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THE INVERSION OF A CONVOLUTION TRANSFORM WHOSE KERNEL IS A GENERALIZED BATMAN k-FUNCTION

By

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Summary. In the case of some functions it is possible to invert a certain convolution transform by a similar convolution transform. Ta Li [6] and Buschman [3] have proved this for Tchebicheff and Legendre polynomials. In this article we consider a similar problem involving the generalised Bateman k-function [7] as its kernel. The technique used here is that of Widder [8].

1. Introduction. The Bateman k-function was first introduced Bateman [1] in the form $k_n(x)=\int_0^{\pi/2} \cos(x \tan\theta - n\theta)\,d\theta$ and then by Chakravarty [4] in the form $\overset{u}{\underset{n}{k}}(x)=\frac{2}{\pi}\int_0^{\pi/2} 2^u \cos^u\theta \cos(x\tan\theta - n\theta)\,d\theta$, where $u>-1$. Recently, Srivastava [7] generalised the function $\overset{u}{\underset{n}{k}}(x)$ of Chakravarty in the form

$$(1.1)\qquad \overset{u,\,v}{\underset{n}{k}}(x)=\frac{2}{\pi}\int_0^{\pi/2}\sin^u\theta\cos^v\theta\cos(x\tan\theta-n\theta)\,d\theta$$

and made a systematic study of this funetion. We will take in general n, u and v to be non-negative integers.

The classical Laplace transform of $f(t)$ as defined by the equation

$$(1.2)\qquad F(p)=p\int_0^{\infty} e^{-pt} f(t)\,dt, \qquad R(p)>0$$

will be symbolically denoted by $F(p)\div f(t)$.

The following results will be used in the sequel :

$$(1.3) \qquad k_{2n}^{2m,\,2l}(x) \div 2(-1)^m p^{2m+1} \; (1-p)^{n-l-m-1}/(1+p)^{n+l+m+1}$$

provided $R(p)>0$, $(n-l)$ is a positive integer and $(n-l-m-1)>0$

$$(1.4) \qquad e^{-x/2}\, k_{2n}^{2m,\,2l}\left(\frac{x}{2}\right)$$

$$=\sum_{r=0}^{m} (-1)^{n-l-1} \binom{m}{r} \frac{2^{-2l-2r}}{\Gamma(n+l+r+1)} \cdot D^{n-l-r-1} \; (e^{-x}\, x^{n+l+r})$$

provided $n>l+m$; $n>-l$ and $2l>-1$.

Also from (1.3) we can deduce

$$(1{\cdot}5) \qquad e^{-x/2}\, k_{2n}^{2m,\,2l}\left(\frac{x}{2}\right)$$

$$\div p(2p+1)^{2m} \; (-1)^{n-l-1}\, 2^{-2l-2m} \frac{p^{n-l-m}}{(p+1)^{n+l+m+1}}$$

$$(1.6) \qquad e^{ax}\, k_{2n}^{2m,\,2l}\left(\frac{x}{2}\right)$$

$$\div 2^{2m+2} \; (-1)^m \, p(p-a)^{2m} \frac{(1-2p+2a)^{n-l-m-1}}{(1+2p-2a)^{n+l+m+1}}$$

and

$$(1.7) \qquad e^{ax}\, k_{2n}^{2m,\,2l}\left(\frac{x}{2}\right)$$

$$\div -2^{2m+2} \; (-1)^m \; p(p-a)^{2m} \frac{(1+2p-2a)^{n-l-m-1}}{(1-2p+2a)^{n+l+m+1}}$$

being valid under the same conditions as in (1.3).

To prove (1.3) and (1.4) see Srivastava [7].

The object of this paper is to obtain an inversion formula of the convolution transform whose kernel is the generalised Bateman function $k_{n\,(x)}^{u,\,v}$ as defined by (1.1).

2. Theorem 1. Let $n>l$, $g(x)$ be a function of x such that its first $(2l+1)$ differential coefficients are continuous in $0\leqslant x<\infty$ and vanish at $x=0$, $f(x)$ be a function defined by

$$(2.1)\qquad f(x)=\int_0^x e^{a(x-t)}\, k_{2n}^{2m,\,2l}\,[\tfrac{1}{2}(x-t)]\, g(t)\, dt$$

and

$$(2.2)\qquad \overset{*}{F}_{m}(p)=2^{4l}\,(p-a)^{-4m}\,(p-a-\tfrac{1}{2})^{4l+4m+4}\, F(p)$$

where

$$(2.3)\qquad F(p)\div f(x)$$

Then $f(x)$ and its $(4l+3)$ differential coefficients with respect to x are continuous in $0\leqslant x<\infty$ and vanish at $x=0$, and $g(x)$ is given by

$$(2.4)\qquad g(x)=-\int_0^x e^{a(x-t)}\, k_{2n+4l+4m+4}^{2m,\,2l}\left(\frac{t-x}{2}\right)\overset{*}{f}_{m}(t)\, dt \text{ where}$$

$\overset{*}{f}_{m}(x)$ is the inverse Laplace transform of $\overset{*}{F}_{m}(p)$.

Proof. From (1.1) it is easy to verify

$$(2.5)\qquad e^{a(x-t)}\, k_{2n}^{2m,\,2l}\,[\tfrac{1}{2}(x-t)]=e^{(a-\frac{1}{2})(x-t)}$$

$$\sum_{r=0}^{m}(-1)^r\binom{m}{r}\frac{2^{-2l-2r}}{\Gamma(2n-k)}$$

$$\sum_{\nu=0}^{k}\binom{k}{\nu}(k-2n+1)_{\nu}\cdot(x-t)^{2n-k-\nu-1}$$

where $k=n-l-r-1$.

Since the right-hand side of (2.5) is a polynomial in $(x-t)$ and the lowest degree of $(x-t)$ in this expression is $(2l+1)$, hence the first $2l$ differential coefficients of $e^{a(x-t)}\, k_{2n}^{2m,\,2l}\,[\tfrac{1}{2}(x-t)]$ with respect to x are continuous in $0\leqslant x<\infty$ and vanish at $x=t$.

Now differentiating (2.1) μ times with respect to x we obtain

$$(2.6) \qquad f^{(\mu)}(x)=\int_0^x D^{\mu}\Big[e^{a(x-t)}\, k_{2n}^{2m,\,2l} \;[\tfrac{1}{2}(x-t)]\, g(t)\Big]dt$$

for $\mu=0, 1, 2, \ldots, (2l+1)$.

Since $g(x)$ and its first $(2l+1)$ differential coefficients with respect to x are continuous in $0 \leqslant x < \infty$ and vanish at $x=0$ it is obvious that on differenting (2.6) further, $f(x)$ and all its first $(4l+3)$ differential coefficients are also continuous in $0 \leqslant x < \infty$ and vanish at $x=0$.

Clearly, if $F(p) \div f(t)$, then

$$(2.7) \qquad p^s\, F(p) \div D^s f(x) \qquad s=0, 1, 2, \ldots, (4l+4).$$

If we assume $G(p) \div g(t)$, then the equation (2.1) is transformed to

$$F(p)=\frac{(-1)^m}{p}\cdot 2^{2m+2}\cdot \Big[p(p-2)^{2m}\cdot \frac{(1-2p+2a)^{n-l-m-1}}{(1+2p-2a)^{n+l+m+1}}\cdot \quad G(p) \Big]$$

which may also be written in the form

$$(2.8) \quad G(p)=\Bigg[(-1)^m\, 2^{2m+2}\, \frac{(1+2p-2a)^{n+2l+2m+2-(l+m+1)}}{(1-2p+2a)^{n+2l+2m+2+(l+m+1)}}\cdot$$

$$(p-2a)^{2m}\Bigg]\, [2^{4l}\, (p-a)^{-4m}\, [(p-a-\tfrac{1}{2})^{4l+4m+4}\, F(p)]$$

The result (2.4) follows, after a little simplification, on inverting (2.8) with the help of a result of Erdelyi [4, p. 131 (20)] and the equation (2.2).

Theorem 2. Let $n>l$, $g(x)$ be a function of x such that its first $(2l+1)$ differential coefficients with respect to x are continuous in $0 \leqslant x < \infty$ and vanish at $x=0$, $f(x)$ be a function defined by

$$(2.9) \qquad f(x)=\int_0^x e^{a(x-t)}\, k_{2n+4l+4m+4}^{2m,\,2l}\, [\tfrac{1}{2}(t-x)]\, g(t)\, dt$$

and

$$(2.10) \qquad \overset{*}{F}_m(p)=2^{4l}\, (p-a)^{-4m}\, (p-a-\tfrac{1}{2})^{4l+4m+4}\, F(p)$$

where

(2.11) $\quad F(p) \div f(x)$

Then $f(x)$ and all its first $(4l+3)$ differential coefficients with respect to x are continuous in $0 \leqslant x < \infty$ and vanish at $x=0$, and $g(x)$ is given by

$$(2.12) \qquad g(x) = -\int_0^\infty e^{a(x-t)}\, k_{2n}^{2m,\,2l}\left[\tfrac{1}{2}(x-t)\right] \overset{*}{f}_m(t)$$

where $\overset{*}{f}_m(x)$ is the inverse Laplace transform of $\overset{*}{F}_m(p)$.

Proof. The proof of this theorem is similar to that of Theorem 1.

Also on using (2.7) we can write

$$(p-a-\tfrac{1}{2})^{4l+4m+4}\, F(p) \div (D-a-\tfrac{1}{2})^{4l+4m+4} f(t)$$

If we take $m=0$ and use the above equation then Theorems 1 and 2 are reduced to the results of Bhartiya [2] for the inversion of a convolution transform whose kernel is the function $k_n^u(x)$ of Chakravarti.

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