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(Dedicated to Honor Professor H.M. Srivastava on His 80th Birth Anniversary Celebrations)

SUBCLASS OF MEROMORPHIC FUNCTIONS WITH POSITIVE COEFFICIENTS INVOLVING POLYLOGARITHM FUNCTION

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Abstract

In this paper, we introduce and study a new subclass of meromorphic functions with positive coefficients involving the polylogaritham function and obtain coefficient estimates, growth and distortion theorem, radius of convexity, integral transforms, convex linear combinations and convolution properties for the class $\sigma_{c,p}(\alpha, \beta, \lambda)$.
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1 Introduction

Historically,the classical polylogarithm function was invented in 1696, by Leibnitz and Bernoulli, as mentioned in [3]. For $|z| < 1$ and c a natural number with $c \geq 2$, the polylogarithm function (which is also known as Jonquiere's function) is defined by the absolutely convergent series:

$$Li_c(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^c}. \tag{1.1}$$

Later on, many mathematicians studied the polylogarithm function such as Euler, Spence, Abel, Lobachevsky, Rogers, Ramanujan and many others [6], where they discovered many functional identities by using polylogarithm function. However, the work employing polylogarithm has been stopped many decades later. During the past four decades, the work using polylogarithm has again been intensified vividly due to its importance in many fields of mathematics, such as complex analysis, algebra, geometry, topology, and mathematical physics (quantum field theory) [5, 7, 9]. In [10], Ponnusamy and Sabapathy discussed the geometric mapping properties of the generalized polylogarithm. Recently, Al-Shaqsi and Darus [1] generalized Ruscheweyh and Salagean operators, using polylogarithm functions on class A of analytic functions in the open unit disk $U = \{z : |z| < 1\}$. By making use of the generalized operator they introduced certain new subclasses of A and investigated many related polylogarithm function to define a multiplier transformation on the class A in U [2]. To the best of our knowledge, no research work has discussed the polylogarithm function conjunction with meromorphic functions. Thus, in this present paper, we redefine the polylogarithm function to be on meromorphic type. Let $\sum$ denote the class of functions of the form

$$f(z) = \frac{1}{z} + \sum_{m=0}^{\infty} a_m z^m, \tag{1.2}$$

which are analytic in the punctured open unit disk

$$U^* := \{z : z \in C,\ 0 < |z| < 1\} = U \setminus \{0\}. \tag{1.3}$$

A function f in $\sum$ is said to be meromorphically starlike of order δ if and only if

$$\Re\Big\{-\frac{zf'(z)}{f(z)}\Big\} > \delta;\ (z \in U^*), \tag{1.4}$$

for some $\delta\ (0 \le \delta < 1)$. We denote by $\sum(\delta)$ the class of all meromorphically starlike order δ. Furthermore, a function f in $\sum$ is said to be meromorphically convex of order δ if and only if

$$\Re\Big\{-\Big(1 + \frac{zf''(z)}{f'(z)}\Big)\Big\} > \delta;\ (z \in U^*), \tag{1.5}$$

for some δ, $(0 \le \delta < 1)$. We denote by $\sum_k(\delta)$ the class of all meromorphically convex order δ. For functions $f \in \sum$ given by (1.2) and $g \in \sum$

$$g(z) = \frac{1}{z} + \sum_{m=0}^{\infty} b_m z^m, \tag{1.6}$$

we define the Hadamard product (or convolution) of f and g by

$$(f * g)(z) = \frac{1}{z} + \sum_{m=0}^{\infty} a_m b_m z^m. \tag{1.7}$$

Let $\sum_p$ be the class of functions of the form

$$f(z) = \frac{1}{z} + \sum_{m=0}^{\infty} a_m z^m;\ a_m \ge 0 \tag{1.8}$$

which are analytic and univalent in U^*. Liu and Srivastava [8] defined a function $h_p(\alpha_1, \cdots, \alpha_q; \beta_1, \cdots, \beta_s; z)$ by multiplying the well known generalized hypergeometric function ${}_qF_s$, with z^{-p} as follows:

$$h_p(\alpha_1, \cdots, \alpha_q; \beta_1, \cdots, \beta_s; z) = z^{-p}\ {}_qF_s(\alpha_1, \cdots, \alpha_q; \beta_1, \cdots, \beta_s; z), \tag{1.9}$$

where $\alpha_1, \cdots, \alpha_q; \beta_1, \cdots, \beta_s$ are complex parameters and $q \le s + 1$, $p \in N$. Analogous to Liu and Srivastava work [8] and corresponding to a function $\phi_c(z)$ given by

$$\phi_c(z) = z^{-2} Li_c(z) = \frac{1}{z} + \sum_{m=0}^{\infty} \frac{1}{(m+2)^c} z^m. \tag{1.10}$$

We consider a linear operator $\Omega_c f(z) : \sum \to \sum$ which is defined by the following Hadamard product (or Convolution):

$$\begin{aligned} \Omega_c f(z) &= \phi_c(z) * f(z) \\ &= \frac{1}{z} + \sum_{m=0}^{\infty} \frac{1}{(m+2)^c} a_m z^m. \end{aligned} \tag{1.11}$$

Next, we define the linear operator $\mathfrak{D}_c f(z) : \sum \to \sum$ as follows:

$$\mathfrak{D}_c f(z) = \{\Omega_c f(z) - \frac{1}{2^c} a_0\} = \frac{1}{z} + \sum_{m=1}^{\infty} \frac{1}{(m+2)^c} a_m z^m. \tag{1.12}$$

For function f in the class $\sum_p$, we define a linear operator $\mathfrak{D}^n_{c,\,\lambda} f(z)$ as follows

$$\begin{aligned}
\mathfrak{D}^0_{c,\,\lambda} f(z) &= f(z),\\
\mathfrak{D}^1_{c,\,\lambda} f(z) &= (1-\lambda)\mathfrak{D}_c f(z) + \lambda \frac{(z^2 \mathfrak{D}_c f(z))'}{z} \ \lambda \geq 0,\\
&= (1+\lambda)\mathfrak{D}_c f(z) + \lambda z (\mathfrak{D}_c f(z)') = \mathfrak{D}_{c,\,\lambda} f(z),\\
\mathfrak{D}^2_{c,\,\lambda} f(z) &= \mathfrak{D}_{c,\,\lambda} f(z)(\mathfrak{D}^1_{c,\,\lambda} f(z)),\\
&\ \vdots\\
\mathfrak{D}^n_{c,\,\lambda} f(z) &= \mathfrak{D}_{c,\,\lambda} f(z)(\mathfrak{D}^{n-1}_{c,\,\lambda} f(z)),\\
&= \frac{1}{z} + \sum_{m=1}^{\infty} \frac{[1+\lambda(m+1)]^n}{(m+2)^c} a_m z^m \text{ for } n = 1, 2, \cdots .
\end{aligned} \tag{1.13}$$

Now, by making use of operator $\mathfrak{D}^n_{c,\,\lambda} f(z)$, we define a new subclass of functions in $\sum_p$ as follows.

***Definition* 1.1.** *For* $-1 \leq \alpha < 1$, $\beta \geq 1$, *and* $\lambda \geq 0$ *we let* $\sigma_{c,p}(\alpha, \beta, \lambda)$ *be the subclass of* $\sum_p$ *consisting of functions of the form* (1.8) *and satisfying the analytic criterion*

$$-Re\left\{\frac{z(\mathfrak{D}^n_{c,\,\lambda} f(z))'}{\mathfrak{D}^n_{c,\,\lambda} f(z)} + \alpha\right\} > \beta \left|\frac{z(\mathfrak{D}^n_{c,\,\lambda} f(z))'}{\mathfrak{D}^n_{c,\,\lambda} f(z)} + 1\right|. \tag{1.14}$$

$\mathfrak{D}^n_{c,\,\lambda} f(z)$ *is given by* (1.13). *The main object of the paper is to study some usual properties of the geometric function theory such as coefficient bounds, growth and distortion properties, radius of convexity, convex linear combination and convolution properties, and integral operators for the class* $\sigma_{c,p}(\alpha, \beta, \lambda)$.

2 Coefficient inequality

***Theorem* 2.1.** *A function* f *of the form* (1.8) *is in* $\sigma_{c,p}(\alpha, \beta, \lambda)$ *if*

$$\sum_{m=1}^{\infty} \frac{[1+\lambda(m+1)]^n[(1+\beta)(m+1)+1-\alpha]}{(m+2)^c} |a_m| \leq 1-\alpha, \tag{2.1}$$

$-1 \leq \alpha < 1$, $\beta \geq 1$ *and* $\lambda \geq 0$.

Proof. It is sufficient to show that

$$\beta\left|\frac{z(\mathfrak{D}^n_{c,\,\lambda} f(z))'}{\mathfrak{D}^n_{c,\,\lambda} f(z)} + 1\right| + Re\left\{\frac{z(\mathfrak{D}^n_{c,\,\lambda} f(z))'}{\mathfrak{D}^n_{c,\,\lambda} f(z)} + 1\right\} \leq 1-\alpha.$$

We have

$$\beta\left|\frac{z(\mathfrak{D}^n_{c,\,\lambda} f(z))'}{\mathfrak{D}^n_{c,\,\lambda} f(z)} + 1\right| + Re\left\{\frac{z(\mathfrak{D}^n_{c,\,\lambda} f(z))'}{\mathfrak{D}^n_{c,\,\lambda} f(z)} + 1\right\}$$

$$\leq (1+\beta)\left|\frac{z(\mathfrak{D}^n_{c,\,\lambda} f(z))'}{\mathfrak{D}_c f(z)}+1\right|$$

$$\leq \frac{(1+\beta)\sum\limits_{m=1}^{\infty}\frac{[1+\lambda(m+1)]^n}{(m+2)^c}(m+1)|a_m||z|^m}{\frac{1}{|z|}-\sum\limits_{m=1}^{\infty}\frac{[1+\lambda(m+1)]^n}{(m+2)^c}|a_m||z|^m}.$$

Letting $z \to 1$ along the real axis, we obtain

$$\leq \frac{(1+\beta)\sum\limits_{m=1}^{\infty}\frac{[1+\lambda(m+1)]^n}{(m+2)^c}(m+1)|a_m|}{1-\sum\limits_{m=1}^{\infty}\frac{[1+\lambda(m+1)]^n}{(m+2)^c}|a_m|}.$$

The last expression is bounded by $(1-\alpha)$ if

$$\sum_{m=1}^{\infty}\frac{[1+\lambda(m+1)]^n[(1+\beta)(m+1)+1-\alpha]}{(m+2)^c}|a_m| \leq 1-\alpha.$$

Hence the theorem is proved.

***Corollary* 2.1.** *Let the function f defined by* (1.8) *be in the class* $\sigma_{c,p}(\alpha,\beta,\lambda)$. *Then*

$$a_m \leq \sum_{m=1}^{\infty}\frac{(m+2)^c(1-\alpha)}{[1+\lambda(m+1)]^n[(1+\beta)(m+1)+1-\alpha]},\ (m \geq 1). \tag{2.2}$$

Equality holds for the functions of the form

$$f_m(z) = \frac{1}{z}+\sum_{m=1}^{\infty}\frac{(m+2)^c(1-\alpha)}{[1+\lambda(m+1)]^n[(1+\beta)(m+1)+1-\alpha]}z^m. \tag{2.3}$$

3 Distortion Theorems

***Theorem* 3.1.** *Let the function f defined by* (1.8) *be in the class* $\sigma_{c,p}(\alpha,\beta,\lambda)$. *Then for* $0 < |z| = r < 1$,

$$\frac{1}{r}-\frac{3^c(1-\alpha)}{[1+2\lambda]^n(3+2\beta-\alpha)}r \leq |f(z)| \leq \frac{1}{r}+\frac{3^c(1-\alpha)}{[1+2\lambda]^n(3+2\beta-\alpha)}r \tag{3.1}$$

with equality for the function

$$f(z) = \frac{1}{z}+\frac{3^c(1-\alpha)}{[1+2\lambda]^n(3+2\beta-\alpha)}z. \tag{3.2}$$

Proof. Suppose f is in $\sigma_{c,p}(\alpha,\beta,\lambda)$. In view of ***Theorem* 2.1**, we have

$$\frac{[1+2\lambda]^n(3+2\beta-\alpha)}{3^c}\sum_{m=1}^{\infty}a_m \leq \sum_{m=1}^{\infty}\frac{[(1+\beta)(m+1)+1-\alpha][1+\lambda(m+1)]^n}{(m+2)^c} \leq (1-\alpha)$$

which evidently yields

$$\sum_{m=1}^{\infty}a_m \leq \frac{3^c(1-\alpha)}{[1+2\lambda]^n(3+2\beta-\alpha)}.$$

Consequently, we obtain

$$|f(z)| = \left|\frac{1}{z}+\sum_{m=1}^{\infty}a_m z^m\right|$$

$$\leq |\frac{1}{z}| + \sum_{m=1}^{\infty} a_m |z|^m \leq \frac{1}{r} + r\sum_{m=1}^{\infty} a_m$$

$$\leq \frac{1}{r} + \frac{3^c(1-\alpha)}{[1+2\lambda]^n(3+2\beta-\alpha)} r.$$

Also

$$|f(z)| = \left|\frac{1}{z} + \sum_{m=1}^{\infty} a_m z^m\right|$$

$$\geq \left|\frac{1}{z}\right| - \sum_{m=1}^{\infty} a_m |z|^m$$

$$\geq \frac{1}{r} - r\sum_{m=1}^{\infty} a_m$$

$$\geq \frac{1}{r} - \frac{3^c(1-\alpha)}{[1+2\lambda]^n(3+2\beta-\alpha)} r.$$

Hence the results (3.1) follow.

***Theorem* 3.2.** *Let the function f defined by* (1.8) *be in the class $\sigma_{c,p}(\alpha,\beta,\lambda)$. Then for $0 < |z| = r < 1$,*

$$\frac{1}{r^2} - \frac{3^c(1-\alpha)}{[1+2\lambda]^n(3+2\beta-\alpha)} \leq |f'(z)| \leq \frac{1}{r^2} + \frac{3^c(1-\alpha)}{[1+2\lambda]^n(3+2\beta-\alpha)}.$$

The result is sharp, the extremal function being of the form (2.3).

Proof. From Theorem 2.1, we have

$$\frac{[1+2\lambda]^n(3+2\beta-\alpha)}{3^c} \sum_{m=1}^{\infty} m a_m \leq \sum_{m=1}^{\infty} \frac{[(1+\beta)(m+1)+1-\alpha][1+\lambda(m+1)]^n}{(m+2)^c} \leq (1-\alpha)$$

which evidently yields

$$\sum_{m=1}^{\infty} m a_m \leq \frac{3^c(1-\alpha)}{[1+2\lambda]^n(3+2\beta-\alpha)}$$

Consequently, we obtain

$$|f(z)| = \frac{1}{r^2} + \sum_{m=1}^{\infty} m a_m r^{m-1}$$

$$\leq \frac{1}{r^2} + \sum_{m=1}^{\infty} m a_m$$

$$\leq \frac{1}{r^2} + \frac{3^c(1-\alpha)}{[1+2\lambda]^n(3+2\beta-\alpha)}.$$

Also

$$|f(z)| \geq \frac{1}{r^2} - \sum_{m=1}^{\infty} m a_m r^{m-1}$$

$$\geq \frac{1}{r^2} - \sum_{m=1}^{\infty} m a_m$$

$$\geq \frac{1}{r^2} - \frac{3^c(1-\alpha)}{[1+2\lambda]^n(3+2\beta-\alpha)}.$$

This completes the proof.

4 Class Preserving Integral Operators

In this section we consider the class preserving integral operators of the form (1.8).

***Theorem* 4.1.** *Let the function f be defined by* (1.8) *be in the class $\sigma_{c,p}(\alpha,\beta,\lambda)$. Then*

$$F(z) = \mu z^{-\mu-1}\int_0^z t^{\mu} f(t)dt = \frac{1}{z} + \sum_{m=1}^{\infty}\frac{\mu}{\mu+m+1}a_m z^m,\ \mu > 0 \tag{4.1}$$

belongs to the class $\sigma[\delta(\alpha,\beta,\lambda,m,\mu)]$, where

$$\delta(\alpha,\beta,\lambda,m,\mu) = \frac{[1+2\lambda]^n(3+2\beta-\alpha)(\mu+2) - 3^c\mu(1-\alpha)}{[1+2\lambda]^n(3+2\beta-\alpha)(\mu+2) + 3^c\mu(1-\alpha)}. \tag{4.2}$$

The result is sharp for $f(z) = \frac{1}{z} + \frac{3^c(1-\alpha)}{[1+2\lambda]^n(3+2\beta-\alpha)}z$.

Proof. Suppose $f(z) = \frac{1}{z} + \sum\limits_{m=1}^{\infty} a_m z^m$ is in $\sigma_{c,p}(\alpha,\beta,\lambda)$.

We have

$$F(z) = \mu z^{-\mu-1}\int_0^z t^{\mu} f(t)dt = \frac{1}{z} + \sum_{m=1}^{\infty}\frac{\mu}{\mu+m+1}a_m z^m,\ \mu > 0$$

It is sufficient to show that

$$\sum_{m=1}^{\infty}\frac{m+\delta}{1-\delta}\frac{\mu a_m}{m+\mu+1} \le 1. \tag{4.3}$$

Since $f(z)$ is in $\sigma_{c,p}(\alpha,\beta,\lambda)$, we have

$$\sum_{m=1}^{\infty}\frac{[(1+\beta)(m+1)+1-\alpha][1+\lambda(m+1)]^n}{(m+2)^c(1-\alpha)}|a_m| \le 1. \tag{4.4}$$

Thus (4.3) will be satisfied if

$$\frac{(m+\delta)\mu}{(1-\delta)(m+\mu+1)} \le \frac{[(1+\beta)(m+1)+1-\alpha][1+\lambda(m+1)]^n}{(m+2)^c(1-\alpha)},\quad \text{for each } m \tag{4.5}$$

or

$$\delta \le \frac{[1+\lambda(m+1)]^n[(1+\beta)(m+1)+(1-\alpha)](\mu+m+1) - m\mu(1-\alpha)(m+2)^c}{[1+\lambda(m+1)]^n[(1+\beta)(m+1)+(1-\alpha)](\mu+m+1) + m\mu(1-\alpha)(m+2)^c}$$

$$G(m) = \frac{[1+\lambda(m+1)]^n[(1+\beta)(m+1)+(1-\alpha)](\mu+m+1) - m\mu(1-\alpha)(m+2)^c}{[1+\lambda(m+1)]^n[(1+\beta)(m+1)+(1-\alpha)](\mu+m+1) + m\mu(1-\alpha)(m+2)^c}.$$

Then $G(m+1) - G(m) > 0$, for each m. Hence $G(m)$ is increasing function of m. Since

$$G(1) = \frac{[1+2\lambda]^n(3+2\beta-\alpha)(\mu+2) - 3^c\mu(1-\alpha)}{[1+2\lambda]^n(3+2\beta-\alpha)(\mu+2) + 3^c\mu(1-\alpha)}.$$

The result follows.

5 Convex Linear Combinations and Convolution Properties

***Theorem* 5.1.** *If the function f is in $\sigma_{c,p}(\alpha,\beta,\lambda)$ then $f(z)$ is meromorphically convex of order δ $(0 \le \delta < 1)$ in $|z| < r = r(\alpha,\beta,\lambda,\delta)$ where*

$$r(\alpha,\beta,\lambda,\delta) = \inf_{n\ge1}\left\{\frac{[1+\lambda(m+1)]^n(1-\delta)[(1+\beta)(1+m)+1-\alpha]}{(m+2)^c(1-\alpha)m(m+2-\delta)}\right\}^{\frac{1}{m+1}}$$

The result is sharp.

Proof. Let $f(z)$ is in $\sigma_{c,p}(\alpha,\beta,\lambda)$. Then by *Theorem* 2.1, we have

$$\sum_{m=1}^{\infty}\frac{[1+\lambda(m+1)]^n[(1+\beta)(m+1)+1-\alpha]}{(m+2)^c}|a_m| \le (1-\alpha). \tag{5.1}$$

It is sufficient to show that $\left|2+\frac{zf''(z)}{f'(z)}\right| \le 1-\delta$, for $|z| < r = r(\alpha,\beta,\lambda,\delta)$, where $r(\alpha,\beta,\lambda,\delta)$ is specified in the statement of the theorem. Then

$$\left|2+\frac{zf''(z)}{f'(z)}\right| = \left|\frac{\sum_{m=1}^{\infty} m(m+1)a_m z^{m-1}}{\frac{-1}{z^2}+\sum_{m=1}^{\infty} m a_m z^{m-1}}\right| \le \sum_{m=1}^{\infty}\frac{m(m+1)a_m|z|^{m+1}}{1-\sum_{m=1}^{\infty} m a_m|z|^{m+1}}.$$

This will be bounded by $(1-\delta)$ if

$$\sum_{m=1}^{\infty}\frac{m(m+2-\delta)}{1-\delta}a_m|z|^{m+1} \le 1. \tag{5.2}$$

By (5.1), it follow that (5.2) is true if

$$\frac{m(m+2-\delta)}{1-\delta}|z|^{m+1} \le \frac{[1+\lambda(m+1)]^n[(1+\beta)(m+1)+1-\alpha]}{(m+2)^c(1-\alpha)},\ m\ge1$$

or

$$|z| \le \left\{\frac{[1+\lambda(m+1)]^n(1-\delta)[(1+\beta)(1+m)+1-\alpha]}{(m+2)^c(1-\alpha)m(m+2-\delta)}\right\}^{\frac{1}{m+1}}. \tag{5.3}$$

Setting $|z| = r(\alpha,\beta,\lambda,\delta)$ in (5.3), the result follows.
The result is sharp for the function

$$f_m(z) = \frac{1}{z}+\frac{(m+2)^c(1-\alpha)}{[1+\lambda(m+1)]^n[(1+\beta)(m+1)+1-\alpha]}z^m,\ (m\ge1).$$

***Theorem* 5.2.** *Let $f_0(z) = \frac{1}{z}$ and*

$$f_m(z) = \frac{1}{z}+\frac{(m+2)^c(1-\alpha)}{[1+\lambda(m+1)]^n[(1+\beta)(m+1)+1-\alpha]}z^m,\ (m\ge1).$$

Then $f(z) = \frac{1}{z}+\sum_{m=1}^{\infty}a_m z^m$ is in the class $\sigma_{c,p}(\alpha,\beta,\lambda)$ if and only if it can be expressed in the form $f(z) = \lambda_0 f_0(z)+\sum_{m=1}^{\infty}\lambda_m f_m(z)$, where $\lambda_0 \ge 0, \lambda_m \ge 0$ $(m\ge1)$ and $\lambda_0+\sum_{m=1}^{\infty}\lambda_m = 1$.

Proof. Let $f(z) = \lambda_0 f_0(z)+\sum_{m=1}^{\infty}\lambda_m f_m(z)$ with $\lambda_0 \ge 0, \lambda_m \ge 0$ $(m\ge1)$ and

$$\lambda_0+\sum_{m=1}^{\infty}\lambda_m = 1.$$

Then

$$f(z) = \lambda_0 f_0(z) + \sum_{m=1}^{\infty} \lambda_m f_m(z)$$

$$= \frac{1}{z} + \sum_{m=1}^{\infty} \lambda_m \frac{(m+2)^c(1-\alpha)}{[1+\lambda(m+1)]^n[(1+\beta)(m+1)+1-\alpha]} z^m.$$

Since

$$\sum_{m=1}^{\infty} \frac{[1+\lambda(m+1)]^n[(1+\beta)(m+1)+1-\alpha]}{(m+2)^c(1-\alpha)} \lambda_m \frac{(m+2)^c(1-\alpha)}{[1+\lambda(m+1)]^n[(1+\beta)(m+1)+1-\alpha]}$$

$$= \sum_{m=1}^{\infty} \lambda_m = 1 - \lambda_0 \le 1.$$

By *Theorem* 2.1, f is in the class $\sigma_{c,p}(\alpha, \beta, \lambda)$.

Conversely suppose that the function f is in the class $\sigma_{c,p}(\alpha, \beta, \lambda)$, since

$$a_m \le \frac{(m+2)^c(1-\alpha)}{[1+\lambda(m+1)]^n[(1+\beta)(m+1)+1-\alpha]}, \ (m \ge 1)$$

$$\lambda_m = \frac{[1+\lambda(m+1)]^n[(1+\beta)(m+1)+1-\alpha]}{(m+2)^c(1-\alpha)} a_m,$$

and $\lambda_0 = 1 - \sum_{m=1}^{\infty} \lambda_m$, it follows that $f(z) = \lambda_0 f_0(z) + \sum_{m=1}^{\infty} \lambda_m f_m(z)$.

This completes the proof of the *Theorem*.

For the functions $f(z) = \frac{1}{z} + \sum_{m=1}^{\infty} a_m z^m$ and $g(z) = \frac{1}{z} + \sum_{m=1}^{\infty} b_m z^m$ belongs to $\sum_p$ we denote by $(f * g)(z)$ the convolution of $f(z)$ and $g(z)$ or

$$(f * g)(z) = \frac{1}{z} + \sum_{m=1}^{\infty} a_m b_m z^m.$$

***Theorem* 5.3.** *If the functions* $f(z) = \frac{1}{z} + \sum_{m=1}^{\infty} a_m z^m$ *and* $g(z) = \frac{1}{z} + \sum_{m=1}^{\infty} b_m z^m$ *are in the class* $\sigma_{c,p}(\alpha, \beta, \lambda)$, *then*

$$(f * g)(z) = \frac{1}{z} + \sum_{m=1}^{\infty} a_m b_m z^m$$

is in the class $\sigma_{c,p}(\alpha, \beta, \lambda)$.

Proof. Suppose $f(z)$ and $g(z)$ are in $\sigma_{c,p}(\alpha, \beta, \lambda)$. By ***Theorem* 2.1**, we have

$$\sum_{m=1}^{\infty} \frac{[1+\lambda(m+1)]^n[(1+\beta)(m+1)+1-\alpha]}{(m+2)^c(1-\alpha)} a_m \le 1$$

$$\sum_{m=1}^{\infty} \frac{[1+\lambda(m+1)]^n[(1+\beta)(m+1)+1-\alpha]}{(m+2)^c(1-\alpha)} b_m \le 1.$$

Since $f(z)$ and $g(z)$ are regular are in E, so is $(f * g)(z)$. Furthermore,

$$\sum_{m=1}^{\infty} \frac{[1+\lambda(m+1)]^n[(1+\beta)(m+1)+1-\alpha]}{(m+2)^c(1-\alpha)} a_m b_m$$

$$\leq \left\{ \frac{[1+\lambda(m+1)]^n[(1+\beta)(m+1)+1-\alpha]}{(m+2)^c(1-\alpha)} \right\}^2 a_m b_m$$

$$\leq \Big(\sum_{m=1}^{\infty} \frac{[1+\lambda(m+1)]^n[(1+\beta)(m+1)+1-\alpha]}{(m+2)^c(1-\alpha)} a_m \Big)$$

$$\Big(\sum_{m=1}^{\infty} \frac{[1+\lambda(m+1)]^n[(1+\beta)(m+1)+1-\alpha]}{(m+2)^c(1-\alpha)} b_m \Big)$$

$$\leq 1.$$

Hence by *Theorem* 2.1, $(f * g)(z)$ is in the class $\sigma_{c,p}(\alpha, \beta, \lambda)$.

6 Neighborhoods for the class $\sigma_{c,p}(\alpha, \beta, \lambda)$

Neighborhoods for the class $\sigma_{c,p}(\alpha, \beta, \lambda)$ which we define as follows:

***Definition* 6.1.** *A function $f \in \sum_p$ is said to in the class $\sigma_{c,p}(\alpha, \beta, \lambda, \gamma)$ if there exists a function $g \in \sigma_{c,p}(\alpha, \beta, \lambda)$ such that*

$$\left| \frac{f(z)}{g(z)} - 1 \right| < 1 - \gamma, \; z \in U, \; (0 \leq \gamma < 1). \tag{6.1}$$

Following the earlier works on neighborhoods of analytic functions by Goodman [4] *and Ruschweyh* [11]*, we define the $\delta-$neighborhhod of a function $f \in \sum_p$ by*

$$N_\delta(f) := \Big\{ g \in \sum_p : \; g(z) = \frac{1}{z} + \sum_{m=1}^{\infty} b_m z^m : \; \sum_{m=1}^{\infty} m|a_m - b_m| \leq \delta \Big\}. \tag{6.2}$$

***Theorem* 6.1.** *If $g \in \sigma_{c,p}(\alpha, \beta, \lambda)$ and*

$$\gamma = 1 - \frac{\delta(3+2\beta-\alpha)(1+2\lambda)}{(3+2\beta-\alpha)(2+2\beta) - 3^c(1-\alpha)}. \tag{6.3}$$

Then $N_\delta(g) \subset \sigma_{c,p}(\alpha, \beta, \lambda, \gamma)$.

Proof. Let $f \in N_\delta(g)$. Then we find from (6.2) that

$$\sum_{m=1}^{\infty} m|a_m - b_m| \leq \delta, \tag{6.4}$$

which implies the coefficient inequality

$$\sum_{m=1}^{\infty} |a_m - b_m| \leq \delta, \; (m \in N). \tag{6.5}$$

Since $g \in \sigma_{c,p}(\alpha, \beta, \lambda)$, we have $\sum_{m=1}^{\infty} b_m < \frac{3^c(1-\alpha)}{(1+2\lambda)^n(3+2\beta-\alpha)}$. So that

$$\left| \frac{f(z)}{g(z)} - 1 \right| \leq \frac{\sum_{m=1}^{\infty} |a_m - b_m|}{1 - \sum_{m=1}^{\infty} b_m} \leq \frac{\delta(1+2\lambda)(3+3\beta-\alpha)}{(1+2\lambda)(3+3\beta-\alpha) - 3^c(1-\alpha)} = 1 - \gamma$$

provided γ is given by (6.3). Hence, by definition $f \in \sigma_{c,p}(\alpha, \beta, \lambda, \gamma)$ for γ given by (6.3), which completes the proof.

***Remark* 6.1.**

(i.) *For $\lambda = 0$ in the results mentioned in all the sections above the class are the same as those of Venkateswarlu et al:* [13].

(ii.) *For $\lambda = 0$ and $\beta = 1$ in the results mentioned in all the sections above the class are the same as those of Thirupathi Reddy et al:* [12].

7 Conclusion

This research has introduced a new linear differential operator related to polylogarithm function and studied some properties were studied. Accordingly, some results related to closure theorems have also been considered, inviting future research for this field of study.

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