Then for a sequence of values of r tending to infinity,*

$$\mu_h^{-1} \mu_{f\circ g}(r) < \mu_h^{-1} \mu_f \left(\exp^{[q-1]} (\log r)^{\delta}\right).$$

***Theorem* 3.20.** *Let f, g, h and k be any four entire functions such that $\lambda_{\log k}^{[m]}(g) > 0$ and $\rho_{\log h}^{[p]}(f) < \infty$ where $p(> 1), m(> 1)$ are two integers. Also $\lambda_{\log}^{[q]}(g) < \delta < \infty$ where $q(> 2)$ be any two integers. Then for a sequence of values of r tending to infinity,*

$$\log^{[p-1]} \mu_h^{-1} \mu_{f\circ g}(r) < \log^{[m-1]} \mu_k^{-1} \mu_g \left(\exp^{[q-1]} (\log r)^{\delta}\right).$$

We may state the following theorem in view of ***Theorems* 3.13** and **3.17**,

***Theorem* 3.21.** *Let f, g and h be any three entire functions such that $0 < \lambda_{\log h}^{[p]}(f) \le \rho_{\log h}^{[p]}(f) < \infty$ and $\lambda_{\log}^{[q]}(g) < \delta < \rho_{\log}^{[q]}(g)$ where $p(> 1)$ and $q(> 2)$ are any two integers. Then we get,*

$$\liminf_{r\to\infty} \frac{M_h^{-1} M_{f\circ g}(r)}{M_h^{-1} M_f \left(\exp^{[q-1]} (\log r)^{\delta}\right)} \le 1 \le \limsup_{r\to\infty} \frac{M_h^{-1} M_{f\circ g}(r)}{M_h^{-1} M_f \left(\exp^{[q-1]} (\log r)^{\delta}\right)}.$$

Proof. The proof is omitted.

Again in view of ***Theorems* 3.14** and **3.18**, we obtain the following theorem

***Theorem* 3.22.** *Let f, g, h and k be any four entire functions such that $0 < \lambda_{\log h}^{[p]}(f) \le \rho_{\log h}^{[p]}(f) < \infty$, $0 < \lambda_{\log k}^{[m]}(g) \le \rho_{\log k}^{[m]}(g) < \infty$ and $0 < \lambda_{\log}^{[m]}(g) < \delta < \rho_{\log}^{[m]}(g) < \infty$ where $p(> 1), q(> 2)$ and $m(> 1)$ are any three integers. Then we get,*

$$\liminf_{r\to\infty} \frac{\log^{[p-1]} M_h^{-1} M_{f\circ g}(r)}{\log^{[m-1]} M_k^{-1} M_g \left(\exp^{[q-1]} (\log r)^{\delta}\right)} \le 1 \le \limsup_{r\to\infty} \frac{\log^{[p-1]} M_h^{-1} M_{f\circ g}(r)}{\log^{[m-1]} M_k^{-1} M_g \left(\exp^{[q-1]} (\log r)^{\delta}\right)}.$$

Proof. The proof is omitted.

Similarly one can state the following two theorems without their proofs using the ***Theorems* 3.15, 3.19** and the ***Theorems* 3.16, 3.20** respectively.

***Theorem* 3.23.** *Let f, g and h be any three entire functions such that $0 < \lambda^{[p]}_{\log h}(f) \leq \rho^{[p]}_{\log h}(f) < \infty$ and $\lambda^{[q]}_{\log}(g) < \delta < \infty$ where $p(> 1)$ and $q(> 2)$ are any two integers. Then for a sequence of values of r tending to infinity,*

$$\liminf_{r\to\infty} \frac{\mu_h^{-1}\mu_{f\circ g}(r)}{\mu_h^{-1}\mu_f\left(\exp^{[q-1]}(\log r)^{\delta}\right)} \leq 1 \leq \limsup_{r\to\infty} \frac{\mu_h^{-1}\mu_{f\circ g}(r)}{\mu_h^{-1}\mu_f\left(\exp^{[q-1]}(\log r)^{\delta}\right)}.$$

***Theorem* 3.24.** *Let f, g, h and k be any four entire functions such that $0 < \lambda^{[p]}_{\log h}(f) \leq \rho^{[p]}_{\log h}(f) < \infty$, $0 < \lambda^{[m]}_{\log k}(g) \leq \rho^{[m]}_{\log k}(g) < \infty$ and $0 < \lambda^{[m]}_{\log}(g) < \delta < \rho^{[m]}_{\log}(g) < \infty$ where $p(> 1), q(> 2)$ and $m(> 1)$ are any three integers. Then we get,*

$$\liminf_{r\to\infty} \frac{\log^{[p-1]}\mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[m-1]}\mu_k^{-1}\mu_g\left(\exp^{[q-1]}(\log r)^{\delta}\right)} \leq 1 \leq \limsup_{r\to\infty} \frac{\log^{[p-1]}\mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[m-1]}\mu_k^{-1}\mu_g\left(\exp^{[q-1]}(\log r)^{\delta}\right)}.$$

***Theorem* 3.25.** *Let f, g, h, k, l and b be any six entire functions such that $\lambda^{[m]}_{\log b}(l) > 0, \rho^{[p]}_{\log h}(f) < \infty$ and $\rho^{[q]}_{\log}(g) < \lambda^{[n]}_{\log}(k)$ where p, q, m, n are all positive integers with $p \geq 1$, $m \geq 1$ and $n \geq q \geq 2$. Then*

$$\textit{(i)}\ \lim_{r\to\infty} \frac{M_b^{-1}M_{l\circ k}(r)}{M_h^{-1}M_{f\circ g}(r)} = \infty \textit{ if } p = m$$

$$\textit{(ii)}\ \lim_{r\to\infty} \frac{M_b^{-1}M_{l\circ k}(r)}{\log^{[p-m]} M_h^{-1}M_{f\circ g}(r)} = \infty \textit{ if } p > m$$

and

$$\textit{(iii)}\ \lim_{r\to\infty} \frac{\log^{[m-p]} M_b^{-1}M_{l\circ k}(r)}{M_h^{-1}M_{f\circ g}(r)} = \infty \textit{ if } p < m.$$

Proof. Since $M_b^{-1}(r)$ is an increasing function of r, then for a sequence of values of r tending to infinity we have from the first part of ***Lemma* 2.1**,

$$\log^{[m]} M_b^{-1}M_{l\circ k}(r) \geq \left(\lambda^{[m]}_{\log b}(f) - \varepsilon\right)\exp^{[n-3]}\left(\log\frac{r}{2}\right)^{\lambda^{[n]}_{\log}(k)-\varepsilon} + O(1). \tag{3.21}$$

Since $\rho^{[q]}_{\log}(g) < \lambda^{[n]}_{\log}(k)$, choose $\varepsilon(> 0)$ such that

$$\rho^{[q]}_{\log}(g) + \varepsilon < \lambda^{[n]}_{\log}(k) - \varepsilon. \tag{3.22}$$

Case I: Suppose $p = m$. Now combining (3.4) and (3.21) and in view of (3.22) we get for all sufficiently large values of r that,

$$\frac{M_b^{-1}M_{l\circ k}(r)}{M_h^{-1}M_{f\circ g}(r)} \geq \frac{\exp^{[m]}\left[\left(\lambda^{[m]}_{\log b}(f) - \varepsilon\right)\exp^{[n-3]}\left(\log\frac{r}{2}\right)^{\lambda^{[n]}_{\log}(k)-\varepsilon} + O(1)\right]}{\exp^{[m]}\left[\left(\rho^{[m]}_{\log h}(f) + \varepsilon\right)\exp^{[q-3]}(\log r)^{\rho^{[q]}_{\log}(g)+\varepsilon}\right]}$$

then

$$\lim_{r\to\infty} \frac{M_b^{-1}M_{l\circ k}(r)}{M_h^{-1}M_{f\circ g}(r)} = \infty.$$

(i) is proved.

Case II: Suppose $p > m$.Now combining (3.4) and (3.21) and in view of (3.22) we get for all sufficiently large values of r that,

$$\frac{M_b^{-1} M_{l\circ k}(r)}{\log^{[p-m]} M_h^{-1} M_{f\circ g}(r)} \geq \frac{\exp^{[m]}\left[\left(\lambda_{\log b}^{[m]}(f) - \varepsilon\right)\exp^{[n-3]}\left(\log \frac{r}{2}\right)^{\lambda_{\log}^{[n]}(k)-\varepsilon} + O(1)\right]}{\exp^{[m]}\left[\left(\rho_{\log h}^{[p]}(f) + \varepsilon\right)\exp^{[q-3]}(\log r)^{\rho_{\log}^{[q]}(g)+\varepsilon}\right]}$$

then

$$\lim_{r\to\infty} \frac{M_b^{-1} M_{l\circ k}(r)}{\log^{[p-m]} M_h^{-1} M_{f\circ g}(r)} = \infty.$$

(ii) is proved.

Case III: Suppose $p < m$. Similarly combining (3.4) and (3.21) and in view of (3.22) we get for all sufficiently large values of r that,

$$\frac{\log^{[m-p]} M_b^{-1} M_{l\circ k}(r)}{M_h^{-1} M_{f\circ g}(r)} \geq \frac{\exp^{[p]}\left[\left(\lambda_{\log b}^{[m]}(f) - \varepsilon\right)\exp^{[n-3]}\left(\log \frac{r}{2}\right)^{\lambda_{\log}^{[n]}(k)-\varepsilon} + O(1)\right]}{\exp^{[p]}\left[\left(\rho_{\log h}^{[p]}(f) + \varepsilon\right)\exp^{[q-3]}(\log r)^{\rho_{\log}^{[q]}(g)+\varepsilon}\right]}$$

then

$$\lim_{r\to\infty} \frac{\log^{[m-p]} M_b^{-1} M_{l\circ k}(r)}{M_h^{-1} M_{f\circ g}(r)} = \infty.$$

(iii) is proved.

***Theorem* 3.26.** *Let f, g, h, k, l and b be any six entire functions such that $\lambda_{\log b}^{[m]}(l) > 0$, $\rho_{\log h}^{[p]}(f) < \infty$ and $\rho_{\log}^{[q]}(g) < \lambda_{\log}^{[n]}(k)$ where p, q, m, n are all positive integers with $p \geq 1$, $m \geq 1$ and $n \geq q \geq 2$. Then*

$$(i)\ \lim_{r\to\infty} \frac{\mu_b^{-1}\mu_{l\circ k}(r)}{\mu_h^{-1}\mu_{f\circ g}(r)} = \infty \text{ if } p = m$$

$$(ii)\ \lim_{r\to\infty} \frac{\mu_b^{-1}\mu_{l\circ k}(r)}{\log^{[p-m]} \mu_h^{-1}\mu_{f\circ g}(r)} = \infty \text{ if } p > m$$

and

$$(iii)\ \lim_{r\to\infty} \frac{\log^{[m-p]} \mu_b^{-1}\mu_{l\circ k}(r)}{\mu_h^{-1}\mu_{f\circ g}(r)} = \infty \text{ if } p < m.$$

Proof. We can prove the theorem similarly as ***Theorem* 3.26** and with the help of ***Lemmas* 2.2,2.3** and **2.5**.

***Remark* 3.7.** *If we replace "limit" by "limit superior" and $\rho_{\log}^{[q]}(g) < \lambda_{\log}^{[n]}(k)$ by $\rho_{\log}^{[q]}(g) < \rho_{\log}^{[n]}(k)$ then **Theorems** 3.25 and 3.26 also hold.*

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