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NON-AXISYMMETRIC THERMOELASTIC STRESS DISTRIBUTION IN A SOLID CONTAINING A PENNY SHAPED CRACK

by

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Abstract. The non-axisymmetric thermal stress distribution in an infinite elastic solid containing a penny shaped stress free crack with prescribed heat flux is investigated. The problem is formulated in terms of a complete set of harmonic functions and leads to a set of two mixed, coupled boundary value problems. The coupling is through two arbitrary constants which are determined by the condition that the displacements be finite near the edge of the crack. The problems are solved using a standard Hankel transform technique and expressions for the stress intensity factors are given.

1. Introduction. In engineering an important class of problems concerns the evaluation of the thermal stresses set up in a heated elastic solid containing cracks. The calculations of the thermal stresses in an infinite space, in which an axially symmetric heat flux across the faces of a penny shaped crack is prescribed, was first carried out by Olesiak and Sneddon [1]. Later on Florence and Goodier [2] considered the linear thermoelastic problem of uniform heat flow disturbed by a penny shaped insulated crack and the solution for the infinite solid was obtained in the closed form by the method of dual integral equations. Recently Shail [3] has treated some thermoelastic stress distribution in an infinite solid and a thick plate containing penny shaped crack. Shail, using representations due to Williams [4], reduces the problem to a determination of single harmonic function, and derives the complete solution for the axisymmetric case only.

Here the thermoelastic problem, with an odd temperature distribution, is solved using a representation in terms of a complete set of stress functions due to Youngdahl [5]. This representation permits us to reduce the problem to a determination of two harmonic functions satisfying mixed boundary conditions on the plane $z=0$. The non-axisymmetric case in completely solved using the dual integral equation approach discussed by Sneddon [6]. The solution involves two arbitrary constants which are so chosen that the displacements are finite at the edge of the crack.

In section two the problem is formulated and the mixed boundary conditions, involving the two arbitrary constants, are derived. In section three a solution to the mixed boundary value problems of the previous section is presented. The two arbitrary constants are calculated in section four and the stress-intensity factors are derived in section five in terms of the prescribed heat flux.

2. **Formulation of the Problem.** We consider an infinite, isotropic, elastic solid containing a penny shaped, stress free crack. With a suitable normalization the crack can be described in terms of cylindrical coordinates (ρ, θ, z) by $z=0$, $\rho<1$. In the absence of body forces and heat sources within the medium the steady-state equations of thermoelasticity are

$$(1-2\eta)\nabla^2\bar{u}+\text{grad}\,[\Delta-2\alpha(1+\eta)T]=0, \tag{1}$$

and

$$\nabla^2 T=0, \tag{2}$$

where $\bar{u}$ is the displacement vector, $\nabla=\text{div}\,\bar{u}$, η is Poisson's ratio, α is the linear expansion of the solid and T is the temperature.

The problem considered here is that of non-axisymmetric solution of (1) and (2) when the heat flux $\partial T/\partial z$ is prescribed on the crack surface. Following Youngdahl [5], the three displacements and six stresses can be represented in terms of three functions $\bar{\phi}_1$, ϕ_2 and ϕ_3 by

$$U_\rho=\frac{\partial\bar{\phi}_1}{\partial\rho}-\frac{1}{\rho}\frac{\partial\phi_2}{\partial\theta}, \quad U_\theta=\frac{1}{\rho}\frac{\partial\bar{\phi}_1}{\partial\theta}+\frac{\partial\phi_2}{\partial\rho},$$

$$U_z=\frac{\partial}{\partial z}(\bar{\phi}_1+\phi_3),$$

$$\frac{1}{2\mu}\sigma_{\rho\rho}=-\alpha\,\frac{(1+\eta)}{1-\eta}T+\frac{\eta}{2(1-\eta)}\,\frac{\partial^2\phi_3}{\partial z^2}+\frac{\partial^2\bar{\phi}_1}{\partial\rho^2}-\frac{1}{\rho}\,\frac{\partial^2\phi_2}{\partial\rho\partial\theta}+\frac{1}{\rho^2}\,\frac{\partial\phi_2}{\partial\theta},$$

$$\frac{1}{2\mu}\sigma_{\theta\theta}=-\alpha\left(\frac{1+\eta}{1-\eta}\right)T+\frac{\eta}{2(1-\eta)}\,\frac{\partial^2\phi_3}{\partial z^2}+\frac{1}{\rho^2}\,\frac{\partial^2\bar{\phi}_1}{\partial\theta_2}+\frac{1}{\rho}\,\frac{\partial^2\phi_2}{\partial\rho\partial\theta}$$
$$+\frac{1}{\rho}\,\frac{\partial\bar{\phi}_1}{\partial\rho}-\frac{1}{\rho^2}\,\frac{\partial\phi_2}{\partial\theta},$$

$$\frac{1}{2\mu}\,\sigma_{zz}=-\alpha\left(\frac{1+\eta}{1-\eta}\right)T+\frac{\eta}{2(1-\eta)}\,\frac{\partial^2\phi_3}{\partial z^2}+\frac{\partial^2}{\partial z^2}\,(\bar{\phi}_1+\phi_3),$$

$$\frac{1}{\mu}\sigma_{\rho\theta}=\frac{2}{\rho}\,\frac{\partial^2\bar{\phi}_1}{\partial\rho\partial\theta}-\frac{2}{\rho^2}\,\frac{\partial\bar{\phi}_1}{\partial\theta}+\frac{\partial^2\phi_2}{\partial\rho^2}-\frac{1}{r}\,\frac{\partial\phi_2}{\partial\rho}-\frac{1}{\rho^2}\,\frac{\partial^2\phi_2}{\partial\theta^2},$$

$$\frac{1}{\mu}\sigma_{\rho\partial}=2\,\frac{\partial^2\bar{\phi}_1}{\partial\rho\partial z}-\frac{1}{r}\,\frac{\partial^2\phi_2}{\partial z\partial\theta}+\frac{\partial^2\phi_3}{\partial\rho\partial z},$$

$$\frac{1}{\mu}\sigma_{\theta z}=\frac{2}{\rho}\,\frac{\partial^2\bar{\phi}_1}{\partial z\partial\theta}+\frac{\partial^2\phi_2}{\partial\rho\partial z}+\frac{1}{\rho}\,\frac{\partial^2\phi_3}{\partial z\partial\theta},\tag{3}$$

where μ is the shear modulus. The functions $\bar{\phi}_1$, ϕ_2 and ϕ_3 are to satisfy

$$\nabla^2\phi_2=\nabla^2\phi_3=0,\tag{4}$$

$$\nabla^2\bar{\phi}_1=-\frac{1}{2(1-\eta)}\,\frac{\partial^2\phi_3}{\partial z^2}+\alpha\left(\frac{1+\eta}{1-\eta}\right)T\tag{5}$$

Let us define the temperature T in terms of a function ϕ_0 by

$$\alpha\left(\frac{1+\eta}{1-\eta}\right)T=-\frac{1}{2(1-\eta)}\,\frac{\partial^2\phi_0}{\partial z^2},\tag{6}$$

where

$$\nabla^2\phi_0=0.\tag{7}$$

The function ϕ_0 will be determined later from (6) and (7) in terms of the prescribed heat flux on the crack.

By using (6) the most general solution of (5) for $\bar{\phi}_1$ in terms of ϕ_3 and ϕ_0 can be written as

$$\bar{\phi}_1=\phi_1-\frac{z}{4(1-\eta)}\,\frac{\partial}{\partial z}\,(\phi_3+\phi_0)\tag{8}$$

where $\nabla^2\phi_1=0.$ (9)

Writing any quantity of interest, p, in the form

$$p\ (\rho, \theta, z) = p^{(n)}(\rho, z)\ e^{in\theta},\ n \neq 0 \quad \text{(for non-axisymmetric case)}$$

and substituting (6) and (8) into (3), we obtain the following representation for the displacements and stresses in terms of the three unknown functions $\phi_1^{(n)}$, $\phi_2^{(n)}$ and $\phi_3^{(n)}$, and the (assumed) known function $\phi_0^{(n)}$:

$$U_\rho^{(n)} = \frac{\partial \phi_1}{\partial \rho} - \frac{in}{\rho}\ \phi_2^{(n)} - \frac{z}{4\ (1-\eta)}\ \frac{\partial^2}{\partial\rho\partial z}\ (\phi_3^{(n)} + \phi_0^{(n)}), \tag{10}$$

$$U_\theta^{(n)} = \frac{in}{\rho}\phi_1^{(n)} + \frac{\partial\phi_2^{(n)}}{\partial\rho} - \frac{in\ z}{4\rho\ (1-\eta)}\ \frac{\partial}{\partial z}\ (\phi_3^{(n)} + \phi_0^{(n)}), \tag{11}$$

$$U_z^{(n)} = \frac{\partial\phi_1^{(n)}}{\partial z} - \frac{1}{4\ (1-\eta)}\ \frac{\partial}{\partial z}\ (\phi_3^{(n)} + \phi_0^{(n)}) + \frac{\partial\phi_3^{(n)}}{\partial z} - \frac{z}{4\ (1-\eta)}\ \frac{\partial^2}{\partial z^2}\ (\phi_3^{(n)} + \phi_0^{(n)}), \tag{12}$$

$$\frac{1}{2\mu}\ \sigma_{\theta\theta}^{(n)} = \frac{1}{2(1-\eta)}\ \frac{\partial^2\phi_0^{(n)}}{\partial z^2} + \frac{\eta}{2(1-\eta)}\ \frac{\partial^2\phi_3^{(n)}}{\partial z^2} - \frac{n^2}{\rho^2}\ \phi_1^{(n)} + \frac{in}{\rho}\ \frac{\partial\phi_2}{\partial\rho} + \frac{1}{\rho}\ \frac{\partial\phi_2}{\partial\rho} - \frac{in}{\rho}\phi_2^{(n)} + \frac{n^2 z}{4\rho^3\ (1-\eta)}\ \frac{\partial}{\partial z}\ (\phi_3^{(n)} + \phi_0^{(n)}) - \frac{z}{4\rho\ (1-\eta)}\ \frac{\partial^2}{\partial\rho\partial z}\ (\phi_3^{(n)} + \phi_0^{(n)}) \tag{13}$$

$$\frac{1}{2\mu}\sigma_{\rho\rho}^{(n)} = \frac{1}{2\ (1-\eta)}\ \frac{\partial^2\phi_0^{(n)}}{\partial z^2} + \frac{\eta}{2\ (1-\eta)}\ \frac{\partial^2\phi_3^{(n)}}{\partial z^2} + \frac{\partial^2\phi_1^{(n)}}{\partial\rho^2} - \frac{in}{\rho}\ \frac{\partial\phi_2^{(n)}}{\partial\rho} + \frac{in}{\rho^2}\ \phi_2^{(n)} - \frac{z}{4\ (1-\eta)}\ \frac{\partial^3}{\partial\rho^2\partial z}\ (\phi_3^{(n)} + \phi_0^{(n)}) \tag{14}$$

$$\frac{1}{2\mu}\ \sigma_{zz}^{(n)} = \frac{1}{2\ (1-\eta)}\ \frac{\partial^2\phi_0^{(n)}}{\partial z^2} + \left(1 + \frac{\eta}{2\ (1-\eta)}\right)\frac{\partial^2\phi_3^{(n)}}{\partial z^2} + \frac{\partial^2\phi_1^{(n)}}{\partial z^2} - \frac{1}{2\ (1-\eta)}\ \frac{\partial^2}{\partial z^2}\ (\phi_3^{(n)} + \phi_0^{(n)}) - \frac{z}{4(1-\eta)}\ \frac{\partial^3}{\partial z^3}\ (\phi_8^{(n)} + \phi_3^{(n)}), \tag{15}$$

$$\frac{1}{\mu}\sigma^{(n)}_{\rho\theta}=\frac{2\,in}{\rho}\frac{\partial\phi^{(n)}_1}{\partial\rho}-\frac{2\,in}{\rho^2}\phi^{(n)}_1+\frac{\partial^2\phi^{(n)}_2}{\partial\rho^2}-\frac{1}{\rho}\frac{\partial\phi^{(n)}_2}{\partial\rho^2}+\frac{n^2}{\rho^2}\phi^{(n)}_2$$

$$-\frac{inz}{2\rho(1-\eta)}\frac{\partial^2}{\partial\rho\partial z}\left(\phi^{(n)}_3+\phi^{(n)}_0\right)+\frac{inz}{2\rho^2(1-\eta)}\frac{\partial}{\partial z}\left(\phi^{(n)}_3+\phi^{(n)}_2\right) \quad (16)$$

$$\frac{1}{\mu}\sigma^{(n)}_{\rho z}=\frac{2\partial^2\phi^{(n)}_1}{\partial\rho\partial z}-\frac{1}{2(1-\eta)}\frac{\partial^2}{\partial\rho\partial z}\left(\phi^{(n)}_3+\phi^{(n)}_0\right)-\frac{in}{\rho}\frac{\partial\phi^{(n)}_2}{\partial z}+\frac{\partial^2\phi^{(n)}_3}{\partial\rho\partial z}$$

$$-\frac{z}{2(1-\eta)}\frac{\partial^3}{\partial\rho\partial z^2}\left(\phi^{(n)}_3+\phi^{(n)}_0\right) \quad (17)$$

$$\frac{1}{\mu}\sigma^{(n)}_{\theta z}=\frac{2\,in}{\rho}\frac{\partial\phi^{(n)}_1}{\partial z}-\frac{in}{2\rho(1-\eta)}\frac{\partial}{\partial z}\left(\phi^{(n)}_3+\phi^{(n)}_0\right)+\frac{\partial^2\phi^{(n)}_2}{\partial\rho\partial z}+\frac{in}{\rho}\frac{\partial\phi^{(n)}_3}{\partial z}$$

$$-\frac{inz}{2\rho(1-\eta)}\frac{\partial^2}{\partial z^2}\left(\phi^{(n)}_3+\phi^{(n)}_0\right) \quad (18)$$

where

$$\left(\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}+\frac{\partial^2}{\partial z^2}-\frac{n^2}{\rho^2}\right)\phi^{(n)}_{0,\,1,\,2,\,3}=0. \quad (19)$$

Since $\partial T^{(n)}/\partial Z$ is prescribed on the stress-free crack, it can be verified that the quantities $T^{(n)}$, $U^{(n)}_\rho$, $U^{(n)}_\theta$ and $\sigma^{(n)}_{zz}$ must be odd functions of Z while $U^{(n)}_z$, $\sigma^{(n)}_{\rho\rho}$, $\sigma^{(n)}_{\theta\theta}$, $\sigma^{(n)}_{\rho\theta}$, $\sigma^{(n)}_{\rho z}$ and $\sigma^{(n)}_{\theta z}$ must be even in Z. It is easily seen from equations (10—19) that these conditions are met by assuming $\phi^{(n)}_{0,\,1,\,2,\,3}$ all to be odd in Z. This reduces the problem to a consideration of equation (19) in the half space $Z>0$ with the following boundary conditions on $Z=0$:

$\rho>1$:

$$\left.\begin{aligned} &U^{(n)}_\rho\ (\rho,\,0)=0,\\ &U^{(n)}_\theta\ (\rho,\,0)=0,\\ &\sigma^{(n)}_{zz}\ (\rho,\,0)=0,\\ &T^{(n)}\ (\rho,\,0)=0,\end{aligned}\right\} \quad (20)$$

$\rho<1$:

$$\left.\begin{aligned} \sigma^{(n)}_{\rho z}\ (\rho, 0)&=0,\\ \sigma^{(n)}_{\theta z}\ (\rho, 0)&=0,\\ \sigma^{(n)}_{zz}\ (\rho, 0)&=0,\\ \frac{\partial T^{(n)}}{\partial Z}\ (\rho, 0)&=\tau^{(n)}\ (\rho), \end{aligned}\right\} \qquad (21)$$

where $\tau\ (\rho)=\tau^{(n)}(\rho)e^{in\theta}$ is the prescribed heat flux.

Equation (20) derives from oddness, in Z, of $U^{(n)}_{\rho}$, $U^{(n)}_{\theta}$, $U^{(n)}_{zz}$ and $T^{(n)}$ and their continuity across $Z=0$, $\rho>1$, while (21) derives from the boundry. conditions on the stress–free crack.

Applying third of the boundary conditions (20) and (21) to equation (15) leads to

$$\frac{\partial^2}{\partial z^2}\left(\tfrac{1}{2}\phi^{(n)}_3(\rho, 0)+\phi^{(n)}_1(\rho,\ 0)\right)=0,\ 0<\rho<\infty \qquad (22)$$

Since by (19), we can replace

$$\frac{\partial^2}{\partial z^2}\ \text{by}\ \left(\frac{n^2}{\rho^2}-\frac{\partial^2}{\partial \rho^2}-\frac{1}{\rho}\frac{\partial}{\partial \rho}\right),$$

the equation (20), after integration, yields, for $n\neq 0$,

$$\tfrac{1}{2}\phi^{(n)}_3(\rho,\ 0)+\phi^{(n)}_1(\rho,\ 0)=\alpha_n\ \rho^{|n|}+\beta_n\ \rho^{-|n|},\ 0<\rho<\infty. \qquad (23)$$

If potentials $\phi^{(n)}_{0,\,1,\,2,\,3}$ are finite as $\rho\to 0$ or $\rho\to\infty$, we must choose $\alpha_n=\beta_n=0$ in (23). Then the quantity $\left[\tfrac{1}{2}\phi^{(n)}_3\ (\rho,\ z)+\phi^{(n)}_1(\rho, z)\right]\times e^{in\theta}$ is harmonic in $z>0$ (by 19) and vanishes on $z=0$ (by 23) with $\alpha_n=\beta_n=0$). Therefore, by the uniqueness of this Dirichlet problem, we must have

$$\phi^{(n)}_3\ (\rho,\ z)\equiv-2\phi^{(n)}_1\ (\rho,\ z) \qquad (24)$$

If we now substitute (10) and (11) into the first two of the conditions in (20) we find

$$\frac{\partial \phi_1^{(n)}}{\partial \rho}(\rho, 0) - \frac{in}{\rho}\phi_2^{(n)}(\rho, 0) = 0, \ \rho > 1,$$

$$\frac{\partial \phi_2}{\partial \rho}(\rho, 0) + \frac{in}{\rho}\phi_1^{(n)}(\rho, 0) = 0, \ > \rho 1. \tag{25}$$

Integration of the above equations, together with the requirement of boundedness at $\rho \to \infty$ gives

$$\phi_1^{(n)}(\rho, 0) = (|n| \, |in) \, A_n \, \rho^{-|n|}, \ \rho > 1,$$

$$\phi_2^{(n)}(\rho, 0) = A_n \, \rho^{-|n|}, \ \rho > 1. \tag{26}$$

Again substituting (17) and (18) into first two conditions of (21), and using (24), we get

$$\frac{1}{2(1-\eta)} \frac{\partial^2}{\partial \rho \partial z}\left(\phi_0^{(n)}(\rho, 0) - 2\phi_1^{(n)}(\rho, 0)\right) + \frac{in}{\rho}\frac{\partial \phi_2}{\partial z}(\rho, 0) = 0,$$
$$0 < \rho < 1,$$

$$\frac{\partial^2 \phi_2^{(n)}}{\partial \rho \partial z}(\rho, 0) - \frac{in}{2\rho(1-\eta)}\frac{\partial}{\partial z}\left(\varphi_0^{(n)}(\rho, 0) - 2\phi_1^{(n)}(\rho, 0)\right) = 0,$$
$$0 < \rho < 1. \tag{27}$$

Equation (27), together with the condition of finiteness as $\rho \to 0$, yields

$$\frac{\partial \phi_1^{(n)}}{\partial z}(\rho, 0) = -\frac{|n|(1-\eta)}{in} B_n \, \rho^{|n|} + \tfrac{1}{2}\frac{\partial \phi_0^{(n)}}{\partial z}(\rho, 0), \ \rho < 1,$$

$$\frac{\partial \varphi_2^{(n)}}{\partial z}(\rho, 0) = B_n \, \rho^{|n|}, \ \rho < 1. \tag{28}$$

The constants A_n and B_n in (26) and (28) will be determined from the requirement that the displacements be finite on $z=0$ near the edge of the crack.

Finally, substituting the expression for $T^{(n)}(\rho, z)$ in terms of $\phi_0^{(n)}(\rho, z)$ given by (6) into the last of the conditions of equations (20) and (21), we arrive at

$$\frac{\partial^2 \phi_0^{(n)}}{\partial z^2}(\rho, 0)=0, \ \rho>1 \tag{29}$$

$$\frac{\partial^3 \phi_0^{(n)}}{\partial z^3}(\rho, 0)=-2\alpha(1+\eta)\, \tau^{(n)}(\rho), \ \rho<1.$$

3. Solution for Potentials. A complete solution of potentials $\phi_{0,1,2}(\rho, z)$ are to satisfy (19) in $z>0$ together with the mixed boundary conditions on $z=0$ given by (26), (28) and (29). These are standard problems which can be solved using Hankel transforms (see [6]). Hence the results are

$$\phi_0^{(n)}(\rho, z)=\left(\frac{\pi}{2}\right)^{1/2}\int_0^1 t^{1/2-|n|}\, G_0^{(n)}(t)\, dt.$$

$$\int_0^\infty \xi^{-3/2} J_{|n|}(\xi\rho)\, J_{|n|+1/2}(\xi t)\, e^{-\xi z}\, d\xi, \tag{30}$$

$$\phi_{1,2}^{(n)}(\rho, z)=\left(\frac{\pi}{2}\right)^{1/2}\int_1^\infty t^{1/2-|n|}\, H_{1,2}^{(n)}(t)\, dt.$$

$$\int_0^\infty \xi^{1/2} J_{|n|}(\xi\rho)\, J_{|n|+1/2}(\xi t)\, e^{-\xi z}\, d\xi$$

$$-\left(\frac{\pi}{2}\right)^{1/2}\int_0^1 t^{|n|+1/2}\, G_{1,2}^{(n)}(t)\, dt.$$

$$\int_0^\infty \xi^{1/2} J_{|n|}(\xi\rho)\, J_{|n|+1/2}(\xi t)\, e^{-\xi z}\, d\xi, \tag{31}$$

where

$$H_1^{(n)}(t)=\frac{|n|}{in}\,\frac{2\pi(2|n|-1)!\,A_n}{4^{|n|}[(|n|-1)!]^2}\, t^{-2|n|}, \tag{32}$$

$$H_2^{(n)}(t)=\frac{2\pi(2|n|-1)\,!\,A_n}{4^{|n|}[(|n|-1)\,!]^2}\,t^{-2|n|}\,, \tag{33}$$

$$G_0^{(n)}(t)=2\alpha(1+\eta)\int_0^t \frac{\lambda^{1+|n|}\,\tau^{(n)}(\lambda)}{(t^2-\lambda^2)^{1/2}}\,d\lambda, \tag{34}$$

$$G_1^{(n)}(t)=-\frac{|n|}{in}\,\frac{4^{|n|}[(|n|)\,!]^2\,(1-\eta)}{(2|n|+1)!}\,B_n\,t^{1+2|n|}$$

$$+\frac{1}{2}\int_0^t \frac{\lambda^{1+|n|}}{(t^2-\lambda^2)^{1/2}}\,\frac{\partial\phi_0}{\partial z}(\lambda,\,0)\,d\lambda \tag{35}$$

$$G_2^{(n)}(t)=\frac{4^{|n|}[(|n|)\,!]^2}{(2|n|+1)!}\,B_n\,t^{1+2|n|}. \tag{36}$$

In transferming the results given in [6] to the forms (30—36) use was made of the following integrals :

$$\int_0^1 \frac{\lambda^{2|n|+1}}{(1-\lambda^2)^{1/2}}\,d\lambda=\frac{4^{|n|}\,[(|n|)!]^2}{(2|n|+1)!},$$

$$\int_1^\infty \frac{\lambda^{1-2|n|}}{(\lambda^2-1)^{1/2}}\,d\lambda=\frac{2\pi\,(2|n|-2)!}{4^{|n|}\,[(|n|-1)!]^2}.$$

The solution of (35) can be written simply in terms of $G_0^{(n)}(t)$ in the following form

$$G_1^{(n)}(t)=-\frac{4^{|n|}|n|}{in}\,\frac{[(|n|)!]^2\,(1-\eta)}{(2n+1)!}\,B_n\,t^{1+2|n|}\;-\frac{\pi t^{2|n|+1}}{(8|n|+4)}\times$$

$$\int_t^1 \lambda^{-2|n|}\,G_0^{(n)}(\lambda)\,d\lambda-\frac{\pi}{(8|n|+4)}\int_0^1 \lambda\,G_0^{(n)}(\lambda)\,d\lambda. \tag{37}$$

4. Edge Conditions. We desire that the displacements and temperature remain finite on $z=0$ near the edge of the crack. Since $U_\rho^{(n)}$, $U_\theta^{(n)}$ and $T^{(n)}$ already vanish on $z=0$, $\rho>1$, we need only require $U_\rho^{(n)}$, $U_\theta^{(n)}$ and $T^{(n)}$ to be finite on $z=0$ as $\rho\uparrow 1$ and $U_z^{(n)}$ to be finite on $z=0$ as $\rho \to 1$ from either direction. From equations (6) and (10) to (12), it is clearly sufficient to demand that

$$\frac{\partial^2 \phi_0^{(n)}}{\partial z^2}(\rho, 0),\ \phi_{1,2}^{(n)}(\rho, 0) \text{ and } \frac{\partial \phi_{1,2}^{(n)}}{\partial \rho}(\rho, 0);$$

be finite as $\rho \uparrow 1$ and

$$\frac{\partial \phi_1^{(n)}}{\partial z}(\rho, 0)$$

be finite as $\rho \downarrow 1$ (where we have used the finiteness of

$$\frac{\partial \phi_1^{(n)}}{\partial z}(\rho, 0)$$

as $\rho \uparrow 1$ implied by equation (28).

We can evaluate $\dfrac{\partial^2 \phi_0^{(n)}(\rho, 0)}{\partial z^2}$ from (30) and $\phi_{1,2}^{(n)}(\rho, 0)$ from (31). Using the result [7].

$$\int_0^\infty \xi^{1/2} J_{|n|}(\xi\rho) J_{|n|+1/2}(\xi t)\, d\xi = \left(\frac{2}{\pi}\right)^{1/2} \frac{t^{-|n|-1/2} \rho^{|n|} H(t-\rho)}{(t^2-\rho^2)^{1/2}}, \tag{38}$$

where $H(t-\rho)$ is the Heaviside function,

$$\frac{\partial^2 \phi_0^{(n)}}{\partial z^2}(\rho, 0) = \rho^{|n|} \int_\rho^1 \frac{t^{-2|n|} G_0^{(n)}(t)\, dt}{(t^2-\rho^2)^{1/2}},\ \rho<1, \tag{39}$$

$$\phi_{1,2}^{(n)}(\rho, 0) = \rho^{|n|} \int_1^\infty \frac{t^{-2|n|} H_{1,2}^{(n)}(t)}{(t^2-\rho^2)^{1/2}}\, dt - \rho^{|n|} \int_\rho^1 \frac{G_{1,2}^{(n)}(t)\, dt}{(t^2-\rho^2)^{1/2}},\ \rho<1. \tag{40}$$

Now we evaluate $\left(\frac{\partial\phi^{(n)}_{1,2}}{\partial z}\right)(\rho,\ 0)$, for $\rho>1$, we must proceed somewhat differently. From (31),

$$\frac{\partial\phi^{(n)}_{1,2}}{\partial z}(\rho,z)=-\left(\frac{\pi}{2}\right)^{1/2}\int_1^\infty t^{-|n|+1/2}H^{(n)}_{1,2}(t)\,dt\times\int_0^\infty \xi^{3/2}J_{|n|}(\xi\rho)\,J_{|n|+1/2}(\xi t)\,e^{-\xi z}\,d\xi+\left(\frac{\pi}{2}\right)^{1/2}\int_0^1 t^{|n|+1/2}G^{(n)}_{1,2}(t)\,dt\times\int_0^\infty \xi^{3/2}J_{|n|}(\xi\rho)\,J_{|n|+1/2}(\xi t)\,e^{-\xi z}\,d\xi \qquad (41)$$

Since ([7])

$$\int_0^\infty \xi^{3/2}J_{|n|}(\xi\rho)\,J_{|n|+1/2}(\xi t)\,d\xi=-\left(\frac{2}{\pi}\right)^{1/2}\rho^{-|n|}\,t^{|n|+1/2}\,(\rho^2-t^2)^{-3/2}\,H(\rho-t), \qquad (42)$$

setting $z=0$ will lead to a non-integrable singularity in the first interval on the right hand side of (41). To get around this difficulty we observe from (32) and (33) that we can write

$$G^{(n)}_{1,2}(t)=t^{-2|n|}\,G_{1,2}(1)$$

Substituting this value in (41), we have

$$\frac{\partial\phi^{(n)}_{1,2}}{\partial z}(\rho,0)=-\left(\frac{\pi}{2}\right)^{1/2}\int_1^\infty t^{-|n|+1/2}H^{(n)}_{1,2}(t)\,dt\times\int_0^\infty \xi^{3/2}J_{|n|}(\xi\rho)\,J_{|n|+1/2}(\xi t)\,d\xi+\left(\frac{\pi}{2}\right)^{1/2}G^{(n)}_{1,2}(1)\int_0^\infty \xi^{3/2}J_{|n|}(\xi\rho)\,d\xi\times\int_0^1 t^{-|n|+1/2}J_{|n|+1/2}(\xi t)\,d\xi. \qquad (43)$$

Using the results [8], p. 683.

$$\int_0^1 t^{-|n|+1/2} J_{|n|+1/2}(\xi t)\, d\xi = \xi^{|n|-3/2} \;/\; 2^{|n|-1/2}$$

and $$\Gamma(|n|+1/2) - \xi^{-1} J_{|n|-1/2}(\xi),$$

$$\int_0^\infty \xi^{|n|} J_{|n|}(\xi\rho)\, d\xi = 2^{|n|} \rho^{-|n|-1}\, \Gamma(|n|+1/2)/\Gamma(1/2),$$

the equation (43) becomes

$$\frac{\partial \phi_{1,2}^{(n)}}{\partial z}(\rho, 0) = \rho^{-|n|} \int_1^\infty \frac{t\, H_{1,2}^{(n)}(t)}{(\rho^2 - t^2)^{3/2}} dt + \rho^{-|n|-1} G_{1,2}^{(n)}(1) - \rho^{-|n|} \frac{G_{1,2}^{(n)}(1)}{(\rho^2-1)^{1/2}}. \tag{44}$$

From (39), (40) and (44) we easily arrive at the following edge behavior :

$$\phi_{1,2}^{(n)}(\rho, 0) = 0\,(1), \quad \rho \uparrow 1,$$

$$\frac{\partial \phi_{1,2}^{(n)}}{\partial \rho}(\rho, 0) = \left(H_{1,2}^{(n)}(1) - G_{1,2}^{(n)}(1)\right)(1-\rho^2)^{-1/2} + 0\,(1), \quad \rho \uparrow 1,$$

$$\frac{\partial \phi_{1,2}^{(n)}}{\partial z}(\rho, 0) = \left(H_{1,2}^{(n)}(1) - G_{1,2}^{(n)}(1)\right)(\rho^2-1)^{-1/2} + \rho^{-1} G_{1,2}^{(n)}(1) + \left[\frac{d}{dt} H_{1,2}^{(n)}(1)\right](\rho^2-1)^{1/2} + \ldots\ldots, \quad \rho \downarrow 1, \tag{45}$$

$$\frac{\partial^2 \phi_0^{(n)}}{\partial z^2}(\rho, 0) = 0\,(1), \quad \rho \uparrow 1.$$

The requirement that the displacements remain bounded near the edge then leads to

$$H_{1,2}^{(n)}(1) - G_{1,2}^{(n)}(1) = 0. \tag{46}$$

These conditions, together with equations (32—37) serve to determine the following expressions for the unknown constants A_n and B_n in terms of the heat flux

$$A_n = -\frac{4^{|n|}\,[(|n|-1)!]^2\,\alpha\,(1+\eta)}{(2|n|-1)!\;|n|(2-\eta)\,(8|n|+4)} \times \int_0^1 x^{1+|n|}\,(1-x^2)^{1/2}\,\tau^{(n)}(x)\,dx,$$

$$B_n = -\frac{(2|n|+1)!\,\pi\alpha\,(1+\eta)}{4^{|n|}\,|n|[|n|]^2\,(2-\eta)\,(4|n|+2)} \times \int_0^1 x^{1+|n|}\,(1-x^2)^{1/2}\,\tau^{(n)}(x)\,dx.$$

5. Stress Intensity Factors. We know that the stress intensity factors are given by

$$N_1 = \lim_{\rho\to 1} \sqrt{2}\,(\rho-1)^{1/2}\,\sigma_{\rho z}^{(n)}(\rho, 0),$$

$$N_2 = \lim_{\rho\to 1} \sqrt{2}\,(\rho-1)^{1/2}\,\sigma_{\theta z}^{(n)}(\rho, 0). \tag{47}$$

From (17) and (18), it is clear that the only terms in $\sigma_{\rho z}^{(n)}(\rho, 0)$ and $\sigma_{\theta z}^{(n)}(\rho, 0)$ which are singular as $\rho\to 1$ are the terms involving $(\partial^2/\partial\rho\partial z)\,\phi_{1,2}^{(n)}(\rho, 0)$. Since $\phi_3^{(n)} = -2\phi_1^{(n)}$, we get

$$N_1 = \mu \lim_{\rho\to 1} \sqrt{2}\,(\rho-1)^{1/2}\,(\partial^2/\partial\rho\partial z)\,\phi_1^{(n)}(\rho, 0),$$

$$N_2 = \mu \lim_{\rho\to 1} \sqrt{2}\,(\rho-1)^{1/2}\,(\partial^2/\partial\sigma\partial z)\,\phi_2^{(n)}(\rho, 0). \tag{48}$$

We can compute $(\partial^2\phi_{1,2}/\partial\rho\partial z)(\rho, 0)$ $(\rho>1)$ from (45). Doing this and using (35), (36) and (37) to evaluate $\left(\frac{d}{dt}\right) H_{1,2}^{(n)}(1)$, we arrive at the following results for stress intensity factors:

$$N_1 = \frac{\mu\pi\alpha|n|\ (1+\eta)}{in\ (2-\eta)\ (2|n|+1)} \int_0^1 x^{1+|n|}\ (1-x^2)^{1/2}\ \tau^{(n)}\ (x)\ dx,$$

$$N_2 = \frac{\mu\pi\alpha\ (1+\eta)}{(2-\eta)\ (2|n|+1)} \int_0^1 x^{1+|n|}\ (1-x^2)^{1/2}\ \tau^{(n)}\ (x)\ dx.$$

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