Jñānābha, Vol. 50(1) (2020), 179-188
(Dedicated to Honor Professor H.M. Srivastava on His 80th Birth Anniversary Celebrations)

HYPERGEOMETRIC REPRESENTATIONS OF SOME MATHEMATICAL FUNCTIONS VIA MACLAURIN SERIES

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(*Received : May* 27, 2020 *; Revised: June* 07, 2020)
DOI: https://doi.org/10.58250/jnanabha.2020.50118

Abstract

In this paper, by using Maclaurin series of given mathematical function and expressing the coefficient of the general term of corresponding Maclaurin series in the form of Pochhammer symbols, we obtain the hypergeometric forms of following functions:

$$\frac{\sin^{-1}(x)}{\sqrt{(1-x^2)}},\ [\sin^{-1}(x)]^2,\ \sin^{-1}(x),\ \frac{\sinh^{-1}(x)}{\sqrt{(1+x^2)}},\ [\sinh^{-1}(x)]^2,\ \sinh^{-1}(x) \text{ and } \ell n\{e(1-x)^{\frac{1}{x}}\}^{-\frac{2}{x}}.$$

2010 Mathematics Subject Classifications: 33Cxx, 34A35, 41A58, 33B10.
Keywords and phrases: Hypergeometric functions, Leibnitz theorem, Maclaurin series, Pochhammer symbol.

1 Introduction and Preliminaries

In our investigations, we shall use the following standard notations:
$\mathbb{N} := \{1,2,3,\cdots\}; \mathbb{N}_0 := \mathbb{N}\bigcup\{0\}; \mathbb{Z}_0^- := \mathbb{Z}^-\bigcup\{0\} = \{0,-1,-2,-3,\cdots\}$.

The symbols $\mathbb{C},\mathbb{R},\mathbb{N},\mathbb{Z},\mathbb{R}^+$ and $\mathbb{R}^-$ denote the sets of complex numbers, real numbers, natural numbers, integers, positive and negative real numbers respectively.

Pochhammer symbol:
The Pochhammer symbol (or the *shifted* factorial) $(\lambda)_\nu$ $(\lambda,\nu\in\mathbb{C})$[13, p.22 eq(1), p.32 Q.N.(8) and Q.N.(9)], see also [15, p.23, eq(22) and eq(23)], is defined by

$$(\lambda)_\nu := \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} = \begin{cases} 1 & (\nu=0;\ \lambda\in\mathbb{C}\backslash\{0\}) \\ \prod_{j=0}^{n-1}(\lambda+j) & (\nu=n\in\mathbb{N};\ \lambda\in\mathbb{C}) \\ \frac{(-1)^k n!}{(n-k)!} & (\lambda=-n;\ \nu=k;\ n,k\in\mathbb{N}_0;\ 0\leqq k\leqq n) \\ 0 & (\lambda=-n;\ \nu=k;\ n,k\in\mathbb{N}_0;\ k>n) \\ \frac{(-1)^k}{(1-\lambda)_k} & (\nu=-k;\ k\in\mathbb{N};\ \lambda\in\mathbb{C}\backslash\mathbb{Z}), \end{cases}$$

it being understood conventionally that $(0)_0 = 1$ and assumed tacitly that the Gamma quotient exists.

Generalized hypergeometric function of one variable:
A natural generalization of the Gaussian hypergeometric series ${}_2F_1[\alpha,\beta;\gamma;z]$, is accomplished by

introducing any arbitrary number of numerator and denominator parameters. Thus, the resulting series

$$(1.1)\qquad {}_pF_q\left[\begin{array}{c}(\alpha_p);\\ \\ (\beta_q);\end{array}\; z\right] = {}_pF_q\left[\begin{array}{c}\alpha_1,\alpha_2,\ldots,\alpha_p;\\ \\ \beta_1,\beta_2,\ldots,\beta_q;\end{array}\; z\right] = \sum_{n=0}^{\infty}\frac{(\alpha_1)_n(\alpha_2)_n\ldots(\alpha_p)_n}{(\beta_1)_n(\beta_2)_n\ldots(\beta_q)_n}\,\frac{z^n}{n!}\,,$$

is known as the generalized hypergeometric series, or simply, the generalized hypergeometric function. Here p and q are positive integers or zero and we assume that the variable z, the numerator parameters $\alpha_1,\alpha_2,\ldots,\alpha_p$ and the denominator parameters $\beta_1,\beta_2,\ldots,\beta_q$ take on complex values, provided that

$$\beta_j \neq 0,-1,-2,\ldots\;;\;\; j=1,2,\ldots,q.$$

Supposing that none of the numerator and denominator parameters is zero or a negative integer, we note that the ${}_pF_q$ series defined by equation (1.1):

(i) converges for $|z| < \infty$, if $p \leq q$,
(ii) converges for $|z| < 1$, if $p = q+1$,
(iii) diverges for all z, $z \neq 0$, if $p > q+1$,
(iv) converges absolutely for $|z| = 1$, if $p = q+1$, and $\Re(\omega) > 0$,
(v) converges conditionally for $|z| = 1 (z \neq 1)$, if $p = q+1$ and $-1 < \Re(\omega) \leqq 0$,
(vi) diverges for $|z| = 1$, if $p = q+1$ and $\Re(\omega) \leqq -1$,

where by convention, a product over an empty set is interpreted as 1 and

$$(1.2)\qquad \omega := \sum_{j=1}^{q}\beta_j - \sum_{j=1}^{p}\alpha_j,$$

$\Re(\omega)$ being the real part of complex number ω.

Relation between inverse hyperbolic and inverse trigonometric functions:

$$(1.3)\qquad \sin^{-1}(i\theta) = i\sinh^{-1}(\theta) \;\; or \;\; \sinh^{-1}(i\theta) = i\sin^{-1}(\theta).$$

Leibnitz theorem:

The *nth* derivative of the product of two functions, is given by

$$(1.4)\qquad D^n[U(x)\,T(x)] = ({}^nC_0)(D^n\,U)(D^0\,T) + ({}^nC_1)(D^{n-1}\,U)(D^1\,T) + ({}^nC_2)(D^{n-2}\,U)(D^2\,T) + \cdots + +({}^nC_{n-1})(D^1\,U)(D^{n-1}\,T) + ({}^nC_n)(D^0\,U)(D^n\,T).$$

Maclaurin series :

Suppose *nth* derivative of $y(x)$, w.r.t. x is denoted by $D^n y = \frac{d^n y}{dx^n} = y_n$.

Then

$$y(x) = (y)_0 + x(y_1)_0 + \frac{x^2}{2!}(y_2)_0 + \frac{x^3}{3!}(y_3)_0 + \frac{x^4}{4!}(y_4)_0 + \frac{x^5}{5!}(y_5)_0 + \cdots$$

$$(1.5)\qquad = \sum_{n=0}^{\infty}\frac{x^n}{n!}(y_n)_0,$$

$$(1.6)\qquad = \sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}(y_{2n})_0 + \sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!}(y_{2n+1})_0,$$

where,

$$(y_m)_0 = \left(\frac{d^m y}{dx^m}\right)_{x=0} .$$

$$(\alpha)_{2n} = 2^{2n}\left(\frac{\alpha}{2}\right)_n\left(\frac{\alpha+1}{2}\right)_n . \tag{1.7}$$

The present article is organized as follows. In ***Section*** **3**, we have derived the hypergeometric forms of some functions involving arcsine function and logarithmic function by using Maclaurin series. In ***Section*** **4**, we have given hypergeometric forms of some inverse hyperbolic sine function as special cases. For hypergeometric forms of other mathematical functions and functions of mathematical physics, one can refer the literature [1],[2],[3],[4],[5],[6],[7],[8],[14] and [16], where the proof of hypergeometric forms of related functions are not given. So we are interested to give the proof of hypergeometric forms of the functions mentioned in ***Section*** **2**. For some recent related work, the interested readers can consult the papers by Qureshi, et al.[9, 10, 11, 12].

2 Some Hypergeometric Forms Involving Arcsine Function and Logarithmic Function

When $|x| < 1$, then following hypergeometric forms hold true:

$$\frac{\sin^{-1}(x)}{\sqrt{(1-x^2)}} = x\ {}_2F_1\left[\begin{matrix}1,\ 1; \\ \frac{3}{2};\end{matrix}\ x^2\right]. \tag{2.1}$$

$$[\sin^{-1}(x)]^2 = x^2\ {}_3F_2\left[\begin{matrix}1,\ 1,\ 1; \\ 2,\ \frac{3}{2};\end{matrix}\ x^2\right]. \tag{2.2}$$

$$\sin^{-1}(x) = x\ {}_2F_1\left[\begin{matrix}\frac{1}{2},\ \frac{1}{2}; \\ \frac{3}{2};\end{matrix}\ x^2\right]. \tag{2.3}$$

$$\ell n\{e(1-x)^{\frac{1}{x}}\}^{-\frac{2}{x}} = {}_2F_1\left[\begin{matrix}1,\ 2; \\ 3;\end{matrix}\ x\right]. \tag{2.4}$$

3 Proof of Hypergeometric Forms

Proof of hypergeometric form (2.1)***:***

Consider the following function

$$y = y(x) = \frac{\sin^{-1}(x)}{\sqrt{(1-x^2)}}, \tag{3.1}$$

that is

$$\sqrt{(1-x^2)}\ y = \sin^{-1}(x).$$

Put $x = 0$ in the equation (3.1), we get

$$(y)_0 = 0 . \tag{3.2}$$

Differentiate the equation (3.1) w.r.t. x and use product rule, after simplification we get

$$(1-x^2)y_1 - xy = 1, \tag{3.3}$$

$$(y_1)_0 = 1 . \tag{3.4}$$

Again differentiate the equation (3.3) w.r.t. x and use product rule, after simplification we have

(3.5) $$(1 - x^2)y_2 - 3xy_1 - y = 0,$$

(3.6) $$(y_2)_0 = 0\ .$$

Now differentiate the equation (3.3) n-times w.r.t. x and apply Leibnitz theorem we obtain

(3.7) $$D^n[(1 - x^2)y_1] - D^n[xy] = D^n[1];\ n \geq 2,$$

$$(1 - x^2)y_{n+1} - (2n + 1)xy_n - n^2y_{n-1} = 0;\ n \geq 2\ .$$

Put $x = 0$ in the equation (**??**), we get

(3.8) $$(y_{n+1})_0 = n^2(y_{n-1})_0;\ n \geq 2\ .$$

Put $n = 2, 3, 4, 5, 6, 7, 8, 9$ in the equation (3.8), we get

(3.9) $$(y_3)_0 = (2)^2(1),$$

(3.10) $$(y_5)_0 = (4)^2(2)^2(1),$$

(3.11) $$(y_7)_0 = (6)^2(4)^2(2)^2(1),$$

(3.12) $$(y_9)_0 = (8)^2(6)^2(4)^2(2)^2(1).$$

Using the equation (3.8), we can write the recurrence relation:

(3.13) $$(y_m)_0 = (m - 1)^2(y_{m-2})_0; m \geq 2\ .$$

When $m = 2n$, then

(3.14) $$(y_{2n})_0 = (2n - 1)^2(y_{2n-2})_0 = 0\ .$$

When $m = 2n + 1$, then

(3.15) $$\begin{aligned}(y_{2n+1})_0 &= (2n)^2(y_{2n-1})_0\\ &= (2n)^2(2n - 2)^2(2n - 4)^2 \cdots (8)^2(6)^2(4)^2(2)^2(1)\\ &= \{2^n(1 \times 2 \times 3 \times 4 \times \cdots \times n)\}^2\\ &= 4^n(n!)^2.\end{aligned}$$

Now using Maclaurin series, we get

$$y = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}(y_{2n})_0 + \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n + 1)!}(y_{2n+1})_0\ .$$

The function $\frac{\sin^{-1}(x)}{\sqrt{(1-x^2)}}$ is an odd function so that the even coefficients of its Maclaurin expansion vanish. That is

$$\begin{aligned}y &= 0 + \sum_{n=0}^{\infty} \frac{x^{2n+1}\ 4^n\ (n!)^2}{(2n + 1)!}\\ &= x \sum_{n=0}^{\infty} \frac{4^n\ (1)_n\ (1)_n\ x^{2n}}{(1)_{2n+1}}\\ &= x \sum_{n=0}^{\infty} \frac{4^n\ (1)_n\ (1)_n\ x^{2n}}{(2)_{2n}}\\ &= x \sum_{n=0}^{\infty} \frac{(1)_n\ (1)_n\ x^{2n}}{(\frac{3}{2})_n\ n!}.\end{aligned}$$

On using the definition (1.1), we get the required hypergeometric form (2.1).

Proof of hypergeometric form* (2.2)*:

Consider the following function

$$y = y(x) = [\sin^{-1}(x)]^2. \tag{3.16}$$

Put $x = 0$ in the equation (3.16), we get

$$(y)_0 = 0\,. \tag{3.17}$$

Differentiate the equation (3.16) w.r.t. x, we get

$$\sqrt{(1-x^2)}y_1 = 2[\sin^{-1}(x)], \tag{3.18}$$

$$(y_1)_0 = 0\,. \tag{3.19}$$

Differentiate the equation (3.18) w.r.t. x and use product rule, after simplification we get

$$(1-x^2)y_2 - xy_1 = 2, \tag{3.20}$$

$$(y_2)_0 = 2. \tag{3.21}$$

Again differentiate the equation (3.20) w.r.t. x and use product rule, after simplification we have

$$(1-x^2)y_3 - 3xy_2 - y_1 = 0. \tag{3.22}$$

Now differentiate the equation (3.20) n-times w.r.t. x and apply Leibnitz theorem we obtain

$$D^n[(1-x^2)y_2] - D^n[xy_1] = D^n[2]; n \geq 2, \tag{3.23}$$

$$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2y_n = 0; n \geq 2.$$

Put $x = 0$ in the equation (3.23), we get

$$(y_{n+2})_0 = n^2(y_n)_0; n \geq 2\,. \tag{3.24}$$

Put $n = 2, 3, 4, 5, 6, 7, 8$ in the equation (3.24), we get

$$(y_4)_0 = (2)^2(2), \tag{3.25}$$

$$(y_6)_0 = (4)^2(2)^2(2), \tag{3.26}$$

$$(y_8)_0 = (6)^2(4)^2(2)^2(2), \tag{3.27}$$

$$(y_{10})_0 = (8)^2(6)^2(4)^2(2)^2(2)\,. \tag{3.28}$$

Using the equation (3.24), we can write the recurrence relation:

$$(y_m)_0 = (m-2)^2(y_{m-2})_0; m \geq 2\,. \tag{3.29}$$

When $m = 2n$, then

$$\begin{aligned} (y_{2n})_0 &= (2n-2)^2(y_{2n-2})_0 \\ &= (2n-2)^2(2n-4)^2 \cdots (8)^2(6)^2(4)^2(2)^2(2) \\ &= 2\{2^{n-1}\,(n-1)!\}^2 \\ &= (2)^{2n-1}\,\{(n-1)!\}^2. \end{aligned} \tag{3.30}$$

When $m = 2n+1$, then

$$(y_{2n+1})_0 = (2n-1)^2(y_{2n-1})_0 = 0. \tag{3.31}$$

Now using Maclaurin series, we get

$$y = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}(y_{2n})_0 + \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}(y_{2n+1})_0.$$

The function $[\sin^{-1}(x)]^2$ is an even function so that the odd coefficients of its Maclaurin expansion vanish. That is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}(y_{2n})_0 + 0 \\ &= 0 + \sum_{n=1}^{\infty} \frac{x^{2n}}{(2n)!}(y_{2n})_0 \\ &= \sum_{n=1}^{\infty} \frac{x^{2n}\ (2)^{2n-1}\{(n-1)!\}^2}{(2n)!}. \end{aligned} \tag{3.32}$$

Replacing n by $n+1$ in equation (3.32), we get

$$\begin{aligned} y &= \sum_{n=0}^{\infty} \frac{x^{2n+2}\ (2)^{2n+1}\{n!\}^2}{(2n+2)!} \\ &= x^2 \sum_{n=0}^{\infty} \frac{(1)_n(1)_n\ (2)^{2n+1}\ x^{2n}}{(1)_{2n+2}} \\ &= x^2 \sum_{n=0}^{\infty} \frac{(1)_n(1)_n\ (2)^{2n+1}\ x^{2n}}{(1)_2\ (3)_{2n}} \\ &= x^2 \sum_{n=0}^{\infty} \frac{(1)_n(1)_n(1)_n\ x^{2n}}{(2)_n\ (\frac{3}{2})_n\ n!}, \end{aligned}$$

on using the definition (1.1), we get the required hypergeometric form (2.2).

Proof of hypergeometric form (2.3)***:***

Consider the following function

$$y = y(x) = \sin^{-1}(x). \tag{3.33}$$

Put $x = 0$ in the equation (3.33), we get

$$(y)_0 = 0. \tag{3.34}$$

Differentiate the equation (3.33) w.r.t. x, we get

$$y_1 = y_1(x) = \frac{1}{\sqrt{(1-x^2)}},$$

that is

$$\sqrt{(1-x^2)}\ y_1 = 1, \tag{3.35}$$

$$(y_1)_0 = 1\ . \tag{3.36}$$

Differentiate the equation (3.35) w.r.t. x and use product rule, after simplification we get

$$(1-x^2)y_2 - xy_1 = 0, \tag{3.37}$$

$$(y_2)_0 = 0. \tag{3.38}$$

Again differentiate the equation (3.37) w.r.t. x and use product rule, after simplification we have

$$(1-x^2)y_3 - 3xy_2 - y_1 = 0, \tag{3.39}$$

$$(y_3)_0 = 1\ . \tag{3.40}$$

Now differentiate the equation (3.37) n-times w.r.t. x, and apply Leibnitz theorem we obtain

$$D^n[(1-x^2)y_2] - D^n[xy_1] = 0; n \geq 2, \tag{3.41}$$

$$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2 y_n = 0; n \geq 2\ .$$

Put $x = 0$ in the equation (**??**), we get

$$(y_{n+2})_0 = n^2 (y_n)_0; n \geq 2\ . \tag{3.42}$$

Put $n = 2, 3, 4, 5, 6, 7, 8$ in the equation (3.42),we get

$$(y_5)_0 = (3)^2(1), \tag{3.43}$$

$$(y_7)_0 = (5)^2(3)^2(1), \tag{3.44}$$

$$(y_9)_0 = (7)^2(5)^2(3)^2(1). \tag{3.45}$$

Using the equation (3.42), we can write the recurrence relation:

$$(y_m)_0 = (m-2)^2 (y_{m-2})_0; m \geq 2\ . \tag{3.46}$$

When $m = 2n$, then

$$(y_{2n})_0 = (2n-2)^2 (y_{2n-2})_0 = 0\ . \tag{3.47}$$

When $m = 2n+1$, then

$$\begin{aligned} (y_{2n+1})_0 &= (2n-1)^2 (y_{2n-1})_0 \\ &= (2n-1)^2(2n-3)^2(2n-5)^2 \cdots (7)^2(5)^2(3)^2(1) \\ &= \{(1)(3)(5)(7)\cdots(2n-5)(2n-3)(2n-1)\}^2. \end{aligned} \tag{3.48}$$

Now divide and Multiply R.H.S. of the equation (3.48) by $[(2)(4)(6)\cdots(2n-4)(2n-2)(2n)]^2$, we get

$$\begin{aligned} (y_{2n+1})_0 &= \frac{\{(1)(2)(3)(4)(5)(6)(7)\cdots(2n-5)(2n-4)(2n-3)(2n-2)(2n-1)(2n)\}^2}{[(2)(4)(6)\cdots(2n-4)(2n-2)(2n)]^2} \\ &= \frac{\{(2n)!\}^2}{[2^n(1\times 2\times 3\times\cdots\times n)]^2} \\ &= \frac{(2n)!\ (2n)!}{4^n\ (n!)^2}. \end{aligned} \tag{3.49}$$

Now using Maclaurin series, we get

$$y = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}(y_{2n})_0 + \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}(y_{2n+1})_0\ .$$

The function $\sin^{-1}(x)$ is an odd function so that the even coefficients of its Maclaurin expansion vanish. That is

$$y = 0 + \sum_{n=0}^{\infty} \frac{x^{2n+1}\ (2n)!\ (2n)!}{(2n+1)!\ 4^n\ (n!)^2}$$

$$= x \sum_{n=0}^{\infty} \frac{(1)_{2n}\,(1)_{2n}\,x^{2n}}{(1)_{2n+1}\,4^n\,(1)_n\,n!}$$
$$= x \sum_{n=0}^{\infty} \frac{(1)_{2n}\,(1)_{2n}\,x^{2n}}{(2)_{2n}\,4^n\,(1)_n\,n!}$$
$$= x \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n\,(\frac{1}{2})_n\,x^{2n}}{(\frac{3}{2})_n\,n!},$$

On using the definition (1.1), we get the required hypergeometric form (2.3).

Proof of hypergeometric form* (2.4)*:

Let,

$$y = \ell n\{e(1-x)^{\frac{1}{x}}\}^{-\frac{2}{x}}$$
$$= -\frac{2}{x}\ell n\{e(1-x)^{\frac{1}{x}}\}$$
$$= -\frac{2}{x}\{\ell n\, e + \ell n(1-x)^{\frac{1}{x}}\}$$
$$= -\frac{2}{x}\{1 + \frac{1}{x}\ell n(1-x)\}$$
$$= 1 + 2(\frac{x}{3} + \frac{x^2}{4} + \frac{x^3}{5} + \cdots)$$
$$= 1 + 2\sum_{n=1}^{\infty} \frac{x^n}{(n+2)}$$
$$= 1 + 2\sum_{n=1}^{\infty} \frac{(2)_n\,x^n}{(2)_n\,(2+n)}$$
$$= \sum_{n=0}^{\infty} \frac{(1)_n\,(2)_n\,x^n}{(3)_n\,n!},$$

on using the definition (1.1), we get the required hypergeometric form (2.4).

4 Hypergeometric Forms Involving Inverse Hyperbolic Sine Function

Replacing x by (ix) in both sides of equations (2.1), (2.2), (2.3) and using the relation (1.3), we obtain the following hypergeometric forms of the functions involving inverse hyperbolic sine function.

When $|x| < 1$, then following hypergeometric forms hold true:

$$\frac{\sinh^{-1}(x)}{\sqrt{(1+x^2)}} = x\;{}_2F_1\left[\begin{array}{c} 1,\ 1; \\ \frac{3}{2}; \end{array} -x^2\right]. \tag{4.1}$$

$$[\sinh^{-1}(x)]^2 = x^2\;{}_3F_2\left[\begin{array}{c} 1,\ 1,\ 1; \\ 2,\ \frac{3}{2}; \end{array} -x^2\right]. \tag{4.2}$$

$$\sinh^{-1}(x) = x\;{}_2F_1\left[\begin{array}{c} \frac{1}{2},\ \frac{1}{2}; \\ \frac{3}{2}; \end{array} -x^2\right]. \tag{4.3}$$

5 Conclusion

In our present investigation, we derived the hypergeometric forms of some functions involving arcsine function, inverse hyperbolic sine function and logarithmic function by using Maclaurin series. Moreover, the results derived in this paper are expected to have useful applications in wide range of problems of Mathematics, Statistics and Physical sciences. Similarly, we can derive the hypergeometric forms of other functions in an analogous manner.

Acknowledgement. The authors are very much thankful to the Editor and referee for their valuable suggestions (use of even and odd function in Maclaurin expansion) to improve the paper in its present form.

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