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*(Dedicated to Honor Professor H.M. Srivastava on His 80$^{th}$ Birth Anniversary Celebrations)*

# THE THEORETICAL OVERVIEW OF THE HARTLEY TRANSFORM AND THE GENERALIZED *R*-FUNCTION

By

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**Abstract**

In this paper the *R*-functions have been mentioned in connection with integral operator named as Hartely transform. The Hartley transform is a mathematical transformation which is closely related to the better known Fourier transform. The properties that differentiate the Hartley Transform from its Fourier counterpart are that the forward and the inverse transforms are identical and also that the Hartley transform of a real signal is a real function of frequency. The Whitened Hartley spectrum, which stems from the Hartley transform, is a bounded function that encapsulates the phase content of a signal. The Whitened Hartley spectrum, unlike the Fourier phase spectrum, is a function that does not suffer from discontinuities or wrapping ambiguities. An overview on how the Whitened Hartley spectrum encapsulates the phase content of a signal more efficiently compared with its Fourier counterpart as well as the reason that phase unwrapping is not necessary for the Whitened Hartley spectrum, are provided in this study. Moreover, in this study, we deal with the function which is significant generalization of Fox's $H$-function which was introduced by Hartley and Lorenzo and later on modified by Jain et al.

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## 1 Introduction

### 1.1 Fox's $H$-function

The $H$-function series introduced by Fox [4] will be represented and defined in the following manner

$$H_{p,q}^{m,n}\left\{\begin{array}{l}(a_1,A_1)(a_2,A_2)...(a_p,A_p)\\(b_1,B_1)(b_2,B_2)...(b_q,B_q)\end{array}|x\right\}=\frac{1}{2\pi i}\int_L\frac{\prod_{j=1}^m\Gamma(b_j-B_js)\prod_{j=1}^n\Gamma(1-a_j+A_js)}{\prod_{j=m+1}^q\Gamma(1-b_j+B_js)\prod_{j=n+1}^p\Gamma(a_j-A_js)}x^sds,$$

where $L$ is a suitable contour.

### 1.2 Hartley transform

The Hartley transform is an integral transformation that maps a real-valued temporal or spacial function into a real-valued frequency function via the kernel, $cas(vx) \equiv cos(vx) + sin(vx)$. This novel symmetrical formulation of the traditional Fourier transform, attributed to Ralph Vinton Lyon Hartley [5], leads to a parallelism that exists between the function of the original variable and that of its transform. Furthermore, the Hartley transform permits a function to be decomposed into two independent sets of sinusoidal components; these sets are represented in terms of positive and negative frequency components, respectively. This is in contrast to the complex exponential, $exp(jwx)$, used in classical Fourier analysis. For periodic power signals, various mathematical forms of the familiar Fourier series come to mind. For a periodic energy and power signals of either finite or infinite duration, the Fourier integral can be used. In either case, signal and systems analysis and design in the frequency domain using the Hartley transform may be deserving of increased awareness due necessarily to the existence of a fast algorithm that can substantially lessen the computational burden when compared to the classical complex-valued Fast Fourier Transform ($FFT$). Perhaps one of Hartley's most long-lasting contributions was a more symmetrical Fourier integral originally developed for steady-state and transient analysis of telephone transmission system problems. Although this transform remained in a quiescent state for over 40 years, the Hartley transform was rediscovered more than a decade ago by Wang [12],[14] and Bracewell [1],[3] who authored definitive treatises on the subject.

The Hartley transform of a function $f(x)$ can be expressed as either

$$H(v) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) cas(vx) dx, \tag{1.1}$$

$$H(f) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) cas(2\pi f x) dx. \tag{1.2}$$

Here the integral kernel, known as the *cosine* and *sine* or *cas* function, is defined as

$$cas(vx) = cos(vx) + sin(vx),$$
$$cas(vx) = \sqrt{2} sin(vx + \frac{\pi}{4}),$$
$$cas(vx) = \sqrt{2} cos(vx - \frac{\pi}{4}).$$

The Hartley transform has the convenient property of being its own inverse

$$f = \{H\{Hf\}\}.$$

### 1.3 Generalized Functions for the Fractional Calculus($R$-function)

It is of significant usefulness to develop a generalized function which when fractionally differ integrated (by any order) returns itself. Such a function would greatly ease the analysis of fractional order differential equations. To end this process the following was proposed by Hartley and Lorenzo,[7]. The $R$-function is unique in that it contains all of the derivatives and integrals of the $F$-function. The $R$-function has the Eigen property that is it returns itself on $q^{th}$ order differ-integration. Special cases of the $R$-function also include the exponential function, the sine, cosine, hyperbolic sine and hyperbolic cosine functions. The value of the $R$-function is clearly demonstrated in the dynamic thermocouple problem where it enables the analyst to directly inverse transform the Laplace domain solution, to obtain the time domain solution, and is defined as follows

(1.3)
$$R_{q,v}[a,c,t] = \sum_{n=0}^{\infty} \frac{a^n (t-c)^{(n+1)q-1-v}}{\Gamma((n+1)q-v)}.$$

The more compact notation

$$R_{q,v}[a,t-c] = \sum_{n=0}^{\infty} \frac{a^n (t-c)^{(n+1)q-1-v}}{\Gamma((n+1)q-v)}.$$

When $c = 0$, we get

$$R_{q,v}[a,t] = \sum_{n=0}^{\infty} \frac{a^n (t)^{(n+1)q-1-v}}{\Gamma((n+1)q-v)}.$$

Put $v = q-1$, we get Mittag-Leffler function

$$R_{q,q-1}[a,t] = \sum_{n=0}^{\infty} \frac{a^n (t)^{(nq)}}{\Gamma(nq+1)} = E(at^q).$$

Taking $a = 1, v = q-\beta$

$$R_{q,q-\beta}[1,t] = \sum_{n=0}^{\infty} \frac{1^n (t)^{(n+1)q-1-q+\beta}}{\Gamma((n+1)q-q+\beta)}.$$

$$\Rightarrow R_{q,q-\beta}[1,t] = t^{\beta-1} E_{q,\beta}(t^q).$$

## 2 Main Result

In this section, the authors have derived the Hartley transform of $R$- functions as follows

***Theorem* 2.1.** *The Hartley transform H of R-functions*

$$H\{R_{q,v}[a,0,t]\} = \frac{1}{\Gamma((n+1)q-v)}[1+(-1)^n]cos(\frac{n\pi}{2})\Gamma(n+1).$$

***Proof.*** The Hartley transform of $R$- functions in terms of Fox's $H$-function is given by

$$H\{R_{q,v}[a,c,t]\} = H\left\{\sum_{n=0}^{\infty} \frac{a^n (t-c)^{(n+1)q-1-v}}{\Gamma((n+1)q-v)}\right\},$$

or

$$H\{R_{q,v}[a,c,t]\} = \sum_{n=0}^{\infty} (a)^n H\left\{\frac{(t-c)^{(n+1)q-1-v}}{\Gamma((n+1)q-v)}\right\}.$$

Taking $c = 0$, we get

$$H\{R_{q,v}[a,0,t]\} = \sum_{n=0}^{\infty} (a)^n H\left\{\frac{(t)^{(n+1)q-1-v}}{\Gamma((n+1)q-v)}\right\}, Re((n+1)q-v) > 0$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{(t)^{(n+1)q-1-v}}{\Gamma((n+1)q-v)} cas(vt)dt,$$

$$= \frac{1}{\Gamma((n+1)q-v)} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (t)^{(n+1)q-1-v} \{cos(vt) + sin(vt)\}dt,$$

$$= \frac{1}{\Gamma((n+1)q-v)} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (t)^{(n+1)q-1-v} \sqrt{2} sin(vt + \frac{\pi}{4})dt.$$

This gives

$$H\{R_{q,v}[a,0,t]\} = \frac{1}{\Gamma((n+1)q-v)}[1+(-1)^n]cos(\frac{n\pi}{2})\Gamma(n+1).$$

This proves the ***Theorem* 2.1**.

**Special Case**: Putting $c = 0$ and $v = q - 1$, we get Mittag-Leffler function as special case of the above result, the result follows as

***Theorem* 2.2.** *The Hartley transform of Fox-Wright function in terms Fox's H-function*

$$H\left\{p\Psi q\left[\begin{array}{c}(a_1,A_1)(a_2,A_2)(a_3,A_3)...(a_p,A_p)\\(b_1,B_1)(b_2,B_2)(b_3,B_3)...(b_q,B_q)\end{array}|z\right]\right\}$$

$$= \frac{1}{s}H^{1,p}_{1,q}\left[\begin{array}{c}(1-a_1,-A_1)(1-a_2,-A_2)(1-a_3,-A_3)...(1-a_p,-A_p)\\(1-b_1,-B_1)(1-b_2,-B_2)(1-b_3,-B_3)...(1-b_q,-B_q)\end{array}|S\right].$$

***Proof.*** The Hartley transform of Fox-Wright function in terms Fox's $H$-function is given by

$$H\left\{p\Psi q\left[\begin{array}{c}(a_1,A_1)(a_2,A_2)...(a_p,A_p)\\(b_1,B_1)(b_2,B_2)...(b_q,B_q)\end{array}|z\right]\right\}.$$

This implies

$$H\left\{p\Psi q\left[\begin{array}{c}(a_1,A_1)(a_2,A_2)...(a_p,A_p)\\(b_1,B_1)(b_2,B_2)...(b_q,B_q)\end{array}|z\right]\right\}$$

$$=\left\{\sum_{n=0}^{\infty}\frac{\Gamma(a_1+nA_1)\Gamma(a_2+nA_2)...\Gamma(a_p+nA_p)}{\Gamma(b_1+nB_1)\Gamma(b_2+nB_2)...\Gamma(b_q+nB_q)}\right\}H\left\{\frac{z^n}{n!}\right\}$$

$$=\left\{\sum_{n=0}^{\infty}\frac{\Gamma(a_1+nA_1)\Gamma(a_2+nA_2)...\Gamma(a_p+nA_p)}{\Gamma(b_1+nB_1)\Gamma(b_2+nB_2)...\Gamma(b_q+nB_q)}\right\}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{z^n}{n!}cas(vz)dz$$

$$=\left\{\sum_{n=0}^{\infty}\frac{\Gamma(a_1+nA_1)\Gamma(a_2+nA_2)...\Gamma(a_p+nA_p)}{\Gamma(b_1+nB_1)\Gamma(b_2+nB_2)...\Gamma(b_q+nB_q)}\right\}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{z^n}{n!}\{cos(vz)+sin(vz)\}dz$$

$$=\left\{\sum_{n=0}^{\infty}\frac{\Gamma(a_1+nA_1)\Gamma(a_2+nA_2)...\Gamma(a_p+nA_p)}{\Gamma(b_1+nB_1)\Gamma(b_2+nB_2)...\Gamma(b_q+nB_q)}\right\}\frac{1}{\Gamma(n+1)}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}z^n\sqrt{2}sin(vz+\frac{\pi}{4})dz$$

$$=\left\{\sum_{n=0}^{\infty}\frac{\Gamma(a_1+nA_1)\Gamma(a_2+nA_2)...\Gamma(a_p+nA_p)}{\Gamma(b_1+nB_1)\Gamma(b_2+nB_2)...\Gamma(b_q+nB_q)}\right\}\frac{-1}{\Gamma(n+1)}[1+(-1)^n]cos(\frac{n\pi}{2})\Gamma(n+1)$$

$$H\left\{p\Psi q\left[\begin{array}{c}(a_1,A_1)(a_2,A_2)...(a_p,A_p)\\(b_1,B_1)(b_2,B_2)...(b_q,B_q)\end{array}|z\right]\right\}$$

$$=\frac{1}{s}\left\{\sum_{n=0}^{\infty}\frac{\Gamma(1-(1-a_1)+nA_1)\Gamma(1-(1-a_2)+nA_2)...\Gamma(1-(1-a_p)+nA_p)}{\Gamma(1-(1-b_1)+nB_1)\Gamma(1-(1-b_2)+nB_2)...\Gamma(1-(1-b_q)+nB_q)}\right\}s^{-n}$$

$$=\frac{1}{s}H^{1,p}_{1,q}\left[\begin{array}{c}(1-a_1,-A_1)(1-a_2,-A_2)(1-a_3,-A_3)...(1-a_p,-A_p)\\(1-b_1,-B_1)(1-b_2,-B_2)(1-b_3,-B_3)...(1-b_q,-B_q)\end{array}|S\right].$$

This proves the ***Theorem* 2.2**.

## 3 Application of the Hartley Transform via the Fast Hartley Transform

The discretized versions of the continuous Fourier and Hartley transform integrals may be put in an amenable form for digital computation. Consider the discrete Fourier transform (*DFT*) and inverse *DFT* (*IDFT*) of a periodic function of period *NT* seconds.

- The *DHT* requires only half the memory storage for real data arrays vs. complex data arrays.
- For a sequence of length $N$, the *DHT* performs $O(N \log 2N)$ real operations vs. the *DFT* $O(N \log 2N)$ complex operations.
- The *DHT* performs fewer operations that may lead to fewer truncation and rounding errors from computer finite word length.
- The *DHT* is its own inverse (i.e., it has a self-inverse) For reasons of computational advantage either occurring through waveform symmetry or simply the use of real quantities, the Hartley transform is recommended as a serious alternative to the Fourier transform for frequency-domain analysis. The salient disadvantage of the Hartley approach is that Fourier amplitude and phase information is not readily interpreted. This is not a difficulty in many applications because this information is typically used as an intermediate stage toward a final goal. Due to the cited advantages above, it is clear that the Hartley transform has much to offer when engineering applications warrant digital filtering of real-valued signals. In particular, the *FHT* should be used when either the computation time is to be minimized; for example, in real-time signal processing. The minimization of computing time includes many other issues, such as memory allocation, real vs. complex variables, computing platforms, and so forth. However, when one is interested in computing the Hartley transform or the convolution or correlation integral, the Hartley transform is the method of choice. In general, most engineering applications based on the *FFT* can be reformulated in terms of the all-real *FHT* in order to realize a computational advantage. This is due primarily to the vast amounts of research within the past decade on *FHT* algorithm development as evidenced in [16]. A voluminous number of applications exist for the Hartley transform *H* some of which are listed below
- Fast convolution, correlation, interpolation and extrapolation, finite-impulse response and multidimensional filter design.

## 4 Conclusion.

In this paper, an overview of the Hartley transform is presented, the relationship between the Hartley transform and the Fourier transform is provided and the Hartley transform properties are analyzed. More importantly, the Whitened Hartley spectrum is defined, its properties for phase spectral estimation are highlighted, its short time analysis is provided and its advantages compared with the Fourier phase spectrum are underlined. The properties of the Whitened Hartley spectrum are also demonstrated via an example involving time-delay measurement. Summarizing, the Whitened Hartley spectrum is proposed as an alternative to the Fourier phase spectrum for applications related to phase spectral processing. Specifically, the Whitened Hartley spectrum, unlike its Fourier counterpart, does not convey extrinsic discontinuities since it is not using the inverse tangent function, whereas the discontinuities of the signal in the phase spectrum which are caused because of intrinsic characteristics of the signal can be compensated. Finally, it is important to mention that the phase spectrum which is developed via the Whitened Hartley spectrum does not only have important advantages compared with the Fourier phase spectrum but it is also very straightforward in terms of its implementation and processing.

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