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ON THE ABSOLUTE NöRLUND SUMMABILITY OF A SERIES RELATED TO A FOURIER SERIES

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1. Let $\sum_{n=0}^{\infty} a_n$ be an infinite series with partial sums S_n, and let $\{p_n\}$ be a sequence of constants with $P_n = p_0 + p_1 + \ldots p_n \neq 0$ for all $n \geqslant 0$. The series $\sum_{n=0}^{\infty} a_n$ is called summable $|N, p_n|$ if

$$\sum_{n=0}^{\infty} \left| t_n - t_{n+1} \right| < \infty ,$$

where
$$t_n = \frac{1}{P_n} \sum_{k=0}^{n} p_{n-k} S_k.$$

It is known that when

$$p_n = \binom{n+\alpha-1}{\alpha-1}, \ \alpha > 0,$$

the Nörlund mean reduces to the familiar Cesàro mean of order α.

2. Let $f(t)$ be a periodic function with period 2π and integrable in the Lebesgue sense over $(-\pi, \pi)$. We assume as we may without loss of generality, that the Fourier series of $f(t)$ is

$$\sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) = \Sigma A_n(t).$$

We use the following notations throughout :

$$\phi(t) = \tfrac{1}{2}\{f(x+t) + f(x-t) - 2S\};$$

where S is a function of x,

$$\chi(t)=\frac{\phi(t)}{\left(\log\frac{2\pi}{t}\right)^{1+\delta}};$$

$$\sigma_n(x)=\sum_{k=1}^{n} p_{n-k}\frac{kA_k(x)}{\{\log\ (k+1)\}^{\delta}};$$

$$N(n,u)=\sum_{k=1}^{n} p_{n-k}\frac{\sin ku}{\{\log\ (k+1)\}^{\delta}};$$

$$\alpha(n,t)=\int_t^{\pi}\left(\log\frac{2\pi}{u}\right)^{1+\delta}\cos nu\, du;$$

$$Q(n,t)=\int_t^{\pi}\left(\log\frac{2\pi}{u}\right)^{1+\delta}\frac{d}{du}\ N(n,u)\, du;$$

$$\tau=\left[\frac{1}{t}\right],$$

and $$m=\left[\frac{n}{2}\right].$$

Recently Ray [1] has proved the following theorem for absolute Cesáro summability of a Fourier series :

THEOREM A. If

$$\frac{\phi(t)}{\left\{\log\left(\frac{2\pi}{t}\right)\right\}^{\delta}} \tag{1}$$

is of bounded variation in $(0, \pi)$

and $$\int_0^{\pi}\frac{|\phi(t)|\,dt}{t\left\{\log\left(\frac{2\pi}{t}\right)\right\}^{1+\delta}}<\infty,\quad \delta>0 \tag{2}$$

then $\Sigma\frac{A_n(x)}{\{\log\ (n+1)\}^{\delta}}$ is summable $|C,\rho|$, $\rho>0$.

The object of this paper is to extend the above result for absolute Nörlund summability. We prove the following theorem :

THEOREM 1. If the conditions (1) and (2) of the theorem A hold, then the series

$$\Sigma\frac{A_n(x)}{\{\log\ (n+1)\}^{\delta}},\qquad \delta>0$$

is summable $|N, p_n|$, where $\{p_n\}$ is a monotonic sequence of non-negative numbers such that

$$\left\{\frac{(n+1)\,p_n}{P_n}\right\}\in BV \tag{3}$$

$$P_k \sum_{n=k}^{\infty} \frac{1}{(n+1)\, P_n} < \infty . \tag{4}$$

We have proved elsewhere* that if $\{p_n\}$ is non-decreasing sequence then the condition (4) is satisfied. Thus we have the following result as a necessary consequence of the above theorem :

THEOREM 2. Let $\{p_n\}$ be a non-decreasing sequence of non-negative numbers. If the conditions (1) and (2) of Theorem A and (3) of Theorem 1 hold, then the series

$$\Sigma \frac{A_n(x)}{\{\log (n+1)\}^{\delta}}, \qquad \delta>0$$

is summable $|N, p_n|$.

3. We require the following lemmas for the proof of Theorem 1.

Lemma 1. If $\phi(t)$ satisfies the conditions (1) and (2) of the Theorem A, then

$$A_n(x)=O\left\{\frac{(\log n)^{\delta}}{n}\right\}.$$

Proof. We have

$$\frac{\pi}{2} A_n(x)=\int_0^{\pi}\phi(t)\cos nt\, dt$$

$$=\int_0^{\frac{1}{n\log n}}+\int_{\frac{1}{n\log n}}^{\pi}$$

$$=I_1+I_2$$

say. Now

$$I_1=\int_0^{\frac{1}{n\log n}} \frac{\phi(t)}{t\left\{\log\frac{2\pi}{t}\right\}^{1+\delta}}\, t\left\{\log\left(\frac{2\pi}{t}\right)\right\}^{1+\delta}\cos nt\, dt.$$

* L. B. Singh [2] (Lemma 1)

$$\leqslant C \frac{1}{n \log n}\left\{ \log (2\pi\, n \log n)\right\}^{1+\delta} \int_0^{\frac{1}{n \log n}} \frac{|\phi(t)|}{t\left\{\log \frac{2\pi}{t}\right\}^{1+\delta}}\, dt$$

$$\leqslant C\, \frac{(\log n)^{\delta}}{n}.$$

Again

$$I_2 = \int_{\frac{1}{n \log n}}^{\pi} \frac{\phi(t)}{\left(\log \frac{2\pi}{t}\right)^{\delta}} \left(\log \frac{2\pi}{t}\right)^{\delta} \cos nt\, dt.$$

$$= \{\log (2\pi n \log n)\}^{\delta} \int_{\frac{1}{n \log n}}^{\xi} \frac{\phi(t)}{\left(\log \frac{2\pi}{t}\right)^{\delta}} \cos nt\, dt$$

by mean value theorem where ξ is some number in between $\frac{1}{n \log n}$ and π. Now since $\frac{\phi(t)}{\left(\log \frac{2\pi}{t}\right)^{\delta}} \in BV$ in (O, π), the integral on the right side is of order $\frac{1}{n}$.

Thus

$$I_2 = O\, \frac{(\log n)^{\delta}}{n}.$$

This proves the lemma.

Lemma 2. The series

$$\sum \frac{A_n(x)}{\{\log (n+1)\}^{\delta}} \tag{5}$$

is convergent, if $\phi(t)$ satisfies conditions (1) and (2) of Theorem A.

Proof. A necessary consequence of Theorem A stated above is that the series (5) is summable (C, ρ), $\rho > 0$. By Lemma 1, we observe that

$$\frac{A_n(x)}{\{\log (n+1)\}^{\delta}} = O\left(\frac{1}{n}\right),$$

hence proof of the lemma follows from a well known Tauberian Theorem.

Lemma 3.* If $\eta>0$, then necessary and sufficient conditions that (i) $\chi(t)\log\left(\frac{2\pi}{t}\right)$ should be of bounded variation in $(0,\eta)$ and (ii) $\frac{|\chi(t)|}{t}$ should be integrable in $(0,\eta)$, are that

$$\int_0^\eta \log\frac{2\pi}{t}\,|d\,\chi(t)|<\infty$$

and

$$\chi(+0)=0.$$

Lemma 4.† $$\int_0^\pi \left\{\log\left(\frac{2\pi}{t}\right)\right\}^\beta \cos nt\,dt\sim\pi/2\,\frac{\beta(\log n)^{\beta-1}}{n}$$

for all β except $\beta=0$.

4. **Proof of the theorem.** By a lemma due to Bhatt and Nand Kishore [4] and Lemma 2, it follows that to prove our theorem it is sufficient to show that

$$\sum\frac{|\sigma_n(x)|}{(n+1)P_{n-1}} \tag{6}$$

is convergent.

Now $$A_n(x)=\frac{2}{\pi}\int_0^\pi \chi(t)\left\{\log\left(\frac{2\pi}{t}\right)\right\}^{1+\delta}\cos nt\,dt$$

$$=-\frac{2}{\pi}\int_0^\pi \chi(t)\,\alpha'(n,t)\,dt.$$

$$=\frac{2}{\pi}\int_0^\pi \alpha(n,t)d\,\chi(t),$$

since by Lemma 3, $\chi(+0)=0$.

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* R. Mohanty [3] (Lemma 10).

† B. K. Ray [1] (Lemma 3(ii)).

Hence

$$\sigma_n(x)=\sum_{k=1}^{n} \frac{p_{n-k}\, k}{\{\log (k+1)\}^{\delta}} \frac{2}{\pi}\int_0^{\pi}\left\{\int_t^{\pi}\left(\log\frac{2\pi}{u}\right)^{1+\delta}\cos ku\, du\right\}d\chi(t).$$

$$=\frac{2}{\pi}\int_0^{\pi}\left\{\int_t^{\pi}\left(\log\frac{2\pi}{u}\right)^{1+\delta}\frac{d}{du}N(n,u)\,du\right\}d\chi(t).$$

$$=\frac{2}{\pi}\int_0^{\pi} Q(n,t)d\,\chi(t).$$

Thus we have,

$$\sum\frac{|\sigma_n(x)|}{(n+1)P_{n-1}}\leqslant\int_0^{\pi}\left\{\sum\frac{|Q(n,t)|}{(n+1)\,P_{n-1}}\right\}|d\chi(t)| \qquad (7)$$

Now in order to prove (6) we first show that

$$\sum\frac{|Q(n,t)|}{(n+1)P_{n-1}}=O\left(\log\frac{2\pi}{t}\right) \qquad (8)$$

uniformly in $0<t\leqslant\pi$.

Write $\sum\frac{|Q(n,t)|}{(n+1)P_{n-1}}=\sum_1^{\tau}+\sum_{\tau+1}^{\infty}$, say

Writing $Q(n,t)=\int_t^{\pi}\left(\log\frac{2\pi}{u}\right)^{1+\delta}\frac{d}{du}N(n,u)\,du$

$$=\int_0^{\pi}-\int_0^{t}$$

$$=M(n,\pi)-M(n,t),\text{ say,}$$

we have

$$M(n,\pi)=\int_0^{\pi}\left(\log\frac{2\pi}{u}\right)^{1+\delta}\sum_{k=1}^{n}p_{n-k}\frac{k\cos ku}{\{\log(k+1)\}^{\delta}}du$$

$$=\sum_{k=1}^{n}\frac{p_{n-k}\,k}{\{\log(k+1)\}^{\delta}}\int_0^{\pi}\left\{\log\left(\frac{2\pi}{u}\right)\right\}^{1+\delta}\cos ku\,du$$

$$=O\left\{\sum_{k=1}^{n}\frac{p_{n-k}\,k}{\{\log(k+1)\}^{\delta}}\,\frac{(\log k)^{\delta}}{k}\right\}$$

$$=O(P_n).$$

by lemma 4 and also

$$M(n,t)=\int_0^t\left(\log\frac{2\pi}{u}\right)^{1+\delta}\frac{d}{du}\left\{\sum_{k=1}^{n}\frac{p_{n-k}\sin ku}{\{\log(k+1)\}^{\delta}}\right\}du$$

$$\leqslant C\int_0^t\left(\log\frac{2\pi}{u}\right)^{1+\delta}\frac{n\,P_n}{(\log n)^{\delta}}\,du$$

$$=O\left(\frac{n\,P_n}{(\log n)^{\delta}}\right)t\left(\log\frac{2\pi}{t}\right)^{1+\delta}.$$

Thus we observe that

$$\sum_1^{\tau}\frac{|Q(n,t)|}{(n+1)\,P_{n-1}}\leqslant\sum_1^{\tau}\frac{|M(n,\pi)|}{(n+1)\,P_{n-1}}+\sum_1^{\tau}\frac{|M(n,t)|}{(n+1)\,P_{n-1}}$$

$$=O\left(\sum_1^{\tau}\frac{1}{n}\right)+O\left\{\sum_1^{\tau}\frac{1}{(\log n)^{\delta}}\;t\left(\log\frac{2\pi}{t}\right)^{1+\delta}\right\}$$

$$=O\left(\log\frac{2\pi}{t}\right). \tag{9}$$

Finally, by mean value theorem with $t<T_1<\pi$

$$\sum_{\tau+1}^{\infty}\frac{|Q(n,t)|}{(n+1)\,P_{n-1}}$$

$$=\sum_{\tau+1}^{\infty}\frac{1}{(n+1)\,P_{n-1}}\left|\int_t^{\pi}\left(\log\frac{2\pi}{u}\right)^{1+\delta}\frac{d}{du}\,N(n,u))\,du\right|$$

$$=\sum_{\tau+1}^{\infty}\frac{1}{(n+1)\,P_{n-1}}\left(\log\frac{2\pi}{t}\right)^{1+\delta}\left|\int_t^{T_1}\frac{d}{du}N(n,u)\,du\right| \tag{10}$$

$$\leqslant \sum_{\tau+1}^{\infty} \frac{1}{(n+1)\,P_{n-1}} \left(\log \frac{2\pi}{t}\right)^{1+\delta}$$

$$\left| \left\{ \sum_{k=1}^{n} p_{n-k} \frac{\sin kt}{\{\log (k+1)\}^{\delta}} - \sum_{k=1}^{n} p_{n-k} \frac{\sin kT_1}{\{\log (k+1)\}^{\delta}} \right\} \right|$$

$$\leqslant C \sum_{\tau+1}^{\infty} \frac{1}{(n+1)\,P_{n-1}} \left(\log \frac{2\pi}{t}\right)^{1+\delta} \left| \sum_{k=1}^{n} p_{n-k} \frac{\sin kt}{\{\log (k+1)\}^{\delta}} \right|$$

since summation involved is from $\tau+1$ and as such $\sin kT_1$ will behave like $\sin kt$ where $T_1 > t$.

Case 1. Suppose $\{p_n\}$ is non-increasing. We have for $0<\alpha<1$,

$$\left| \sum_{k=1}^{n} p_{n-k} \frac{\sin kt}{\{\log (k+1)\}^{\delta}} \right| \leqslant \left| \sum_{k=1}^{m} \frac{p_{n-k}\, k^{\alpha}}{\{\log (k+1)\}^{\delta}} \frac{\sin kt}{k^{\alpha}} \right|$$

$$+ \left| \sum_{m+1}^{n} \frac{p_{n-k} \sin kt}{\{\log (k+1)\}^{\delta}} \right|$$

$$= O\left(\frac{p_{n-m}\, n^{\alpha}}{\{\log (n+1)\}^{\delta}} \right) \max_{1 \leqslant \mu \leqslant m} \left| \sum_{1}^{\mu} \frac{\sin kt}{k^{\alpha}} \right|$$

$$+ O\, \frac{1}{\{\log (m+1)\}^{\delta}} \max_{1 \leqslant \mu \leqslant n} \left| \sum_{1}^{\mu} p_{n-k} \sin kt \right|$$

$$= O\left(\frac{p_{n-m}\, n^{\alpha}\, \tau^{1-\alpha}}{\{\log (n+1)\}^{\delta}} \right) + O\left(\frac{P_{\tau}}{\{\log (m+1)\}^{\delta}} \right)$$

by Abel's lemma and McFadden's lemma, [5], Thus $\sum_{\tau+1}^{\infty} \frac{|Q(n,t)|}{(n+1)P_{n-1}} \leqslant C\left\{\log\left(\frac{2\pi}{t}\right)\right\}^{1+\delta} \times$

$$\sum_{\tau+1}^{\infty} \frac{1}{(n+1)P_{n-1}} \frac{p_{n-m}\, n^{\alpha}}{\{\log(n+1)\}^{\delta}} \tau^{1-\alpha}$$

$$+C\left(\log\frac{2\pi}{t}\right)^{1+\delta} \sum_{\tau+1}^{\infty} \frac{1}{(n+1)P_{n-1}} \frac{P_{\tau}}{\{\log(n+1)\}^{\delta}}$$

$$\leqslant C\log\left(\frac{2\pi}{t}\right)\tau^{1-\alpha}\sum_{\tau+1}^{\infty}\frac{n^{\alpha}}{n^{2}}$$

$$+C\log\left(\frac{2\pi}{t}\right)P_{\tau}\sum_{\tau+1}^{\infty}\frac{1}{(n+1)P_{n-1}}$$

$$=O\left\{\log\left(\frac{2\pi}{t}\right)\right\}. \tag{11}$$

Case 2. Suppose $\{p_n\}$ is monotonic non-decreasing. We have

$$\sum_{\tau+1}^{\infty}\frac{|Q(n,t)|}{(n+1)P_{n-1}} = \sum_{\tau+1}^{\infty}\frac{\left\{\log\left(\frac{2\pi}{t}\right)\right\}^{1+\delta}}{(n+1)P_{n-1}}$$

$$\left|\sum_{k=1}^{n}\frac{p_{n-k}\,k^{\alpha}}{\{\log(k+1)\}^{\delta}}\frac{\sin kt}{k^{\alpha}}\right|$$

$$=\sum_{\tau+1}^{\infty}\frac{\left(\log\frac{2\pi}{t}\right)}{(n+1)P_{n-1}}O(n^{\alpha}p_n\,\tau^{1-\alpha})$$

$$=O\left(\log\frac{2\pi}{t}\right)(\tau^{1-\alpha})\sum_{\tau+1}^{\infty}\frac{n^{\alpha}p_n}{(n+1)P_{n-1}}$$

$$=O\left(\log\frac{2\pi}{t}\right). \tag{12}$$

Proof of (8) follows from (9) (11) and (12).

Substituting (8) in the inequality (7) we observe that

$$\sum \frac{|\sigma_n(x)|}{(n+1)\, P_{n-1}} \leqslant C \int_0^{\pi} \left(\log \frac{2\pi}{t}\right) |d\chi(t)|,$$

$$\leqslant C,$$

by lemma 3.

This completes the proof of the Theorem 1.

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