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(Dedicated to Honor Professor H.M. Srivastava on His 80th Birth Anniversary Celebrations)

ANALYTICAL SOLUTIONS FOR TIME-FRACTIONAL CAUCHY REACTION-DIFFUSION EQUATIONS USING ITERATIVE LAPLACE TRANSFORM METHOD

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Abstract

In the present work, the iterative Laplace transform method *(ILTM)* is implemented to derive approximate analytical solutions for the time-fractional Cauchy reaction-diffusion equations *(CRDEs)* within the Caputo fractional derivative. The proposed technique is an elegant amalgam of the Iterative method and the Laplace transform method. The *ILTM* produces the solution in a rapid convergent series which may lead to the solution in a closed form. The obtained analytical outcomes with the help of the proposed technique are examined graphically.

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1 Introduction

Fractional calculus is a branch of mathematical analysis which is concerned with derivatives and integrals of arbitrary orders.It has attracted the great attention of scientists and engineers from a long time ago and has resulted in many applications being created. Since the 1990's, fractional calculus has been rediscovered and adapted in a growing number of fields such as biology, mathematical physics, electrochemistry, signal processing, chemical physics, electromagnetics, acoustics, viscoelasticity, material science, probability and statistics, engineering, physics fluid dynamics and other areas of sciences and technology. In recent years, many researchers have paid attention to investigating solutions of fractional differential equations using numerous techniques such as the Laplace decomposition method *(LDM)* [8], the Homotopy Analysis method *(HAM)* [12], the Homotopy perturbation method *(HPM)* [18], the Homotopy perturbation transform method *(HPTM)* [10, 11], Sumudu transform iterative method *(STIM)* [17], q-homotopy analysis transform method *(q-HATM)* [16] etc.

In 2006, Daftardar-Gejji and Jafari introduced the iterative technique for numerically examining nonlinear functional equations [6, 7]. Since then, iterative technique has been used to find a solution for several non-linear differential equations of arbitrary orders [3] and the viewing of fractional Boundary Value problems [5]. Jafari et al. [9] used Laplace transform together with iterative method, became a well-known technique called iterative Laplace transform method for solving a system of partial differential equations of fractional order. Recently, fractional heat and wave like equations [15], fractional Navier-Stokes equations [2] and fractional Zakharov-Kuznetsov Equations [1] are solved successfully using the iterative Laplace transform method.

In the present study, the following time-fractional Cauchy reaction-diffusion equation in operator form is considered as[11]

$$D_t^{\alpha} w(\xi,t) = \upsilon \frac{\partial^2 w(\xi,t)}{\partial \xi^2} + p(\xi,t)w(\xi,t), \xi \in \mathbb{R}, t > 0, 0 < \alpha \leq 1, \tag{1.1}$$

with initial condition $w(\xi,0) = w_0(\xi)$, where $\upsilon > 0$ is diffusion coefficient, $w(\xi,t)$ and $p(\xi,t)$ denote the concentration and the reaction parameter, respectively. In particular for $\alpha = 1$, time-fractional Cauchy reaction-diffusion equation reduces to classical Cauchy reaction-diffusion equation.
The main objective of the present paper is to extend the work of the *ILTM* technique to investigate approximate analytical solutions for the time-fractional Cauchy reaction-diffusion equations and to conclude accuracy, efficiency, simplicity of the proposed technique.

2 Preliminaries and Basic Definitions

In this section, we give certain basic definitions, notations and properties of fractional calculus with Laplace transform theory, which are used further in this paper.

***Definition* 2.1.** *The fractional derivative of a function in the sense of the Caputo is presented as* [4]

$$\begin{aligned} D_t^{\alpha} w(\xi,t) &= \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\rho)^{m-\alpha-1} w^{(m)}(\xi,\rho)d\rho, \ m-1 < \alpha \leq m, m \in \mathbb{N}, \\ &= J_t^{m-\alpha} D^m w(\xi,t). \end{aligned} \tag{2.1}$$

Here $D^m \equiv \dfrac{d^m}{dt^m}$ *and* J_t^{α} *stands for the Riemann-Liouville fractional integral operator of order* $\alpha > 0$*, defined as* [13]

$$J_t^{\alpha} w(\xi,t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\rho)^{\alpha-1} w(\xi,\rho)d\rho, \ \rho > 0, \tag{2.2}$$

where $\Gamma(.)$ *is the well-known Gamma function.*

***Definition* 2.2.** *The Laplace transform of a function* $f(\xi)$, $\xi > 0$ *is expressed as* [13, 14]

$$L[f(\xi)] = F(s) = \int_0^{\infty} e^{-s\xi} f(\xi)d\xi, \tag{2.3}$$

where s is real or complex number.

***Definition* 2.3.** *The Laplace transform of Caputo fractional derivative is presented in following manner* [13, 14]

$$L[D_t^{\alpha} w(\xi,t)] = s^{\alpha} L[w(\xi,t)] - \sum_{k=0}^{m-1} w^{(k)}(\xi,0)s^{\alpha-k-1}, m-1 < \alpha \leq m, m \in \mathbb{N}, \tag{2.4}$$

where $w^{(k)}(\xi,0)$ *is the k-order derivative of* $w(\xi,t)$ *with respect to t at* $t = 0$.

***Definition* 2.4.** *The Mittag-Leffler function* $E_{\alpha}(z)$ *is defined by the following series representation as* [13]

$$E_{\alpha}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \alpha \in \mathbb{C}, Re(\alpha) > 0. \tag{2.5}$$

3 Basic Idea of Iterative Laplace Transform Method

In order to elucidate the solution procedure of this method [9], we take the subsequent fractional partial differential equation having the prescribed initial conditions can be expressed in the form of an operator as

$$D_t^{\alpha} w(\xi, t) + Rw(\xi, t) + Nw(\xi, t) = g(\xi, t), \; m-1 < \alpha \le m, m \in \mathbb{N}, \tag{3.1}$$

$$w^{(k)}(\xi, 0) = h_k(\xi), \; k = 0, 1, 2, ..., m-1, \tag{3.2}$$

where $D_t^{\alpha} w(\xi, t)$ is the Caputo fractional derivative of order $\alpha, m-1 < \alpha \le m$, defined by equation (2.1), R is a linear operator and may include other fractional derivatives of order less than α, N is a non-linear operator which may include other fractional derivatives of order less than α and $g(\xi, t)$ is a known analytic function.

Applying the Laplace transform on both sides of equation (3.1), we have

$$L[D_t^{\alpha} w(\xi, t)] + L[Rw(\xi, t) + Nw(\xi, t)] = L[g(\xi, t)]. \tag{3.3}$$

Using equation (2.4), we obtain

$$L[w(\xi, t)] = \frac{1}{s^{\alpha}} \sum_{k=0}^{m-1} s^{\alpha-1-k} w^k(\xi, 0) + \frac{1}{s^{\alpha}} L[g(\xi, t)] - \frac{1}{s^{\alpha}} L[Rw(\xi, t) + Nw(\xi, t)]. \tag{3.4}$$

On taking inverse Laplace transform on equation (3.4), we have

$$w(\xi, t) = L^{-1}\Big[\frac{1}{s^{\alpha}}\Big(\sum_{k=0}^{m-1} s^{\alpha-1-k} w^k(\xi, 0) + L[g(\xi, t)]\Big)\Big] - L^{-1}\Big[\frac{1}{s^{\alpha}} L[Rw(\xi, t) + Nw(\xi, t)]\Big]. \tag{3.5}$$

Further, we apply the iterative method introduced by Daftardar-Gejji and Jafari [6], which represents a solution $w(\xi, t)$ in infinite series of components

$$w(\xi, t) = \sum_{i=0}^{\infty} w_i(\xi, t). \tag{3.6}$$

As R is a linear operator, so we have

$$R\Big(\sum_{i=0}^{\infty} w_i(\xi, t)\Big) = \sum_{i=0}^{\infty} R[w_i(\xi, t)], \tag{3.7}$$

and the non-linear operator N is decomposed as

$$N\Big(\sum_{i=0}^{\infty} w_i(\xi, t)\Big) = N[w_0(\xi, t)] + \sum_{i=0}^{\infty} \Big[N\Big(\sum_{k=0}^{i} w_k(\xi, t)\Big) - N\Big(\sum_{k=0}^{i-1} w_k(\xi, t)\Big)\Big], \tag{3.8}$$

Substituting the results given by equations from (3.6) to (3.8) in the equation (3.5), we get

$$\begin{aligned} \sum_{i=0}^{\infty} w_i(\xi, t) &= L^{-1}\Big[\frac{1}{s^{\alpha}}\Big(\sum_{k=0}^{m-1} s^{\alpha-1-k} w^k(\xi, 0) + L[g(\xi, t)]\Big)\Big] \\ &- L^{-1}\Big[\frac{1}{s^{\alpha}} L\Big[\sum_{i=0}^{\infty} R[w_i(\xi, t)] + N[w_0(\xi, t)] + \sum_{i=1}^{\infty}\Big(N\Big(\sum_{k=0}^{i} w_k(\xi, t)\Big) - N\Big(\sum_{k=0}^{i-1} w_k(\xi, t)\Big)\Big)\Big]\Big]. \end{aligned} \tag{3.9}$$

We have defined the recurrence relations as

$$\begin{cases} w_0(\xi, t) &= L^{-1}\Big[\frac{1}{s^{\alpha}}\Big(\sum_{k=0}^{m-1} s^{\alpha-1-k} w^k(\xi, 0) + L[g(\xi, t)]\Big)\Big] \\ w_1(\xi, t) &= -L^{-1}\Big[\frac{1}{s^{\alpha}} L\Big[\sum_{i=0}^{\infty} R[w_i(\xi, t)] + N[w_0(\xi, t)]\Big]\Big] \\ w_{m+1}(\xi, t) &= L^{-1}\Big[\frac{1}{s^{\alpha}} L\Big[R[w_m(\xi, t)] - \Big(N\Big(\sum_{k=0}^{m} w_k(\xi, t)\Big) \\ & \quad -N\Big(\sum_{k=0}^{m-1} w_k(\xi, t)\Big)\Big)\Big]\Big], m \ge 1 \end{cases} \tag{3.10}$$

Proceeding in the same manner the rest of components of the *ILTM* solution can be obtained. Finally, we approximate the analytical solution $w(\xi, t)$ in truncated series form is given by

$$w(\xi, t) \cong \lim_{N\to\infty} \sum_{m=0}^{N} w_m(\xi, t). \tag{3.11}$$

The above series solutions generally converge very rapidly. A classical approach of convergence of this type of series is already presented by Jafari [7] and Daftardar-Gejji and Jafari [6]

4 Solutions of the time-fractional Cauchy Reaction-Diffusion equations

In this section, we apply the *ILTM* technique for solving time-fractional Cauchy reaction-diffusion equations with initial conditions.

***Example* 4.1.** In this example, the following linear time-fractional Cauchy reaction-diffusion equation is considered as [11]

$$D_t^{\alpha} w(\xi, t) = \frac{\partial^2 w(\xi, t)}{\partial x^2} - w(\xi, t), 0 < \alpha \leq 1, \tag{4.1}$$

subject to the initial condition

$$w(\xi, 0) = e^{-\xi} + \xi. \tag{4.2}$$

Taking the Laplace transform on the both sides of equation (4.1), and making use of the result given by equation (4.2), we have

$$L[w(\xi, t)] = \frac{(e^{-\xi} + \xi)}{s} + \frac{1}{s^{\alpha}} L\Big[\frac{\partial^2 w}{\partial \xi^2} - w\Big]. \tag{4.3}$$

Operating with inverse Laplace transform on both sides of equation (4.3) gives

$$w(\xi, t) = e^{-\xi} + \xi + L^{-1}\Big[\frac{1}{s^{\alpha}} L\Big[\frac{\partial^2 w}{\partial \xi^2} - w\Big]\Big]. \tag{4.4}$$

Now, applying the iterative method, substituting the equations (3.6) to (3.8) into equation (4.4) and applying equation (3.10), we determine the components of the *ILTM* solution as follows

$$w_0(\xi, t) = e^{-\xi} + \xi, \tag{4.5}$$

$$w_1(\xi, t) = L^{-1}\Big[\frac{1}{s^{\alpha}} L\Big[\frac{\partial^2 w_0}{\partial \xi^2} - w_0\Big]\Big] = \xi \frac{(-t^{\alpha})}{\Gamma(\alpha + 1)}, \tag{4.6}$$

$$w_2(\xi, t) = L^{-1}\Big[\frac{1}{s^{\alpha}} L\Big[\frac{\partial^2 w_1}{\partial \xi^2} - w_1\Big]\Big] = \xi \frac{(-t^{\alpha})^2}{\Gamma(2\alpha + 1)}, \tag{4.7}$$

$$w_3(\xi, t) = L^{-1}\Big[\frac{1}{s^{\alpha}} L\Big[\frac{\partial^2 w_2}{\partial \xi^2} - w_2\Big]\Big] = \xi \frac{(-t^{\alpha})^3}{\Gamma(3\alpha + 1)}. \tag{4.8}$$

Proceeding in the same manner the rest of components $w_m(\xi, t)$ for $m \geq 4$ can be obtained. Thus, the approximate analytical solution in the series form can be obtained as

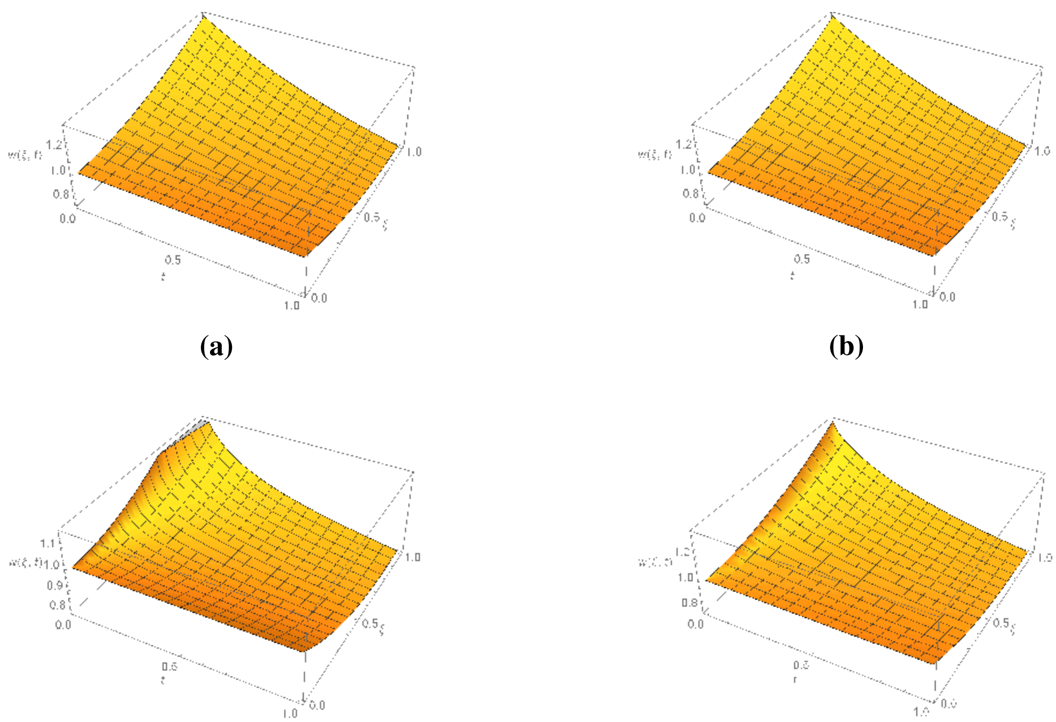
$$\begin{aligned} w(\xi, t) \cong \lim_{N\to\infty} \sum_{m=0}^{N} w_m(\xi, t) &= e^{-\xi} + \xi + \xi \frac{(-t^{\alpha})}{\Gamma(\alpha + 1)} + \xi \frac{(-t^{\alpha})^2}{\Gamma(2\alpha + 1)} + \xi \frac{(-t^{\alpha})^3}{\Gamma(3\alpha + 1)} + ..., \\ &= e^{-\xi} + \xi E_{\alpha}(-t)^{\alpha}. \end{aligned} \tag{4.9}$$

which is the same result was obtained by Kumar [11] using *HPTM*.

***Remark* 4.1.** For $\alpha = 1$, the result in equation (4.9) reduces to the following exact form

$$w(\xi, t) = e^{-\xi} + \xi e^{-t}. \tag{4.10}$$

This result was achieved earlier by Yildirim [18] using the *HPM* method.



(a) (b) (c) (d)

***Figure* 4.1:** The surface shows the solution $w(\xi, t)$ for ***Example* 4.1.**, when (a) The exact solution, (b) The approximate solution for $\alpha = 1$, (c) The approximate solution for $\alpha = 0.5$, (d) The approximate solution for $\alpha = 0.75$.

***Example* 4.2.** Consider the following time fractional linear Cauchy reaction-diffusion equation [11]

$$D_t^{\alpha} w(\xi, t) = \frac{\partial^2 w(\xi, t)}{\partial \xi^2} - (1 + 4\xi^2) w(\xi, t), 0 < \alpha \leq 1, \tag{4.11}$$

with the initial condition

$$w(\xi, 0) = e^{\xi^2}. \tag{4.12}$$

Taking the Laplace transform of the equation (4.11), and making use of the result given by equation (4.12), we have

$$L[w(\xi, t)] = \frac{e^{\xi^2}}{s} + \frac{1}{s} L\Big[\frac{\partial^2 w}{\partial \xi^2} - (1 + 4\xi^2) w\Big]. \tag{4.13}$$

Applying inverse Laplace transform to the equation (4.13), we obtain

$$w(\xi, t) = e^{\xi^2} + L^{-1}\Big[\frac{1}{s^{\alpha}} L\Big[\frac{\partial^2 w}{\partial \xi^2} - (1 + 4\xi^2) w\Big]\Big]. \tag{4.14}$$

Now,applying the iterative method, substituting the equations (3.6) to (3.8) into equation (4.14) and applying equation (3.10), we determine the components of the *ILTM* solution as follows

$$w_0(\xi, t) = e^{\xi^2}, \tag{4.15}$$

$$w_1(\xi,t) = L^{-1}\Big[\frac{1}{s^\alpha}L\Big[\frac{\partial^2 w_0}{\partial \xi^2} - (1+4\xi^2)w_0\Big]\Big] = e^{\xi^2}\frac{t^\alpha}{\Gamma(\alpha+1)}, \tag{4.16}$$

$$w_2(\xi,t) = L^{-1}\Big[\frac{1}{s^\alpha}L\Big[\frac{\partial^2 w_1}{\partial \xi^2} - (1+4\xi^2)w_1\Big]\Big] = e^{\xi^2}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}, \tag{4.17}$$

$$w_3(\xi,t) = L^{-1}\Big[\frac{1}{s^\alpha}L\Big[\frac{\partial^2 w_2}{\partial \xi^2} - (1+4\xi^2)w_2\Big]\Big] = e^{\xi^2}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}. \tag{4.18}$$

Proceeding in the same manner the rest of components $w_m(\xi,t)$ for $m \geq 4$ can be obtained. Thus the approximate analytical solution in the series form can be obtained as

$$w(\xi,t) \cong \lim_{N\to\infty}\sum_{m=0}^{N} w_m(\xi,t) = e^{\xi^2} + e^{\xi^2}\frac{t^\alpha}{\Gamma(\alpha+1)} + e^{\xi^2}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + e^{\xi^2}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + ... \tag{4.19}$$
$$= e^{\xi^2}E_\alpha(t^\alpha).$$

which is the same result was obtained by Kumar [11] using *HPTM*.

***Remark* 4.2.** For $\alpha = 1$, the result in equation (4.19) reduces to the following exact form

$$w(\xi,t) = e^{\xi^2+t}. \tag{4.20}$$

This result was achieved earlier by Yildirim [18] using the *HPM* method.

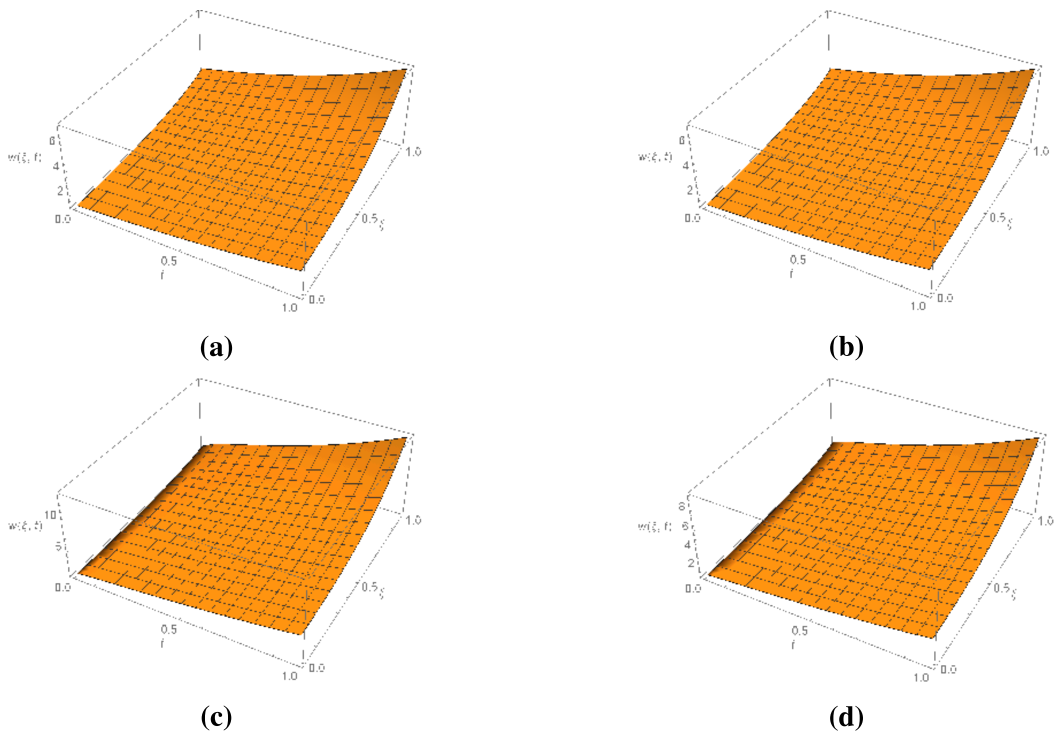


***Figure* 4.2:** The surface shows the solution $w(\xi,t)$ for ***Example* 4.2**, when (a) The exact solution, (b) The approximate solution for $\alpha = 1$, (c) The approximate solution for $\alpha = 0.5$, (d) The approximate solution for $\alpha = 0.75$.

***Example* 4.3.** Consider the following linear Cauchy reaction-diffusion equation involving time-fractional derivative as [11]

$$D_t^\alpha w(\xi,t) = \frac{\partial^2 w(\xi,t)}{\partial \xi^2} - (4\xi^2 - 2t + 2)w(\xi,t), 0 < \alpha \leq 1, \tag{4.21}$$

subject to initial condition

$$w(\xi,0) = e^{\xi^2}. \tag{4.22}$$

Taking the Laplace transform of the equation (4.21), and making use of the result given by equation (4.22), we have

$$L[w(\xi,t)] = \frac{e^{\xi^2}}{s} + \frac{1}{s^\alpha}L\Big[\frac{\partial^2 w}{\partial \xi^2} - (4\xi^2 - 2t + 2)w\Big]. \tag{4.23}$$

Applying inverse Laplace transform to the equation (4.23), we obtain

$$w(\xi,t) = e^{\xi^2} + L^{-1}\Big[\frac{1}{s^\alpha}L\Big[\frac{\partial^2 w}{\partial \xi^2} - (4\xi^2 - 2t + 2)w\Big]\Big]. \tag{4.24}$$

Now, applying the the iterative method, substituting the equations (3.6) to (3.8) into equation (4.24) and applying equation (3.10), we determine the components of the *ILTM* solution as follows

$$w_0(\xi,t) = e^{\xi^2}, \tag{4.25}$$

$$w_1(\xi,t) = L^{-1}\Big[\frac{1}{s^\alpha}L\Big[\frac{\partial^2 w_0}{\partial \xi^2} - (4\xi^2 - 2t + 2)w_0\Big]\Big] = 2e^{\xi^2}\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}, \tag{4.26}$$

$$w_2(\xi,t) = L^{-1}\Big[\frac{1}{s^\alpha}L\Big[\frac{\partial^2 w_1}{\partial \xi^2} - (4\xi^2 - 2t + 2)w_1\Big]\Big] = 4e^{\xi^2}\frac{(\alpha+2)t^{2\alpha+2}}{\Gamma(2\alpha+3)}, \tag{4.27}$$

$$w_3(\xi,t) = L^{-1}\Big[\frac{1}{s^\alpha}L\Big[\frac{\partial^2 w_2}{\partial \xi^2} - (4\xi^2 - 2t + 2)w_2\Big]\Big] = 8e^{\xi^2}\frac{(\alpha+2)(2\alpha+3)t^{3\alpha+3}}{\Gamma(3\alpha+4)}, \tag{4.28}$$

$$w_4(\xi,t) = L^{-1}\Big[\frac{1}{s^\alpha}L\Big[\frac{\partial^2 w_2}{\partial \xi^2} - (4\xi^2 - 2t + 2)w_2\Big]\Big] = 16e^{\xi^2}\frac{(\alpha+2)(2\alpha+3)(3\alpha+4)t^{4\alpha+4}}{\Gamma(4\alpha+5)}. \tag{4.29}$$

Proceeding in the same manner the rest of components $w_m(\xi,t)$ for $m \geq 5$ can be obtained. Thus, the approximate analytical solution in the series form can be obtained as

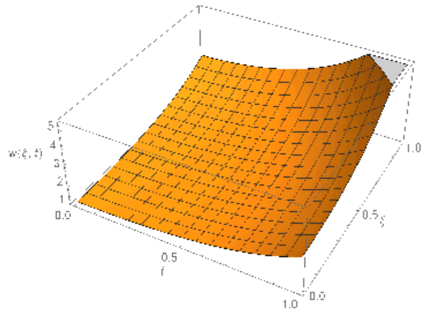
$$\begin{aligned} w(\xi,t) \cong \lim_{N\to\infty}\sum_{m=0}^{N} w_m(\xi,t) &= e^{\xi^2} + 2e^{\xi^2}\frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + 4e^{\xi^2}\frac{(\alpha+2)t^{2\alpha+2}}{\Gamma(2\alpha+3)} \\ &+ 8e^{\xi^2}\frac{(\alpha+2)(2\alpha+3)t^{3\alpha+3}}{\Gamma(3\alpha+4)} + 16e^{\xi^2}\frac{(\alpha+2)(2\alpha+3)(3\alpha+4)t^{4\alpha+4}}{\Gamma(4\alpha+5)} + ... \end{aligned} \tag{4.30}$$

which is the same result was obtained by Kumar [11] using *HPTM*.

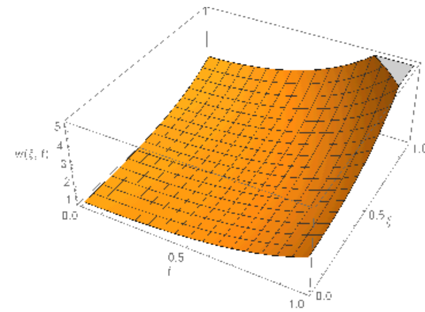
***Remark* 4.3.** For $\alpha = 1$, the result in equation (4.30) reduces to the following exact form

$$w(\xi,t) = e^{\xi^2+t^2}. \tag{4.31}$$

This result was obtained earlier by Yildirim [18] by using the method of *HPM*.



(a)



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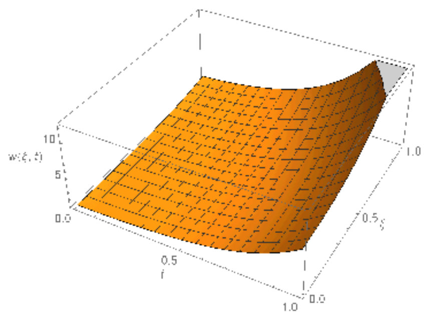

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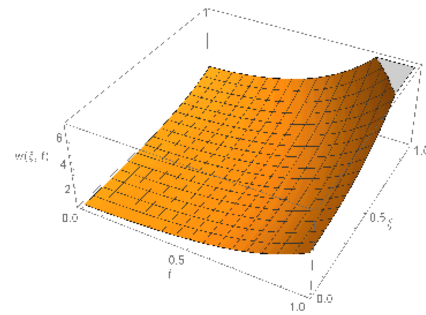

(d)

***Figure* 4.3:** The surface shows the solution $w(\xi, t)$ for ***Example* 4.3**, when (a) The exact solution, (b) The approximate solution for $\alpha = 1$, (c) The approximate solution for $\alpha = 0.5$, (d) The approximate solution for $\alpha = 0.75$.

***Example* 4.4.** Finally, we consider the following time-fractional linear Cauchy reaction-diffusion equation as [11]

$$D_t^\alpha w(\xi, t) = \frac{\partial^2 w(\xi, t)}{\partial \xi^2} + 2tw(\xi, t), 0 < \alpha \leq 1, \tag{4.32}$$

with the given initial condition

$$w(\xi, 0) = e^\xi. \tag{4.33}$$

Taking the Laplace transform of the equation (4.32), and making use of the result given by equation (4.33), we have

$$L[w(\xi, t)] = \frac{e^\xi}{s} + \frac{1}{s^\alpha} L\Big[\frac{\partial^2 w}{\partial \xi^2} + 2tw\Big]. \tag{4.34}$$

Applying inverse Laplace transform to the equation (4.34), we obtain

$$w(\xi, t) = e^\xi + L^{-1}\Big[\frac{1}{s^\alpha} L\Big[\frac{\partial^2 w}{\partial \xi^2} + 2tw\Big]\Big]. \tag{4.35}$$

Now, applying the iterative method, substituting the equations (3.6) to (3.8) into equation (4.35) and applying equation (3.10), we determine the components of the *ILTM* solution as follows

$$w_0(\xi, t) = e^\xi, \tag{4.36}$$

$$w_1(\xi, t) = L^{-1}\Big[\frac{1}{s^\alpha} L\Big[\frac{\partial^2 w_0}{\partial \xi^2} + 2tw_0\Big]\Big] = e^\xi\Big(\frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{2t^{\alpha+1}}{\Gamma(\alpha+2)}\Big), \tag{4.37}$$

$$w_2(\xi, t) = L^{-1}\Big[\frac{1}{s^\alpha} L\Big[\frac{\partial^2 w_1}{\partial \xi^2} + 2tw_1\Big]\Big] = e^\xi\Big(\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{2(\alpha+2)t^{2\alpha+1}}{\Gamma(2\alpha+2)} + \frac{4(\alpha+2)t^{2\alpha+2}}{\Gamma(2\alpha+3)}\Big), \tag{4.38}$$

$$(4.39)\qquad w_3(\xi,t) = L^{-1}\Big[\frac{1}{s^\alpha}L\Big[\frac{\partial^2 w_2}{\partial \xi^2} + 2tw_2\Big]\Big] = e^{\xi}\Big(\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \frac{6(\alpha+1)t^{3\alpha+1}}{\Gamma(3\alpha+2)} + \frac{4(\alpha+2)(2\alpha+3)t^{3\alpha+2}}{\Gamma(3\alpha+3)} + \frac{8(\alpha+2)(2\alpha+3)t^{3\alpha+3}}{\Gamma(3\alpha+4)}\Big).$$

Proceeding in the same manner the rest of components $w_m(\xi,t)$ for $m \geq 4$ can be obtained. Thus,the approximate analytical solution in the series form can be obtained as

$$(4.40)\qquad w(\xi,t) \cong \lim_{N\to\infty}\sum_{m=0}^{N} w_m(\xi,t) = e^{\xi} + e^{\xi}\Big(\frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{2t^{\alpha+1}}{\Gamma(\alpha+2)}\Big) + e^{\xi}\Big(\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{2(\alpha+2)t^{2\alpha+1}}{\Gamma(2\alpha+2)} + \frac{4(\alpha+2)t^{2\alpha+2}}{\Gamma(2\alpha+3)}\Big) + e^{\xi}\Big(\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \frac{6(\alpha+1)t^{3\alpha+1}}{\Gamma(3\alpha+2)} + \frac{4(\alpha+2)(2\alpha+3)t^{3\alpha+2}}{\Gamma(3\alpha+3)} + \frac{8(\alpha+2)(2\alpha+3)t^{3\alpha+3}}{\Gamma(3\alpha+4)}\Big) + \ldots$$

which is the same result was obtained by Kumar [11] using *HPTM*.

***Remark* 4.4.** For $\alpha = 1$, the result in equation (4.40) reduces to the following exact form

$$(4.41)\qquad w(\xi,t) = e^{\xi+t+t^2}.$$

This result was obtained earlier by Yildirim [18] by using the method of *HPM*.

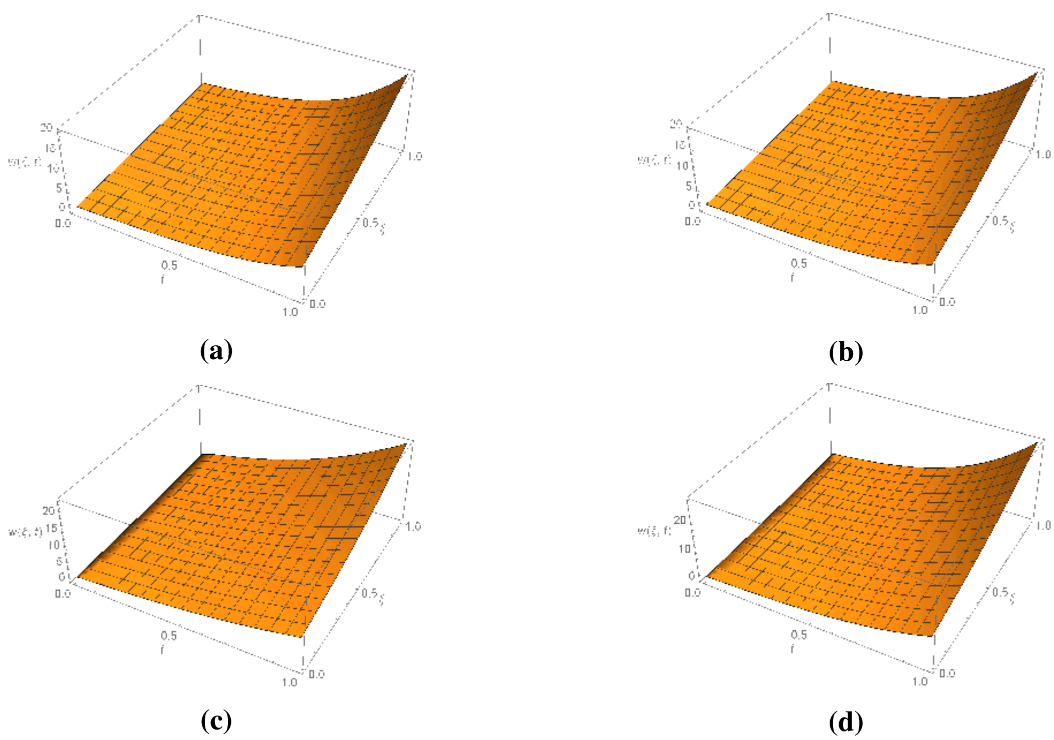

***Figure* 4.4:** The surface shows the solution $w(\xi,t)$ for ***Example* 4.4**, when (a) The exact solution, (b) The approximate solution for $\alpha = 1$, (c) The approximate solution for $\alpha = 0.5$, (d) The approximate solution for $\alpha = 0.75$.

5 Conclusion

In this study, the approximate analytical solutions for the time-fractional Cauchy reaction-diffusion equations were determined by using an effective and straight procedure of the iterative Laplace transform method *(ILTM)*. The fractional derivatives were described in the Caputo Sense. The graphical representation of the obtained solutions has been done successfully. The present method has proved to be an effective and straightforward procedure as compared with other analytical and numerical techniques to find approximate analytical solutions of the fractional partial differential equations and it can be utilized to investigate analytical solutions of more problems of the partial differential equations of fractional order.

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