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BOUNDARY LAYER GROWTH OVER AN OSCILLATING CIRCULAR CYLINDER

by

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ABSTRACT

In present paper, the boundary layer equations have been worked out for an impulsive motion of a circular cylinder. The stream function has been assumed in terms of some functions of y and the cylinder is set into motion with usual velocity $V(t)=Ae^{ct}$. In section A, the case of constant suction is considered and in section B, the case of constant injection is considered. Expressions for the velocity components u, v in the boundary layer have been deduced and cases of small and large frequency of oscillations have been discussed.

It has been found in the case of small frequency of oscillations for both suction and injection that (i) the phase angles α_1, α_2 of the velocity component u are respectively equal to the phase angles β_1, β_2 of the velocity component v and (ii) the amplitudes $|M_1|$, $|M_2|$ of the velocity component u are respectively twice the amplitudes $|N_1|$ and $|N_2|$ of the velocity component v. In the case of large frequency of oscillations, the amplitudes and the phase angles of the velocity components u, v are found to be functions of y. This is also found to be true for either suction or injection. Some variations are shown graphically.

1. Introduction. The problem of boundary layer growth on a circular cylinder started impulsively from rest was taken up by Blausius [1] and the time of separation of the boundary layer upto second approximation has also been calculated. Goldstein and Rosenhead [2] calculated the time of separation upto third approximation for the above cited problem. Later on Watson [3] has considered the boundary layer growth for the unsteady two dimensional flow and similar solutions have also been discussed. The boundary layer growth with suction when the velocity of the cylinder varies with time t as (i) $V(t)=At^n$ and (ii) $V(t)=Ae^{ct}$, $c>0$, where A and c are constants have been considered by Nanda [4]. Lal [5] taking the velocity of the cylinder for case (ii), deduced some expressions for the constant suction velocity when the motion is oscillatory.

In the present study, the boundary layer equations when the velocity of the cylinder is of the form $V(t)=Ae^{ct}$, but the motion is oscillatory have been considered for constant suction velocity in section A and for constant injection velocity in section B. The velocity components u, v in the boundary layer have been deduced for small and large frequency of oscillations for both the cases of suction and injection. Some interesting results were obtained for the amplitudes and phase angles of the velocity components u, v in the case of small and large frequency of oscillations. The variations of the phase angles and amplitudes have been graphically represented.

When the frequency of oscillation is small, the amplitudes of the velocity components u, v are found to increase with the increase in the frequency of oscillation. Also the phase angles of the velocity components u, v are found to lead by angles α_1, α_2 and β_1, β_2 respectively over the wall fluctuations in the case of small frequency of oscillations. When the frequency of oscillation is large, the amplitudes $|R_1|$ and $|R_2|$ of the velocity component u are functions of y and are found to increase with the increase in y. These are found to be true for both the cases of suction or injection.

SECTION A

2. Basic Equations. The boundary layer equations and the equation of continuity in two dimensional flow are given by

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\frac{\partial U}{\partial t}+U\frac{\partial U}{\partial x}+\nu\frac{\partial^2 u}{\partial y^2} \tag{1}$$

$$\frac{\partial p}{\partial y}=0 \tag{2}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{3}$$

where $U(x, t)$ is the velocity just outside the boundary layer.

The boundary conditions are

$$y=0 \;:\; u=0,\ v=-v_0$$
$$y=\infty \;:\; u=U(x,t) \tag{4}$$

where v_0 is constant nonzero negative suction velocity.

3. Application to circular Cylinder. Let the fluid be at rest at time $t=0$ and the cylinder be set into motion with velocity

$$V(t)=Ae^{ct},\ c>0 \tag{5}$$

The velocity components u and v of the fluid are given by

$$u=\frac{\partial\psi}{\partial y},\qquad v=\frac{-\partial\psi}{\partial x} \tag{6}$$

where ψ, the stream function is given by

$$\psi_0=v_0x+\sqrt{\frac{\nu}{c}}\;U(x)\,e^{ct}\Big[X_0(s)+S\,\frac{dU}{dx}\,X_1(s)$$
$$+S^2\Big\{\Big(\frac{dU}{dx}\Big)^2X_{21}(s)+U\,\frac{d^2U}{dx^2}\,X_{22}(s)\Big\}+\ldots\Big] \tag{7}$$

where $s=\sqrt{\dfrac{c}{\nu}}\,y,\ S=\displaystyle\int_{-\infty}^{t} V(t)\,dt$ (8)

Substituting the above equations into (1) and equating the coefficients of various powers of S, Nanda [4] obtained the following set of equations

$$\left.\begin{aligned}X_0'''+KX_0''-X_0'+1&=0\\ X_1'''+KX_1''-2X_1'+1-X_0'^2+X_0X_0'&=0\\ X_{21}'''+KX_{21}''-3X_{21}'-2X_0'X_1'+X_0X_0''+X_1X_0''&=0\\ X_{22}'''+KX_{22}''-3X_{22}'-X_0'X_1'+X_1X_0''&=0\end{aligned}\right\} \tag{9}$$

where $$K=v_0/\sqrt{\nu c} \tag{10}$$

The boundary conditions reduce to

$$\begin{aligned}s=0 &: X_0=X_0'=X_1=X_1'=\ldots\ldots\ldots=0\\ s=\infty &: X_0'=1,\ X_1'=X_{21}'=\ldots\ldots\ldots=0\end{aligned} \tag{11}$$

The solutions of these equations are

$$X_0(s)=s+\frac{1}{h}\,(e^{-hs}-1) \tag{12}$$

$$X_1(s)=\frac{1+Kh}{m}\left[1-e^{-ms}-se^{-hs}+K(e^{-hs}-1)\right] \tag{13}$$

where $$h=\tfrac{1}{2}(K+\sqrt{K^2+4}) \tag{14}$$

and $$m=\tfrac{1}{2}(K+\sqrt{K^2+8}) \tag{15}$$

4. **Discussions.** Two cases have been discussed by Lal [5] (i) when K is small and (ii) when K is large and these are given below :

Case (i).

$$X_0\simeq s^2\left(\frac{1}{2}+\frac{K}{4}+\frac{K^2}{16}-\frac{K^4}{256}+\ldots\ldots\right)$$
$$-s^3\left(\frac{1}{6}+\frac{K}{6}+\frac{K^2}{12}+\frac{K^3}{48}-\frac{K^5}{768}+\ldots\ldots\right) \tag{16}$$

$$X_1\simeq s+s^2(-0{\cdot}71+0{\cdot}25K+0{\cdot}46K^2+0{\cdot}25K^3$$
$$+0{\cdot}0545K^4+0{\cdot}0063K^6)\ldots\ldots \tag{17}$$

Case (ii).

$$X_0\simeq\frac{1}{2}s^2\left(K+\frac{1}{K}-\frac{1}{K^3}\right) \tag{18}$$

$$X_1\simeq s^2\left(\frac{2}{K}+\frac{3}{K^3}-\frac{3}{2}\frac{1}{K^5}\right) \tag{19}$$

5. **Application to Oscillatory Motion.** To get the oscillatory motion, c is to be replaced by $i\omega$ in the above equation. From equation (10), it can be observed that if c is small K is large and if c is large, K is small. At the same time c and ω are decreasing or increasing together.

Case (i). When ω is small

When the frequency of oscillations ω are small, we get from equations (18) and (19)

$$X_0(s)\simeq 0{\cdot}35y^2|A_1|(\cos\alpha+i\sin\alpha) \tag{20}$$

where
$$\left.\begin{aligned} |A_1|&=\sqrt{A_{1r}^2+A_{1i}^2},\quad \alpha=\tan^{-1}(A_{1i}/A_{1r})\\ A_{1r}&=\nu_0\,\nu^{-3/2}\,\omega^{1/2}-\nu_0^{-1}\,\nu^{-1/2}\,\omega^{3/2}+\nu_0^{-3}\,\nu^{1/2}\,\omega^{5/2}\\ A_{1i}&=\nu_0\,\nu^{-3/2}\,\omega^{1/2}+\nu_0^{-1}\,\nu^{-1/2}\,\omega^{3/2}+\nu_0^{-3}\,\nu^{1/2}\,\omega^{5/2}\end{aligned}\right\} \tag{21}$$

Similarly, we get from equation (19)

$$X_1(s)\simeq y^2|B_1|\,(\cos\beta+i\sin\beta) \tag{22}$$

where

$$\left.\begin{array}{l} |B_1|=\sqrt{B_{1r}^2+B_{1i}^2},\ \beta=\tan^{-1}(B_{1i}/B_{1r}) \\ B_{1r}=-(1{\cdot}42\ v_0^{-1}\ \nu^{1/2}\ \omega^{3/2}+2{\cdot}13\ v_0^{-3}\ \nu^{1/2}\ \omega^{5/2}+1{\cdot}065 v_0^{-5}\ \nu^{3/2}\ \omega^{7/2}) \\ B_{1i}=1{\cdot}42\ v_0^{-1}\ \nu^{1/2}\ \omega^{3/2}+2{\cdot}13\ v_0^{-3}\ \nu^{1/2}\ \omega^{5/2}+1{\cdot}065\ v_0^{-5}\ \nu^{3/2}\ \omega^{7/2} \end{array}\right\} \tag{23}$$

We have the expressions for u, v inside the boundary layer from equations (6) and (7) as

$$u \cong e^{ct}\, U(x)\left[X'_0(s)+S\frac{dU}{dx}X_1'(s)\right] \tag{24}$$

$$v \cong -v_0+\sqrt{\frac{\nu}{c}}\ e^{ct}\frac{dU}{dx}\left[X_0(s)+X_1(s)\ S\frac{dU}{dx}\right] + \sqrt{\frac{\nu}{c}}\ e^{ct}\ UX_1 S\frac{d^2U}{dx^2} \tag{25}$$

In the case of oscillatory motion, when the fluctuations are small, equation (24) gives u

$$u \cong U(x)\ y\left[|M_1|\ e^{i(\omega t+\alpha_1)}+|M_2|)\ e^{i(\omega t+\alpha_2)}\frac{dU}{dx}\right] \tag{26}$$

$$\text{where } \left.\begin{array}{l} |M_1|=\sqrt{M_{1r}^2+M_{1i}^2},\ \alpha_1=\tan^{-1}(M_{1i}/M_{1r}) \\ M_{1r}=v_0\ \nu^{-1}+v_0^{-3}\ \nu\ \omega^2,\ M_{1i}=v_0^{-1}\ \omega \end{array}\right\} \tag{27}$$

$$\text{and}\quad |M_2|=\sqrt{M_{2r}^2+M_{2i}^2},\ \alpha_2=\tan^{-1}(M_{2i}/M_{2r}) \tag{28}$$

$$M_{2r}=4v_0^{-1}+3v_0^{-5}\ \nu^2\ \omega^2,\ M_{2i}=6v_0^{-3}\ \nu\omega.$$

and $A=1$, for convenience of calculations.

Similarly equation (25) gives

$$v \cong -v_0+y^2\frac{dU}{dx}\ |N_1|\ e^{i(\omega t+\beta_1)} + y^2\left[\left(\frac{dU}{dx}\right)^2+U\frac{d^2U}{dx^2}\right]|N_2|\ e^{i(\omega t+\beta_2)} \tag{29}$$

$$\text{where } \left.\begin{array}{l} |N_1|=\sqrt{N_{1r}^2+N_{1i}^2},\ \beta_1=\tan^{-1}(N_{1i}/N_{1r}) \\ N_{1r}=-\frac{1}{2}(v_0\ \nu^{-1}+v_0^{-3}\ \nu\ \omega^2),\ N_{1i}=-\frac{1}{2}v_0^{-1}\ \omega \end{array}\right\} \tag{30}$$

$$\text{and } \left.\begin{array}{l} |N_2|=\sqrt{N_{2r}^2+N_{2i}^2},\ \beta_2=\tan^{-1}(N_{2i}/N_{2r}) \\ N_{2r}=-2v_0^{-1}-\frac{3}{2}v_0^{-5}\ \nu^2\ \omega^2,\ N_{1i}=-3v_0^{-3}\ \nu\ \omega \end{array}\right\} \tag{31}$$

We observe that the cylinder is moving with velocity $A \cos t$ and the expressions for u, v are obtained by taking real parts ofequations (26) and (29). We can easily observe from equations (27) and (30)

$$|M_1| = 2\,|N_1|, \qquad \alpha_1 = \beta_1 \tag{32}$$

Similarly from equations (28) and (31)

$$|M_2| = 2\,|N_2|, \qquad \alpha_2 = \beta_2. \tag{33}$$

Case (ii). When ω is large

When the frequency of oscillations ω are large, we get from equation (16)

$$X_0(s) \cong y^2 (\cos \xi + i \sin \xi)\,|C_1| + y^3 (\cos \zeta + i \sin \zeta)\,|D_1| \tag{34}$$

$$\left.\begin{aligned} \text{where}\quad |C_1| &= \sqrt{C_{1r}^2 + C_{1i}^2}, \qquad \xi = \tan^{-1}(C_{1i}/C_{2r}) \\ C_{1r} &= -0{\cdot}00288\, v_0^4\, \nu^{-3}\, \omega^{-1} + 0{\cdot}0625\, v_0^2\, \nu^{-2} \\ &\qquad + 0{\cdot}18\, v_0\, \nu^{-3/2}\, \omega^{1/2} \\ C_{1i} &= 0{\cdot}00288\, v_0^4\, \nu^{-3}\, \omega^{-1} + 0{\cdot}18 v_0\, \omega^{1/2}\, \nu^{-3/2} \\ &\qquad + 0{\cdot}5\, \nu^{-1}\, \omega \end{aligned}\right\} \tag{35}$$

$$\left.\begin{aligned} \text{and}\quad |D_1| &= \sqrt{D_{1r}^2 + D_{1i}^2}, \qquad \zeta = \tan^{-1}(D_{1i}/D_{1r}) \\ D_{1r} &= 0{\cdot}11857\, \nu^{-3/2}\, \omega^{3/2} - 0{\cdot}5893\, v_0^2\, \nu^{-3/2}\, \omega^{1/2} \\ &\qquad - 0{\cdot}021\, v_0^3\, \nu^{-3} \\ D_{1i} &= -(0{\cdot}0013\, v_0^5 \nu^{-4}\, \omega^{-1} + 0{\cdot}05893\, v_0^3\, \nu^{-3/2}\, \omega^{1/2} \\ &\qquad + 0{\cdot}167\, v_0\, \nu\, \omega^{-2} + 0{\cdot}11857\, \nu^{-3/2}\, \omega^{3/2}) \end{aligned}\right\} \tag{36}$$

Similarly, we get from equation (17)

$$X_1(s) \cong y\, e^{i\pi/4}\, \nu^{-1/2}\, \omega^{1/2} + y^2 (\cos \phi + i \sin \phi)\,|E_1| \tag{37}$$

$$\left.\begin{aligned} \text{where}\quad |E_1| &= \sqrt{E_{1r}^2 + E_{1i}^2}, \qquad \phi = \tan^{-1}(E_{1i}/E_{1r}) \\ E_{1r} &= -0{\cdot}0063\, v_0^6\, \nu^{-4}\, \omega^{-2} + 0{\cdot}18\, v_0^3\, \nu^{-5/2} \\ &\qquad + 0{\cdot}46\, v_0^2\, \nu^{-2} + 0{\cdot}18\, v_0\, \nu^{-3/2}\, \omega^{1/2} \\ E_{1i} &= -0{\cdot}0545\, v_0^4\, \nu^{-3}\, \omega^{-1} - 0{\cdot}18\, v_0^{-3}\, \nu^{-5/2}\, \omega^{-1/2} \\ &\qquad + 0{\cdot}18\, v_0\, \nu^{-3/2}\, \omega^{-1/2} - 0{\cdot}71\, \nu^{-1}\, \omega \end{aligned}\right\} \tag{38}$$

In the case of oscillatory motion, when the frequency of oscillations ω are large, we get from equation (24)

$$u \cong U(x)\,|R_1|\, e^{i(\omega t + q_1)} + U(x) \frac{dU}{dx}\,|R_2|\, e^{i(\omega t + q_2)} \tag{39}$$

$$\left.\begin{aligned} &\text{where } |R_1|=\sqrt{R_{1r}^2+R_{1i}^2}, \qquad q_1=\tan^{-1}(R_{1i}/R_{1r}) \\ &R_{1r}=y(0{\cdot}707\,\nu^{-1/2}\,\omega^{1/2}+0{\cdot}5v_0\,\nu^{-1}+0{\cdot}088\,v_0^2\,\nu^{-3/2}\,\omega^{-1/2}) \\ &\qquad -y^2\,(0{\cdot}354\,v_0\,\nu^{-3/2}\,\omega^{-1/2}+0{\cdot}25\,v_0^2\,\nu^{-2}) \\ &R_{1i}=y\,(0.707\,\nu^{-1/2}\,\omega^{1/2}-0{\cdot}088\,v_0^2\,\nu^{-3/2}\,\omega^{1/2}) \\ &\qquad -y^2\,(0{\cdot}5\,\nu^{-1}\,\omega^{-1}+0{\cdot}354\,v_0\,\nu^{-3/2}\,\omega^{-1/2}) \end{aligned}\right\} \tag{40}$$

$$\left.\begin{aligned} &\text{and } |R_2|=\sqrt{R_{2r}^2+R_{2i}^2}, \qquad q_2=\tan^{-1}(R_{2i}/R_{2r}) \\ &R_{2r}=-\nu^{1/2}\,\omega^{1/2}\,y \\ &R_{2i}=-\omega^{-1}+y\,(\nu^{-1/2}\,\omega^{1/2}-0{\cdot}5v_0\,\nu^{-1}\,\omega^{-1} \\ &\qquad +0{\cdot}92\,v_0^2\,\nu^{-3/2}\,\omega^{-3/2}) \end{aligned}\right\} \tag{41}$$

Similarly, we get from equation (25) in the case of large frequency of oscillations

$$v\cong -v_0+y^2\frac{dU}{dx}\,|L_1|\;e^{i(\omega t+\Phi_1)}+\left[\left(\frac{dU}{dx}\right)^2+U\,\frac{d^2U}{dx^2}\right]|L_2|\;e^{i(\omega t+\Phi_2)} \tag{42}$$

$$\left.\begin{aligned} &|L_1|=\sqrt{L_{1r}^2+L_{1i}^2}, \qquad \phi_1=\tan^{-1}(L_{1i}/L_{1r}) \\ &L_{1r}=y^3\,(0{\cdot}118\,v_0\,\nu^{-3/2}\,\omega^{1/2}+0{\cdot}5\,v_0^2\,\nu^{-2}) \\ &\qquad -y^2\,(0{\cdot}354\,\nu^{-1/2}\,\omega^{1/2}+0{\cdot}25\,v_0\,\nu^{-1} \\ &\qquad +0{\cdot}044\,v_0^2\,\nu^{-3/2}\,\omega^{1/2}) \\ &L_{1i}=y^3\,(0{\cdot}167\,\nu^{-1}\,\omega+0{\cdot}118\,v_0\,\nu^{-3/2}\,\omega^{1/2}) \\ &\qquad -y^2(0{\cdot}354\,\nu^{-1/2}\,\omega^{1/2}-0{\cdot}044\,v_0^2\,\nu^{-3/2}\,\omega^{1/2}) \end{aligned}\right\} \tag{43}$$

$$\left.\begin{aligned} &\text{and } |L_2|=\sqrt{L_{2r}^2+L_{2i}^2}, \qquad \phi_2=\tan^{-1}(L_{2i}/L_{2r}) \\ &L_{2r}=y^2(0{\cdot}505\,\nu^{-1/2}\,\omega^{-1/2}-0{\cdot}25v_0\,\nu^{-1} \\ &\qquad -0{\cdot}325v_0^2\,\nu^{-3/2}\,\omega^{-1/2}) \\ &L_{2i}=y\,\omega^{-1}+y^2(0{\cdot}502\,\nu^{-1/2}\,\omega^{-1/2}+0{\cdot}325v_0^2\,\nu^{-3/2}\omega^{-1/2}) \end{aligned}\right\} \tag{44}$$

SECTION B

When there is constant injection at the wall of the cylinder, the basic equations (1), (2) and (3) remain the same. But the boundary conditions change over to

$$y=0\;:\; u=0,\quad v=+v_0$$
$$y=\infty:\; u=U(x,t)$$

where v_0 is constant non-zero injective velocity. The streamfunction assumes the form

$$\psi=-v_0\, x+\sqrt{\frac{\nu}{c}}\,.\,U(x)\,e^{ct}\left[X_0(s)+S\,\frac{dU}{dx}\,X_1(s)\right.$$
$$\left.+S^2\left\{\left(\frac{dU}{dx}\right)^2 X_{21}(s)+U\,\frac{d^2U}{dx^2}\,X_{22}(s)\right\}+\ldots\right]$$

With the substitution of equations (5) and (6) in equation (1), with the above boundary conditions, we get the same set of differential equations as (9) with the same reduced boundary conditions (11). We get the same set of solutions (16) to (18) in terms of K, where K stands here as

$$K=-v_0/\sqrt{\nu c} \tag{46}$$

To get the oscillatory motion, we replace c by $i\omega$ in these equations. From equation (46) it is seen in the case of injection also that if c is small, K is large and if c is large K is small. At the same time c and ω are decreasing or increasing together.

Case (1). When the frequency of oscillations ω are small, we get from equation (18)

$$X_0(s)\cong 0{\cdot}35y^2\,|A_1|\,(\cos\alpha+i\sin\alpha) \tag{47}$$

where
$$\left.\begin{aligned}|A_1|&=\sqrt{A_{1r}^2+A_{1i}^2},\qquad \alpha=\tan^{-1}(A_{1i}/A_{1r})\\ A_{1r}&=-v_0\,\nu^{-3/2}\,\omega^{1/2}+v_0^{-1}\,\nu^{-1/2}\,\omega^{3/2}-v_0^{-3}\,\nu^{1/2}\,\omega^{5/2}\\ A_{1i}&=-v_0\,\nu^{-3/2}\,\omega^{1/2}-v_0^{-1}\,\nu^{-1/2}\,\omega^{3/2}-v_0^{-3}\,\nu^{1/2}\,\omega^{5/2}\end{aligned}\right\} \tag{48}$$

Similarly, we get from equation (19) when the frequency of oscillations are small

$$X_1(s)\cong y^2\,|B_1|\,(\cos\beta+i\sin\beta) \tag{49}$$

where

$$\left.\begin{aligned}|B_1|&=\sqrt{B_{1r}^2+B_{1i}^2},\ \beta=\tan^{-1}(B_{1i}/B_{1r})\\ B_{1r}&=1{\cdot}42v_0^{-1}\,\nu^{-1/2}\,\omega^{3/2}+2{\cdot}13v_0^{-3}\,\nu^{1/2}\,\omega^{5/2}+1{\cdot}065v_0^{-5}\,\nu^{3/2}\,\omega^{7/2}\\ B_{1i}&=-(1{\cdot}42v_0^{-1}\,\nu^{-1/2}\,\omega^{3/2}-2{\cdot}13v_0^{-3}\,\nu^{1/25}\,\omega^{/2}+1{\cdot}065v_0^{-5}\,\nu^{3/2}\,\omega^{7/2}\end{aligned}\right\} \tag{50}$$

When there is constant injection we get the velocity components u, v in the case of small fluctuations, from equations (6) and (7) as

$$u\cong U(x)y\left[|M_1|e^{i(\omega t+\alpha_1)}+|M_2|e^{i(\omega t+\alpha_2)}\frac{dU}{dx}\right] \tag{51}$$

where
$$\left.\begin{aligned}|M_1|&=\sqrt{M_{1r}^2+M_{1i}^2},\quad \alpha_1=\tan^{-1}(M_{1i}/M_{1r})\\ M_{1r}&=-(v_0\,\nu^{-1}+v_0^{-3}\,\nu\omega^2),\ M_{1i}=-v_0^{-1}\omega\end{aligned}\right\} \tag{52}$$

and $$\left.\begin{aligned} |M_2| &= \sqrt{M_{2r}^2+M_{2i}^2},\ \alpha_2=\tan^{-1}(M_{2i}/M_{2r}) \\ M_{2r} &= -(4v_0^{-1}+3v_0^{-5}\,\nu^2\omega^2),\ M_{2i}=-6v_0^{-3}\nu\omega \end{aligned}\right\} \quad (53)$$

and $A=1$ for convenience of calculations.

Similarly,

$$v \cong +v_0+y^2\frac{dU}{dx}|N_1|e^{i(\omega_1+\beta_1)}+y^2\left[\left(\frac{dU}{dx}\right)^2+U\frac{d^2U}{dx^2}\right]|N_2|e^{i(\omega t+\beta_2)} \quad (54)$$

where $$\left.\begin{aligned} |N_1| &= \sqrt{N_{1r}^2+N_{1i}^2},\ \beta_1=\tan^{-1}(N_{1i}/N_{1r}) \\ &= N_{1r}=\tfrac{1}{2}(v_0\,\nu^{-1}+v_0^{-3}\,\nu\omega^2),\ N_{1i}=\tfrac{1}{2}v_0^{-1}\omega \end{aligned}\right\} \quad (55)$$

and $$\left.\begin{aligned} |N_2| &= \sqrt{N_{2r}^2+N_{2i}^2},\ \beta_2=\tan^{-1}(N_{2i}/N_{2r}) \\ N_{2r} &= \tfrac{3}{2}v_0^{-5}\,\nu^2\omega^2+2v_0^{-1},\ N_{2i}=3v_0^{-3}\,\nu\omega \end{aligned}\right\} \quad (56)$$

Thus, we observe in the case of small frequency of oscillations ω, the amplitudes and phase angles of different terms in the case of injection are exactly equal to the corresponding amplitudes and phase angles in the case of suction.

Just as in the case of suction, we can easily observe in the case of injection also from equations (52), (55) and (53), (56)

$$|M_1|=2\,|N_1| \quad , \quad \alpha_1=\beta_1 \quad (57)$$

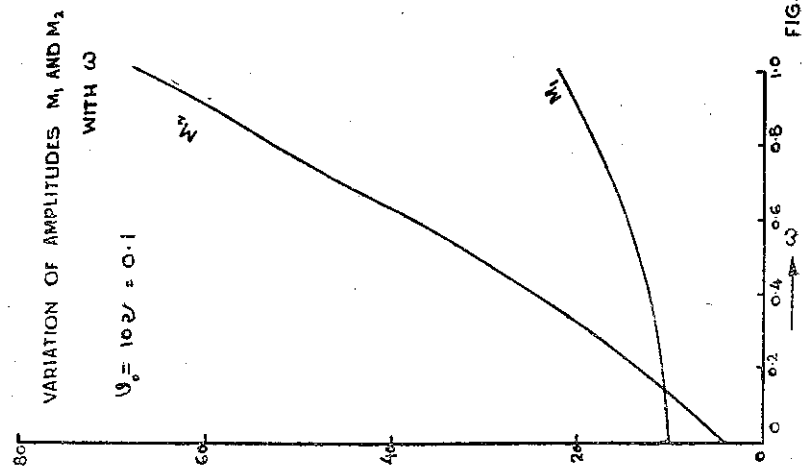
$$|M_2|=2\,|N_2| \quad , \quad \alpha_2=\beta_2 \quad (58)$$

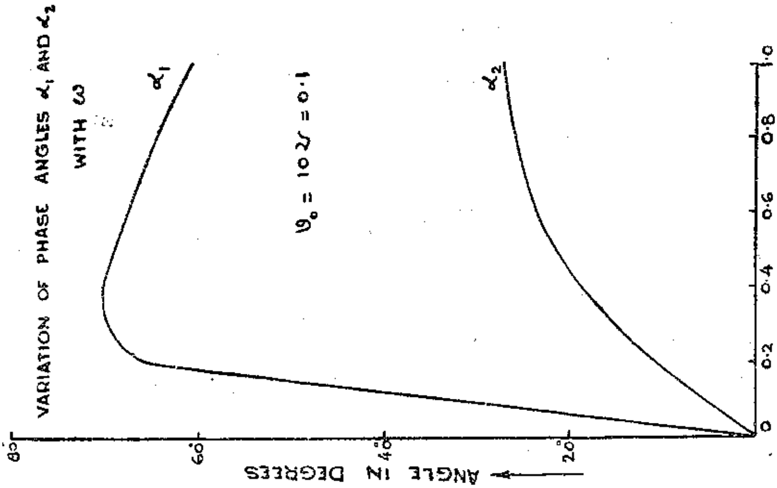
which are exactly equal to equations (32) and (33) of the constant suction case.

Thus when the fluctuations are small, for either suction or injection, the amplitudes $|M_1|$ and $|M_2|$ of the velocity component u are twice the amplitudes $|N_1|$ and $|N_2|$ of the velocity component v respectively. Also, the phase angles α_1, and α_2 of the velocity component u are respectively equal to the phase angles β_1 and β_2 of the velocity component v.

The variations of the amplitudes $|M_1|$ and $|M_2|$ in the case of constant suction with the frequency of oscillations ω are shown in Fig. [1], when $v_0=0{\cdot}1$, $\nu=0{\cdot}01$. We can easily observe that both the amplitudes $|M_1|$ and $|M_2|$ increase with the frequency of oscillation. The same hence will be true for $|N_1|$ and $|N_2|$ also.

In Fig. [2], the variation of the phase angles α_1, α_2 with the frequency of oscillation ω are shown for the case of suction, when


VARIATION OF AMPLITUDES M_1 AND M_2 WITH ω
$\vartheta_0 = 10\nu = 0{\cdot}1$
M_1
M_2
0
20
40
60
80
0
0·2
0·4
0·6
0·8
1·0
ω
FIG.1


VARIATION OF PHASE ANGLES α_1 AND α_2 WITH ω
$\vartheta_0 = 10\nu = 0{\cdot}1$
α_1
α_2
ANGLE IN DEGREES
20°
40°
60°
80°
0
0·2
0·4
0·6
0·8
1·0
ω
Fig. 2

$\nu_0=0{\cdot}1$, $\nu=0{\cdot}01$. The phase angle α_1 is found here to increase with the frequency of oscillation ω, the increase being slow from $\omega=0.8$. The phase angle α_2 is found to increase with the frequency of oscillation ω upto $\omega=0{\cdot}4$ and then is found to decrease. Since the values α_1 and α_2 are positive, the fluctuations lead by angles α_1 and α_2 for small frequency of oscillations. The same will be true for the phase angles β_1 and β_2.

The angles α_1 and α_2 are positive and hence β_1 and β_2 are positive This implies that the velocity u leads by angles α_1,α_2 and the velocity v leads by angles β_1, β_2 over the wall fluctuations when the frequency of oscillation is small in the case of either suction or injection.

Case (2). When the frequency of oscillations ω are large, we get from equation (16)

$$X_0(s)\cong y^2(\cos\xi+i\sin\xi)\,|C_1|+y^3(\cos\zeta+i\sin\zeta)\,|D_1| \tag{55}$$

where

$$\left.\begin{aligned}
&|C_1|=\sqrt{C_{1r}^2+C_{1i}^2},\ \xi=\tan^{-1}(C_{1i}/C_{1r})\\
&C_{1r}=-0{\cdot}00288\nu_0^4\,\nu^{-3}\,\omega^{-1}+0{\cdot}0625\nu_0^{\,2}\nu^{-2}-0{\cdot}18\nu_0\nu^{-3/2}\,\omega^{1/2}\\
&C_{1i}=0{\cdot}00288\nu_0^4\,\nu^{-3}\,\omega^{-1}-0{\cdot}18\nu_0\,\nu^{-3/2}\,\omega^{1/2}+0{\cdot}5\,\nu^{-1}\,\omega
\end{aligned}\right\} \tag{56}$$

and

$$\left.\begin{aligned}
&|D_1|=\sqrt{D_{1r}^2+D_{1i}^2},\ \zeta=\tan^{-1}(D_{1i}/D_{1r})\\
&D_{1r}=0{\cdot}11857\,\nu^{-3/2}\,\omega^{3/2}-0{\cdot}5893\nu_0^2\,\nu^{-3/2}\,\omega^{1/2}\\
&\qquad\qquad+0.021\nu_0^3\,\nu^{-3}\\
&D_{1i}=0{\cdot}0013\nu_0^5\,\nu^{-4}\,\omega^{-1}-0{\cdot}5893\nu_0^2\,\nu^{-3/2}\,\omega^{1/2}\\
&\qquad\qquad+0{\cdot}167\nu_0\,\nu\omega^{-2}-0{\cdot}11857\,\nu^{-3/2}\,\omega^{3/2}
\end{aligned}\right\} \tag{57}$$

Similarly, we get from equation (17)

$$X_1(s)\cong y\,e^{i\pi/4}\,\nu^{-1/2}\,\omega^{1/2}+y^2(\cos\varphi+i\sin\phi)\,|E_1| \tag{58}$$

where

$$\left.\begin{aligned}
&|E_1|=\sqrt{E_{1r}^2+E_{1i}^2},\ \phi=\tan^{-1}(E)_{1i}/E_{1r}\\
&E_{1r}=-0{\cdot}0063\nu_0^6\,\nu^{-4}\,\omega^{-2}-0{\cdot}18\nu_0^3\,\nu^{-5/2}\,\omega^{-1/2}+0{\cdot}46\nu_0^2\,\nu^{-2}\\
&\qquad\qquad-0{\cdot}18\nu_0\,\nu^{-3/2}\,\omega^{-1/2}\\
&E_{1i}=-0{\cdot}0545\nu_0^4\,\nu^{-3}\,\omega^{-1}+0{\cdot}18\nu_0^3\,\nu^{-5/2}\,\omega^{-1/2}\\
&\qquad\qquad-0{\cdot}18\nu_0\,\nu^{-3/2}\,\omega^{-1/2}-0{\cdot}71\,\nu^{-1}\omega
\end{aligned}\right\} \tag{59}$$

When there is constant injection, we get the velocity components u, v in the case of large fluctuations, from equations (6) and (7) as

$$u\cong U(x)|R_1|e^{i(\omega t+q_1)}+U(x)\frac{dU}{dx}|R_2|e^{i(\omega t+q_2)} \tag{60}$$

$$\left.\begin{aligned} \text{where}\quad |R_1| &= \sqrt{R_{1r}^2 + R_{1i}^2},\ q_1 = \tan^{-1}(R_{1i}/R_{1i}) \\ R_{1r} &= y(0{\cdot}707\nu^{-1/2}\omega^{1/2} - 0{\cdot}5v_0\nu^{-1} + 0{\cdot}088v_0^2\nu^{-3/2}\omega^{-1/2}) \\ &\quad + y^2(0{\cdot}354v_0\nu^{-3/2}\omega^{-1/2} - 0{\cdot}25v_0^2\nu^{-2}) \\ R_{1i} &= y(0{\cdot}707\nu^{-1/2}\omega^{1/2} - 0{\cdot}088v_0^2\nu^{-3/2}\omega^{1/2}) \\ &\quad - y^2(0{\cdot}5\nu^{-1}\omega^{-1} - 0{\cdot}354v_0\nu^{-3/2}\omega^{-1/2}) \end{aligned}\right\} \quad (61)$$

$$\left.\begin{aligned} \text{and}\quad |R_2| &= \sqrt{R_{2r}^2 + R_{2i}^2},\ q_2 = \tan^{-1}(R_{2i}/R_{2r}) \\ R_{2r} &= -\nu^{1/2}\omega^{1/2}y \\ R_{2i} &= -\omega^{-1} + y(\nu^{-1/2}\omega^{1/2} + 0{\cdot}5v_0\nu^{-1}\omega^{-1} \\ &\quad + 0{\cdot}92v_0^2\nu^{-3/2}\omega^{-3/2}) \end{aligned}\right\} \quad (62)$$

The variation of the amplitudes with the distance y from the cylinder has been represented graphically in Fig. [3] for both suction

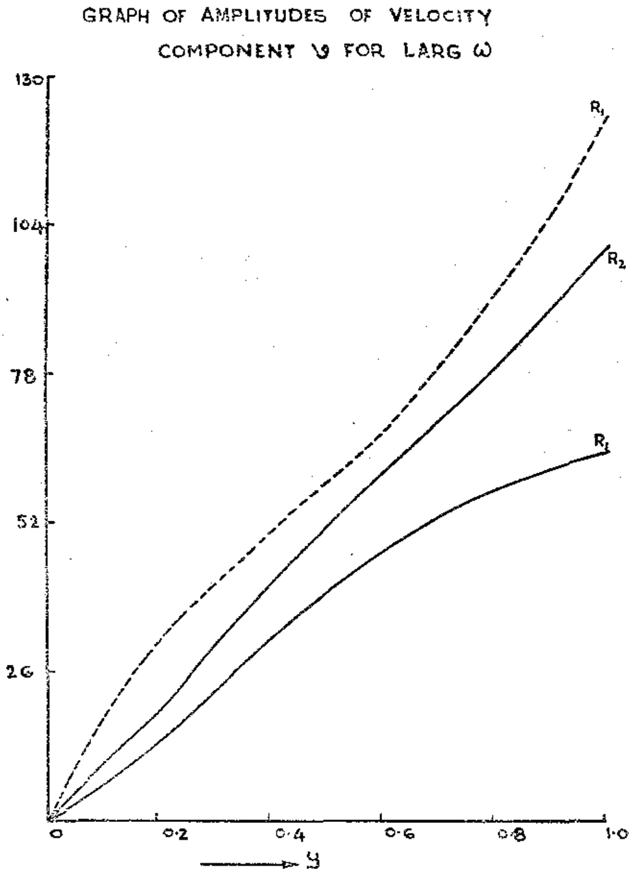


FIG. 3

and injection. It is found from the figure that the amplitude $|R_1|$ increases with y, the increase being more for the case of injection. It has been observed that the amplitude $|R_2|$ comes out to be approximately equal and hence the variation of $|R_2|$ is shown for the case of suction only.

Now at the wall of the cylinder, i.e., $y \to 0$ the velocity component u reduces to

$$u \cong U(x) \frac{dU}{dx} \frac{1}{\omega} e^{i(\omega t + \frac{1}{2}\pi)} \tag{63}$$

This is true for both suction and injection.

Similarly,

$$v \cong +v_0 + y^2 \frac{dU}{dx} |L_1| e^{i(\omega t + \Phi_1)} + y^2 \left[\left(\frac{dU}{dx}\right)^2 + U \frac{d^2U}{dx^2} \right] |L_2| e^{i(\omega t + \Phi_2)} \tag{64}$$

where

$$\left.\begin{aligned} |L_1| &= \sqrt{L_{1r}^2 + L_{1i}^2},\ \phi_1 = \tan^{-1}(L_{1i}/L_{1r}) \\ L_{1r} &= y^3(0{\cdot}5v_0^2\, \nu^{-2} - 0{\cdot}118 v_0\, \nu^{-3/2}\omega^{1/2}) \\ &\quad - y^2(0{\cdot}354\nu^{-1/2}\omega^{1/2} - 0{\cdot}25 v_0\, \nu^{-1} \\ &\quad + 0{\cdot}044 v_0^2\, \nu^{-3/2}\omega^{1/2}) \\ L_{1i} &= y^3(0{\cdot}167\nu^{-1}\omega - 0{\cdot}118 v_0\, \nu^{-3/2}\omega^{1/2}) \\ &\quad - y^2(0{\cdot}354\nu^{-1/2}\omega^{1/2} - 0{\cdot}044 v_0^2\, \nu^{-3/2}\omega^{1/2}) \end{aligned}\right\} \tag{65}$$

and

$$\left.\begin{aligned} |L_2| &= \sqrt{L_{2r}^2 + L_{2i}^2},\ \phi_2 = \tan^{-1}(L_{2i}/L_{2r}) \\ L_{2r} &= y^2(0{\cdot}502\nu^{-1/2}\omega^{-1/2} + 0{\cdot}25 v_0\, \nu^{-1} \\ &\quad - 0{\cdot}32 v_0^2\, \nu^{-3/2}\omega^{-1/2}) \\ L_{2i} &= y\omega^{-1} + y^2(0{\cdot}502\nu^{-1/2}\omega^{-1/2} + 0{\cdot}325 v_0^2\, \nu^{-3/2}\omega^{-1/2}) \end{aligned}\right\} \tag{66}$$

The variations of the phase angles q_1, ϕ_1 and ϕ_2 with y have been represented graphically in Fig. [4] for both suction and injection. The phase angle q_1 is found to increase with y in both the cases of suction and injection. The phase angle q_2 is approximately constant in both the cases. The variation of ϕ_1 and ϕ_2 with y is comparatively little in both the cases. We observe also that the amplitudes and phase angles are functions of y when the frequency of oscillations ω are large for either suction or injection.

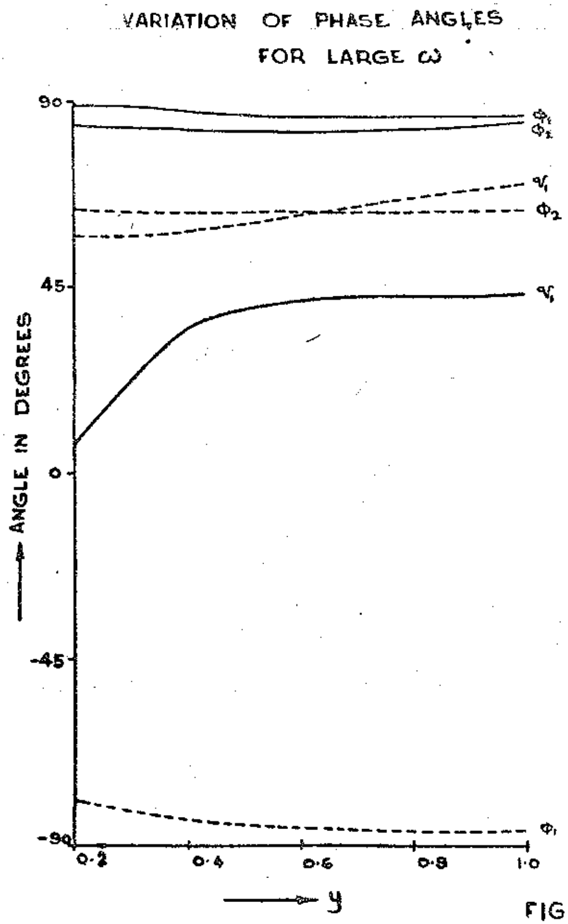


FIG. 4

REFERENCES

[1] Blausius, H. "*Grenzschichten in Flusig Keiten mit Kleiner Reibung*", ZAMP, vol. 56 (1908) 1.

[2] Goldstein, S. and Rosenhead, L. "*Boundary Layer Growth*" Proc. Camb. Phil. Soc., vol. 32 (1936), 392-401.

[3] Watson, E. J. "*Boundary Layer Growth*" Proc. Roy. Soc., vol. 231, ser. A (1955), 104-116.

[4] Nanda, R. S. "*Boundary Layer Growth suction*" Ph. D. Thesis, I. I. T. Kharagpur (1960), 50-66 (Chap. III).

[5] Lal, K. "*Development of unsteady Boundary Layer Theory*" Istanbul Teknik universities Bulteni vol. 22 (1969), 17-24.