ENERGY OF SOME GRAPHS OF PRIME GRAPH OF A RING

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Abstract
Let \( R \) be a commutative ring and \( PG(R) \) is a graph whose vertices are all the elements of ring \( R \) and two vertices are adjacent if their product is zero. In this article, we study the energy of 1-Quasitotal and 2-Quasitotal Prime Graph of a Ring \( \mathbb{Z}_p \) and also find the energy of \( PG_1(\mathbb{Z}_p) \) and \( PG_2(\mathbb{Z}_p) \), \( p \) prime. A General SCILAB Software code for our calculation is also presented.

Keywords and phrases: Ring, Prime graph of a Ring \( PG(R) \), Quasi-total graph, Energy.

1 Introduction
The study of graph theory for a commutative ring began when Beck in [3] introduced the notion of zero divisor of the graph. The graphs \( \Gamma_1(R) \) and \( \Gamma_2(R) \) are defined by R. Sen Gupta et al. in [4]. Another graph structure associated to a ring called prime graph was introduced by Satyanarayana et al. [2]. Prime graph is defined as a graph whose vertices are all elements of the ring and any two distinct vertices \( x, y \in R \) are adjacent if and only if \( xRy = 0 \) or \( yRx = 0 \). This graph is denoted by \( PG(R) \). Pawar and Joshi in [10] gave a simple formulation for finding the degrees of vertices of prime graph \( PG(R) \) as well as its complement \( (PG(R))^c \). Also the number of triangles in \( PG(R) \) and \( (PG(R))^c \) have been calculated using simple combinatorial approach. We have introduced the prime graphs \( PG_1(R) \) in [9] and \( PG_2(R) \) in [8] of a ring and discussed all the results related to degree of vertices, Eulerianity, planarity and girth.

In third section of this paper we give definition and some examples of 1-Quasitotal and 2-Quasitotal Prime Graph of a Ring \( \mathbb{Z}_n \). In last four sections we find the energy of 1-Quasitotal and 2-Quasitotal Prime Graph of a Ring \( \mathbb{Z}_p \) and also find the energy of \( PG_1(\mathbb{Z}_p) \) and \( PG_2(\mathbb{Z}_p) \), where \( p \) is prime and give a general SCILAB software code for finding the energy of any Graph.

2 Preliminary Definitions
Here we are listing some preliminary definitions. For basic terminology and definitions the reader is referred to [2], [5].
Definition 2.1. [4] For a ring $R$, a simple undirected graph $G = (V, E)$ is said to be a graph $\Gamma_1(R)$ if all the nonzero elements of $R$ as vertices, and two distinct vertices $a$ and $b$ are adjacent if and only if either $a \cdot b = 0$ or $b \cdot a = 0$ or $a + b$ is a unit.

Definition 2.2. [4] For a ring $R$, a simple undirected graph $G = (V, E)$ is said to be a graph $\Gamma_2(R)$ if all the nonzero elements of $R$ as vertices, and two distinct vertices $a$ and $b$ are adjacent if and only if either $a \cdot b = 0$ or $b \cdot a = 0$ or $a + b$ is a zero divisor (including zero).

Definition 2.3. [9] The prime graph $PG_1(R)$ is a graph with all the elements of a ring $R$ as vertices, and any two distinct vertices $x, y$ are adjacent if and only if $x \cdot y = 0$ or $y \cdot x = 0$ or $x + y \in U(R)$, the set of all units of $R$.

Definition 2.4. [8] The prime graph $PG_2(R)$ is a graph with all the elements of a ring $R$ as vertices, and any two distinct vertices $x, y$ are adjacent if and only if $x \cdot y = 0$ or $y \cdot x = 0$ or $x + y \in Z(R)$, the set of all zero divisors of $R$.

Definition 2.5. [6] The Energy of the prime graph of a ring $PG(\mathbb{Z}_n)$ is defined as the sum of the absolute values of all the eigen values of its adjacency matrix $M(PG(R))$. i.e. if $\lambda_1, \lambda_2, ..., \lambda_n$ are $n$ eigen values of $M(PG(R))$, then the energy of $PG(\mathbb{Z}_n)$ is -

$$E(PG(R)) = \sum_{i=1}^{n} |\lambda_i|.$$

3 1-Quasitotal and 2-Quasitotal Prime graph of a Ring

From the definitions of satyanarayana Bhavanari and his co-authors in [1], we have define here Quasitotal graphs of prime graph of a ring.

Definition 3.1. Let $PG(R)$ be a prime graph of a ring with vertex set $V(PG(R))$ and edge set $E(PG(R))$. The 1-Quasitotal graph of prime graph of a ring, (denoted by $Q_1(PG(R))$) and is defined as follows:

The vertex set of $Q_1(PG(R))$, that is $V(Q_1(PG(R))) = V(PG(R)) \cup E(PG(R))$. Two vertices $a, b$ in $V(Q_1(PG(R)))$ are adjacent if they satisfy one of the following conditions:

1. $a, b$ are in $V(PG(R))$ and $ab \in E(PG(R))$
2. $a, b$ are in $E(PG(R))$ and $a, b$ are incident in $PG(R)$.

Example 3.1. Consider $\mathbb{Z}_n$, the ring of integers modulo $n$.

Let $R = \mathbb{Z}_3$. The vertex set $V(PG(R)) = \{0, 1, 2\}$. Since, 0R1 = 0, 0R2 = 0 and edge set $E(PG(R)) = \{01, 02\}$. So, the vertex set $V(Q_1(PG(R))) = \{v_1, v_2, v_3, e_1, e_2\}$ and edge set $E(Q_1(PG(R))) = \{v_1v_2, v_1v_3, e_1e_2\}$ and the graph is as shown in figure below-

![Figure 3.1: Q_1(PG(\mathbb{Z}_3))](image-url)
1. $Q_1(PG(R))$ is a graph without loops and multiple edges, i.e. the graph is simple.

2. The graph of $Q_1(PG(\mathbb{Z}_p))$, $p$ prime, is a disconnected graph containing two components - the first component is itself $PG(\mathbb{Z}_p)$ and the other component is a complete graph $K_{p-1}$ on $p - 1$ vertices.

**Definition 3.2.** Let $PG(R)$ be a prime graph of a ring with vertex set $V(PG(R))$ and edge set $E(PG(R))$. The 2-Quasitotal graph of prime graph of a ring, (denoted by $Q_2(PG(R))$) and is defined as follows:

The vertex set of $Q_2(PG(R))$, that is $V(Q_2(PG(R))) = V(PG(R)) \cup E(PG(R))$. Two vertices $a, b$ in $V(Q_2(PG(R)))$ are adjacent in $Q_2(PG(R))$ in case one of the following holds:

1. $a, b$ are in $V(PG(R))$ and $ab \in E(PG(R))$
2. $a$ is in $V(PG(R))$; $b$ is in $E(PG(R))$; and $a, b$ are incident in $PG(R)$.

**Example 3.2.** Consider $\mathbb{Z}_n$, the ring of integers modulo $n$.

Let $R = \mathbb{Z}_3$. So, the vertex set $V(Q_2(PG(\mathbb{Z}_3))) = \{v_1, v_2, v_3, e_1, e_2\}$ and edge set $E(Q_2(PG(\mathbb{Z}_3))) = \{v_1v_2, v_1v_3, v_1e_1, v_1e_2, v_2e_1, v_3e_2\}$ and the graph is as shown in figure below-

![Figure 3.2: $Q_2(PG(\mathbb{Z}_3))$](image)

1. $Q_2(PG(R))$ is a simple graph, i.e without multiple edges and loops.

2. The graph of $Q_2(PG(\mathbb{Z}_p))$, $p$ prime, is a connected graph containing $p - 1$ number of triangles having the vertex zero is a common vertex.

4 **Energy of $Q_1(PG(\mathbb{Z}_p))$**

**Example 4.1.** For $p = 2$, the adjacency matrix of $Q_1(PG(\mathbb{Z}_2))$ is

$$M(Q_1(PG(\mathbb{Z}_2))) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$  

The eigen values are $-1, 0, 1$. Therefore, $E(Q_1(PG(\mathbb{Z}_2))) = 2$.

**Example 4.2.** For $p = 3$, the adjacency matrix of $Q_1(PG(\mathbb{Z}_3))$ is

$$M(Q_1(PG(\mathbb{Z}_3))) = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$  

Therefore, $E(Q_1(PG(\mathbb{Z}_3))) = 4.8284$. 

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From the SCILAB Software we found here some values of Energy of $Q_1(PG(Z_p))$ given in Table 4.1.

<table>
<thead>
<tr>
<th>Sr.No.</th>
<th>$n$</th>
<th>Graph</th>
<th>Energy</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>$Q_1(PG(Z_2))$</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
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<td>4</td>
<td>7</td>
<td>$Q_1(PG(Z_7))$</td>
<td>14.8989</td>
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<tr>
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<td>11</td>
<td>$Q_1(PG(Z_{11}))$</td>
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<tr>
<td>6</td>
<td>13</td>
<td>$Q_1(PG(Z_{13}))$</td>
<td>28.9282</td>
</tr>
</tbody>
</table>

As per the above discussion we conclude the following Theorem -

**Theorem 4.1.** If $p$ is a prime number then energy of $Q_1(PG(Z_p))$ is $(2p - 4) + 2\sqrt{p - 1}$.

5 Energy of $Q_2(PG(Z_p))$

**Example 5.1.** For $p = 2$, the adjacency matrix of $Q_2(PG(Z_2))$ is

$$M(Q_2(PG(Z_2))) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$ 

The eigen values are $-1, -1, 2$. Therefore, $E(Q_2(PG(Z_2))) = 4$.

**Example 5.2.** For $p = 3$, the adjacency matrix of $Q_2(PG(Z_3))$ is

$$M(Q_2(PG(Z_3))) = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix}.$$ 

Therefore, $E(Q_2(PG(Z_3))) = 7.1231$.

From the SCILAB Software we found here some values of Energy of $Q_2(PG(Z_p))$ given in the Table 5.1.

<table>
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<th>Sr.No.</th>
<th>$n$</th>
<th>Graph</th>
<th>Energy</th>
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</thead>
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<td>2</td>
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<tr>
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<td>11</td>
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<tr>
<td>6</td>
<td>13</td>
<td>$Q_2(PG(Z_{13}))$</td>
<td>32.8488</td>
</tr>
</tbody>
</table>

As per the above discussion we conclude the following Theorem -

**Theorem 5.1.** If $p$ is a prime number then energy of $Q_2(PG(Z_p))$ is $(2p - 3) + \sqrt{7p + (p - 7)}$. 

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6 Energy of $PG_1(\mathbb{Z}_p)$

**Example 6.1.** For $p = 2$, the adjacency matrix of $PG_1(\mathbb{Z}_2)$ is

$$M(PG_1(\mathbb{Z}_2)) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$  

The eigen values are $-1, 1$. Therefore, $E(PG_1(\mathbb{Z}_2)) = 2$.

**Example 6.2.** For $p = 3$, the adjacency matrix of $PG_1(\mathbb{Z}_3)$ is

$$M(PG_1(\mathbb{Z}_3)) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$  

Therefore, $E(PG_1(\mathbb{Z}_3)) = 2.8284$.

From the SCILAB Software we found here some values of Energy of $PG_1(\mathbb{Z}_p)$ given in the

<table>
<thead>
<tr>
<th>Sr.No.</th>
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</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>$PG_1(\mathbb{Z}_2)$</td>
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<tr>
<td>2</td>
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<td>$PG_1(\mathbb{Z}_{13})$</td>
<td>22.1655</td>
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</table>

As per the above discussion we conclude the following Theorem -

**Theorem 6.1.** If $p$ is an odd prime number then energy of $PG_1(\mathbb{Z}_p)$ is $(p - 3) + \sqrt{(p - 1)^2 + 4}$.

7 Energy of $PG_2(\mathbb{Z}_p)$

**Example 7.1.** For $p = 2$, the adjacency matrix of $PG_2(\mathbb{Z}_2)$ is

$$M(PG_2(\mathbb{Z}_2)) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$  

The eigen values are $-1, 1$. Therefore, $E(PG_2(\mathbb{Z}_2)) = 2$.

**Example 7.2.** For $p = 3$, the adjacency matrix of $PG_2(\mathbb{Z}_3)$ is

$$M(PG_2(\mathbb{Z}_3)) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$  

Therefore, $E(PG_2(\mathbb{Z}_3)) = 4$.

From the SCILAB Software we found here some values of Energy of $PG_2(\mathbb{Z}_p)$ given in the
Table 7.1

<table>
<thead>
<tr>
<th>Sr.No.</th>
<th>( n )</th>
<th>Graph</th>
<th>Energy</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>( PG_2(\mathbb{Z}_2) )</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>( PG_2(\mathbb{Z}_3) )</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>5</td>
<td>( PG_2(\mathbb{Z}_5) )</td>
<td>7.1231</td>
</tr>
<tr>
<td>4</td>
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<td>( PG_2(\mathbb{Z}_7) )</td>
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<td>13</td>
<td>( PG_2(\mathbb{Z}_{13}) )</td>
<td>18</td>
</tr>
</tbody>
</table>

As per the above discussion we conclude the following Table 7.1.

**Theorem 7.1.** If \( p \) is an odd prime number then energy of \( PG_2(\mathbb{Z}_p) \) is \((p - 2) + \sqrt{3p} + (p - 3)\).

**General Scilab software code to find Energy of a Graph:**

(1) \( A = [...;...;...] \): To create a matrix that has multiple rows, separate, the rows with semicolons.

(2) \( \text{poly}(A,x) \): Gives the polynomial of matrix \( A \) in variable \( x \).

(3) \( \text{spec}(A) \): Gives the Eigen Values of matrix \( A \).

(4) \( \text{abs(spec(A))} \): Gives absolute values of Eigen values of matrix \( A \).

(5) \( \text{sum(abs(spec(A)))} \): Gives the Energy of a Graph.

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**References**


