Abstract

The concept of Boehmian was motivated by the so called regular operators introduced by T.K.Boehme. The construction of Boehmians is similar to the construction of field of quotients. Several integral transforms have been extended to various class of Boehmians. We study here Kamal transform and extend it to Strong Boehmian space. This Kamal tranform is 1-1 and continuous in the space of Boehmians. Inverse Kamal transform is also defined.

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1 Introduction


The Kamal transform of \( f(t) \) is defined by [11]

\[
K[f(t)] = F(v) = \int_{0}^{\infty} f(t)e^{-tv} \, dt \quad J_1 \leq v \leq J_2
\]
over the set of functions

\[ \mathcal{A} = \{ f(t) : \exists M, J_1, J_2 > 0 \ | f(t) < Me^{[j/L]} \text{ if } t \in (-1)^j \times [0, \infty) \}. \]

We denote the usual convolution of \( f \) and \( g \) by

\[ (f * g)(x) = \int_{\mathbb{R}} f(x-t) g(t) \, dt. \]

The Kamal transform of the convolution product is given by

\[ K(f * g) = K(f) \cdot K(g). \]

**General properties of Kamal transforms:**

1. If \( \alpha, \beta \in \mathbb{R} \) and \( K[f] = F(\nu) \) and \( K[g] = G(\nu) \) then Kamal transform is linear.

\[
K[\alpha f + \beta g] = \int_0^\infty (\alpha f + \beta g)(t) e^{-t/\nu} \, dt \\
= \alpha \int_0^\infty f(t) e^{-t/\nu} \, dt + \beta \int_0^\infty g(t) e^{-t/\nu} \, dt \\
= \alpha K[f] + \beta K[g] \\
= \alpha F(\nu) + \beta G(\nu).
\]

2. If \( f(t) = e^{at}, \sin at, \cos at \) then corresponding \( K[f(t)] \) is given by

\[
F(\nu) = \frac{\nu}{1 - av}, \quad G(\nu) = \frac{av^2}{1 + a^2v^2}.
\]

For more properties see [11].

**2 Strong Boehmians**

We study Strong Boehmians [4, 9] and General Boehmians [5]-[8]. Let \( I_+ \) is the set of positive real numbers and \( \mathcal{F} \) denote the Schwartz space of test functions \( \phi \) with compact supports over \( I_+ \) and \( \eta(\Omega) \) be the space of all infinitely differentiable functions over \( \Omega \) where \( \Omega = [1, \infty) \times I_+ \). The dual of \( \eta(\Omega) \) is \( \eta'(\Omega) \) consists of distributions of compact supports. Let \( f \in \eta(\Omega) \) and \( \phi \in \mathcal{F} \) the convolution of \( f \) and \( \phi \) is given by

\[ (f \# \phi)(x) = \int_{I_+} f(\alpha, t) \phi(x - t) \, dt, \]

where \( \alpha \in [1, \infty) \).

Let \( \mu(I_+) \) be the subset of \( \mathcal{F} \) of the test functions such that

\[ \int_{I_+} \phi(x) \, dx = 1. \]

The pair \( (f, \phi) \) or \( (f/\phi) \) of functions such that \( f \in \eta(\Omega), \phi \in \mu(I_+) \) is said to be quotient of function denoted by \( (f, \phi) \) or \( (f/\phi) \) if and only if

\[ \{ f(\alpha, x) \# (\beta \phi(\beta x)) \} = \{ f(\beta, x) \# (\alpha \phi(\alpha x)) \}, \]

for all \( \alpha, \beta \in [1, \infty) \) or we define

\[ f(\alpha, x) \# d_\beta \phi(x) = f(\beta, x) \# d_\alpha \phi(x), \]
where
\[
\begin{align*}
 d_\beta \phi(x) &= \beta \phi(\beta x), \\
 d_\alpha \phi(x) &= \alpha \phi(\alpha x).
\end{align*}
\]

We use both the definitions whenever we required. Two quotients \((f, \phi)\) and \((g, \psi)\) are said to be equivalent that is \((f, \phi) \sim (g, \psi)\) if and only if
\[
(2.4) \quad f(\alpha, x) \# \beta \psi(\beta x) = g(\beta, x) \# \alpha \phi(\alpha x),
\]
\(\alpha, \beta \in [1, \infty)\).

Let the set be denoted by
\[
(2.5) \quad \mathcal{B} = \{(f, \phi)| \forall \ f \in \eta(\Omega), \phi \in \mu(I_+)\}.
\]

Then the equivalence class \([((f, \phi)]\) containing \((f, \phi)\) is called Strong Boehmian. The space of all such Boehmians is denoted by \(\mathcal{L}(\eta, \mu, \#)\) is called as space of Strong Boehmians. Following conclusions are given in [9]

1. Let \(\phi, \psi \in \mu(I_+)\) then \(\phi \# \psi \in \mu(I_+)\),
2. Let \(f \in \eta(\Omega)\) and \(\phi \in \mu(I_+)\) then \(f \# \phi \in \eta(\Omega)\),
3. Let \((f, \phi) \in \mathcal{B}\) and \(\psi \in \mu(I_+)\) then
\[
(2.6) \quad (f \# \psi, \phi \# \psi) \in \mathcal{B} \quad \text{and} \quad (f, \phi) \sim (f \# \psi, \phi \# \psi).
\]
4. If \(\phi \in \mu(I_+)\) then for \(\alpha \geq 1\)
\[
\alpha \phi(\alpha x) \in \mu(I_+),
\]
5. Let \((f, \phi) \in \mathcal{B}, z > 0\) and \(h(\alpha, x) = f(\alpha + z, x)\) and \(\psi = z\phi(zx)\) then
\[
(2.7) \quad (g, \psi) \in \mathcal{B} \quad \text{and} \quad (g, \psi) \sim (f, \phi).
\]

Further the operation of addition and scalar multiplication in \(\mathcal{L}(\eta, \mu, \#)\) are defined in the usual notation as,
\[
(2.8) \quad \frac{f}{\phi} + \frac{g}{\psi} = \frac{f \# \psi + g \# \phi}{\phi \# \psi}, \quad \lambda \cdot \frac{f}{\phi} = \frac{\lambda f}{\phi}, \quad \frac{f}{\phi} \# \psi = \frac{f \# \psi}{\phi}.
\]

The above operations are well defined in \(\mathcal{L}\) and hence \(\mathcal{L}\) is a vector space.

Let
\[
(2.9) \quad D^p = (\frac{\partial}{\partial x_1})^{p_1}(\frac{\partial}{\partial x_2})^{p_2}(\frac{\partial}{\partial x_3})^{p_3} \cdots (\frac{\partial}{\partial x_N})^{p_N}
\]

where \(P = (p_1, p_2, \cdots, p_N)\) and \(p_1, p_2, \cdots, p_N\) are nonnegative integers for \(\frac{f}{\phi} \in \mathcal{L}(\eta, \mu, \#)\) define
\[
D^p(\frac{f}{\phi}) = \frac{D^p f}{\phi}, \quad \text{where} \quad D^p \text{ is well defined operation on } \mathcal{L}. \quad \text{A sequence of Strong Boehmians } \{y_n\}
\]
is said to converge to a Strong Boehmian \(y\) if \(y = \frac{f}{\phi}\) and \(y_n = \frac{f_n}{\phi}\) for some \(f, f_n \in \eta \& \phi \in \mu(I_+),\)
\(n \in \mathbb{N}\) and \(f_n \rightarrow f\) uniformly on compact subset of \(\Omega\) as \(n \rightarrow \infty\).

3 General Construction of Boehmians
Mikusinski introduced a new class of generalised function space called Boehmian space, which is suitable for extending integral transforms. The construction of Boehmian space and its convergence is given in [15] The construction of Boehmians consists of following elements:

1. A set \(\Gamma\),
2. Commutative semi group\((S, \oplus)\),
3. An operation $\star : \Gamma \times S \to \Gamma$ such that for each $x \in \Gamma$ and $\phi_1, \phi_2 \in \Delta \subset S$ \( x \star (\phi_1 \otimes \phi_2) = (x \star \phi_1) \otimes (x \star \phi_2) \),

4. (a) A collection $\Delta \subset S$ such that if $x, y \in \Gamma$, $\phi_n \in \Delta$ $x \star \phi_n = y \star \phi_n$ \( \forall n \Rightarrow x = y \).

(b) If $\phi_n \in \Delta$ and $\psi_n \in \Delta$ then $\phi_n \otimes \psi_n \in \Delta$, $\Delta$ is a set of all delta sequences.

Consider

\[ (3.1) \quad B = \{(x_n, \phi_n) : x_n \in \Gamma, \phi_n \in \Delta, x_n \star \phi_m = x_m \star \phi_n \quad \forall m, n \in \mathbb{N}\}. \]

If $(x_n, \phi_n), (y_n, \psi_n) \in B \quad x_n \star \psi_m = y_m \star \phi_n \quad \forall m, n \in \mathbb{N}$ we say that $(x_n, \phi_n) \sim (y_n, \psi_n).$ The relation $\sim$ is an equivalence relation in $B$. The space of equivalence classes in $B$ is denoted by $\mathcal{L}_\beta(\Gamma, S, \Delta)$. Elements of $\mathcal{L}_\beta(\Gamma, S, \Delta)$ are called General Boehmians.

We define a mapping which is a canonical mapping between $\Gamma$ and $\mathcal{L}_\beta$ as $x \to x \star \phi_n / \phi_n$.

In $\mathcal{L}_\beta(\Gamma, S, \Delta)$ there are two type convergences

1. A sequence $q_n$ in $\mathcal{L}_\beta(\Gamma, \rho, \Delta)$ is said to be $\delta$ convergent to $q$ in $\mathcal{L}_\beta(\Gamma, S, \Delta)$ denoted by $q_n \overset{\delta}{\to} q$ if there exist a delta sequence $\delta_n$ such that $(q_n \star \delta_n), (q \star \delta_n) \in \Gamma$ and for all $k, n \in \mathbb{N}$ $(q_n \star \delta_n) \to (q \star \delta_n)$ as $n \to \infty$ in $\Gamma$.

2. A sequence $q_n$ in $\mathcal{L}_\beta(\Gamma, S, \Delta)$ is said to be $\Delta$ convergent to $q$ in $\mathcal{L}_\beta(\Gamma, S, \Delta)$ denoted by $q_n \overset{\Delta}{\to} q$ if there exist $(\delta_n) \in \Delta$ such that $(q_n - q) \star \delta_n \in \Gamma$ \( \forall n \in \mathbb{N} \) and $(q_n - q) \star 0$ as $n \to \infty$ in $\Gamma$.

Following lemma is an equivalent statement for $\delta$- convergence given by [17]

**Lemma 3.1.** $q_n \overset{\delta}{\to} q$ (as $n \to \infty$) in $\mathcal{L}_\beta(\Gamma, S, \Delta)$ if and only if there exist $f_{n,k}, f_k \in \Gamma$ and $\delta_k \in \Delta$ such that

\[ q_n = [f_{n,k}/\delta_k] \quad q = [f_n/\delta_k] \]

and for each $k \in \mathbb{N}$ $f_{n,k} \to f_k$ as $n \to \infty$ on $\Gamma$.

4 **Kamal Transform of Strong Boehmians**

**Theorem 4.1 (Convolution theorem).** Let $f \in \eta(\Omega)$ and $\phi \in \mu(I_\omega)$ then

\[ (4.1) \quad K(f \# \beta \psi)(\beta x) = K(f(\beta x)) \cdot \beta K(\psi(\beta x)) = \hat{f}(\beta x) \cdot \beta \hat{\psi}(\beta \xi). \]

If onwards we define

\[ f(\alpha, x) \# \beta \psi(\beta x) = f(\alpha, x) \# d_\beta \psi(x), \]

where,

\[ d_\beta \psi(x) = \beta \psi(\beta x) \]

Then,

\[ K(f \# d_\beta \psi)(\xi) = \hat{f}(\alpha, \xi) \cdot d_\beta \hat{\psi}(\xi), \]

where $\hat{f}$ and $\hat{\psi}$ are Kamal transforms of $f$ and $\psi$. 

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**Proof.** By using definition of Kamal transform, Fubini’s theorem,

\[(4.2) \quad K(f(\alpha, t) \# d_\alpha \psi(x))(\xi) = \int_{0}^{\infty} f(\alpha, t) \, dt \int_{0}^{\infty} d_\alpha \psi(x - t) e^{-x/x} \, dx \]

\[= \int_{0}^{\infty} f(\alpha, t) \, dt \int_{0}^{\infty} \alpha \psi(\alpha x - at) e^{-x/x} \, dx. \]

\[(4.3) \quad \text{Put } \alpha x - at = z \quad \text{i.e. } x = \frac{z}{\alpha} + t \quad \text{and } dx = \frac{dz}{\alpha} \text{ to get} \]

\[K(f(\alpha, t) \# d_\alpha \psi(x))(\xi) = \int_{0}^{\infty} f(\alpha, t) \, dt \int_{0}^{\infty} \alpha \psi(z) e^{-(\frac{z}{\alpha} + t)/\xi} \, dz \]

\[= \int_{0}^{\infty} f(\alpha, t) \, dt \int_{0}^{\infty} \alpha \psi(z) e^{-\frac{z}{\alpha}} \cdot e^{-t/\xi} \, dz \]

\[= \int_{0}^{\infty} e^{-\frac{t}{\xi}} f(\alpha, t) \, dt \int_{0}^{\infty} \alpha \psi(z) e^{-\frac{z}{\alpha}} \, dz = \hat{\phi}(\alpha, \xi) \cdot d_\alpha \hat{\psi}(\xi), \]

which completes the proof of the Theorem.

Now we define the images of Kamal transform of Strong Boehmians.

**Definition 4.1.** Let $\Delta_{1}(I_{+})$ or $\Delta_{1}$ be a set of delta sequences such that $\psi_{n} \in \mu(I_{+})$ and suppose

$\psi_{n} \subseteq (0, \gamma_{n}), \gamma_{n} > 0, \gamma_{n} \to 0$ as $n \to \infty$.

Let $m(I_{+})$ be the set of images of Kamal transforms of all $\mu(I_{+})$ elements and $\Delta_{2}(I_{+})$ be the set of Kamal transform of all delta sequences from $\Delta_{1}$ for $f \in \eta(\Omega)$ and $\hat{\psi} \in m(I_{+})$ we define the operation $\otimes$ as

\[f(\alpha, \xi) \otimes \hat{\psi}(\xi) = f(\alpha, \xi) \cdot d_\alpha \hat{\psi}(\xi) \quad \text{as } \alpha \in [1, \infty), \]

from which we see that

\[(4.4) \quad f \otimes \hat{\psi} \in \eta(\Omega) \quad \text{as } f \in \eta(\Omega) \quad \text{and } d_\alpha \hat{\psi} \in m(I_{+}). \]

**Lemma 4.1.** 1. If $\phi_{n}, \psi_{n} \in \Delta_{2}(I_{+})$ then $\psi_{n} \otimes \phi_{n} \in \Delta_{2}(I_{+})$ \quad $\forall n \in N$

2. Let $f, g \in \eta(\Omega)$ and $\psi_{n} \in \Delta_{2}(I_{+})$ such that

\[(4.5) \quad f(c, \xi) \otimes \hat{\psi}_{n}(\xi) = g(d, \xi) \otimes \hat{\psi}_{n}(\xi) \quad \text{then } f(c, \xi) = g(d, \xi) \quad \forall c, d \in [1, \infty). \]

**Proof.** 1. $\phi_{n}, \psi_{n} \in \Delta_{2}(I_{+})$ we find the sequences $\phi_{n}, \psi_{n} \in \Delta_{1}(I_{+})$.

Since $\phi_{n} \# \psi_{n} \in \Delta_{1}(I_{+})$, we get

\[(4.6) \quad K(\phi_{n} \# \psi_{n}) = \hat{\phi}_{n}(\xi) \cdot \hat{\psi}_{n}(\xi) = \hat{\phi}_{n} \otimes \hat{\psi}_{n} \in \Delta_{2}(I_{+}) \quad \forall n \in N \quad \text{[by(4.4)].} \]

2. Let $\psi_{n} \in \Delta_{2}(I_{+})$ where $\hat{\psi}_{n}$ is delta sequence $d_{n} \hat{\psi}_{n} \to 1$ hence $\hat{\psi}_{n} \to 1$ as $n \to \infty$

\[(4.7) \quad f(c, \xi) \otimes \hat{\psi}_{n}(\xi) = f(c, \xi) \cdot d_{n} \hat{\psi}_{n}(\xi) \to f(c, \xi) \quad \text{as } n \to \infty, \]

\[(4.8) \quad g(d, \xi) \otimes \hat{\psi}_{n}(\xi) = g(d, \xi) \cdot d_{n} \hat{\psi}_{n}(\xi) \to g(d, \xi) \quad \text{as } n \to \infty, \]

from (4.7) and (4.8) $f(c, \xi) = g(d, \xi)$ \quad $\forall \xi \in I_{+}$ \quad $c, d \in [1, \infty)$, which completes the proof.
Lemma 4.2. The mapping $\eta(\Omega) \otimes m(I_\ast) \to m(I_\ast)$ defined by

\begin{equation}
(4.9) \quad f(\alpha, \xi) \otimes \hat{\phi}_n(\xi) = f(\alpha, \xi) \, d_\alpha \hat{\phi}_n(\xi)
\end{equation}

satisfies the following properties:

1. $\hat{\phi}_n \otimes \hat{\psi}_n = \hat{\psi}_n \otimes \hat{\phi}_n$ for every $(\hat{\phi}_n), (\hat{\psi}_n) \in \Delta_2(I_\ast)$ then

\begin{equation}
(f + g) \otimes \hat{\phi}_n = f \otimes \hat{\phi}_n + g \otimes \hat{\phi}_n,
\end{equation}

2. If $f \in \eta(\Omega), \ (\hat{\phi}_n), (\hat{\psi}_n) \in \Delta_2(I_\ast)$ then

\begin{equation}
(f \otimes \hat{\phi}_n) \otimes \hat{\psi}_n = f \otimes (\hat{\phi}_n \otimes \hat{\psi}_n),
\end{equation}

3. If $f \in \eta(\Omega), \ (\hat{\phi}_n), (\hat{\psi}_n) \in \Delta_2(I_\ast)$ then

\begin{equation}
(f \otimes \hat{\phi}_n) \otimes \hat{\psi}_n = f \otimes (\hat{\phi}_n \otimes \hat{\psi}_n).
\end{equation}

Theorem 4.2. The following are true

1. If $f_n \to f$ in $\eta(\Omega)$ and $\hat{\psi} \in m(I_\ast)$ then $f_n \otimes \hat{\psi} \to f \otimes \hat{\psi}$ as $n \to \infty$,

2. If $f_n \to f$ in $\eta(\Omega)$ and $\hat{\psi}_n \in \Delta_2(I_\ast)$ then $f_n \otimes \hat{\psi}_n \to f$ as $n \to \infty$.

Proof. 1. $\hat{\psi} \in m(I_\ast)$, $f_n, f \in \eta(\Omega)$ then

\begin{equation}
(4.10) \quad |D^\xi_n(f_n(\alpha, \xi) \otimes \hat{\psi}(\xi) - f(\alpha, \xi) \otimes \hat{\psi}(\xi))| = |D^\xi_n \cdot d_\alpha \hat{\psi}(\xi)(f_n - f)(\alpha, \xi)| \to 0.
\end{equation}

As $n \to \infty$ in $\eta(\Omega)$, therefore $f_n \otimes \hat{\psi} \to f \otimes \hat{\psi}$.

2. $\hat{\phi}_m \Delta_2(I_\ast)$ then $d_\alpha \hat{\phi}_m(\xi) \to 1$ as $n \to \infty$ implies

\begin{equation}
(4.11) \quad |D^\xi_n(f_n \otimes \hat{\psi}_n(\xi) - f(\alpha, \xi))| \to |D^\xi_n(f_n(\alpha, \xi) - f(\alpha, \xi))| \to 0 \text{ as } n \to \infty.
\end{equation}

Hence $f_n \otimes \hat{\psi}_n \to f$, which completes the proof.

The General Boehmian space $L_{\hat{\phi}}(\eta, m, \Delta_2, \otimes)$ or $L_{\hat{\phi}}$ is constructed. We give some properties of sum, scalar multiplication, differentiation as

\begin{equation}
(4.12) \quad \left[ \frac{f_n}{\hat{\phi}_n} \right] + \left[ \frac{g_n}{\hat{\psi}_n} \right] = \left[ \frac{f_n \otimes \hat{\psi}_n + g_n \otimes \hat{\phi}_n}{\hat{\phi}_n \otimes \hat{\psi}_n} \right], \alpha \left[ \frac{f_n}{\hat{\phi}_n} \right] = \left[ \frac{\alpha f_n}{\hat{\phi}_n} \right].
\end{equation}

\begin{equation}
(4.13) \quad \left[ \frac{f_n}{\hat{\phi}_n} \right] \otimes \left[ \frac{g_n}{\hat{\psi}_n} \right] = \left[ \frac{f_n \otimes g_n}{\hat{\phi}_n \otimes \hat{\psi}_n} \right], D^\alpha \left[ \frac{f_n}{\hat{\phi}_n} \right] = \left[ \frac{D^\alpha f_n}{\hat{\phi}_n} \right].
\end{equation}

Now we are concerned with the Strong Boehmians which are described by the set $(\eta, \#)$ and the subset $(\mu, \#)$ with the family $\Delta_1$ of delta sequences such a space is denoted by $L(\eta, (\mu, \#), \Delta_1, \#)$ or simply by $L$. This space preserve the operation of addition, scalar multiplication, differentiation and the convolution.

Definition 4.2. Let $f \in \eta(\Omega)$ and $\phi \in \mu(I_\ast)$ we define the Kamal transform of the Strong Boehmians $[f_n/\phi_n]$ in $L$ by

\begin{equation}
(4.14) \quad \hat{\gamma} \left[ \frac{f_n}{\phi_n} \right] = \left[ \frac{\hat{f}_n}{d_\phi \phi_n} \right] \in L_{\hat{\phi}} \text{ where } L_{\hat{\phi}} \text{ is General Boehmians.}
\end{equation}

Theorem 4.3. The Kamal transform $\hat{\gamma} : L \to L_{\hat{\phi}}$ is well defined.
Proof. Let \( \left[ \frac{f_n}{\phi_n} \right], \left[ \frac{g_n}{\psi_n} \right] \in \mathcal{L} \) are such that \( \left[ \frac{f_n}{\phi_n} \right] = \left[ \frac{g_n}{\psi_n} \right] \). Then

\[
f_n(\alpha, x) \# d_\beta \psi_n(x) = g_n(\beta, x) \# d_\alpha \phi_n(x).
\]

Apply the convolution theorem on both sides of (4.15)

\[
\hat{g}_n \hat{d}_\beta \psi_n = \hat{f}_n \hat{d}_\alpha \phi_n
\]

Hence

\[
(4.16)
\]

which completes the proof.

**Theorem 4.4.** \((\psi_n), (\phi_n) \in \Delta_1(I_\mathcal{L})\) and \(f, g \in \eta(\Omega)\) then mapping \(\tilde{\gamma} : \mathcal{L} \to \mathcal{L}_\beta\) is one-one.

**Proof.** Now by (4.16)

\[
\tilde{\gamma} \left[ \frac{f_n}{\phi_n} \right] = \tilde{\gamma} \left[ \frac{g_n}{\psi_n} \right] \text{ in } \mathcal{L}_\beta.
\]

Therefore

\[
\begin{align*}
\hat{f}_n(\alpha, \xi) \otimes \hat{\psi}_n(\xi) &= \hat{g}_n(\beta, \xi) \otimes \hat{\phi}_n(\xi), \\
\hat{f}_n(\alpha, \xi) d_\beta \hat{\psi}_n(\xi) &= \hat{g}_n(\beta, \xi) d_\alpha \hat{\phi}_n(\xi), \\
K(f_n(\alpha, x) \# d_\beta \psi_n(x)) &= K(g_n(\beta, x) \# d_\alpha \phi_n(x)).
\end{align*}
\]

Since Kamal transform is one-one.

\[
f_n(\alpha, x) \# d_\beta \psi_n(x) = g_n(\beta, x) \# d_\alpha \phi_n(x)
\]

\[
\Rightarrow \quad \frac{f_n(\alpha, x) \# d_\beta \psi_n(x)}{\phi_n(x)} = \frac{g_n(\beta, x) \# d_\alpha \phi_n(x)}{\psi_n(x)}
\]

\[
\Rightarrow \quad \left[ \frac{f_n}{\phi_n} \right] = \left[ \frac{g_n}{\psi_n} \right],
\]

which completes the proof.

**Theorem 4.5.** \(\tilde{\gamma} : \mathcal{L} \to \mathcal{L}_\beta\) is continuous with respect to \(\mu\) convergence.

**Proof.** Let \(y_n \to y \in \mathcal{L}\) by using the convergence concept in \(\mu\) in \(\mathcal{L}[9], [\text{Theorem (2.6)}]\), we have \(\phi\) for all \(y_n\) such that \(y_n = \left[ \frac{f_n}{\phi} \right] \quad y = \left[ \frac{f}{\phi} \right]\) and \(f_n \to f\) as \(n \to \infty\)

Hence \(\hat{f}_n \to \hat{f}\) as \(n \to \infty\)

\[
\Rightarrow \quad \frac{\hat{f}_n \# d_\alpha \phi}{\phi_n} \to \frac{\hat{f} \# d_\alpha \phi}{\phi} \quad \text{as} \quad n \to \infty
\]

Therefore \(\tilde{\gamma}y_n \to \tilde{\gamma}y\) as \(n \to \infty\) in \(\mathcal{L}_\beta\).

**Definition 4.3.** Let \(z = \left[ \frac{f_n}{d_\alpha \phi_n} \right] \in \mathcal{L}_\beta\) then we can define \(\tilde{\gamma}^{-1}\) of \(\tilde{\gamma}\) by

\[
\tilde{\gamma}^{-1}z = \left[ \frac{f_n}{\phi_n} \right] \in \mathcal{L}.
\]

We can prove that \(\tilde{\gamma}^{-1}\) is well defined, linear, continuous w.r.t. \(\delta\) convergence.
Theorem 4.6. The mapping $\tilde{\gamma}^{-1} : \mathcal{L}_b \to \mathcal{L}$ is well defined.

Proof. Let
\[
[f_n/d_\phi n] = [\hat{h}_m/d_\phi n] \quad a, b \in [1, \infty). 
\]
Then
\[
\hat{f}_n(a, \xi) \otimes \hat{\phi}_m(\xi) = \hat{h}_m(b, \xi) \otimes \hat{\phi}_n(\xi).
\]
By (4.4)
\[
(4.17) \quad \hat{f}_n(a, \xi) d_b \hat{\phi}_m(\xi) = \hat{h}_m(b, \xi) d_a \hat{\phi}_n(\xi),
\]
therefore by Theorem (4.1)
\[
K(f_n(a, x) \# d_b \psi_m(x)) = K(h_m(b, x) \# d_a \phi_n(x)).
\]
Hence
\[
(4.18) \quad f_n(a, x) \# d_b \psi_m(x) = h_m(b, x) \# d_a \phi_n(x),
\]
which completes the proof.

Theorem 4.7. The mapping $\tilde{\gamma}^{-1} : \mathcal{L}_b \to \mathcal{L}$ is linear

Proof. Let $[\hat{f}_n/d_\phi n], [\hat{h}_n/d_\phi n] \in \mathcal{L}_b \quad \& \quad c \in [1, \infty).$
Then by (4.4)
\[
(4.19) \quad \tilde{\gamma}^{-1}([\hat{f}_n/d_\phi n] + [\hat{h}_n/d_\phi n]) = \tilde{\gamma}^{-1}\left\{ \frac{\hat{f}_n(a, \xi) d_b \hat{\phi}_n(\xi) + \hat{h}_n(b, \xi) d_a \hat{\phi}_n(\xi)}{d_a \hat{\phi}_n(\xi) \# d_b \hat{\phi}_n(\xi)} \right\} 
= \left[ \frac{f_n(a, x) \# d_b \psi_n(x) + h_n(b, x) \# d_a \phi_n(x)}{d_a \phi_n(x) \# d_b \psi_n(x)} \right] 
= \left[ \frac{f_n}{\phi_n} + \frac{h_n}{\psi_n} \right] 
= \tilde{\gamma}^{-1}[\hat{f}_n/d_\phi n] + \tilde{\gamma}^{-1}[\hat{h}_n/d_\phi n].
\]
We can also prove that
\[
\tilde{\gamma}^{-1}[c \hat{f}_n/d_\phi n] = c \tilde{\gamma}^{-1}[\hat{f}_n/d_\phi n],
\]
which completes the proof.

Theorem 4.8. The mapping $\tilde{\gamma}^{-1} : \mathcal{L}_b \to \mathcal{L}$ is continuous w.r.t. $\delta$ convergence.

Proof. Let $x_n \to x$ in $\mathcal{L}_b$
Therefore,
\[
x_n = [\hat{h}_{n,k}/d_\phi k] \quad x = [\hat{h}_k/d_\phi k]
\]
and
\[
\hat{h}_{n,k} \to \hat{h}_k \quad \text{as} \quad n \to \infty.
\]
Applying inverse Kamal transform
\[
h_{n,k} \to h_k \quad \text{as} \quad n \to \infty
\]
\[
\Rightarrow h_n/\phi_n \to f/\phi_n \quad \text{as} \quad n \to \infty,
\]
which completes the proof.
5 Conclusion
In this paper we defined the Strong Boehmians for Kamal transform and defined a mapping from Strong Boehmians to General Boehmians. Also we defined the convolution and inverse transform from General Boehmian to Strong Boehmians. An attempt is made to define Strong Boehmian with some references.

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References