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(Dedicated to Honor Professor H.M. Srivastava on His 80th Birth Anniversary Celebrations)

KAMAL TRANSFORM OF STRONG BOEHMIANS

By

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Abstract

The concept of Boehmian was motivated by the so called regular operators introduced by T.K.Boehme. The construction of Boehmians is similar to the construction of field of quotients. Several integral transforms have been extended to various class of Boehmians. We study here Kamal transform and extend it to Strong Boehmian space. This Kamal tranform is 1-1 and continuous in the space of Boehmians. Inverse Kamal transform is also defined.
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1 Introduction

Boehmians have an algebraic character of Mikusinski operators and do not have any restriction on the support. Here we discuss the Kamal transform defined by [11] on certain space of Strong Boehmians. Definition and some properties of Kamal transforms are given. The Kamal transform was introduced by Abdelilah Kamal [11] and many properties are discussed in [10, 13]. Some application related to population growth and decay of Kamal transform are given in[3]. Khandelwal [13] discussed Kamal transform and Kamal decomposition method for solving system of non linear PDE. Also Alomari and Kilicman[7] studied generalized Hartley-Hilbert and Fourier-Hilbert transform and extended them to a class of Boehmians. Al-omari [6] studied the distributional Elzaki transform and gave the extension to Boehmian space. The application of Natural transform and Boehmians [5] is also studied by Al-omari. Sudhansh Aggarwal [2, 3] gave application of Kamal transform for solving voltera integral equation, population growth & decay problems. S.K.Q Al-omari [6] gave application and the relation between Boehmians and Elzaki transform. E.R. Dill & P. Mikusinski [9] defined the concept of Strong Boehmians & its applications. The concept of Mikusinski operators was defined by T.K. Boehme[8]. R. Roopkumar & E.R. Negrin [17] discussed the unified extension of Stieltjes and Poission transform to Boehmians.

The Kamal transform of *f(t)* is defined by [11]

(1.1) $$K[f(t)] = F(v) = \int_0^\infty f(t)e^{-t/v}\,\mathrm{d}t \qquad J_1 \le v \le J_2$$

over the set of functions

$$\mathscr{A} = \{f(t) : \exists M,\ J_1, J_2 > 0 \quad |f(t)| < Me^{|t|/J_j} \quad \text{if} \quad t \in (-1)^j \times [0,\infty)\}. \tag{1.2}$$

We denote the usual convolution of f and g by

$$(f * g)(x) = \int_{R^+} f(x-t)\, g(t)\, \mathrm{d}t. \tag{1.3}$$

The Kamal transform of the convolution product is given by

$$K(f * g) = K(f) \cdot K(g). \tag{1.4}$$

General properties of Kamal transforms:

1. If $\alpha,\beta \in \mathbb{R}$ and $K[f] = F(v)$ and $K[g] = G(v)$ then Kamal transform is linear.

$$\begin{aligned} K[\alpha f + \beta g] &= \int_0^\infty (\alpha f + \beta g)\,(t)\, e^{-t/v}\, \mathrm{d}t \\ &= \alpha \int_0^\infty f(t)\, e^{-t/v}\, \mathrm{d}t + \beta \int_0^\infty g(t)\, e^{-t/v}\, \mathrm{d}t \\ &= \alpha K[f] + \beta K[g] \\ &= \alpha F(v) + \beta G(v). \end{aligned}$$

2. If $f(t) = e^{at}, \sin at, \cos at$ then corresponding $K[f(t)]$ is given by $\dfrac{v}{1-av}, \dfrac{av^2}{1+a^2v^2}, \dfrac{v}{1+a^2v^2}$
   For more properties see [11].

2 Strong Boehmians

We study Strong Boehmians [4, 9] and General Boehmians [5]-[8]. Let I_+ is the set of positive real numbers and $\mathscr{F}$ denote the Schwartz space of test functions ϕ with compact supports over I_+ and $\eta(\Omega)$ be the space of all infinitely differentiable functions over Ω where Ω= [1,∞)$\times$ I_+. The dual of $\eta(\Omega)$ is $\eta'(\Omega)$ consists of distributions of compact supports. Let $f \in \eta(\Omega)$ and $\phi \in \mathscr{F}$ the convolution of f and ϕ is given by

$$(f\#\phi)(x) = \int_{I_+} f(\alpha, t)\, \phi(x-t) \mathrm{d}t, \tag{2.1}$$

where $\alpha \in [1,\infty)$.

Let $\mu(I_+)$ be the subset of $\mathscr{F}$ of the test functions such that

$$\int_{I_+} \phi(x)\, \mathrm{d}x = 1. \tag{2.2}$$

The pair (f,ϕ) or (f/ϕ) of functions such that $f \in \eta(\Omega)$,$\phi \in \mu(I_+)$ is said to be quotient of function denoted by (f,ϕ) or (f/ϕ) if and only if

$$\{f(\alpha, x)\}\#\{\beta\phi\,(\beta x)\} = \{f(\beta, x)\}\#\{\alpha\phi\,(\alpha x)\}, \tag{2.3}$$

for all $\alpha,\beta \in [1,\infty)$
or we define

$$f(\alpha, x) \,\#\, d_\beta\, \phi(x) = f(\beta, x) \,\#\, d_\alpha\, \phi(x),$$

where

$$d_\beta\,\phi(x) = \beta\,\phi(\beta x),$$
$$d_\alpha\,\phi(x) = \alpha\,\phi(\alpha x).$$

We use both the definitions whenever we required. Two quotients (f,ϕ) and (g,ψ) are said to be equivalent that is $(f,\phi)\sim(g,\psi)$ if and only if

$$f(\alpha,x)\#\beta\psi\,(\beta x) = g\,(\beta,x)\,\#\,\alpha\,\phi(\alpha x), \tag{2.4}$$

$\alpha,\beta\in[1,\infty)$.

Let the set be denoted by

$$\mathscr{B} = \{(f,\phi)|\forall f\in\eta(\Omega)\,,\phi\in\mu(I_+)\}. \tag{2.5}$$

Then the equivalence class $[(f,\phi)]$ containing (f,ϕ) is called Strong Boehmian. The space of all such Boehmians is denoted by $\mathscr{L}(\eta,\mu,\#)$ is called as space of Strong Boehmians. Following conclusions are given in [9]

1. Let $\phi,\,\psi\in\mu(I_+)$ then $\phi\,\#\,\psi\in\mu(I_+)$,
2. Let f $\in\eta(\Omega)$ and $\phi\in\mu(I_+)$ then f $\#\,\phi\in\eta(\Omega)$,
3. Let (f,ϕ) $\in\mathscr{B}$ and $\psi\in\mu(I_+)$ then

$$(f\#\psi,\phi\#\psi)\in\mathscr{B}\quad\text{and}\quad(f,\phi)\sim(f\#\psi,\phi\#\psi). \tag{2.6}$$

4. If $\phi\in\mu(I_+)$ then for $\alpha\geq1$ $\quad\alpha\phi(\alpha\text{ x})\in\mu(I_+)$,
5. Let $(f,\phi)\in\mathscr{B}, z>0$ and $h(\alpha,x) = f(\alpha+z,x)$ and $\psi = z\phi(zx)$ then

$$(g,\psi)\in\mathscr{B}\quad\text{and}\quad(g,\psi)\sim(f,\phi). \tag{2.7}$$

Further the operation of addition and scalar multiplication in $\mathscr{L}(\eta,\mu,\#)$ are defined in the usual notation as,

$$\frac{f}{\phi}+\frac{g}{\psi} = \frac{f\#\psi+g\#\phi}{\phi\#\psi},\ \lambda\cdot\frac{f}{\phi} = \frac{\lambda f}{\phi},\quad\frac{f}{\phi}\#\psi = \frac{f\#\psi}{\phi}. \tag{2.8}$$

The above operations are well defined in $\mathscr{L}$ and hence $\mathscr{L}$ is a vector space.

Let

$$D^P = (\frac{\partial}{\partial x_1})^{P_1}(\frac{\partial}{\partial x_2})^{P_2}(\frac{\partial}{\partial x_3})^{P_3}\cdots(\frac{\partial}{\partial x_N})^{P_N} \tag{2.9}$$

where $P = (p_1,p_2,\cdots,p_N)$ and $p_1,p_2,\cdots,p_N$ are nonnegative integers for $\frac{f}{\phi}\in\mathscr{L}(\eta,\mu,\#)$ define $D^p(\frac{f}{\phi}) = \frac{D^p f}{\phi}$, where D^p is well defined operation on $\mathscr{L}$. A sequence of Strong Boehmians $\{y_n\}$ is said to converge to a Strong Boehmian y if $y = \frac{f}{\phi}$ and $y_n = \frac{f_n}{\phi}$ for some $f, f_n\in\eta$ & $\phi\in\mu(I_+)$, $n\in\mathbb{N}$ and $f_n\to f$ uniformly on compact subset of Ω as n $\to\infty$.

3 General Construction of Boehmians

Mikusinski introduced a new class of generalised function space called Boehmian space, which is suitable for extending integral transforms. The construction of Boehmian space and its convergence is given in [15] The construction of Boehmians consists of following elements:

1. A set Γ,
2. Commutative semi group(S,$\otimes$),

3. An operation $\bigstar$: $\Gamma \times S \to \Gamma$ such that for each $x \in \Gamma$ and $\phi_1, \phi_2 \in \triangle \subset S$ $x\bigstar(\phi_1 \otimes \phi_2) = (x \bigstar \phi_1) \otimes (x \bigstar \phi_2)$,
4. (a) A collection $\triangle \subset S$ such that if $x, y \in \Gamma$, $\phi_n \in \triangle$ $x \bigstar \phi_n = y\bigstar \phi_n$ $\forall n \Rightarrow x = y$,
   (b) If $\phi_n \in \triangle$ and $\psi_n \in \triangle$ then $\phi_n \otimes \psi_n \in \triangle$, $\triangle$ is a set of all delta sequences.

Consider

$$\mathscr{B} = \{(x_n, \phi_n) : x_n \in \Gamma, \phi_n \in \triangle, x_n \bigstar \phi_m = x_m \bigstar \phi_n \quad \forall m, n \in \mathbb{N}\}. \tag{3.1}$$

If $(x_n, \phi_n), (y_n, \psi_n) \in \mathscr{B}$ $\quad x_n\bigstar\psi_m = y_m\bigstar\phi_n \quad \forall m, n \in N$ we say that $(x_n, \phi_n) \sim (y_n, \psi_n)$.The relation $\sim$ is an equivalence relation in $\mathscr{B}$.The space of equivalence classes in $\mathscr{B}$ is denoted by $\mathscr{L}_{\mathscr{B}}(\Gamma, S, \triangle)$. Elements of $\mathscr{L}_{\mathscr{B}}(\Gamma, S, \triangle)$ are called General Boehmians.We define a mapping which is a canonical mapping between Γ and $\mathscr{L}_{\mathscr{B}}$ as $x \to x \bigstar \phi_n/\phi_n$.
In $\mathscr{L}_{\mathscr{B}}(\Gamma, S, \triangle)$ there are two type convergences

1. A sequence q_n in $\mathscr{L}_{\mathscr{B}}(\Gamma, \rho, \triangle)$ is said to be δ convergent to q in $\mathscr{L}_{\mathscr{B}}(\Gamma, S, \triangle)$ denoted by $q_n \xrightarrow{\delta} q$ if there exist a delta sequence δ_n such that ($q_n \bigstar \delta_n$), ($q \bigstar \delta_n$) $\in \Gamma$ and for all k,n $\in \mathbb{N}$ ($q_n \bigstar \delta_k$)$\to$ ($q \bigstar \delta_k$) as n$\to \infty$ in Γ,
2. A sequence (q_n) in $\mathscr{L}_{\mathscr{B}}(\Gamma, S, \triangle)$ is said to be $\triangle$ convergent to q in $\mathscr{L}_{\mathscr{B}}(\Gamma, S, \triangle)$ denoted by $q_n \xrightarrow{\Delta} q$ if there exist $(\delta_n) \in \triangle$ such that $(q_n - q)\bigstar\ \delta_n \in \Gamma \quad \forall n \in N$ and $(q_n - q)\bigstar\ \delta_n \to 0$ as $n \to \infty$ in Γ.

Following lemma is an equivalent statement for δ- convergence given by [17]

***Lemma* 3.1.** *$q_n \xrightarrow{\delta} q$ (as n$\to \infty$) in $\mathscr{L}_{\mathscr{B}}(\Gamma, S, \triangle)$ if and only if there exist $f_{n,k}, f_k \in \Gamma$ and $\delta_k \in \triangle$ such that*

$$q_n = [f_{n,k}/\delta_k] \qquad q = [f_n/\delta_k]$$

and for each $k \in N$ $f_{n,k} \to f_k$ as $n \to \infty$ on Γ.

4 Kamal Transform of Strong Boehmians

***Theorem* 4.1** (Convolution theorem)**.** *Let $f \in \eta(\Omega)$ and $\phi \in \mu(I_+)$ then*

$$\begin{aligned} K(f \# \beta\, \psi(\beta x))(\xi) &= K(f(x)) \cdot \beta K(\psi(\beta x)) \\ &= \hat{f}(\xi) \cdot \beta\hat{\psi}(\beta\xi). \end{aligned} \tag{4.1}$$

If onwards we define

$$f(\alpha, x) \# \beta\, \psi\, (\beta x) = f(\alpha, x) \#\, d_\beta\, \psi(x),$$

where,

$$d_\beta\, \psi(x) = \beta\, \psi(\beta x)$$

Then,

$$K(f \# d_\beta\ \psi)\,(\xi) = \hat{f}(\alpha, \xi) \cdot d_\beta\, \hat{\psi}(\xi),$$

where $\hat{f}$ and $\hat{\psi}$ are Kamal transforms of f and ψ.

Proof. By using definition of Kamal transform, Fubini's theorem,

$$K(f(\alpha,t)\# d_\alpha\,\psi(x))(\xi) = \int_0^\infty f(\alpha,t)\,\mathrm{d}t \int_0^\infty d_\alpha\,\psi(x-t)e^{-x/\xi}\,\mathrm{d}x \tag{4.2}$$

$$= \int_0^\infty f(\alpha,t)\,\mathrm{d}t \int_0^\infty \alpha\;\psi(\alpha x-\alpha t)\,e^{-x/\xi}\,\mathrm{d}x.$$

$$\textit{Put}\qquad \alpha x-\alpha t = z \quad \text{i.e}\quad x=\frac{z}{\alpha}+t \quad\text{and}\quad dx=\frac{dz}{\alpha}\ \text{to get} \tag{4.3}$$

$$K(f(\alpha,t)\# d_\alpha\psi(x))(\xi) = \int_0^\infty f(\alpha,t)\,\mathrm{d}t \int_0^\infty \alpha\,\psi(z)\;e^{-(\frac{z}{\alpha}+t)/\xi}\;\mathrm{d}z$$

$$= \int_0^\infty f(\alpha,t)\,\mathrm{d}t \int_0^\infty \alpha\,\psi(z)\;e^{-\frac{z}{\alpha\xi}}\;\cdot e^{-\frac{t}{\xi}}\;\mathrm{d}z$$

$$= \int_0^\infty e^{-\frac{t}{\xi}}\;f(\alpha,t)\,\mathrm{d}t \int_0^\infty \alpha\,\psi(z)\;e^{-\frac{z}{\alpha\xi}}\;\mathrm{d}z$$

$$= \hat{f}(\alpha,\xi)\cdot d_\alpha\;\hat{\psi}(\xi),$$

which completes the proof of the Theorem.

Now we define the images of Kamal transform of Strong Boehmians.

***Definition* 4.1.** *Let $\triangle_1(I_+)$ or $\triangle_1$ be a set of delta sequences such that $\psi_n \in \mu(I_+)$ and suppose $\psi_n \subset (0,\gamma_n)$ $\gamma_n > 0$, $\gamma_n \to 0$ as $n\to\infty$.*
Let $m(I_+)$ be the set of images of Kamal transforms of all $\mu(I_+)$ elements and $\triangle_2(I_+)$ be the set of Kamal transform of all delta sequences from $\triangle_1$ for $f \in \eta(\Omega)$ and $\hat{\psi} \in m(I_+)$ we define the operation $\circledast$ as

$$f(\alpha,\xi)\circledast\hat{\psi}(\xi) = f(\alpha,\xi)\;d_a\,\hat{\psi}(\xi)\quad as \in [1,\infty),$$

from which we see that

$$f\circledast\hat{\psi}\;\in\eta(\Omega)\quad as f\in\eta(\Omega)\quad and\quad d_a\cdot\hat{\psi}\in m(I_+). \tag{4.4}$$

***Lemma* 4.1.** 1. *If $\hat{\phi}_n,\hat{\psi}_n \in \triangle_2(I_+)$ then $\hat{\psi}_n\circledast\hat{\phi}_n \in \triangle_2(I_+)$ $\quad\forall n\in N$*
2. Let $f,g\in\eta(\Omega)$ and $\hat{\psi}_n\in\triangle_2(I_+)$ such that

$$f(c,\xi)\circledast\hat{\psi}_n(\xi) = g(d,\xi)\circledast\hat{\psi}_n(\xi) \tag{4.5}$$

$$then\ f(c,\xi)=g(d,\xi)\quad\forall c,d\in[1,\infty).$$

Proof. 1. $\hat{\phi}_n,\hat{\psi}_n\in\triangle_2(I_+)$ we find the sequences $\phi_n,\psi_n\in\triangle_1(I_+)$.

Since $\phi_n\#\psi_n\;\in\triangle_1(I_+)$, we get

$$K(\phi_n\#\psi_n) = \hat{\phi}_n(\xi)\cdot\hat{\psi}_n(\xi) = \hat{\phi}_n\circledast\hat{\psi}_n\in\triangle_2(I_+)\quad\forall n\in N\quad[by(4.4)]. \tag{4.6}$$

2. Let $\hat{\psi}_n\in\triangle_2(I_+)$ where $\hat{\psi}_n$ is delta sequence $d_a\hat{\psi}_n\;\to 1$ hence $\hat{\psi}_n\to 1$ as $n\to\infty$

$$f(c,\xi)\circledast\hat{\psi}_n(\xi) = f(c,\xi)\,d_a\hat{\psi}_n(\xi)\;\to f(c,\xi)\quad\text{as}\quad n\to\infty, \tag{4.7}$$

$$g(d,\xi)\circledast\hat{\psi}_n(\xi) = g(d,\xi)\,d_a\hat{\psi}_n(\xi)\;\to g(d,\xi)\quad\text{as}\quad n\to\infty, \tag{4.8}$$

from (4.7) and (4.8) $f(c,\xi)=g(d,\xi)\quad\forall\xi\in I_+\quad c,d\in[1,\infty)$, which completes the proof.

***Lemma* 4.2.** *The mapping $\eta(\Omega) \circledast m(I_+) \to m(I_+)$ defined by*

$$f(\alpha,\xi) \circledast \hat{\phi}_n(\xi) = f(\alpha,\xi)\ d_a\hat{\phi}_n(\xi) \tag{4.9}$$

satisfies the following properties:

1 $\hat{\phi}_n \circledast \hat{\psi}_n = \hat{\psi}_n \circledast \hat{\phi}_n$ *for every* $(\hat{\phi}_n), (\hat{\psi}_n) \in \triangle_2(I_+)$ *then*

$$(f+g) \circledast \hat{\phi}_n = f \circledast \hat{\phi}_n + g \circledast \hat{\phi}_n,$$

2 *If* $f \in \eta(\Omega)$, $(\hat{\phi}_n)$, $(\hat{\psi}_n) \in \triangle_2(I_+)$ *then*

$$(f \circledast \hat{\phi}_n) \circledast \hat{\psi}_n = f \circledast (\hat{\phi}_n \circledast \hat{\psi}_n),$$

3 *If* $f \in \eta(\Omega)$, $(\hat{\phi}_n)$, $(\hat{\psi}_n) \in \triangle_2(I_+)$ *then*

$$(f \circledast \hat{\phi}_n) \circledast \hat{\psi}_n = f \circledast (\hat{\phi}_n \circledast \hat{\psi}_n).$$

***Theorem* 4.2.** *The following are true*

1 *If* $f_n \to f$ *in* $\eta(\Omega)$ *and* $\hat{\psi} \in m(I_+)$ *then* $f_n \circledast \hat{\psi} \to f \circledast \hat{\psi}$ *as* $n \to \infty$,
2 *If* $f_n \to f$ *in* $\eta(\Omega)$ *and* $\hat{\psi}_n \in \triangle_2(I_+)$ *then* $f_n \circledast \hat{\psi}_n \to f$ *as* $n \to \infty$.

Proof. 1. $\hat{\psi} \in m(I_+)$ f_n, $f \in \eta(\Omega)$ then

$$|D^k_\xi(f_n(\alpha,\xi) \circledast \hat{\psi(\xi)} - f(\alpha,\xi) \circledast \hat{\psi(\xi)})| = |D^k_\xi \cdot d_a\,\hat{\psi}(\xi)\,(f_n - f)(\alpha,\xi)| \to 0. \tag{4.10}$$

As $n \to \infty$ in $\eta(\Omega)$, therefore $f_n \circledast \hat{\psi} \to f \circledast \hat{\psi}$.

2. $\hat{\phi}_m$ $\triangle_2(I_+)$ then $d_a\,\hat{\phi}_n(\xi) \to 1$ as $n \to \infty$ implies

$$|D^k_\xi(f_n \circledast \hat{\psi}_n(\xi) - f(\alpha,\xi)| \to |D^k_\xi(f_n(\alpha,\xi) - f(\alpha,\xi)| \to 0 \text{ as } n \to \infty. \tag{4.11}$$

Hence $f_n \circledast \hat{\psi}_n \to f$, which completes the proof.

The General Boehmian space $\mathscr{L}_{\mathscr{B}}(\eta,\ m,\ \triangle_2,\ \circledast)$ or $\mathscr{L}_{\mathscr{B}}$ is constructed. We give some properties of sum, scalar multiplication, differentiation as

$$[\frac{f_n}{\hat{\phi}_n}] + [\frac{g_n}{\hat{\psi}_n}] = [\frac{f_n \circledast \hat{\psi}_n + g_n \circledast \hat{\phi}_n}{\hat{\phi}_n \circledast \hat{\psi}_n}], \alpha[\frac{f_n}{\hat{\phi}_n}] = [\frac{\alpha f_n}{\hat{\phi}_n}]. \tag{4.12}$$

$$[\frac{f_n}{\hat{\phi}_n}] \circledast [\frac{g_n}{\hat{\psi}_n}] = [\frac{f_n \circledast g_n}{\hat{\phi}_n \circledast \hat{\psi}_n}], D^\alpha[\frac{f_n}{\hat{\phi}_n}] = [\frac{D^\alpha f_n}{\hat{\phi}_n}]. \tag{4.13}$$

Now we are concerned with the Strong Boehmians which are described by the set $(\eta, \#)$ and the subset $(\mu, \#)$ with the family $\triangle_1$ of delta sequences such a space is denoted by $\mathscr{L}(\eta,\ (\mu,\#),\ \triangle_1,\ \#)$ or simply by $\mathscr{L}$.This space preserve the operation of addition, scalar multiplication,differentiation and the convolution.

***Definition* 4.2.** *Let* $f \in \eta(\Omega)$ *and* $\phi \in \mu(I_+)$ *we define the Kamal transform of the Strong Boehmians* $[f_n/\phi_n]$ *in* $\mathscr{L}$ *by*

$$\tilde{\gamma}[\frac{f_n}{\phi_n}] = [\frac{\hat{f}_n}{d_a\hat{\phi}_n}] \in \mathscr{L}_{\mathscr{B}} \quad \textit{where } \mathscr{L}_{\mathscr{B}} \textit{ is General Boehmians.} \tag{4.14}$$

***Theorem* 4.3.** *The Kamal transform* $\tilde{\gamma} : \mathscr{L} \to \mathscr{L}_{\mathscr{B}}$ *is well defined.*

Proof. Let $[\frac{f_n}{\phi_n}], [\frac{g_n}{\psi_n}] \in \mathscr{L}$ are such that $[\frac{f_n}{\phi_n}] = [\frac{g_n}{\psi_n}]$. Then

$$f_n(\alpha, x) \# d_\beta \psi_n(x) = g_n(\beta, x) \# d_\alpha \phi_n(x). \tag{4.15}$$

Apply the convolution theorem on both sides of (4.15)

$$\frac{\hat{g}_n}{d_\beta \hat{\psi}_n} = \frac{\hat{f}_n}{d_\alpha \hat{\phi}_n}$$

Hence

$$\tilde{\gamma}\,[\frac{f_n}{\phi_n}] = \tilde{\gamma}\,[\frac{g_n}{\psi_n}], \tag{4.16}$$

which completes the proof.

***Theorem* 4.4.** *(ψ_n), $(\phi_n) \in \triangle_1(I_+)$ and $f, g \in \eta(\Omega)$ then mapping $\tilde{\gamma} : \mathscr{L} \to \mathscr{L}_{\mathscr{B}}$ is one-one.*

Proof. Now by (4.16)

$$\tilde{\gamma}\,[f_n/\phi_n] = \tilde{\gamma}\,[g_n/\psi_n] \text{ in } \mathscr{L}_{\mathscr{B}}.$$

Therefore

$$\hat{f}_n(\alpha, \xi) \circledast \hat{\psi}_n(\xi) = \hat{g}_n(\beta, \xi) \circledast \hat{\phi}_n(\xi),$$

$$\hat{f}_n(\alpha, \xi)\, d_\beta \hat{\psi}_n(\xi) = \hat{g}_n(\beta, \xi)\, d_\alpha \hat{\phi}_n(\xi),$$

$$K(f_n(\alpha, x) \# d_\beta \psi_n(x)) = K(g_n(\beta, x) \# d_\alpha \phi_n(x)).$$

Since Kamal transform is one-one.

$$f_n(\alpha, x) \# d_\beta \psi_n(x) = g_n(\beta, x) \# d_\alpha \phi_n(x)$$

$$\Rightarrow \quad \frac{f_n(\alpha, x)}{\phi_n(x)} \sim \frac{g_n(\beta, x)}{\psi_n(x)}$$

$$\Rightarrow \quad [\frac{f_n}{\phi_n}] = [\frac{g_n}{\psi_n}],$$

which completes the proof.

***Theorem* 4.5.** *$\tilde{\gamma} : \mathscr{L} \to \mathscr{L}_{\mathscr{B}}$ is continuous with respect to μ convergence.*

Proof. Let $y_n \to y \in \mathscr{L}$ by using the convergence concept in μ in $\mathscr{L}$[[9], [Theorem (2.6)]], we have ϕ for all y_n such that $y_n = [\frac{f_n}{\phi}]$ $\quad y = [\frac{f}{\phi}]$ and $f_n \to f$ as $n \to \infty$

Hence $\hat{f}_n \to \hat{f}$ as $n \to \infty$

$$\Rightarrow \frac{\hat{f}_n}{d_\alpha \hat{\phi}} \to \frac{\hat{f}}{d_\alpha \hat{\phi}} \quad \text{as } n \to \infty$$

Therefore $\tilde{\gamma}\, y_n \to \tilde{\gamma}\, y$ as $n \to \infty$ in $\mathscr{L}_{\mathscr{B}}$.

***Definition* 4.3.** *Let $z = [\hat{f}_n/d_\alpha \hat{\phi}_n] \in \mathscr{L}_{\mathscr{B}}$ then we can define $\tilde{\gamma}^{-1}$ of $\tilde{\gamma}$ by*

$$\tilde{\gamma}^{-1} z = [\frac{f_n}{\phi_n}] \in \mathscr{L}.$$

We can prove that $\tilde{\gamma}^{-1}$ is well defined,linear,continuous w.r.t. δ convergence.

***Theorem* 4.6.** *The mapping $\tilde{\gamma}^{-1} : \mathscr{L}_{\mathscr{B}} \to \mathscr{L}$ is well defined.*

Proof. Let

$$[\hat{f}_n/d_a\hat{\phi}_n] = [\hat{h_m}/d_b\hat{\psi}_n] \qquad a, b \in [1, \infty).$$

Then

$$\hat{f}_n(a,\xi) \circledast \hat{\psi_m}(\xi) = \hat{h_m}(b,\xi) \circledast \hat{\phi}_n(\xi).$$

By(4.4)

$$\hat{f}_n(a,\xi)\, d_b\, \hat{\psi_m}(\xi) = \hat{h_m}(b,\xi)\, d_a\, \hat{\phi}_n(\xi), \tag{4.17}$$

therefore by *Theorem* (4.1)

$$K(f_n(a,x) \# d_b\, \psi_m(x)) = K(h_m(b,x) \# d_a\, \phi_n(x)).$$

Hence

$$f_n(a,x) \# d_b\, \psi_m(x) = h_m(b,x) \# d_a\, \phi_n(x), \tag{4.18}$$

which completes the proof.

***Theorem* 4.7.** *The mapping $\tilde{\gamma}^{-1} : \mathscr{L}_{\mathscr{B}} \to \mathscr{L}$ is linear*

Proof. Let $[\hat{f}_n/d_a\hat{\phi}_n]$, $[\hat{h}_n/d_b\hat{\psi}_n] \in \mathscr{L}_{\mathscr{B}}$ & $c \in [1, \infty)$.
Then by (4.4)

$$\begin{aligned} \tilde{\gamma}^{-1}\{[\hat{f}_n/d_a\hat{\phi}_n] + [\hat{h}_n/d_b\hat{\psi}_n]\} &= \tilde{\gamma}^{-1}\{\frac{\hat{f}_n(a,\xi)\, d_b\hat{\psi}_n(\xi) + \hat{h}_n(b,\xi)\, d_a\hat{\phi}_n(\xi)}{d_a\hat{\phi}_n(\xi) \# d_b\hat{\psi}_n(\xi)}\} \\ &= [\frac{f_n(a,x) \# d_b\psi_n(x) + h_n(b,x) \# d_a\phi_n(x)}{d_a\phi_n(x) \# d_b\psi_n(x)}] \\ &= [\frac{f_n}{\phi_n} + \frac{h_n}{\psi_n}] \\ &= \tilde{\gamma}^{-1}[\hat{f}_n/d_a\hat{\phi}_n] + \tilde{\gamma}^{-1}[\hat{h}_n/d_b\hat{\psi}_n]. \end{aligned} \tag{4.19}$$

We can also prove that

$$\tilde{\gamma}^{-1}[c\hat{f}_n/d_a\hat{\phi}_n] = c\,\tilde{\gamma}^{-1}[\hat{f}_n/d_a\hat{\phi}_n],$$

which completes the proof.

***Theorem* 4.8.** *The mapping $\tilde{\gamma}^{-1} : \mathscr{L}_{\mathscr{B}} \to \mathscr{L}$ is continuous w.r.t. δ convergence.*

Proof. Let $x_n \to x$ in $\mathscr{L}_{\mathscr{B}}$
Therefore,

$$x_n = [\hat{h}_{n,k}/d_a\hat{\phi}_k] \qquad x = [\hat{h}_k/d_a\hat{\phi}_k]$$

and

$$\hat{h}_{n,k} \to \hat{h}_k \qquad \text{as} \quad n \to \infty.$$

Applying inverse Kamal transform

$$\begin{aligned} h_{n,k} &\to h_k \qquad \text{as} \quad n \to \infty \\ \Rightarrow h_n/\phi_n &\to f/\phi_n \qquad \text{as} \quad n \to \infty, \end{aligned}$$

which completes the proof.

5 Conclusion

In this paper we defined the Strong Boehmians for Kamal transform and defined a mapping from Strong Boehmians to General Boehmians. Also we defined the convolution and inverse transform from General Boehmian to Strong Boehmians. An attempt is made to define Strong Boehmian with some references.

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