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ON MULTIPLE INTEGRALS INVOLVING HYPERGEOMETRIC FUNCTIONS OF TWO VARIABLES

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1. **Introduction.** Recently, Srivastava [4] proved, by induction, the identity

$$\sum_{m_1, \ldots, m_n=0}^{\infty} f(m_1+\ldots+m_n) \frac{x_1^{m_1}}{m_1 !} \cdots \frac{x_n^{m_n}}{m_n !}$$

$$= \sum_{m=0}^{\infty} f(m) \frac{(x_1+\ldots+x_n)^m}{m !}, \qquad \ldots(1)$$

provided that the series involved are absolutely convergent.

The object of this paper is to give a simple and direct proof of this identity. We also show that similar identities, of general nature, can be proved with the help of our method. Finally, we use these identities to evaluate multiple integrals involving Appell's and Kampé de Fériet's hypergeometric functions of two variables.

2. To prove identity (1), we let

$$E_{\alpha}^{n} f(\alpha)=f(\alpha+n), \qquad \ldots(2)$$

where E is a difference operator.

Consider

$$I=\sum_{m_1, \ldots, m_n=0}^{\infty} f(\alpha+m_1+\ldots+m_n) \frac{x_1^{m_1}}{m_1 !} \cdots \frac{x_n^{m_n}}{m_n !}.$$

Using (2) we get

$$I=\sum_{m_1,\ldots,m_n=0}^{\infty} E_{\alpha}^{m_1+\ldots+m_n} f(\alpha)\,\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!}$$

$$=\exp\{(x_1+\ldots+x_n)\,E_\alpha\}\,f(\alpha)$$

$$=\sum_{m=0}^{\infty}(x_1+\ldots+x_n)^m\,E_{\alpha}^{m} f(\alpha)$$

$$=\sum_{m=0}^{\infty}\frac{(x_1+\ldots+x_n)^m}{m!}f(\alpha+m),\ \text{by (2)},$$

and on putting $\alpha=0$, we get the desired result (1).

A repeated application of (1) will readily yield the following identity.

$$\sum_{m,\,n=}^{\infty} f(m+n)\,g(m)\,h(n)\,\frac{(x_1+\ldots+x_r)^m}{m!}\,\frac{(y_1+\ldots+y_s)^n}{n!}$$

$$=\sum_{m_1,\ldots,m_r=0}^{\infty}\ \sum_{n_1,\ldots,n_s=0}^{\infty} f(m_1+\ldots+m_r+n_1+\ldots+n_s)$$

$$g(m_1+\ldots+m_r)\,h(n_1+\ldots+n_s)\,\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_r^{m_r}}{m_r!}\cdot\frac{y_1^{n_1}}{n_1!}\cdots\frac{y_s^{n_s}}{n_s!}, \qquad \ldots(3)$$

provided that the series involved converge absolutely.

Evidently, (3) can easily be extended to the multiple series

$$\sum_{m_1,\ldots,m_n=0}^{\infty} f(m_1+\ldots+m_n)\,f_1(m_1)\ldots f_n(m_n)$$

$$\frac{(x_{11}+\ldots+x_{1r})^{m_1}}{m_1!}\cdots\frac{(x_{n1}+\ldots+x_{ns})^{m_n}}{m_n!}$$

$$=\sum_{(m_{1r})=0}^{\infty}\cdots\sum_{(m_{ns})=0}^{\infty} f\left(\sum_{k=1}^{r} m_{1k}+\ldots+\sum_{k=1}^{s} m_{nk}\right)$$

$$f_1\left(\sum_{k=1}^{r} m_{1k}\right)\cdot f_n\left(\sum_{k=1} m_{ns}\right)\prod_{k=1}^{r}\left[\frac{x_{1k}^{m_{1k}}}{m_{1k}!}\right]\cdots\prod_{k=1}^{s}\left[\frac{x_{nk}^{m_{nk}}}{m_{nk}!}\right], \ldots(4)$$

where, for convenience, (m_{ij}) denotes the set $m_{i1}, \ldots, m_{ij}, j \geqslant 1, i=1, \ldots, n$, and the series involved are absolutely convergent.

3. In this section we use the identity (3) to evaluate the following integrals involving Appell's functions $F_1, \ldots, F_4$ (see [1], p. 224).

$$\int_0^\pi \int_0^\pi \int_0^\pi \int_0^\pi F_1(\alpha, \beta, \gamma ; \delta ; \lambda_1+\lambda_2 \cos \psi+\lambda_3 \cos \theta \sin \psi,$$

$$\lambda_4+\lambda_5 \cos \eta+\lambda_6 \cos \phi \sin \eta) \sin \psi \sin \eta \, d\psi \, d\theta \, d\eta \, d\phi$$

$$=\frac{\pi^2(\delta-1)(\delta-2)}{R_1 R_2(\alpha-1)(\alpha-2)(\beta-1)(\gamma-1)} \{F_1(\alpha-2, \beta-1, \gamma-1 ;$$

$$\delta-2 ; \lambda_1+R_1, \lambda_4+R_2)-F_1(\alpha-2, \beta-1, \gamma-1 ; \delta-2 ;$$

$$\lambda_1-R_1, \lambda_4+R_2)-F_1(\alpha-2, \beta-1, \gamma-1 ; \delta-2 ; \lambda_1+R_1,$$

$$\lambda_4-R_2)+F_1(\alpha-2, \beta-1, \gamma-1 ; \delta-2 ; \lambda_1-R_1, \lambda_4-R_2)\}. \quad \ldots(5)$$

where $$R_1^2=\lambda_2^2+\lambda_3^2 ; R_2^2=\lambda_5^2+\lambda_6^2, \quad \ldots(5a)$$

$$\max \{|\lambda_1|+|\lambda_2|+|\lambda_3|, |\lambda_4|+|\lambda_5|+|\lambda_6|,$$
$$|\lambda_1+R_1|, |\lambda_4+R_2|\}<1, \quad \ldots(5b)$$

and $$\delta-2 \neq -1, -2, -3, \ldots, \quad \ldots(5c)$$

$$\int_0^\pi \int_0^\pi \int_0^\pi \int_0^\pi F_2(\alpha, \beta, \gamma ; \delta, \delta' ; \lambda_1+\lambda_2 \cos \psi+\lambda_3 \cos \theta \sin \psi,$$

$$\lambda_4+\lambda_5 \cos \eta+\lambda_6 \cos \phi \sin \eta) \sin \psi \sin \eta \, d\psi \, d\theta \, d\eta d\phi$$

$$=\frac{\pi^2(\delta-1)(\delta'-1)}{R_1 R_2(\alpha-1)(\alpha-2)(\beta-1)(\gamma-1)} \{F_2(\alpha-2, \beta-1, \gamma-1 ;$$

$$\delta-1, \delta'-1 ; \lambda_1+R_1, \lambda_4+R_2)-$$

$$-F_2(\alpha-2, \beta-1, \gamma-1 ; \delta-1, \delta'-1 ; \lambda_1-R_1, \lambda_4+R_2)$$

$$-F_2(\alpha-2, \beta-1, \gamma-1 ; \delta-1, \delta'-1 ; \lambda_1+R_1, \lambda_4-R_2)$$

$$+F_2(\alpha-2, \beta-1, \gamma-1 ; \delta-1, \delta'-1 ; \lambda_1-R_1, \lambda_4-R_2)\}. \quad \ldots(6)$$

where R_1, R_2 are given by (5a) and by analytic continuation

$$(\delta-1), (\delta'-1) \neq -1, -2, -3, \ldots, \quad \ldots(6a)$$

and $$\max \{|\lambda_1|+|\lambda_2|+|\lambda_3|+|\lambda_4|+|\lambda_5|+|\lambda_6|,$$
$$|\lambda_1+R_1|+|\lambda_4+R_2|\}<1. \quad \ldots(6b)$$

$$\int_0^\pi \int_0^\pi \int_0^\pi \int_0^\pi F_3(\alpha, \beta, \gamma, \delta ; \delta' ; \lambda_1+\lambda_2 \cos \psi+\lambda_3 \cos \theta \sin \psi,$$

$$\lambda_4+\lambda_5 \cos \eta+\lambda_6 \cos \phi \sin \eta) \sin \psi \sin \eta \, d\psi \, d\theta \, d\eta d\phi=$$

$$= \frac{\pi^2(\delta'-1)(\delta'-2)}{R_1R_2(\alpha-1)(\beta-1)(\gamma-1)(\delta-1)} \{F_3(\alpha-1, \beta-1, \gamma-1, \delta-1 ; \delta'-2 ; \lambda_1+R_1, \lambda_4+R_2) - F_3(\alpha-1, \beta-1, \gamma-1, \delta-1 ; \delta'-2 ; \lambda_1-R_1, \lambda_4+R_2) - F_3(\alpha-1, \beta-1, \gamma-1, \delta-1 ; \delta'-2 ; \lambda_1+R_1, \lambda_4-R_2) + F_3(\alpha-1, \beta-1, \gamma-1, \delta-1 ; \delta'-2 ; \lambda_1-R_1, \lambda_4-R_2)\}. \quad \ldots(7)$$

where R_1, R_2 are given by (5a) and under the conditions (5b) and

$$\delta'-2 \neq -1, -2, -3, \ldots, . \quad \ldots(7a)$$

$$\int_0^\pi \int_0^\pi \int_0^\pi \int_0^\pi F_4(\alpha, \beta ; \gamma, \delta ; \lambda_1+\lambda_2 \cos\psi+\lambda_3 \cos\theta \sin\psi, \lambda_4+\lambda_5 \cos\eta+\lambda_6 \cos\phi \sin\eta) \sin\psi \sin\eta \, d\psi \, d\theta \, d\eta \, d\phi$$

$$= \frac{\pi^2(\gamma-1)(\delta-1)}{R_1R_2(\alpha-1)(\alpha-2)(\beta-1)(\beta-2)} \{F_4(\alpha-2, \beta-2 ; \gamma-1, \delta-1 ; \lambda_1+R_1, \lambda_4+R_2) - F_4(\alpha-2, \beta-2 ; \gamma-1, \delta-1 ; \lambda_1-R_1, \lambda_4+R_2) - F_4(\alpha-2, \beta-2 ; \gamma-1 ; \delta-1 ; \lambda_1+R_1, \lambda_4-R_2) + F_4(\alpha-2, \beta-2 ; \gamma-1, \delta-1 ; \lambda_1-R_1, \lambda_4-R_2)\}. \quad \ldots(8)$$

where R_1, R_2 are given by (5a) and

$$(\gamma-1), (\delta-1) \neq -1, -2, -3, \ldots, \quad \ldots(8a)$$

and

$$\max\{\sqrt{(|\lambda_1|+|\lambda_2|+|\lambda_3|)}+\sqrt{(|\lambda_4|+|\lambda_5|+|\lambda_6|)}, \sqrt{(|\lambda_1+R_1|)}+\sqrt{(|\lambda_4+R_2|)}\}<1. \quad \ldots(8b)$$

Proof. Making use of the definition

$$F_1(\alpha, \beta, \beta' ; \gamma ; x, y) = \sum_{m, n=0}^{\infty} \frac{(\alpha)_{m+n} (\beta)_m (\beta')_n}{(\gamma)_{m+n} \, m! \, n!} x^m y^n, \quad \ldots(9)$$

and the identity (3) on the left-hand side of (5), and following the same lines as outlined by Srivastava [4], we obtain (5) under the conditions (5a), (5b) and (5c).

Proceeding on similar lines we can prove the remaining results (6), (7) and (8) under the appropriate conditions already stated.

Similarly, using identity (4) we can evaluate the corresponding multiple integrals involving Lauricella functions [2, p. 227-228] or their generalizations introduced by Srivastava and Daoust [3, p. 454]. Note also that formulae (5), (6), (7) and (8) can easily be extended to hold for Kampé de Fériet's hypergeometric functions of two variables or the generalized Kampé de Fériet function (cf., e.g., [3], p. 450).

Indeed, if we let (a) stand for the set or A parameters $a_1, \ldots, a_A$, with similar interpretations for (b), (b′), etc., one of our desired unifications of formulae (5), (6), (7) and (8) is the integral.

$$\int_0^\pi \int_0^\pi \int_0^\pi \int_0^\pi F\left[\begin{matrix}(a):(b);(b');\\(c):(d);(d');\end{matrix}\ \lambda_1+\lambda_2\cos\psi+\lambda_3\cos\theta\sin\psi,\right.$$

$$\left.\lambda_4+\lambda_5\cos\eta+\lambda_6\cos\phi\sin\eta\right]\sin\psi\sin\eta\, d\psi\, d\theta\, d\eta\, d\phi$$

$$=\frac{\pi^2\prod_{j=1}^{C}\{(c_j-1)(c_j-2)\}\prod_{j=1}^{D}\{(d_j-1)\}\prod_{j=1}^{D'}\{(d'_j-1)\}}{R_1R_2\prod_{j=1}^{A}\{(a_j-1)(a_j-2)\}\prod_{j=1}^{B}\{(b_j-1)\}\prod_{j=1}^{B'}\{(b'_j-1)\}}$$

$$\times\left\{F\left[\begin{matrix}(a-2):(b-1);(b'-1);\\(c-2):(d-1);(d'-1);\end{matrix}\ \lambda_1+R_1,\ \lambda_4+R_2\right]\right.$$

$$-F\left[\begin{matrix}(a-2):(b-1);(b'-1);\\(c-2):(d-1);(d'-1);\end{matrix}\ \lambda_1-R_1,\ \lambda_4+R_2\right]$$

$$-F\left[\begin{matrix}(a-2):(b-1);(b'-1);\\(c-2):(d-1);(d'-1);\end{matrix}\ \lambda_1+R_1,\ \lambda_4-R_2\right]$$

$$\left.+F\left[\begin{matrix}(a-2):(b-1);(b'-1);\\(c-2):(d-1);(d'-1);\end{matrix}\ \lambda_1-R_1,\ \lambda_4-R_2\right]\right\} \qquad \ldots(10)$$

where, for convergence of the double series of Kampé de Fériet's function, $A+B\leqslant C+D+1$, $A+B'\leqslant C+D'+1$, the equality in each case holds only if

$$\max\{|\lambda_1+\lambda_2+\lambda_3|+|\lambda_4+\lambda_5+\lambda_6|,\ |\lambda_1-R_1|+|\lambda_4+R_2|,$$
$$|\lambda_1+R_1|+|\lambda_4-R_2|\}<\min\{1,\ 2^{C-A+1}\}. \qquad \ldots(11)$$

Evidently (10) provides a generalization of the formula (22), p. 8 of Srivastava [4].

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