Jñānābha, Vol. 50(1) (2020), 14-19
(Dedicated to Honor Professor H.M. Srivastava on His 80th Birth Anniversary Celebrations)

ENERGY OF SOME GRAPHS OF PRIME GRAPH OF A RING

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(*Received : September* 10, 2019 *; Revised: March* 04, 2020)
DOI: https://doi.org/10.58250/jnanabha.2020.50101

Abstract

Let R be a commutative ring and $PG(R)$ is a graph whose vertices are all the elements of ring R and two vertices are adjacent if their product is zero. In this article, we study the energy of 1-Quasitotal and 2-Quasitotal Prime Graph of a Ring $\mathbb{Z}_p$ and also find the energy of $PG_1(\mathbb{Z}_p)$ and $PG_2(\mathbb{Z}_p)$, p prime. A General SCILAB Software code for our calculation is also presented.

2010 Mathematics Subject Classifications: 05C25, 05C15, 13E15.
Keywords and phrases: Ring, Prime graph of a Ring $PG(R)$, Quasi-total graph, Energy.

1 Introduction

The study of graph theory for a commutative ring began when Beck in [3] introduced the notion of zero divisor of the graph. The graphs $\Gamma_1(R)$ and $\Gamma_2(R)$ are defined by R. Sen Gupta et al. in [4]. Another graph structure associated to a ring called prime graph was introduced by Satyanarayana et al. [2]. Prime graph is defined as a graph whose vertices are all elements of the ring and any two distinct vertices $x, y \in R$ are adjacent if and only if $xRy = 0$ or $yRx = 0$. This graph is denoted by $PG(R)$. Pawar and Joshi in [10] gave a simple formulation for finding the degrees of vertices of prime graph $PG(R)$ as well as it's complement $(PG(R))^c$. Also the number of triangles in $PG(R)$ and $(PG(R))^c$ have been calculated using simple combinatorial approach. We have introduced the prime graphs $PG_1(R)$ in [9] and $PG_2(R)$ in [8] of a ring and discussed all the results related to degree of vertices, Eulerianity, planarity and girth.

In third section of this paper we give definition and some examples of 1-Quasitotal and 2-Quasitotal Prime Graph of a Ring $\mathbb{Z}_n$. In last four sections we find the energy of 1-Quasitotal and 2-Quasitotal Prime Graph of a Ring $\mathbb{Z}_p$ and also find the energy of $PG_1(\mathbb{Z}_p)$ and $PG_2(\mathbb{Z}_p)$, where p is prime and give a general SCILAB software code for finding the energy of any Graph.

2 Preliminary Definitions

Here we are listing some preliminary definitions. For basic terminology and definitions the reader is referred to [2], [5].

***Definition* 2.1.** [4] *For a ring R, a simple undirected graph $G = (V, E)$ is said to be a graph $\Gamma_1(R)$ if all the nonzero elements of R as vertices, and two distinct vertices a and b are adjacent if and only if either $a \cdot b = 0$ or $b \cdot a = 0$ or $a + b$ is a unit.*

***Definition* 2.2.** [4] *For a ring R, a simple undirected graph $G = (V, E)$ is said to be a graph $\Gamma_2(R)$ if all the nonzero elements of R as vertices, and two distinct vertices a and b are adjacent if and only if either $a \cdot b = 0$ or $b \cdot a = 0$ or $a + b$ is a zero divisor (including zero).*

***Definition* 2.3.** [9] *The prime graph $PG_1(R)$ is a graph with all the elements of a ring R as vertices, and any two distinct vertices x, y are adjacent if and only if $x \cdot y = 0$ or $y \cdot x = 0$ or $x + y \in U(R)$, the set of all units of R.*

***Definition* 2.4.** [8] *The prime graph $PG_2(R)$ is a graph with all the elements of a ring R as vertices, and any two distinct vertices x, y are adjacent if and only if $x \cdot y = 0$ or $y \cdot x = 0$ or $x + y \in Z(R)$, the set of all zero divisors of R.*

***Definition* 2.5.** [6] *The Energy of the prime graph of a ring $PG(\mathbb{Z}_n)$ is defined as the sum of the absolute values of all the eigen values of its adjacency matrix $M(PG(R))$. i.e. if $\lambda_1, \lambda_2, ..., \lambda_n$ are n eigen values of $M(PG(R))$, then the energy of $PG(\mathbb{Z}_n)$ is -*

$$E(PG(R)) = \sum_{i=1}^{n} |\lambda_i| .$$

3 1-Quasitotal and 2-Quasitotal Prime graph of a Ring

From the definitions of satyanarayana Bhavanari and his co-authors in [1], we have define here Quasitotal graphs of prime graph of a ring.

***Definition* 3.1.** *Let $PG(R)$ be a prime graph of a ring with vertex set $V(PG(R))$ and edge set $E(PG(R))$. The 1-Quasitotal graph of prime graph of a ring, (denoted by $Q_1(PG(R))$) and is defined as follows:*

The vertex set of $Q_1(PG(R))$, that is $V(Q_1(PG(R))) = V(PG(R)) \bigcup E(PG(R))$. Two vertices a, b in $V(Q_1(PG(R)))$ are adjacent if they satisfy one of the following conditions:

1. *a, b are in $V(PG(R))$ and $ab \in E(PG(R))$*
2. *a, b are in $E(PG(R))$ and a, b are incident in $PG(R)$.*

***Example* 3.1.** *Consider $\mathbb{Z}_n$, the ring of integers modulo n.*

Let $R = \mathbb{Z}_3$. The vertex set $V(PG(R)) = \{0, 1, 2\}$. Since, $0R1 = 0, 0R2 = 0$ and edge set $E(PG(R)) = \{01, 02\}$. So, the vertex set $V(Q_1(PG(R))) = \{v_1, v_2, v_3, e_1, e_2\}$ and edge set $E(Q_1(PG(R))) = \{v_1v_2, v_1v_3, e_1e_2\}$ and the graph is as shown in figure below-

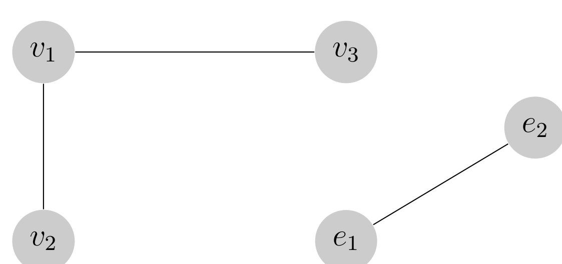

***Figure* 3.1:** $Q_1(PG(\mathbb{Z}_3))$

1. $Q_1(PG(R))$ is a graph without loops and multiple edges, i.e. the graph is simple.
2. The graph of $Q_1(PG(\mathbb{Z}_p))$, p prime, is a disconnected graph containing two components - the first component is itself $PG(\mathbb{Z}_p)$ and the other component is a complete graph K_{p-1} on $p-1$ vertices.

***Definition* 3.2.** *Let $PG(R)$ be a prime graph of a ring with vertex set $V(PG(R))$ and edge set $E(PG(R))$. The 2-Quasitotal graph of prime graph of a ring, (denoted by $Q_2(PG(R))$) and is defined as follows:*

The vertex set of $Q_2(PG(R))$, that is $V(Q_2(PG(R))) = V(PG(R)) \bigcup E(PG(R))$. Two vertices a, b in $V(Q_2(PG(R)))$ are adjacent in $Q_2(PG(R))$ in case one of the following holds:

1. *a, b are in $V(PG(R))$ and $ab \in E(PG(R))$*
2. *a is in $V(PG(R))$; b is in $E(PG(R))$; and a, b are incident in $PG(R)$.*

***Example* 3.2.** *Consider $\mathbb{Z}_n$, the ring of integers modulo n.*

Let $R = \mathbb{Z}_3$. So, the vertex set $V(Q_2(PG(R))) = \{v_1, v_2, v_3, e_1, e_2\}$ and edge set $E(Q_2(PG(R))) = \{v_1v_2, v_1v_3, v_1e_1, v_1e_2, v_2e_1, v_3e_2\}$ and the graph is as shown in figure below-

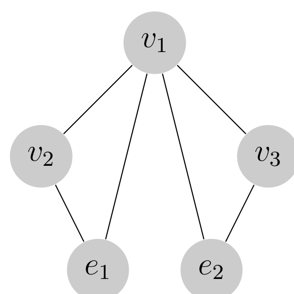

***Figure* 3.2:** $Q_2(PG(\mathbb{Z}_3))$

1. $Q_2(PG(R))$ is a simple graph, i.e without multiple edges and loops.
2. The graph of $Q_2(PG(\mathbb{Z}_p))$, p prime, is a connected graph containing $p-1$ number of triangles having the vertex zero is a common vertex.

4 Energy of $Q_1(PG(\mathbb{Z}_p))$

***Example* 4.1.** *For $p = 2$, the adjacency matrix of $Q_1(PG(\mathbb{Z}_2))$ is*

$$M(Q_1(PG(\mathbb{Z}_2))) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The eigen values are $-1, 0, 1$. Therefore, $E(Q_1(PG(\mathbb{Z}_2))) = 2$.

***Example* 4.2.** *For $p = 3$, the adjacency matrix of $Q_1(PG(\mathbb{Z}_3))$ is*

$$M(Q_1(PG(\mathbb{Z}_3))) = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Therefore, $E(Q_1(PG(\mathbb{Z}_3))) = 4.8284$.

From the SCILAB Software we found here some values of Energy of $Q_1(PG(\mathbb{Z}_p))$ given in

***Table* 4.1**

Sr.No.	n	Graph	Energy
1	2	$Q_1(PG(\mathbb{Z}_2))$	2
2	3	$Q_1(PG(\mathbb{Z}_3))$	4.8284
3	5	$Q_1(PG(\mathbb{Z}_5))$	10
4	7	$Q_1(PG(\mathbb{Z}_7))$	14.8989
5	11	$Q_1(PG(\mathbb{Z}_{11}))$	24.3245
6	13	$Q_1(PG(\mathbb{Z}_{13}))$	28.9282

.

As per the above discussion we conclude the following Theorem -

***Theorem* 4.1.** *If p is a prime number then energy of* $Q_1(PG(\mathbb{Z}_p))$ *is* $(2p-4)+2\sqrt{p-1}$.

5 Energy of $Q_2(PG(\mathbb{Z}_p))$

***Example* 5.1.** *For* $p = 2$*, the adjacency matrix of* $Q_2(PG(\mathbb{Z}_2))$ *is*

$$M(Q_2(PG(\mathbb{Z}_2))) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

The eigen values are $-1, -1, 2$. Therefore, $E(Q_2(PG(\mathbb{Z}_2))) = 4$.

***Example* 5.2.** *For* $p = 3$*, the adjacency matrix of* $Q_2(PG(\mathbb{Z}_3))$ *is*

$$M(Q_2(PG(\mathbb{Z}_3))) = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

Therefore, $E(Q_2(PG(\mathbb{Z}_3))) = 7.1231$.

From the SCILAB Software we found here some values of Energy of $Q_2(PG(\mathbb{Z}_p))$ given in the

***Table* 5.1**

Sr.No.	n	Graph	Energy
1	2	$Q_2(PG(\mathbb{Z}_2))$	4
2	3	$Q_2(PG(\mathbb{Z}_3))$	7.1231
3	5	$Q_2(PG(\mathbb{Z}_5))$	12.7445
4	7	$Q_2(PG(\mathbb{Z}_7))$	18
5	11	$Q_2(PG(\mathbb{Z}_{11}))$	28
6	13	$Q_2(PG(\mathbb{Z}_{13}))$	32.8488

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As per the above discussion we conclude the following Theorem -

***Theorem* 5.1.** *If p is a prime number then energy of* $Q_2(PG(\mathbb{Z}_p))$ *is* $(2p-3)+\sqrt{7p+(p-7)}$.

6 Energy of $PG_1(\mathbb{Z}_p)$

***Example* 6.1.** *For $p = 2$, the adjacency matrix of $PG_1(\mathbb{Z}_2)$ is*

$$M(PG_1(\mathbb{Z}_2)) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

The eigen values are $-1, 1$. Therefore, $E(PG_1(\mathbb{Z}_2)) = 2$.

***Example* 6.2.** *For $p = 3$, the adjacency matrix of $PG_1(\mathbb{Z}_3)$ is*

$$M(PG_1(\mathbb{Z}_3)) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Therefore, $E(PG_1(\mathbb{Z}_3)) = 2.8284$.

From the SCILAB Software we found here some values of Energy of $PG_1(\mathbb{Z}_p)$ given in

***Table* 6.1**

Sr.No.	n	Graph	Energy
1	2	$PG_1(\mathbb{Z}_2)$	2
2	3	$PG_1(\mathbb{Z}_3)$	2.8284
3	5	$PG_1(\mathbb{Z}_5)$	6.4721
4	7	$PG_1(\mathbb{Z}_7)$	10.3245
5	11	$PG_1(\mathbb{Z}_{11})$	18.1980
6	13	$PG_1(\mathbb{Z}_{13})$	22.1655

.

As per the above discussion we conclude the following Theorem -

***Theorem* 6.1.** *If p is an odd prime number then energy of $PG_1(\mathbb{Z}_p)$ is $(p-3) + \sqrt{(p-1)^2 + 4}$.*

7 Energy of $PG_2(\mathbb{Z}_p)$

***Example* 7.1.** *For $p = 2$, the adjacency matrix of $PG_2(\mathbb{Z}_2)$ is*

$$M(PG_2(\mathbb{Z}_2)) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

The eigen values are $-1, 1$. Therefore, $E(PG_2(\mathbb{Z}_2)) = 2$.

***Example* 7.2.** *For $p = 3$, the adjacency matrix of $PG_2(\mathbb{Z}_3)$ is*

$$M(PG_2(\mathbb{Z}_3)) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

Therefore, $E(PG_2(\mathbb{Z}_3)) = 4$.

From the SCILAB Software we found here some values of Energy of $PG_2(\mathbb{Z}_p)$ given in the

Table 7.1

Sr.No.	n	Graph	Energy
1	2	$PG_2(\mathbb{Z}_2)$	2
2	3	$PG_2(\mathbb{Z}_3)$	4
3	5	$PG_2(\mathbb{Z}_5)$	7.1231
4	7	$PG_2(\mathbb{Z}_7)$	10
5	11	$PG_2(\mathbb{Z}_{11})$	15.4031
6	13	$PG_2(\mathbb{Z}_{13})$	18

As per the above discussion we conclude the following ***Table* 7.1**.

***Theorem* 7.1.** *If p is an odd prime number then energy of $PG_2(\mathbb{Z}_p)$ is $(p-2)+\sqrt{3p+(p-3)}$.*

General Scilab software code to find Energy of a Graph:

(1) $A = [...;...;...;...]$: To create a matrix that has multiple rows, separate, the rows with semicolons.
(2) $poly(A, x)$: Gives the polynomial of matrix A in variable x.
(3) $spec(A)$: Gives the Eigen Values of matrix A.
(4) $abs(spec(A))$: Gives absolute values of Eigen values of matrix A.
(5) $sum(abs(spec(A)))$: Gives the Energy of a Graph.

Acknowledgments
The authors would like to express their sincere thanks to the Editors and referee of *Jñānābha* for their valuable suggestions to bring the paper in its present form and also to the parents and family for their inseparable support and prayers.

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