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# INTEGRAL RELATIONS INVOLVING FOX'S H-FUNCTION

*by*

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**Abstract.** In this present paper we evaluate two basic integrals which are then used to prove two triple integral relations involving Fox's H-function [2]. Our results are the generalization of the results given by Dahiya [1, Eq (1), (2)], and Prasad and Ram [3]. On specializing the parameters many new results are derived.

1. In this section we give two lemmas and two main results as follows :

**Lemma 1.** $\int_0^{\pi/2} \cos 2u\phi \, (\cos \phi)^{2\nu} \, (\sin \phi)^{2\nu_1} \, d\phi$

$$= \frac{\Gamma(u+\nu+\frac{1}{2})\,\Gamma(\nu_1+\frac{1}{2})}{2\Gamma(u+\nu+\nu_1+1)} \; {}_3F_2\left[\begin{matrix} \nu_1+\frac{1}{2}, \; -u, \; -u+\frac{1}{2} \\ -u-\nu+\frac{1}{2}, \; \frac{1}{2} \end{matrix}; 1\right] \tag{1}$$

provided that $R(2\nu+1)>0$, $R(2\nu_1+1)>0$, and $u$ is a positive integer.

**Lemma 2.** $\int_0^{\pi} \sin (2u+1)\phi (\cos \phi)^{2\nu} \, (\sin \phi)^{2\nu_1} \, d\phi$

$$= \frac{\Gamma(2u+2)\,\Gamma(u+\nu+\frac{1}{2})\,\Gamma(\nu_1+1)}{\Gamma(u+\nu+\nu_1+\frac{3}{2})\,\Gamma(2u+1)} \; {}_3F_2\left[\begin{matrix} \nu_1+1, \; -u, \; -u+\frac{1}{2} \\ -u-\nu+\frac{1}{2}, \; \frac{3}{2} \end{matrix}; 1\right] \tag{2}$$

provided that $R(\nu+1)>0$, $R(\nu_1+1)>0$, and $u$ is a positive integer.

**Main Results**

$$\text{(i)} \quad \int_0^\infty \int_0^\infty \int_0^\infty (x^2+y^2+z^2)^{-1/2} \frac{x^{2\nu}\ y^{2\nu_1}}{(x^2+y^2)^{\nu+\nu_1+\frac{1}{2}}}$$

$$F\left(\tan^{-1}\frac{\sqrt{x^2+y^2}}{z}\right) \cos 2u\left(\tan^{-1}\frac{y}{x}\right)$$

$$f(x^2+y^2+z^2)\ H_{p,q}^{m,n}\left[\frac{\alpha(x^2+y^2+z^2)^{\rho+\mu+\mu_1}}{x^{2\sigma}\ y^{2\sigma_1}\ z^{2\mu}\,(x^2+y^2)^{\mu_1-(\sigma+\sigma_1)}}\right.$$

$$\left|\begin{matrix}\{(a_p,\ \alpha_p)\}\\ \{(b_q,\ \beta_q)\}\end{matrix}\right]dxdydz$$

$$=\tfrac{1}{2}\sum_{t=0}^{u}(-1)^t\frac{(-u)_t\,(-u+\frac{1}{2})_t}{(\frac{1}{2})_t\qquad t!}\int_0^\infty\int_0^\infty (v^2+w^2)^{-1/2} f(v^2+w^2)$$

$$F\left(\tan^{-1}\frac{w}{v}\right)$$

$$H_{p+1,\,q+2}^{m+2,\,n}\left[\frac{\alpha(v^2+w^2)^{\rho+\mu+\mu_1}}{v^{2\mu}\ w^{2\mu_1}}\right.$$

$$\left|\begin{matrix}\{(a_p,\ \alpha_p)\},\ (u+\nu+\nu_1+1,\ \sigma+\sigma_1)\\ (u+\nu+\frac{1}{2}-t,\ \sigma),\ (\nu_1+\frac{1}{2}+t,\ \sigma_1),\ \{(b_q,\ \beta_q)\}\end{matrix}\right]dvdw \qquad (3)$$

provided that $\rho,\ \mu,\ \mu_1,\ \sigma,\ \sigma_1 \geqslant 0$, $R(2\nu-2\sigma\beta+1)>0$, $R(2\nu_1-2\sigma_1\beta+1)>0$, $f$ and $F$ are two functions such that the integrals exist, $|\arg\alpha|<\frac{1}{2}\lambda\pi$, $\lambda>0$ and $A>0$, $f(z)=0(z^{-\delta})$ for large $z$, where

$$\lambda\equiv\sum_{j=1}^{n}(\alpha_j)-\sum_{j=n+1}^{p}(\alpha_j)+\sum_{j=1}^{m}(\beta_j)-\sum_{j=m+1}^{q}(\beta_j)>0$$

$$A\equiv\sum_{j=1}^{q}(\beta_j)-\sum_{j=1}^{p}(\alpha_j)>0 \text{ and } \beta=\max_{1\leqslant i\leqslant n} R\left(\frac{a_i-1}{\alpha_i}\right)$$

$$\text{(ii)} \quad \int_0^\infty \int_0^\infty \int_0^\infty (x^2+y^2+z^2)^{-1/2} \frac{x^{2\nu}\ y^{2\nu_1}}{(x^2+y^2)^{\nu+\nu_1+\frac{1}{2}}}$$

$$.\ F\left(\tan^{-1}\frac{\sqrt{x^2+y^2}}{z}\right)$$

$$\sin\left\{(2u+1)\tan^{-1}\frac{y}{z}\right\} f(x^2+y^2+z^2).$$

$$. H^{m,n}_{p,q}\left[\frac{\alpha(x^2+y^2+z^2)^{\rho+\mu+\mu_1}}{x^{2\sigma}\ y^{2\sigma_1}\ z^{2\mu}(x^2+y^2)^{\mu_1-(\sigma+\sigma_1)}}\middle|\begin{matrix}\{a_p,\alpha_p)\}\\(b_q,\beta_q)\}\end{matrix}\right]dxdydz$$

$$=\frac{1}{2}\sum_{t=0}^{u}\frac{(-1)^t(-u)_t(-u+\frac{1}{2})_t(2u+1)}{(\frac{3}{2})_t\ t!}\int_0^\infty\int_0^\infty (v^2+w^2)^{-\frac{1}{2}} f(v^2+w^2)$$

$$F\left(\tan^{-1}\frac{W}{v}\right)H^{m+2,n}_{p+1,q+2}\left[\frac{\alpha(w^2+v^2)^{\rho+\mu+\mu_1}}{v^{2\mu}\ w^{2\mu_1}}\middle|\begin{matrix}\{(a_p,\alpha_p)\},(u+\nu+\nu_1+\frac{3}{2},\sigma+\sigma_1)\\(u+\nu+\frac{1}{2}-t,\sigma),(\nu_1+1+t,\sigma_1),\{(b_q,\beta_q)\}\end{matrix}\right]dvdw \qquad (4)$$

provided, that $\rho, \mu, \mu_1, \sigma, \sigma_1 \geqslant, 0$, $R(1+\nu_1-\sigma_1\beta)>0$, $R(1+2\nu-2\sigma\beta)>0$, $f$ and $F$ are two functions such that the integrals exist, $|\arg\alpha|<\frac{1}{2}\lambda\pi$, $\lambda>0$, $A>0$ and $\lambda$, $\mu$, $\beta$ and $A$ are the same as in (3).

**Proof.** For the proof of (1) and (2), we expand $\cos 2u\phi$ and $\sin(2u+1)\phi$ in powers of $\cos\phi$ and $\sin\phi$ ; put the expansions into the integrand of (1) and (2), evaluate each term with the help of gamma function and use the formulae

$$\sqrt{\pi}\,\Gamma(2z)=2^{2z-1}\,\Gamma(z)\,\Gamma(z+\tfrac{1}{2})$$

and

$$\Gamma(z-r+1)=(-1)^r\,\frac{\Gamma(-z)\,\Gamma(z+1)}{\Gamma(-z+r)}. \qquad (5)$$

Now in order to obtain the result (3), we start from the following integral, viz.

$$\int_0^{\frac{\pi}{2}}\cos 2u\phi\,(\cos\phi)^{2\nu}(\sin\phi)^{2\nu_1}$$

$$. H^{m,n}_{p,q}\left[\frac{\alpha r^{2\rho}}{(\cos\theta)^{2\mu}(\sin\theta)^{2\mu_1}(\cos\phi)^{2\sigma}(\sin\phi)^{2\sigma_1}}\middle|\begin{matrix}\{(a_p,\alpha_p)\}\\\{(b_q,\beta_q)\}\end{matrix}\right]d\phi$$

$$=\frac{1}{2}\sum_{t=0}^{u}\frac{(-1)^t(-u)_t(-u+\frac{1}{2})_t}{(\frac{1}{2})_t\ t!}H^{m+2,n}_{p+1,q+2}\left[\frac{\alpha r^{2\rho}}{(\cos\theta)^{2\mu}(\sin\theta)^{2\mu_1}}\middle|\begin{matrix}\{(a_p,\alpha_p)\},(u+\nu+\nu_1+1,\sigma+\sigma_1)\\(u+\nu+\frac{1}{2}-t,\sigma),(\nu_1+\frac{1}{2}+t,\sigma_1),\{(b_q,\beta_q)\}\end{matrix}\right] \qquad (6)$$

provided that, $\rho$, $\mu$, $\mu_1$, $\sigma$, $\sigma_1$, $\geqslant 0$, $R(2\nu-2\sigma\beta+1)>0$, $R(2\nu_1-2\sigma_1\beta+1)>0$, $u$ being a positive integer, $|\arg \alpha|<\frac{1}{2}\lambda\pi$, $\lambda>0$, $A>0$, where $\beta$, $\lambda$ and $A$ are given in (3).

To prove (6), we write the *H*-function as Mellin-Barnes type contour integral and change the order of integration which is justifiable under the given conditions. The left-hand side of (6) is

$$\frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m} \Gamma(b_j-\beta_j s) \prod_{j=1}^{n} \Gamma(1-a_j+\alpha_j s)}{\prod_{j=m+1}^{q} \Gamma(1-b_j+\beta_j s) \prod_{j=n+1}^{p} \Gamma(a_j-\alpha_j s)} \cdot \left(\frac{\alpha r^{2\rho}}{(\cos\theta)^{2\mu}(\sin\theta)^{2\mu_1}}\right)^S ds.$$

$$\cdot \int_0^{\frac{\pi}{2}} (\cos 2u\phi\, (\cos\phi)^{2\nu-2\sigma s} (\sin\phi)^{2\nu_1-2\sigma_1 s}\, d\phi \tag{7}$$

Now evaluating the inner integral in (7) with the help of (1), it reduces to

$$=\frac{1}{2}\sum_{t=0}^{u} (-1)^t \frac{(-u)_t\,(-u+\frac{1}{2})_t}{(\frac{1}{2})_t\; t!}(2\pi i)^{-1}$$

$$\int_L \frac{\prod_{j=1}^{m} \Gamma(b_j-\beta_j s) \prod_{j=1}^{n} \Gamma(1-a_j+\alpha_j s)}{\prod_{j=m+1}^{q} \Gamma(1-b_j+\beta_j s) \prod_{j=n+1}^{q} \Gamma(a_j-\alpha_j s)}$$

$$\frac{\Gamma(u+\nu-\sigma s+\frac{1}{2})\,\Gamma(-u-\nu+\sigma s+\frac{1}{2})\,\Gamma(\nu_1-\sigma_1 s+\frac{1}{2}+t)}{\Gamma(u+\nu+\nu_1-(\sigma+\sigma_1)s+1)\,\Gamma(-u-\nu+\frac{1}{2}+\sigma s+t)}.$$

$$\cdot\left(\frac{\alpha r^{2\rho}}{(\cos\theta)^{2\mu}(\sin\theta)^{2\mu_1}}\right)^S ds$$

Now using the formula given in (5) in above and interpreting the result in the light of the definition of the *H*-function of Fox [2, p. 408], we obtain the right-hand side of (6). Now multiplying both sides of (6) by $f(r^2)\, F(\theta)\, dr\, d\theta$ and integrating with respect to $\theta$ from 0 to $\frac{\pi}{2}$ and $r$ from 0 to $\infty$, we get

$$\int_0^\infty \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{1}{r^2 \sin\theta} f(r^2)\, F(\theta) \cos 2u\phi\, (\cos\phi)^{2\nu} (\sin\phi)^{2\nu_1}.$$

$$.\; H_{p,q}^{m,n}\left[\alpha r^{2\rho} \frac{(\cos\theta)^{-2\mu} (\sin\theta)^{-2\mu_1}}{(\cos\phi)^{2\sigma} \quad (\sin\phi)^{2\sigma_1}} \middle| \begin{matrix} \{(a_p, \alpha_p)\} \\ \{(b_q, \beta_q)\} \end{matrix} \right] r^2 \sin\theta\, dr d\theta d\phi$$

$$= \tfrac{1}{2} \sum_{t=0}^{u} (-1)^t \frac{(-u)_t (-u+\frac{1}{2})_t}{(\frac{1}{2})_t \; t!} \int_0^\infty \int_0^{\frac{\pi}{2}} \frac{1}{r} f(r^2)\, F(\theta).$$

$$.\; H_{p+1,\, q+2}^{m+2,\, n}\left[\frac{\alpha r^{2\rho}}{(\cos\theta)^{2\mu} \; (\sin\theta)^{2\mu_1}}\right.$$

$$\left| \begin{matrix} \{(a_p, \alpha_p)\}, (u+\nu+\nu_1+1, \sigma+\sigma_1) \\ (u+\nu+\frac{1}{2}-t, \sigma), (\nu_1+\frac{1}{2}+t, \sigma_1), \{b_q, \beta_q)\} \end{matrix} \right] r dr d\theta \tag{8}$$

Now using the spherical polar coordinates $x=r \sin\theta \cos\phi$, $y=r \sin\theta \sin\phi$, $z=r\cos\theta$ in the left-hand side of (8), and the polar coordinates $v=r \cos\theta$, $w=r \sin\theta$ in the right-hand side of (8), we obtain the result given in (3).

The result (4) can be obtained on similar lines by using the result given in (2).

**2. Particular cases.** (i) On putting $\nu_1=\sigma_1=0$ and $\mu=0$ in (3) and inserting the summation inside the integration, we obtain

$$\int_0^\infty \int_0^\infty \int_0^\infty \frac{(x^2+y^2+z^2)^{-\frac{1}{2}} x^{2\nu}}{(x^2+y^2)^{\nu+\frac{1}{2}}}$$

$$F\left(\tan^{-1} \frac{\sqrt{x^2+y^2}}{z}\right) \cos 2u \left(\tan^{-1} \frac{y}{x}\right).$$

$$.\; f(x^2+y^2+z^2)\, H_{p,q}^{m,n}\left[\frac{\alpha(x^2+y^2+z^2)^{\rho+\mu_1} x^{-2\sigma}}{(x^2+y^2)^{\mu_1-\sigma}}\right.$$

$$\left| \begin{matrix} \{(a_p, \alpha_p)\} \\ \{(b_q, \beta_q)\} \end{matrix} \right] dx\, dy\, dz$$

$$=\frac{\sqrt{\pi}}{2}\int_0^\infty\int_0^\infty (v^2+w^2)^{-\frac{1}{2}} f(v^2+w^2)\, F\left(\tan^{-1}\frac{w}{v}\right)$$

$$\cdot\left[\frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j s)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+\beta_j s)\prod_{j=n+1}^{p}\Gamma(a_j-\alpha_j s)}\left(\frac{r^2}{(\sin\theta)^{2\mu_1}}\right)^S ds.\right.$$

$$\left.\cdot\sum_{t=0}^{u}(-1)^t\frac{(u+\nu-\sigma s+\frac{1}{2}-t)\,\Gamma(-u+t)\,\Gamma(-u+\frac{1}{2}+t)}{\Gamma(u+\nu-\sigma s+1)\,\Gamma(-u)\,\Gamma(-u+\frac{1}{2})\,t!}\right]dv\,dw.$$

$$=\frac{\sqrt{\pi}}{2}\int_0^\infty\int_0^\infty (v^2+w^2)^{-1/2} f(v^2+w^2)\, F\left(\tan^{-1}\frac{w}{v}\right).$$

$$\left[\frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j s)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+\beta_j s)\prod_{j=n+1}^{n}\Gamma(1-a_j+\alpha_j s)}\right.$$

$$\left.{}_2F_1\left[\begin{matrix}-u, & -u+\frac{1}{2} \\ -u-\nu & +\frac{1}{2}-\sigma s\end{matrix}\,;\,1\right]\left(\frac{\alpha r^{2\rho}}{\sin\theta^{2\mu_1}}\right)^S ds\right]dv\,dw \qquad (9)$$

The sum of the hypergeometric series involved in (9) is written by a known formula

$${}_2F_1\ (a, b, c\ ;\ 1)=\frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)}$$

where $c$ is not a negative integer and $R(c-a-b)>0$. Now using the duplication formula of gamma function on the right hand side of (9) and a formula

$$\Gamma(z)\,\Gamma(1-z)=\frac{\pi}{\sin\pi z},$$

then right-hand side of (9) reduces to

$$=\frac{\sqrt{\pi}}{2}\int_0^\infty\int_0^\infty (v^2+w^2)^{-1/2} f(v^2+w^2)\, F\left(\tan^{-1}\frac{w}{v}\right).$$

$$H^{m+2,\, n}_{p+2,\, q+2}\left[\frac{\alpha(v^2+w^2)^{\rho+\mu_1}}{w^{2\mu_1}}\left|\begin{matrix}\{(a_p,\ \alpha_p)\},\ (\nu\pm u+1,\ \sigma)\\ (\nu+\tfrac{1}{2},\ \sigma),\ (\nu+1,\ \sigma),\ \{(b_q,\ \beta_q)\}\end{matrix}\right.\right] dv\, dw \quad (10)$$

provided that $R(2\nu-2\sigma\beta+1)>0$, $|\arg\alpha|<\frac{1}{2}\lambda\pi$, $\lambda>0$, $A>0$, $f$ and $F$ are two functions such that the integrals exist, where $\lambda$, $u$, $\beta$ and $A$ are given in (3).

This is a known result due to Prasad and Ram [3].

(ii) Putting $\nu=\sigma=0$ and $\mu=0$ in (3) and simplifying further as in (10), we obtain

$$\int_0^\infty\int_0^\infty\int_0^\infty (x^2+y^2+z^2)^{-\frac{1}{2}}(x^2+y^2)^{-\nu_1-\frac{1}{2}}\, y^{2\nu_1}$$

$$\cos 2u\left(\tan^{-1}\frac{y}{x}\right) f(x^2+y^2+z^2)$$

$$F\left(\tan^{-1}\frac{\sqrt{x^2+y^2}}{z}\right) H^{m,\, n}_{p,\, q}\left[\frac{\alpha(x^2+y^2+z^2)^{\rho+\mu_1}}{(x^2+y^2)^{\mu_1-\sigma_1}\, y^{2\sigma_1}}\left|\begin{matrix}\{(a_p,\ \alpha_p)\}\\ \{(b_q,\ \beta_q)\}\end{matrix}\right.\right]$$

$$dx\, dy\, dz$$

$$=\frac{\Gamma(\frac{1}{2}\pm u)}{2\sqrt{\pi}}\int_0^\infty\int_0^\infty (v^2+w^2)^{-1/2} f(v^2+w^2)\ F\left(\tan^{-1}\frac{w}{v}\right)$$

$$H^{m+2,\, n}_{p+2,\, q+2}\left[\frac{\alpha(v^2+w^2)^{\rho+\mu_1}}{w^{2\mu_1}}\left|\begin{matrix}\{(a_p,\ \alpha_p)\},\ (\nu_1\pm u+1,\ \sigma_1)\\ (\nu_1+\tfrac{1}{2},\ \sigma_1),\ (\nu_1+1,\ \sigma_1),\ \{(b_q,\ \beta_q)\}\end{matrix}\right.\right]$$

$$du\, dw \quad (11)$$

provided that $R(2\nu-2\sigma\beta+1)>0$, $|\arg\alpha|<\frac{1}{2}\lambda\pi$, $\lambda>0$, $A>0$, $f$ and $R$ are two functions such that the integrals exist, where $\lambda$, $\beta$, $u$ and $A$ are given in (3).

This is a known result also due to Prasad and Ram [3].

(iii) On taking $F\left(\tan^{-1}\frac{w}{v}\right)=\cos\left(2\mu_0\tan^{-1}\frac{w}{v}\right)\sin^{2\nu_0}\left(\tan^{-1}\frac{w}{v}\right)$

$$=\left(\frac{w^2}{v^2+w^2}\right)^{\nu_0}\cos\left(2\mu_0\tan^{-1}\frac{w}{v}\right);$$

$\mu=0$, and $f(v^2+w^2)=(v^2+w^2)^{1/2}f(v^2+w^2)$ in (3), we obtain

$$\int_0^\infty\int_0^\infty\int_0^\infty \frac{x^{2\nu}y^{2\nu_1}(x^2+y^2+z^2)^{-\nu_0}}{(x^2+y^2)^{\nu+\nu_1+\frac{1}{2}-\nu_0}}\cos\left(2\mu_0\tan^{-1}\frac{\sqrt{x^2+y^2}}{z}\right)$$

$$\cos 2u\left(\tan^{-1}\frac{y}{x}\right)f(x^2+y^2+z^2)\,H^{m,n}_{p,q}$$

$$\left[\frac{\alpha(x^2+y^2+z^2)^{\rho+\mu_1}}{x^{2\sigma}y^{2\sigma_1}(x^2+y^2)^{\mu_1-(\sigma+\sigma_1)}}\middle| \begin{matrix}\{(a_p,\ \alpha_p)\}\\ \{(b_q,\ \beta_q)\}\end{matrix}\right]dx\,dy\,dz$$

$$=\tfrac{1}{2}\sum_{t=0}^{u}(-1)^t\frac{(-u)_t(-u+\frac{1}{2})_t}{(\frac{1}{2})_t\,t\,!}\int_0^\infty\int_0^\infty f(v^2+w_2)\left(\frac{w^2}{w^2+v^2}\right)^{\nu_0}$$

$$\cos\left(2\mu_0\ \tan^{-1}\frac{w}{v}\right)$$

$$.\,H^{m+2,n}_{p+1,q+2}\left[\frac{\alpha(v^2+w^2)^{\rho+\mu_1}}{w^{2\mu_1}}\middle|\right.$$

$$\left.\begin{matrix}\{(a_p,\ \alpha_p)\},\ (u+\nu+\nu_1+1,\ \sigma+\sigma_1)\\ (u+\nu+\frac{1}{2}-t,\ \sigma),\ (\nu_1+\frac{1}{2}+t,\ \sigma_1),\ \{(b_q,\beta_q)\}\end{matrix}\right]dv\,dw \quad (12)$$

provided that $R(2\nu-2\sigma\beta+1)>0$, $R(2\nu_1-2\sigma_1\beta+1)>0$, $\rho,\ \mu_1,\ \sigma,\ \sigma_1\geqslant 0$, $R(\nu_0)>0$, $u$, $\mu_0$ are positive integers, $|\arg\alpha|<\frac{1}{2}\lambda\pi$, $\lambda>0$, $A>0$, $f$ is such that the integrals exist; $A$, $\lambda$ and $\beta$ are given in (3).

Now using the following integral relation which can easily be proved on Dahiya's lines [1, Eq. (1)], viz;

$$\int_0^\infty\int_0^\infty\left(\frac{y^2}{x^2+y^2}\right)^{\nu}\cos\left(2u\tan^{-1}\frac{y}{x}\right)f(x^2+y^2)$$

$$.H^{m,n}_{p,q}\left[\frac{\alpha(x^2+y^2)^{\rho+\sigma_1}}{y^{2\sigma_1}}\middle|\begin{matrix}\{(a_p,\ \alpha_p)\}\\ \{(b_q,\ \beta_q)\}\end{matrix}\right]dx\,dy$$

$$=\frac{\Gamma(\frac{1}{2}\pm u)}{4\sqrt{\pi}}\int_0^\infty f(z)\,H^{m+2,n}_{p+2,q+2}$$

$$\left[\alpha z^{\rho}\middle|\begin{matrix}\{(a_p,\ \alpha_p)\},\ (\nu\pm u+1,\ \sigma)\\ (\nu+\frac{1}{2},\ \sigma),\ (\nu+1,\ \sigma),\ \{(b_q,\ \beta_q)\}\end{matrix}\right]dz \qquad (13)$$

where $R(\nu) > 0$, $u$ being a positive integer, $|\arg \alpha| < \frac{1}{2}\lambda\pi$, $\lambda>0$, $A>0$ ; and $f(z)$ is such that the integral exists ; the right-hand side of (12) becomes

$$\frac{\Gamma(\frac{1}{2}\pm\mu_0)}{8\sqrt{\pi}} \sum_{t=0}^{u} (-1)^t \frac{(-u)_t\,(-u+\frac{1}{2})_t}{(\frac{1}{2})_t\, t\,!} \int_0^\infty f(\eta)\, H^{m+4,\, n}_{p+3,\, q+4}$$

$$\left[ \alpha\eta^\rho \,\middle|\, \begin{matrix} \{(a_p,\, \alpha_p)\},\, (u+\nu+\nu_1+1,\, \sigma+\sigma_1),\, (\nu_0\pm\mu_0+1,\, \mu_1) \\ (\nu+\frac{1}{2},\, \sigma),\, (\nu+1,\, \sigma),\, (\nu_0+\frac{1}{2},\, \mu_1),\, (\nu_0+1,\, \mu_1),\, \{(b_q,\, \beta_q)\} \end{matrix} \right] d\eta \quad (14)$$

provided that $R(2\nu-2\sigma\beta+1)>0$, $R(2\nu_1-2\sigma_1\beta+1)>0$, $R(\nu,\, 0)>0$, $u$, $\mu_0$ being positive integers, $|\arg \alpha|<\frac{1}{2}\lambda\pi$, $\lambda>0$, $A>0$ ; $\lambda$, $\beta$, $A$ are given in (3) and $f(\eta)$ is such that the integral exists.

Now, on taking $f(\eta)=\eta^{-\rho'} e^{-\beta'\eta}$, so that

$$f(x^2+y^2+z^2)=(x^2+y^2+z^2)^{-\rho'}\, e^{-\beta'(x^2+y^2+z^2)}$$

and utilizing the known integral

$$\int_0^\infty x^{-\rho}\, e^{-\beta x}\, H^{m,\, n}_{p,\, q}\left[\alpha x^\sigma \,\middle|\, \begin{matrix} \{(a_p,\, \alpha_p)\} \\ \{(b_q,\, \beta_q)\} \end{matrix}\right] dx$$

$$=\beta^{\rho-1}\, H^{m,\, n+1}_{p+1,\, q}\left[\frac{\alpha}{\beta^\sigma} \,\middle|\, \begin{matrix} (\rho,\, \sigma),\, \{(a_p,\, \alpha_p)\} \\ \{(b_q,\, \beta_q)\} \end{matrix}\right] \quad (15)$$

provided that $|\arg \alpha|<\frac{1}{2}\lambda\pi$, $\lambda>0$, $A>0$, $|\arg \beta|<\frac{1}{2}\pi$, $R(-\rho+\sigma\delta+1)>0$, $\delta=\min R(b_h/\beta_h)$ $(h=1,\, 2,\, \ldots,\, m)$, (14) reduces to

$$\int_0^\infty \int_0^\infty \int_0^\infty \frac{x^{2\nu}\, y^{2\nu_1}\, (x^2+y^2+z^2)^{-\nu_0-\rho'}}{(x^2+y^2)^{\nu+\nu_1\,\frac{1}{2}-\nu_0}}\; e^{-\beta'(x^2+y^2+z^2)}.$$

$$\cos\left(2\mu_0 \tan^{-1}\frac{\sqrt{x^2+y^2}}{z}\right) \cos 2u \left(\tan^{-1}\frac{y}{x}\right) H^{m,\, n}_{p,\, q}$$

$$\left[\frac{\alpha(x^2+y^2+z^2)^{\rho+\mu_1}}{(x^2+y^2)^{\mu_1-(\sigma+\sigma_1)}\, x^{2\sigma}\, y^{2\sigma_1}} \,\middle|\, \begin{matrix} \{(a_p,\, \alpha_p)\} \\ \{(b_q,\, \beta_q)\} \end{matrix}\right] dx\, dy\, dz$$

$$=\frac{\Gamma(\frac{1}{2}\pm\mu_0)}{8\sqrt{\pi}} \sum_{t=0}^{u} (-1)^t \frac{(-u)_t\,(-u+\frac{1}{2})_t}{(\frac{1}{2})_t\, t!}$$

$$\beta^{\rho'-1}\, .\, H^{m+4,\, n+1}_{p+4,\, q+4}\left[\frac{\alpha}{\beta'^\rho} \,\middle|\, \begin{matrix} (\rho',\, \rho),\, \{(a_p,\, \alpha_p)\},\, (u+\nu+\nu_1+1, \sigma+\sigma_1), \\ (\nu+\frac{1}{2},\, \sigma),\, (\nu+1,\, \sigma),\, (\nu_0+\frac{1}{2}, \mu_1), \end{matrix}\right.$$

$$\left.\begin{matrix} (\nu_0\pm\mu_0+1,\, \mu_1) \\ (\nu_0+1,\, \mu_1),\, \{b_q, \beta_q)\} \end{matrix}\right] \quad (16)$$

provided that the conditions given in (14) are satisfied.

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