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(Dedicated to Honor Professor H.M. Srivastava on His 80th Birth Anniversary Celebrations)

N-POLICY FOR $M/E_k/1$ QUEUEING MODEL WITH SERVICE INTERRUPTION

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Abstract

This study deals with an unreliable single service station Erlangian queueing model with k-phase service and l-phase repair under N-policy. The arriving customers follow Poisson process with arrival rates dependent upon the state of the service station, which may be idle, operating, broken down, and under setup or repair states. Due to N-policy the service station turns on only when at least $N(\geq 1)$ customers are accumulated in the system and turns off only when the system becomes empty. While providing service, the service station may breakdown according to Poisson process. An optimal operating N-policy is proposed to minimize the total expected cost. If the service station breakdowns, then it is sent for repair at the repair facility which renders repair after a set up time. After repairing, the service station works as good as before breakdown. Recursive technique and generating functions are employed for solution purpose. Explicit expressions for various performance indices are established. Cost analysis and sensitivity analysis have been done to explore the effects of different parameters.
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1 Introduction

There is an extensive literature on the Erlangian queueing model, which has been studied in various forms by numerous authors. Earlangeian queueing system represent a simple single server with negative exponential inter arrival and service time distribution. In this direction Conolly [4] studied the generalized state dependent Erlangian queue. Wang and Kuo [18] considered the profit analysis of the $M/E_k/1$ machine repair problem with a non-reliable server. The transient solution to $M/E_k/1$ queue was investigated by Griffiths et al. [5]. The transient phase probabilities are obtained in terms of a new generalisation of the modified Bessel function, and the mean waiting time in the queue is evaluated. Kim et al. [10] developed the erlang loss queueing system with batch arrivals operating in a random environment. Sharma [15] analyzed unreliable server $M^x/G/1$ queue with loss and delay, balking and second optional service. After receiving the essential service, the customers may opt for the optional service with some probability or may leave the system.

The congestion situation where the service does not start until some specified number of customers, say N are accumulated in the system during an idle period and once server starts serving, goes on serving till the system become empty is called N-policy. Medhi and Templeton [11] developed a Poisson input queue under N-policy restriction and a general start up time. Optimal NT policies for $M/G/1$ system with a startup and unreliable server was analysed by Ke [9]. Sharma [14] investigated a single unreliable server interdependent loss and delay queueing model with controllable arrival rate under N-policy. When there is no customer present in the system then the server goes on vacation and returns back in the system whenever the specified N ($\geq$ 1) or more customers are accumulated. Sharma [13] developed machine repair problem with spares, balking, reneging and n-policy for vacation. In this system, there are two repairmen, the first repairman is always available for providing service to the failed units while the second repairman goes on vacation when the failed units are less than a threshold value (say N).

One important fact that has been overlooked in most studies is that the server is subject to breakdown while serving a customer. This phenomenon is encountered in the manufacturing system, communication system, computer system and many others. Some queueing problems with breakdowns were studied by Avi-Itzhak and Naor [2], $M/G/1$ queue with breakdown was investigated by Ke [8]. Wang et al. [17] comprised the two randomized policy M/G/1 queues with second optional service, server breakdown and startup.

In the real life congestion situations, the customer may balk from the system due to some reason; one most common reason is long queue. Blackburn [3] considered optimal control of single server queue with balking and reneging. Multi server queue with balking and reneging was investigated by Abou and Hariri [1]. Controllable multi server queue with balking was studied by Jain and Sharma [6]. We incorporate an additional server which is added and removed at pre-specified threshold levels of queue size to control the balking behaviour of the customers. Finite capacity queueing system with queue dependent servers and discouragement was analysed by Jain and Sharma [7]. The service rates of the servers are different and the number of servers in the system changes depending on the queue length. The first server starts service only when N customers are accumulated in the queue and once he starts serving, continues to serve until the system becomes empty. Sharma [12] developed loss and delay multi server state dependent queue with discouragement, additional server and no-passing. The customers arrive according to Poisson process and depart from the system in the same chronological order in which they join the system, due to no-passing restriction.

The present investigation deals with N-policy for Earlangian service queueing model with unreliable service station, ℓ phase repair, setup and balking. Here, we consider the state dependent arrival rates. The steady state results for different states are obtained by using probability generating function and recursive techniques. The rest of the paper is organized as follows. In the next section, the model is described by stating requisite notations and assumptions. In Sections 3 and 4, we construct the steady state equations and obtain the distribution for the queue size, respectively. Some performance measures are given in Section 5. In Section 6, we discuss the optimal N-policy by constructing the cost function. Special cases are deduced in section 7. Sensitivity analysis is carried out in Section 8 by taking numerical illustrations. In the final Section 9, we conclude the investigation.

2 Model Description

Consider an Earlangian queueing system under N-policy restriction and unreliable single service station. The notations and basic assumptions governing the model are stated below.

1. The service station renders service under N-policy restriction, according to which the service station starts service only when there are N-customers are accumulated in the system and keeps providing service until the system becomes empty.
2. The state of the system is defined by (n, i, j); where $n(n = 0, 1, 2...$ denotes the number of customers present in the system, the customer in service is in phase $i(i = 0, 1, 2, \ldots, k)$, and

$j(j = 0, b, d, 1, 2 \ldots, l)$ represents the state of the service station, respectively. The state j of the service station at any time t is stared as follows:

$$j = \begin{cases} 0, & \text{for turn-off state of the service station} \\ b, & \text{for turn-on and busy state of the service station} \\ d, & \text{for brokendown and under setup state of the service station} \\ m, & \text{for } m^{th} \text{ phase repair state of the server where } m = 1, 2, 3, ..., l \end{cases}$$

3. The customers arrive at the service station according to Poisson fashion with rate λ. The customers may balk on finding service station busy, under setup state when broken down, and under phase repair, with balking function (i.e. the joining probabilities) of customers γ_j when $j(j = 0, b, d, 1, 2, \ldots, l)$ denotes the state of the service station.
4. The service time is k-phase Earlangian distributed with service rate μ.
5. The life time and setup time of service station follow negative exponential distribution with mean $\frac{1}{\alpha}$ and $\frac{1}{\delta}$, respectively.
6. It is assumed that the repair time of m^{th} phase is negative exponentially distributed with mean $1/\beta_m(m = 1, 2, 3, \ldots, l)$.
7. After repair the service station performs its duty with same efficiency as before breakdown.
8. The service discipline is first come first served.

3 The Mathematical Formulation

The steady state probabilities for mathematical formulation of the model are given as follows:

$P^0_{0,0}$ The probability that there is no customer present in the system and service station is in turned off state.

$P^0_{n,k}$ The probability that there are $n(1 \leq n \leq N-1)$ customers present in the system but the service of customer is not initiated (i.e. $i = k$) as service station is in turned off state.

$P^b_{n,i}$ The probability that there are $n(\geq 1)$ customers present in the system, the customer is in $i^{th}(i = 1, 2, ..., k)$ phase of service and service station is in turned on busy in rendering service i.e. in operation.

$P^d_{n,i}$ The probability that there are $n(\geq 1)$ customers present in the system, the customer is in $i^{th}(i = 1, 2, ..., k)$ phase of service and the service station is in brokendown and under setup state.

$P^m_{n,i}$ The probability that there are $n(\geq 1)$ customers in the system, customer is in $i^{th}(i = 1, 2, ..., k)$ phase of service and the service station is receiving m^{th} phase repair $m = 1, 2, 3, ..., l$.

The steady states equations governing the model are obtained as follows:

$$\lambda P^0_{1,k} = \lambda P^0_{0,0}, \tag{3.1}$$

$$\lambda P^0_{n,k} = \lambda P^0_{n-1,k}; \;\; 2 \leq n \leq N-1, \tag{3.2}$$

$$\lambda P^0_{0,0} = k\mu P^b_{1,1}, \tag{3.3}$$

$$(\lambda\gamma_0 + \alpha + k\mu)\, P^b_{1,i} = k\mu P^b_{1,i+1} + \beta_l P^l_{1,i}; \;\; 1 \leq i \leq k-1, \tag{3.4}$$

$$(\lambda\gamma_0 + \alpha + k\mu)\, P^b_{1,k} = k\mu P^b_{2,1} + \beta_l P^l_{1,k}, \tag{3.5}$$

$$(\lambda\gamma_0 + \alpha + k\mu)\, P^b_{n,i} = \lambda\gamma_0 P^b_{n-1,i} + k\mu P^b_{n,i+1} + \beta_l P^l_{n,i}; \;\; 2 \leq n \leq N, \;\; 1 \leq i \leq k-1, \tag{3.6}$$

$$(\lambda\gamma_0 + \alpha + k\mu)\, P^b_{n,k} = \lambda\gamma_0 P^b_{n-1,k} + k\mu P^b_{n+1,1} + \beta_l P^l_{n,k}; \;\; 2 \leq n \leq N-1, \tag{3.7}$$

(3.8) $$(\lambda\gamma_0 + \alpha + k\mu) P^b_{N,k} = \lambda\gamma_0 P^b_{N-1,k} + k\mu P^b_{N+1,1} + \beta_l P^l_{N,k} + \lambda P^0_{N-1,k},$$

(3.9) $$(\lambda\gamma_0 + \alpha + k\mu) P^b_{n,i} = \lambda\gamma_0 P^b_{n-1,i} + k\mu P^b_{n,i+1} + \beta_l P^l_{n,i};\ n \geq N+1,\ 2 \leq i \leq k-1,$$

(3.10) $$(\lambda\gamma_0 + \alpha + k\mu) P^b_{n,k} = \lambda\gamma_0 P^b_{n-1,k} + k\mu P^b_{n+1,1} + \beta_l P^l_{n,k};\ n \geq N+1,$$

(3.11) $$(\lambda\gamma_1 + \delta) P^d_{1,i} = \alpha P^b_{1,i};\ 1 \leq i \leq k,$$

(3.12) $$(\lambda\gamma_1 + \delta) P^d_{n,i} = \lambda\gamma_1 P^d_{n-1,i} + \alpha P^b_{n,i};\ n \geq 2,\ 1 \leq i \leq k,$$

(3.13) $$(\lambda\gamma_2 + \beta_1) P^1_{1,i} = \delta P^d_{1,i};\ 1 \leq i \leq k,$$

(3.14) $$(\lambda\gamma_2 + \beta_1) P^1_{n,i} = \lambda\gamma_2 P^1_{n-1,i} + \delta P^d_{n,i};\ n \geq 2,\ 1 \leq i \leq k,$$

(3.15) $$\left(\lambda\gamma_{j+1} + \beta_j\right) P^j_{1,i} = \beta_{j-1} P^{j-1}_{1,i};\ 1 \leq i \leq k,\ j = 2, 3, ..., l,$$

(3.16) $$\left(\lambda\gamma_{j+1} + \beta_j\right) P^j_{n,i} = \lambda\gamma_{j+1} P^j_{n-1,i} + \beta_{j-1} P^{j-1}_{n,i};\ n \geq 2,\ 1 \leq i \leq k,\ j = 2, 3, ...l.$$

4 The Generating Function Method

By using the recursive technique it is not possible to obtain the explicit result for$P^0_{0,0}$, so we employ probability-generating function technique to obtain analytic solution in neat closed form. The partial probability generating functions are defined as

(4.1) $$X_i(z) = \sum_{n=1}^{\infty} P^b_{n,i} z^n;\ 1 \leq i \leq k,\ |z| < 1,$$

(4.2) $$Y^d_i(z) = \sum_{n=1}^{\infty} P^d_{n,i} z^n;\ 1 \leq i \leq k,\ |z| < 1,$$

(4.3) $$Y^j_i(z) = \sum_{n=1}^{\infty} P^j_{n,i} z^n;\ 1 \leq i \leq k,\ |z| < 1,\ \text{ and , } j = 1, 2, ..., l,$$

(4.4) $$G_0(z) = P^0_{0,0} + \sum_{n=1}^{N-1} P^0_{n,0} z^n;\ |z| \leq 1,$$

(4.5) $$G_1(z) = \sum_{i=1}^{k} X_i(z);\ |z| \leq 1,$$

(4.6) $$G_2(z) = \sum_{i=1}^{k} Y^d_i(z);\ |z| \leq 1,$$

(4.7) $$G^j_3(z) = \sum_{i=1}^{k} Y^j_i(z);\ |z| \leq 1,\ j = 1, 2, ..., l.$$

Equations (3.1), (3.2) and (4.3) yield

(4.8) $$G_0(z) = \frac{\left(1 - z^N\right)}{(1 - z)} P^0_{0,0}.$$

On multiplying equation (3.4) by z, equation (3.6) by $z^n (2 \leq n \leq N)$ and equation (3.9) by $z^n (n \geq N+1)$ respectively and summing over all n, we obtain

(4.9) $$X_{i+1}(z) = (1 + a_0 - b_0 z) X_i(z) - r_l Y^l_i(z), 1 \leq i \leq k,$$

$$a_0 = \frac{\lambda\gamma_0 + \alpha}{k\mu}, b_j = \frac{\lambda\gamma_j}{k\mu}(j = 0, 1, 2, ..., l+1), r_j = \frac{\beta_j}{k\mu} \text{ for } (j = 1, 2, ..., l).$$

Again, multiplying equation (3.5) by z, equation (3.7) by $z^n (2 \le n \le N-1)$, equation (3.8) by z^n and equation (3.10) by $z^n (n \ge N+1)$, respectively and summing over all n and simplifying, we obtain

$$X_1(z) = z\,(1 + a_0 - b_0 z)\, X_k(z) - r_l z Y_k^l(z) - pz\left(z^N - 1\right) P_{0,0}^0, \tag{4.10}$$

where $p = \frac{\lambda}{k\mu}$.

Now, multiplying equation (3.11) by z, equation (3.12) by $z^n (n \ge 2)$, summing over all n, we have

$$Y_i^d(z) = \frac{(a_0 - b_0)}{(b_1 + c - b_1 z)} X_i(z);\ \ 1 \le i \le k. \text{ where } c = \frac{\delta}{k\mu}. \tag{4.11}$$

Similarly, multiplying equation (3.13) by z, equation (3.4) by $z^n (n \ge 2)$, summing over all n, we get

$$Y_i^1(z) = \frac{c}{(b_2 + r_1 - b_2 z)} Y_i^d(z);\ \ 1 \le i \le k. \tag{4.12}$$

In the similar manner, we obtain

$$Y_i^2(z) = \frac{r_1}{(b_3 + r_2 - b_3 z)} Y_i^1(z);\ \ 1 \le i \le k. \tag{4.13}$$

In general, multiplying equations (3.15) and (3.16) by the appropriate power of z and summing over all n, one can have

$$Y_i^j(z) = \frac{r_{j-1}}{\left(b_{j+1} + r_j - b_{j+1} z\right)} Y_i^{j-1}(z);\ \ 1 \le i \le k,\ \ j = 3, 4, ..., l. \tag{4.14}$$

By using equation, (4.9), (4.10) and (4.11) in (4.14), we have

$$Y_i^l(z) = \frac{c\,(a_0 - b_0) \prod_{k=1}^{l-1} r_k}{(b_1 + c - b_1 z) \displaystyle\prod_{k=1}^{l} (b_{k+1} + r_k - b_{k+1} z)} X_i(z);\ \ 1 \le i \le k. \tag{4.15}$$

Substituting the value from equation (4.15) in equation (4.9), we obtain

$$X_{i+1}(z) = w(z) X_i(z) \tag{4.16}$$

where

$$w(z) = (1 + a_0 - b_0 z) - \frac{c\,(a_0 - b_0) \displaystyle\prod_{k=1}^{l} r_k}{(b_1 + c - b_1 z) \displaystyle\prod_{k=1}^{l} (b_{k+1} + r_k - b_{k+1} z)}.$$

Here,

$$w(1) = 1, \tag{4.17}$$

(4.18)

$$w'(z) = -b_0 - \frac{c(a_0 - b_0) \displaystyle\prod_{k=1}^{l} r_k \left\{ b_1 \prod_{k=1}^{l} (b_{k+1} + r_k - b_{k+1} z) + (b_1 + c - b_1 z) \left(\sum_{m=1}^{l} b_m \prod_{\substack{k=1 \\ k \ne m}}^{l} (b_{k+1} + r_k - b_{k+1} z) \right) \right\}}{\left\{ (b_1 + c - b_1 z) \displaystyle\prod_{k=1}^{l} (b_{k+1} + r_k - b_{k+1} z) \right\}^2},$$

$$w'(1) = \frac{(b_0 - a_0)\left\{b_1 \prod_{k=1}^{l}(r_k) + c\left(\sum_{m=1}^{l} b_m \prod_{\substack{k=1\\k\neq m}}^{l}(r_k)\right)\right\} - b_0 c \prod_{k=1}^{l}(r_k)}{c\prod_{k=1}^{l}(r_k)}, \tag{4.19}$$

(4.20)

$$w''(z) = \frac{\begin{aligned} & c\,(a_0 - b_0)\prod_{k=1}^{l} r_k \times \left[\left\{(b_1 + c - b_1 z)\prod_{k=1}^{l}(b_{k+1} + r_k - b_{k+1}z)\right\}^2 \left[b_1\left\{\sum_{m=1}^{l} b_m \prod_{\substack{k=1\\k\neq m}}^{l}(b_{k+1} + r_k - b_{k+1}z)\right\}\right.\right. \\ & \left.+2b_1\left(\sum_{m=1}^{l} b_m \prod_{\substack{k=1\\k\neq m}}^{l}(b_{k+1} + r_k - b_{k+1}z)\right) + (b_1 + c - b_1 z)\sum_{p=1}^{l} b_p \left\{\sum_{m=1}^{l} b_m \prod_{\substack{k=1\\k\neq m}}^{l}(b_{k+1} + r_k - b_{k+1}z)\right\}\right] \\ & +2\left\{(b_1 + c - b_1 z)\prod_{k=1}^{l}(b_{k+1} + r_k - b_{k+1}z)\right\}\times \\ & \left.\left[b_1\prod_{k=1}^{l}(b_{k+1} + r_k - b_{k+1}z) + (b_1 + c - b_1 z)\sum_{m=1}^{l} b_m \prod_{\substack{k=1\\k\neq m}}^{l}(b_{k+1} + r_k - b_{k+1}z)\right]^2\right] \end{aligned}}{\left\{(b_1 + c - b_1 z)\prod_{k=1}^{l}(b_{k+1} + r_k - b_{k+1}z)\right\}^4}$$

(4.21)

$$w''(z) = \frac{c\prod_{k=1}^{l} r_k\left\{2b_1\left(\sum_{m=1}^{l} b_m \prod_{\substack{k=1\\k\neq m}}^{l}(r_k)\right) + c\left(\sum_{p=1}^{l} b_p\left\{\sum_{m=1}^{l} b_m \prod_{\substack{k=1\\k\neq m}}^{l}(r_k)\right\}\right)\right\} + 2\left[b_1\prod_{\substack{k=1\\k\neq m}}^{l}(r_k) + c\left\{\sum_{m=1}^{l} b_m \prod_{\substack{k=1\\k\neq m}}^{l}(r_k)\right\}\right]^2}{\left\{c\prod_{k=1}^{l} r_k\right\}^2}.$$

Equation (4.16) gives

$$X_i(z) = w^{i-1}(z)X_1(z);\ 1 \le i \le k. \tag{4.22}$$

On putting $i = k$ in equation (4.21), we have

$$X_k(z) = w^{k-1}(z)X_1(z). \tag{4.23}$$

Substituting the values from equations (4.23) and (4.15) in equation (4.10), and after simplification, we get

$$X_1(z) = \frac{zp\left(1 - z^N\right)}{(1 - zw^k(z))}P_{0,0}^0. \tag{4.24}$$

Using above equation in equation (4.22), we get

$$X_i(z) = \frac{zp\left(1 - z^N\right)w^{i-1}(z)}{(1 - zw^k(z))}P_{0,0}^0. \tag{4.25}$$

Now we have

$$G_1(z) = \frac{zp\left(1 - z^N\right)}{(1 - zw^k(z))}\left(\frac{1 - w^k(z)}{1 - w(z)}\right)P^0_{0,0}, \tag{4.26}$$

$$G_2(z) = \frac{(a_0 - b_0)}{(b_1 + c - b_1 z)}G_1(z), \tag{4.27}$$

$$G_3^j(z) = \frac{c\,(a_0 - b_0)\prod_{k=1}^{l-1} r_k}{(b_1 + c - b_1 z)\prod_{k=1}^{l}(b_{k+1} + r_k - b_{k+1}z)}G_1(z);\ \text{for } j = 1, 2, ..., l. \tag{4.28}$$

Evaluation of $P^0_{0,0}$

To obtain the value of $P^0_{0,0}$, we use the normalizing condition

$$G\,(1) = G_0\,(1) + G_1\,(1) + G_2\,(1) + \sum_{j=1}^{l} G_3^j\,(1) = 1. \tag{4.29}$$

Using equations (4.4), (4.25), (4.26) and (4.27) in equation (4.28) and applying the L-Hospital rule to get the limiting values when z→1, we get

$$G_0\,(1) = NP^0_{0,0,} \tag{4.30}$$

$$G_1\,(1) = \frac{Npk}{(1 + kw'\,(1))}P^0_{0,0}, \tag{4.31}$$

$$G_2\,(1) = \frac{(a_0 - b_0)\,Npk}{c\,(1 + kw'\,(1))}P^0_{0,0}, \tag{4.32}$$

$$G_3^j\,(1) = \frac{(a_0 - b_0)\,Npk}{r_j\,(1 + kw'\,(1))}P^0_{0,0}; \quad j = 1, 2, 3...., l \tag{4.33}$$

and

$$P^0_{0,0} = \frac{1}{N\left[1 + \frac{pk}{(1+kw'(1))} + \frac{(a_0-b_0)pk}{(1+kw'(1))}\left\{\frac{1}{c} + \sum_{j=1}^{l}\frac{1}{r_j}\right\}\right]}. \tag{4.34}$$

5 Performance Measures

In order to derive expressions for various performance measures, we use the generating functions. Let P_I, P_B, P_D and P_R^m denote the long run fraction of time for which service station is idle, busy, breakdown and under setup, and under j^{th} phase repair ($m = 1, 2, 3, \ldots, l$) states respectively. We compute the long run probabilities P_I, P_B, P_D, and $P_R^m(m = 1, 2, 3, \ldots, l)$, respectively in the following manner:

$$P_I = \lim_{z\to 1} G_0(z) = NP^0_{0,0} = \frac{1}{\left[1 + \frac{pk}{(1+kw'(1))} + \frac{(a_0-b_0)pk}{(1+kw'(1))}\left\{\frac{1}{c} + \sum_{j=1}^{l}\frac{1}{r_j}\right\}\right]}, \tag{5.1}$$

$$P_B = \lim_{z\to 1} G_1(z) = \frac{pk}{(1 + kw'\,(1))\left[1 + \frac{pk}{(1+kw'(1))} + \frac{(a_0-b_0)pk}{(1+kw'(1))}\left\{\frac{1}{c} + \sum_{j=1}^{l}\frac{1}{r_j}\right\}\right]}, \tag{5.2}$$

$$P_D = \lim_{z\to 1} G_2(z) = \frac{(a_0 - b_0)\,pk}{c\,(1 + kw'\,(1))\left[1 + \frac{pk}{(1+kw'(1))} + \frac{(a_0-b_0)pk}{(1+kw'(1))}\left\{\frac{1}{c} + \sum_{j=1}^{l}\frac{1}{r_j}\right\}\right]}, \tag{5.3}$$

Similarly

(5.4)

$$P_R^m = \lim_{z\to 1} G_3^m(z) = \frac{(a_0 - b_0)\, pk}{r_m\,(1 + kw'(1))\left[1 + \frac{pk}{(1+kw'(1))} + \frac{(a_0-b_0)pk}{(1+kw'(1))}\left\{\frac{1}{c} + \sum_{j=1}^{l}\frac{1}{r_j}\right\}\right]}; (m = 1,\ 2,\ 3, ..., l).$$

Expected number of customers in the system when the service station is idle, is obtained as

$$E[N_0] = G_0'(z)\,|_{z=1} = \frac{N\,(N-1)\,P_{0,0}^0}{2}. \tag{5.5}$$

Similarly, we can compute the expected number of customers in the system when the service station is busy, breakdown and under setup, and under m^{th} phase ($m = 1,\ 2,\ 3, ..., l$) repair state by the following formulae:

$$E[N_1] = G_1'(z)\,|_{z=1} = \frac{q[MR - LQ]P_{0,0}^0}{2[R]^2}, \tag{5.6}$$

$$E[N_2] = G_2'(z)|_{z=1} = \frac{\begin{array}{l} c'(1)d'(1)[2a'(1)b'(1) + a''(1)b'(1) + a'(1)b''(1)]\times \\ -a'(1)b'(1)[c''(1)d'(1) + c'(1)d''(1)]\end{array}}{2\{c'(1)d'(1)\}^2}, \tag{5.7}$$

$$E[N_{j+1}] = (G_3^j(z))'\big|_{z=1} = \frac{\begin{array}{l} 2c'(1)d'(1)e(1)[2a'(1)b'(1) + a''(1)b'(1) + a'(1)b''(1)] - \times \\ a'(1)b'(1)[c''(1)d'(1)e(1) + 2c'(1)d''(1)e'(1)]\end{array}}{12\{c'(1)d'(1)e(1)\}^2} \tag{5.8}$$

$$(j = 2,\ 3, ..., l + 1),$$

where $a(z) = 1 - z^N, b(z) = 1 - W^k(z), c(z) = 1 - zW^k(z), d(z) = 1 - W(z), e(z) = b_1 + c - b_1 z$.

Then the expected number of customers in the system is given by

$$E[N] = E[N_0] + E[N_1] + E[N_2] + \sum_{j=1}^{l} E[N_{j+2}]. \tag{5.9}$$

6 Optimal N-Policy

Let the expected length of the idle period, busy period, breakdown period under m^{th} phase repair ($m = 1,\ 2,\ 3, ..., l$) period and busy cycle be denoted by $E[I], E[B], E[D], E[R^m] (m = 1, 2, 3, ..., l)$ and $E[C]$ respectively. We have

$$E[C] = E[B] + E[I] + E[D] + \sum_{m=1}^{l} E[R^m]. \tag{6.1}$$

The length of the idle period is the sum of N exponential distributed random variables with mean $1/\lambda$. This implies that

$$E[I] = \frac{N}{\lambda}. \tag{6.2}$$

Also

$$P_I = \frac{E[I]}{E[C]}, P_B = \frac{E[B]}{E[C]}, P_D = \frac{E[D]}{E[C]} \text{ and } P_R^m = \frac{E[R^m]}{E[C]}\ (m = 1,\ 2,\ 3, ..., l). \tag{6.3}$$

Now, we define the following cost elements to determine the optimal value of control parameters N for an Erlangian queueing model with k-phase service and l-phase repair under N-policy restriction.

C_d holding cost per unit time per customer present in the system,
C_u start up cost per unit time for turning the service station on,
C_B shut down cost per unit time for turning the service station off,
C_I cost per unit time of the service station in idle state,
C_D cost per unit time of the service station in broken down state when repairman is under set up state,
C_R^m repair cost per unit time rendering m^{th} phaserepair ($m = 1, 2, 3, ..., l$),

The expected total cost per unit time is formulated as:

$$E(TC) = C_h E[N_s] + (C_u + C_d)\frac{1}{E[C]} + C_I P_I + C_B P_B + C_D P_D + \sum_{m=1}^{l} C_R^{,m} P_R^m. \tag{6.4}$$

The optimal value (say N^*) of the decision variable N, could be determined by setting

$$\frac{d\{E(TC)\}}{dN} = 0. \tag{6.5}$$

In case when N^* is not an integer, then the best positive integer value N^* is achieved by rounding off the N^*.

7 Special Cases

Case I: When $\lambda = \lambda_0, \lambda\gamma_1 = \lambda_1, \lambda\gamma_2 = \lambda_2$, $l = 1$, and $\delta = 0$, then we get the results for this model.

Case II: If $\lambda = \lambda\gamma_1 = \lambda\gamma_2 = \Lambda$, $l = 1$, and $\delta = 0$, then we get results for N-policy $M/M/1$ queueing system with breakdown.

Case III: In case when $\gamma_1 = \gamma_2 = 1$, $l = 1$, $\delta = 0$, $\alpha = 0$, and $\beta = 1$ then our model coincides with the model developed by Wang and Huang [16].

8 Sensitivity Analysis

In order to show the validity of analytical results, we perform extensive numerical experiment by using MATLAB. The effects of different parameters on the average queue length are shown in ***Figs.*** **8.1-8.8**. The numerical results of the expected total cost with the variation of different parameters are presented in ***Tables*** **8.1** and **8.2**.

The effect of arrival rate (λ), failure rate (α), service rate (μ), setup rate (σ), repair rate of first phase (β_i), repair rate of second phase (β_2), optimal threshold parameter N^* and number of phases of service (k) respectively, on $E(TC)$ are examined for different sets of cost elements which are given as follows:

Set 1:	C_U=10,	C_F =5,	C_I=5,	C_B=10,	C_D=2,	C_H=5,	C_{R1}=2	C_{R2}=1
Set 2:	C_U=20,	C_F =5,	C_I=5,	C_B=10,	C_D=2,	C_H=5,	C_{R1}=2	C_{R2}=1
Set 3:	C_U=10,	C_F =10,	C_I=5,	C_B=10,	C_D=2,	C_H=5,	C_{R1}=2	C_{R2}=1
Set 4:	C_U=10,	C_F =5,	C_I=10,	C_B=10,	C_D=2,	C_H=5,	C_{R1}=2	C_{R2}=1
Set 5:	C_U=10,	C_F =5,	C_I=5,	C_B=20,	C_D=2,	C_H=5,	C_{R1}=2	C_{R2}=1
Set 6:	C_U=10,	C_F =5,	C_I=5,	C_B=10,	C_D=4,	C_H=5,	C_{R1}=2	C_{R2}=1
Set 7:	C_U=10,	C_F =5,	C_I=5,	C_B=10,	C_D=2,	C_H=10,	C_{R1}=2	C_{R2}=1
Set 8:	C_U=10,	C_F =5,	C_I=5,	C_B=10,	C_D=2,	C_H=5,	C_{R1}=4	C_{R2}=1
Set 9:	C_U=10,	C_F =5,	C_I=5,	C_B=10,	C_D=2,	C_H=5,	C_{R1}=2	C_{R2}=2

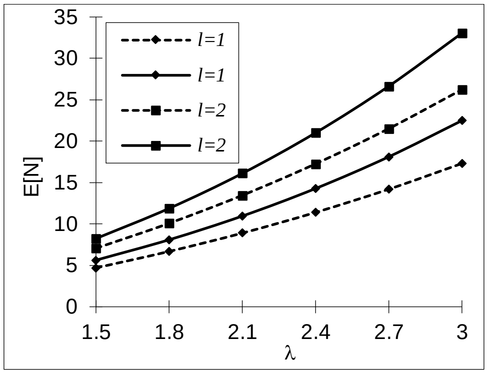

Fig. 8.1: Expected queue length vs. λ

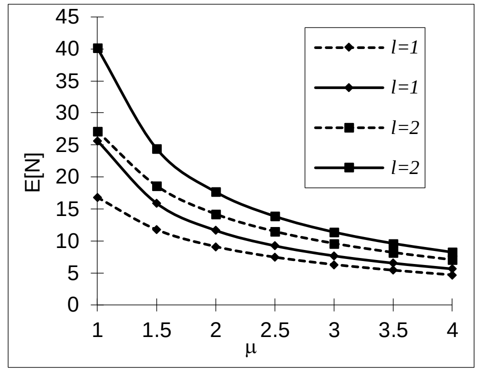

Fig. 8.2: Expected queue length vs. μ

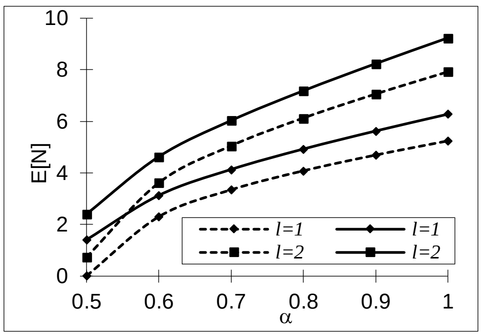

Fig. 8.3: Expected queue length vs. α

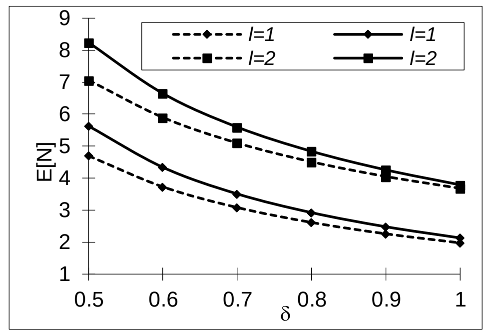

Fig. 8.4: Expected queue length vs. δ

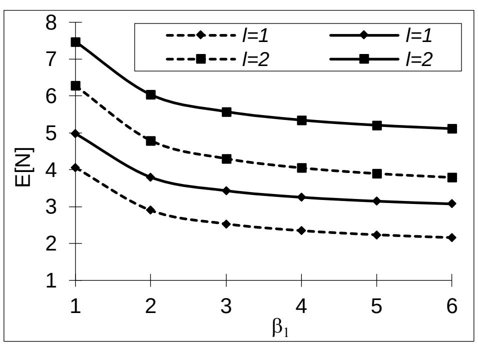

Fig. 8.5: Expected queue length vs. β_1

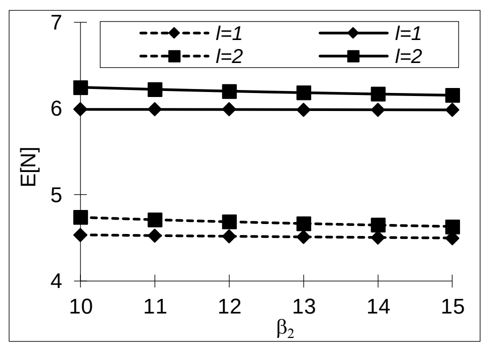

Fig. 8.6: Expected queue length vs. β_2

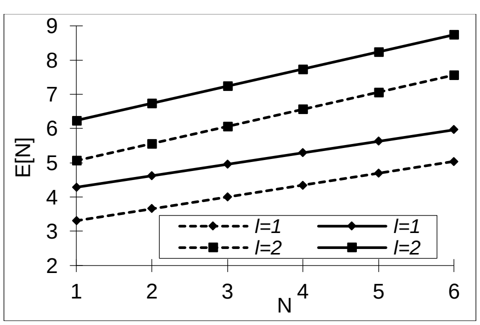

Fig. 8.7: Expected queue length vs. N

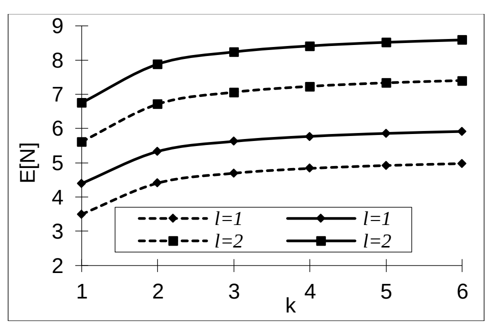

Fig. 8.8: Expected queue length vs. k

From ***Table* 8.1** it is observed that by increasing λ, α, N and k, the expected total cost increases. But as we increase the parameters μ, β_1, β_2 and δ the expected total cost decreases as can be seen from ***Table* 8.2**.

***Table* 8.1:** Effect of parameters (λ, α, k, N) on the expected total cost for different sets of cost elements

λ	SET 1	SET 2	SET 3	SET 4	SET 5	SET 6	SET 7	SET 8	SET 9
1.00	6.69	6.31	6.50	5.76	9.19	7.59	10.69	7.25	6.89
1.20	21.69	20.68	21.18	19.58	24.69	22.77	42.02	22.37	21.94
1.40	30.24	28.39	29.32	26.94	33.74	31.50	60.72	31.03	30.53
1.60	38.08	35.21	36.64	33.59	42.08	39.52	78.28	38.98	38.41
1.80	46.14	42.05	44.10	40.47	50.64	47.76	96.57	47.15	46.51
2.00	54.71	49.22	51.96	47.85	59.71	56.51	116.17	55.83	55.12
α	SET 1	SET 2	SET 3	SET 4	SET 5	SET 6	SET 7	SET 8	SET 9
0.50	7.35	6.89	7.12	6.58	11.10	8.10	11.04	7.82	7.52
0.60	20.56	19.62	20.09	19.00	24.31	21.46	38.65	21.12	20.76
0.70	26.53	25.13	25.83	24.19	30.28	27.58	51.80	27.18	26.77
0.80	30.71	28.84	29.77	27.59	34.46	31.91	61.35	31.46	30.98
0.90	34.16	31.83	33.00	30.27	37.91	35.51	69.48	35.01	34.47
1.00	37.27	34.46	35.86	32.59	41.02	38.77	76.88	38.20	37.61
N	SET 1	SET 2	SET 3	SET 4	SET 5	SET 6	SET 7	SET 8	SET 9
1.00	10.15	1.53	4.31	6.25	13.90	11.50	35.46	10.99	10.46
2.00	21.41	15.57	18.49	17.52	25.16	22.76	49.22	22.25	21.72
3.00	26.83	22.94	24.88	22.94	30.58	28.18	57.14	27.67	27.14
4.00	30.79	27.87	29.33	26.90	34.54	32.14	63.60	31.63	31.10
5.00	34.16	31.83	33.00	30.27	37.91	35.51	69.48	35.01	34.47
6.00	37.25	35.30	36.28	33.36	41.00	38.60	75.06	38.09	37.56
k	SET 1	SET 2	SET 3	SET 4	SET 5	SET 6	SET 7	SET 8	SET 9
1.00	26.94	24.60	25.77	23.05	30.69	28.29	55.03	27.78	27.25
2.00	32.43	30.09	31.26	28.54	36.18	33.78	66.01	33.27	32.74
3.00	34.16	31.83	33.00	30.27	37.91	35.51	69.48	35.01	34.47
4.00	35.02	32.69	33.85	31.13	38.77	36.37	71.19	35.87	35.33
5.00	35.53	33.20	34.36	31.64	39.28	36.88	72.21	36.38	35.84
6.00	35.87	33.54	34.70	31.98	39.62	37.22	72.89	36.72	36.18

***Table* 8.2:** Effect of parameters $(\mu, \delta, \beta_i, \beta_2)$ on the expected total cost for different sets of cost elements.

μ	SET 1	SET 2	SET 3	SET 4	SET 5	SET 6	SET 7	SET 8	SET 9
1.00	102.53	84.19	93.36	71.96	117.53	107.93	238.15	105.91	103.76
2.00	59.29	51.61	55.45	46.50	66.79	61.99	130.37	60.97	59.90
3.00	43.42	39.31	41.37	36.57	48.42	45.22	91.54	44.55	43.83
4.00	34.16	31.83	33.00	30.27	37.91	35.51	69.48	35.01	34.47
5.00	26.11	24.84	25.48	24.00	29.11	27.19	51.24	26.79	26.36
6.00	13.08	12.52	12.80	12.15	15.58	13.98	23.76	13.64	13.28
δ	SET 1	SET 2	SET 3	SET 4	SET 5	SET 6	SET 7	SET 8	SET 9
0.50	34.16	31.83	33.00	30.27	37.91	35.51	69.48	35.01	34.47
0.60	29.18	27.18	28.18	25.85	32.93	30.31	58.67	30.02	29.49
0.70	25.79	24.04	24.92	22.87	29.54	26.76	51.29	26.64	26.10
0.80	23.31	21.73	22.52	20.68	27.06	24.15	45.86	24.15	23.61
0.90	21.37	19.93	20.65	18.97	25.12	22.12	41.63	22.21	21.67
1.00	19.79	18.47	19.13	17.58	23.54	20.46	38.19	20.63	20.10
β_1	SET 1	SET 2	SET 3	SET 4	SET 5	SET 6	SET 7	SET 8	SET 9
1.00	30.87	28.79	29.83	27.40	34.62	32.22	62.25	31.54	31.18
2.00	24.71	23.14	23.92	22.09	28.46	26.06	48.67	25.05	25.02
3.00	22.68	21.27	21.98	20.34	26.43	24.03	44.19	22.91	22.99
4.00	21.64	20.31	20.98	19.43	25.39	22.99	41.89	21.81	21.94
5.00	20.99	19.72	20.35	18.87	24.74	22.34	40.47	21.13	21.30
6.00	20.55	19.31	19.93	18.48	24.30	21.90	39.50	20.66	20.85
β_2	SET 1	SET 2	SET 3	SET 4	SET 5	SET 6	SET 7	SET 8	SET 9
1.00	35.31	32.88	34.09	31.26	39.06	36.66	72.02	36.15	35.64
2.00	29.35	27.43	28.39	26.15	33.10	30.70	58.67	30.19	29.52
3.00	27.50	25.74	26.62	24.57	31.25	28.85	54.49	28.34	27.61
4.00	26.58	24.91	25.74	23.79	30.33	27.93	52.41	27.42	26.66
5.00	26.02	24.40	25.21	23.32	29.77	27.37	51.15	26.86	26.09
6.00	25.65	24.06	24.85	23.01	29.40	27.00	50.31	26.49	25.70

***Figs* 8.1-8.8** depict the effect of parameters $\lambda, \alpha, \mu, \delta, \beta_1, \beta_2, N$ and k respectively, on the average queue length. From all the graphs it is observed that the average queue length is higher for heterogeneous arrival rates in comparison of homogeneous arrival rates. Also, it is noticed that average queue length increases as we increase the number of phases of repair. From ***Figs.* 8.1, 8.2** we examine the effect of arrival rate (service rate) on the average queue length and observed that $E[N]$ increases (decreases) with the increase in λ and μ however the effect of μ on $E[N]$ is more prominent for lower values. Substantially the same effects with respect to failure rate (setup rate) have been seen in ***Figs.* 8.3**, **8.4**. In ***Figs.* 8.5** and **8.6**, the average queue length $E[N]$ is plotted against the parameters β_1 & β_2. As we expect, initially $E[N]$ decreases sharply by increasing β_1 but after some time it becomes almost constant; moreover the decreasing effect due to increment in β_2 is almost not negligible. ***Fig.* 8.7** illustrates the effect of threshold parameter N on the average queue length and we notice that $E[N]$ increases linearly as N increases. From ***Fig.* 8.8** it is seen that initially $E[N]$ increases sharply and then after slowly as we increase k.

From the tables and graphs, overall we conclude that the average queue length increases as

λ, α, N and k increase but decreases as μ, δ, β_1, and β_2 increase, which is in agreement with physical situations.

9 Concluding Remarks

In this paper we have analysed an Erlangian queueing model with phase service and phase repair under *N*-policy. We have employed the generating function approach for computing the steady state probability distribution for various performance measures. The cost analysis facilitated may be helpful to assist the decision makers in determining the optimal value of threshold parameter N, so as to minimize total expected cost per unit time. The incorporation of balking behavior of the customers makes our model more realistic to depict the day-to-day as well as industrial congestion situations.

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