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ON THE GROWTH OF AN ALGEBROID FUNCTION

By

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Introduction. In this note, our main concern is to find a number of growth relations relating to the characteristic function of an algebroid function with respect to certain auxillary functions like counting functions, etc. In section 2, we give certain preliminaries and lemmas needed in the proof of our results. The last section 3 is devoted to the main results of this note. Throughout this paper, we use the symbols employed by Nevanlinna and Selberg ([3] and [4]).

2. **Preliminaries.** Let $f(z)$ be an n-valued algebroid function with n branches in the finite plane defined by the irreducible equation,

$$A_0(z)\, f^n + A_1(z)\, f^{n-1} + \ldots + A_n(z) = 0, \tag{1}$$

where $A_0(z)$, $A_1(z)$, ..., $A_n(z)$ are entire functions without any common zeros, and at least one ratio between the coefficients is transcendental.

The characteristic function $T(r, f)$ for a system of $f = (f_0, f_1, \ldots, f_n)$ of entire functions, without any common zero, is defined by

$$T(r, f) = \frac{1}{2\pi} \int_0^{2\pi} u(r\, e^{i\theta})\, d\theta - u(0),$$

where the real-valued function

$$u(r\, e^{i\theta}) = \max_{0 \leqslant j \leqslant n} \log |f_j(r\, e^{i\theta})|.$$

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Let the system $f_0, f_1, \ldots, f_n$ of entire functions by generated by n-valued algebroid function f, defined be the equation (1). The characteristic function $T(r, f)$ is a convex function with respect to $\log r$ ([1]). Further, the order and lower order of f can be defined by

$$\lim_{r\to\infty} \begin{matrix}\text{Sup}\\ \text{Inf}\end{matrix} \left\{\frac{\log T(r, f)}{\log r}\right\} = \begin{cases}\rho\\ \lambda\end{cases}.$$

LEMMAS

Lemma 1. For all $i \neq j$ $(i, j=0, 1, \ldots, n)$

$$T\left(r, \frac{f_j}{f_i}\right) - 0(1) \leqslant T(r, f) \leqslant \sum_{k\neq j} T\left(r, \frac{f_k}{f_j}\right) + 0(1) \text{ (see Toda [5]).}$$

Lemma 2. Let $\pi(z)$ be a canonical product of genus q and let $n(t)=n\ (t, 0, \pi)$ denote the number of zeros of $\pi(z)$ in the circle $|z| < t$. Then, we have

$$\log M(r, \pi) \leqslant D(q)\, r^q \left\{\int_0^r \frac{n(t)}{t^{q+1}} dt + r \int_r^\infty \frac{n(t)}{t^{q+2}} dt\right\}$$

where $D(q)=2(2+\log(q+1))$ if $q>1$.

(see Nevanlinna [3]).

Lemma 3. Let $F(z)=\sum_{i=0}^{n} a_i f_i(z)$ be a linear combination, homogeneous to constant coefficients, and let

$$N(r, 0, F) = \int_0^r \frac{n(t, 0, F)}{t} dt.$$

Then $\quad N(r, 0, F) < T(r, f) + 0(1)$

(See Cartan [1]).

Lemma 4. If $\phi(r)$ is any function that is positive for all $r \geqslant 0$ and satisfies the condition

$$\lim_{r\to\infty} \text{Sup} \left\{\frac{\log \phi(r)}{\log r}\right\} = \rho,\ \rho < \infty$$

then a proximate order $\rho(r)$ can be constructed with respect to $\phi(r)$, such that

(i) $\lim_{r\to\infty} \rho(r) = \rho,$

(ii) $\lim_{r\to\infty} r\, \rho'(r) \log r = 0,$

(iii) $\phi(r)\leqslant r^{\rho(r)}$, $(r\geqslant r_0)$,

(iv) $\phi(r_n)=r_n^{\rho(r_n)}$, $r_n\to\infty$ with n.

(See Levin [2], p. 57).

3. MAIN RESULTS

Theorem 1. Let f be an n-valued algebroid function of finite and nonintegral order ρ. Then

$$\lim_{r\to\infty} \inf \frac{T(r,f)}{\sum_{i=0}^{n} N(r,f_i)} \leqslant \frac{D(q)}{(q+1-\rho)(\rho-q)}, \tag{2}$$

where $q=[\rho]$, $D(q)=2(2+\log(q+1))$ and

$$N(r,f_i)=\int_0^r \frac{n(t,f_i)}{t}dt.$$

Proof. We represent $f_i(z)$ with the help of Hadamard's representations formula

$$f_i(z)=z^{p_i}\,(P_i(z))\,\pi_i(z),$$

where p_i is positive integer, $\pi_i(z)$ is the canonical product formed with the zeros of $f_i(z)$, and $\rho_i(z)$ is a polynomial of period $q\leqslant\rho$. Therefore

$$T(r,f_i)\leqslant D(q)\left\{r^q\int_0^r \frac{n_i(x)}{x^{q+1}}dx+r^{q+1}\int_r^\infty \frac{n_i(x)}{x^{q+2}}dx\right\}+\mu(q,r), \tag{3}$$

where $\mu(q,r)=0(r^q)+0(\log r)$.

Further, denote

$$\sum_{i=0}^{n} n_i(x)=\sum_{i=0}^{n} n(x,f_i)=n(x),$$

$$\sum_{i=0}^{n} N_i(x)=\sum_{i=0}^{n} N(x,f_i)=N(x),$$

and, by definition of $N_i(x)$,

$$n_i(x)\,d\,x=xd\,N_i(x).$$

From (3), we have

$$T(r,f_i)\leqslant D(q)\left\{qr^q\int_0^r \frac{N_i(x)}{x^{q+1}}\,dx+(q+1)r^{q+1}\int_0^\infty \frac{N_i(x)}{x^{q+2}}dx\right\} +\mu(q,r). \tag{4}$$

Using Lemma 1,

$$T(r,f) \leqslant D(q) \left\{ qr^q \int_0^r \frac{N(x)}{x^{q+1}}\, dx + (q+1)\, r^{q+1} \int_r^\infty \frac{N(x)}{x^{q+2}}\, dx \right\} + \mu(q,r). \quad (5)$$

We assert that

$$D(q) = \rho, \quad (6)$$

and

$$D(q) = \lim_{r\to\infty} \left\{ \frac{\log (N(r))}{\log r} \right\}$$

for if $D(q) < \rho$, then from (5), one has for all $r \geqslant r_0$, the inequality

$$T(r,f) < Kr^\lambda,$$

where K is a constant, $\lambda < \rho$, and so

$$\lim_{r\to\infty} \text{ Sup} \left\{ \frac{\log T(r,f)}{\log r} \right\} \leqslant \lambda < \rho$$

which is a contradiction, and (6) holds. For $0 < \rho < 1$, there exists a proximate order $\rho(r)$ not unique for the function $N(r)$) satisfying the conditions (i) to (iv) of Lemma 4, where $\phi(r)$ can be easily replaced by $N(r)$,

$$\int_0^r \frac{N(x)}{x^{q+1}}\, dx \leqslant 0\,(1) + \int_0^r x^{(x)\, q-1}\, dx \sim \frac{r^{\rho(r)}}{\rho - q} = \frac{N(r)}{\rho - q}$$

Also

$$\int_0^r \frac{N(x)}{x^{q+2}}\, dx \leqslant \int_r^\infty x^{(x)-q-2}\, dx \sim \frac{r^{\rho(r)}}{q+1-\rho} = \frac{N(r)}{q+1-\rho},\ q \leqslant \rho.$$

Therefore, from the inequality (5), we finally get the result of the theorem.

Theorem 2. Let the hypothesis of Theorem 1 be satisfied. Then

$$\lim_{r\to\infty} \text{ Inf } \frac{T(r,f)}{r^q \sum_{i=0}^{n} N(r,f)} \leqslant \frac{D(q)}{q+1-\rho} \quad (7)$$

where $D(q)=2(2+\log(q+1))$ and

$$N(r, f_i)=\int_0^r \frac{n_i(x)}{x^{q+1}}\,dx$$

Proof. We have

$$N_q^{(i)}(r)=N_q(r, f_i)=\int_0^r \frac{n_i(x)}{x^{q+1}}\,dx.$$

Then

$$r^q\int_0^r \frac{n_i(x)}{x^{q+1}}\,dx+r^{q+1}\int_r^\infty \frac{n_i(x)}{x^{q+2}}\,dx.$$

$$r^q N_q^{(i)}(r)+r^{q+1}\int_r^\infty \frac{dN_q^{(i)}(x)}{x}$$

$$=r^q N_q^{(i)}(r)+r^{q+1}\left\{\left[\frac{N_q^{(i)}(x)}{x^{q+1}}\right]_r^\infty+(q+1)\int_r^\infty \frac{N_q^{(i)}(x)}{x^{q+2}}\,dx\right\}$$

$$=r^{q+1}\int_r^\infty \frac{N_q^{(i)}(x)}{x^{q+2}}\,dx.$$

Since, for large x

$$N_q^{(i)}(x)\leqslant 0(1)+\frac{1}{x}\int_0^r x^{\rho-r-q-1}\,dx$$

$$=0(1)+\frac{x^{\rho+r-q-1}}{\rho+r-q},$$

as $x\to\infty$ $(\rho<q+1)$. Then using (3), we get

$$T(r, f_i)\leqslant D(q)\int_r^\infty \frac{N_q^{(i)}(x)}{x^{q+2}}\,dx+\mu(q, r).$$

Let δ and o arbitrary. Then

$$N_q^{(i)}(x)=o(x^{\rho+\delta-q}),\ (x\to\infty),$$

and therefore there exists a sequence $\{x_n\}$, $x_n \to \infty$ with x such that

$$\frac{N_q^{(i)}(x)}{N^{\rho+\delta-q}} \leqslant \frac{N_q^{(i)}(x_n)}{x_n^{\rho+\delta-q}} \quad (x \to x_n).$$

consequently

$$T(x_n, f_i) \leqslant \frac{1+o(1)D(q)\, N_q^{(i)}(x_n)\, x_n^q}{q+1-\rho-\delta}$$

Using lemma 1, we have

$$\lim_{r\to\infty} \text{Inf} \frac{T(r,f)}{r^q \sum_{i=0}^{n} N_q(r, f_i)} \leq \frac{D_{(q)}}{q+1-\rho}$$

Remark 1. If $0<\rho<1$, then $q=0$ and so $D(q)=1$ and the result (7) reduces to

$$\lim_{r\to\infty} \text{Inf} \frac{T(r,f)}{\sum_{i=0}^{n} N(r, f_i)} \leqslant \frac{1}{1-\rho}$$

and $$\lim_{r\to\infty} \text{Inf} \frac{T(r,f)}{\sum_{i=0}^{n} N(r, 0, f_i)} \leqslant \frac{1}{1-\rho},$$

where $$\sum_{i=0}^{n} N(r, 0, f_i) = \sum_{i=0}^{n} N_0(r, f_i)$$

Remark 2. Neither of the results, Theorem 1 and 2, seem to follow from each other. However, we can offer certain possibility under which Theorem 1 follows from Theorem 2, and we find that

$$N_q^{(i)}(r) = \int_0^r \frac{n_i(t)}{t^{q+1}} dt$$

$$= \frac{N_i(r)}{r^q} + q \int_0^r \frac{N_i(t)}{t^{q+1}} dt.$$

For $\delta > 0, \lim_{r\to\infty} \text{Sup} \dfrac{N_i(r)}{r^{\rho-\delta}} = \infty$, and $r \leqslant r_n$ ($r_n \to \infty$ with n). we get

$$\frac{N_i(r)}{r^{\rho-\delta}} \leqslant \frac{N_i(r_n)}{r_n{}^{\rho-}}.$$

Therefore

$$N_q^{(i)}(r_n) \leqslant \frac{N_i(r_n)}{r_n^q} + q\, \frac{N_i(r_n)}{r_n^{q-\delta}} \int_0^{r_n} t^{-\delta-q-1}\, dt$$

$$= \frac{\rho-\delta}{\rho-q-\delta}\, \frac{N_i(r_n)}{r_n^q}.$$

The assumption that $\{r_n\}$ coincides with another sequence (R_n) above for large n or a subsequence $\{r_{n_q}\}$ of $\{r_n\}$ coincides with $\{R_n\}$ for sufficiently large q, gives that

$$\lim_{r\to\infty} \operatorname{Inf} \frac{T(r,f)}{\sum_{i=0}^{n} N(r, f_i)} \leqslant \frac{\rho}{1-\rho} \lim_{r\to\infty} \operatorname{Inf} \frac{T(r,f)}{r^q \sum_{i=0}^{n} N_q^{(i)}(r)}$$

$$\leqslant \frac{D_{(q)}}{(\rho-q)(q+1-\rho)}$$

which is nothing but Theorem 1 of Toda [5]. We have for all $r \leqslant R_n$ of sequence $\{R_n\}$, such that

$$\frac{1}{\sum_{i=0}^{n} N_i(R_n)} \leqslant \frac{\rho-\delta}{\rho-q-\delta}\, \frac{1}{r^q \sum_{i=0}^{n} N_q^{(i)}(R_n)}$$

$$\leqslant \frac{\rho-\delta}{\rho-q-\delta}\, \frac{D_{(q)}+\delta}{(q+1-\rho)\, T(R_n, f)}$$

and so for Theorem 2 to hold good, we should have

$$\lim_{n\to\infty} \frac{T(R_n, f)}{T(r_n, f)} \leqslant 1$$

which is another assumption.

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