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MOTION OF A DUSTY VISCOUS FLUID DOWN AN INCLINED PLANE

by

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Summary. The motion of dusty viscous fluid with uniform distribution of dust particles down an inclined plane under the influence of the exponential pressure gradient has been studied. The velocities of fluid and dust particles are obtained in elegant form and the temperature distribution is studied.

1. Introduction. In recent years the attention of researchers [1-4] in the field of fluid dynamics has been diverted towards the study of the influence of dust particles on viscous fluid flows. Saffmann [4] has studied the stability of the laminar flow of a dusty gas with uniform distribution of dust particles. Michael [2] has considered the Kelvin-Helmoltzs' instability of the dusty gas. Michael and Miller [1] have discussed the motion of dusty gas occupied by the semi-infinite space above a rigid plane boundary. In the present investigation, the flow of a viscous liquid with uniform distribution of dust particles, down an inclined plane under the influence of the exponential pressure gradient is discussed. Analytical expressions for the velocity of fluid and dust particles are obtained in terms of hyperbolic functions. The discharge of the flux per second and maximum velocity of fluid particles are discussed in detail. Finally the distribution of the temperature is studied.

2. Equations of Motion. The equations of motion of a dusty unsteady viscous, incompressible fluid flow are given by 1, 2, 3

$$\frac{\partial \vec{u}}{\partial t}+(\vec{u}.\nabla)\vec{u}=-\frac{1}{\rho}\nabla\vec{p}+\nu\nabla^2\vec{u}+\frac{KN}{\rho}(\vec{v}-\vec{u}) \qquad (1)$$

$$\frac{\partial \vec{v}}{\partial t}+(\vec{v}.\nabla)\vec{v}=\frac{K}{m}(\vec{u}-\vec{v}) \tag{2}$$

$$\text{div } \vec{u}=0 \tag{3}$$

$$\text{div } \vec{v}=0 \tag{4}$$

where $\vec{u}$, $\vec{v}$ denote the velocity vectors of fluid and dust particles respectively, p the fluid pressure, m the mass of the dust particles, N the number density, K the stokes' resistance coefficient (for spherical particles of radius ε is $6\pi\mu\varepsilon$), μ the coefficient of velocity of fluid particle, ρ the density and ν the kinematic coefficient of viscosity.

3. Formulation and Solution of the Problem. In the present investigation we shall study the laminar flow of an unstudy viscous liquid with uniform distribution of dust particles down an inclined plane of inclination θ to the horizontal. The liquid is bounded by a parallel upper surface at a distance h from the plane. Let us choose the origin of co-ordinate system at the bottom of the inclined plane, X-axis opposite to the direction of the flow and along the greatest slope of the plane and Y-axis perpendicular to the plane. Since both the dust and fluid particles move along the inclined plane the motion is symmetric along the greatest slope of the plane. Since the flow is laminar the velocity of both fluid and dust particles can be defined by the following relations.

$$\left.\begin{array}{ll} u=u\ (y,t), & v=w=0 \\ u_1=u_1\ (y,t), & v_1=w_1=0 \end{array}\right\} \tag{5}$$

where u, v, w and u_1, v_1, w_1 are velocity components of fluid and dust particles. Since we have assumed the distribution of dust particles is uniform, the number density of dust particles $N=N_0$ is a constant throughout the motion. Using the equation (5), when the plane is at an inclination θ, equations (1)—(4) can be written as

$$\frac{\partial u}{\partial t}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\frac{\partial^2 u}{\partial y^2}+\frac{KN_0}{\rho}(u_1-u)-g\sin\theta \tag{6}$$

$$\frac{1}{\rho}\frac{\partial p}{\partial y}+g\cos\theta=0 \tag{7}$$

$$-\frac{1}{\rho}\frac{\partial p}{\partial z}=0 \tag{8}$$

$$\frac{\partial u_1}{\partial t}=\frac{K}{m}(u-u_1) \tag{9}$$

from the above equations we express the pressure p as

$$p=-\rho g(x \sin \theta+y \cos \theta)-x\rho \ F(t) \tag{10}$$

where $F(t)$ is a function of t only.

Using equation (10) in (6) and (9) it follows that

$$\frac{\partial u}{\partial t}=F(t)+\nu\frac{\partial^2 u}{\partial y^2}+\frac{KN_0}{\rho}(u_1-u) \tag{11}$$

$$\frac{\partial u}{\partial t}=\frac{u-u_1}{\tau} \tag{12}$$

we shall choose u, u_1 and $F(t)$ respectively

$$\left.\begin{aligned} u(y,t)&=u(y)\ e^{-\lambda^2 t} \\ u_1(y,t)&=u_1(y)\ e^{-\lambda^2 t} \\ F(t)&=\alpha\ e^{-\lambda^2 t} \end{aligned}\right\} \tag{13}$$

where α and λ are the real constants.

Using the values of u, u_1 and $F(t)$, from the above relations one obtains that

$$-\lambda^2 u=\alpha+\nu\frac{\partial^2 u}{\partial y^2}+\frac{KN_0}{\rho}(u_1-u) \tag{14}$$

$$-\lambda^2 u_1\tau=(u-u_1) \tag{15}$$

From the relations (15) we can express the relation between u_1 and u as

$$u_1=\frac{u}{1-\tau\lambda^2} \tag{16}$$

On using the value of u_1 from the above relation, equation (14) yields the following differential equation.

$$\frac{\partial^2 f}{\partial y^2}+A^2 f=0 \tag{17}$$

where

$$f=A^2u+\frac{d}{\nu}$$

$$A^2=\frac{\lambda^2}{\nu}\left(1+\frac{l}{1-\tau\lambda^2}\right)$$

$$l=\frac{m\ N_0}{\rho}$$

$$\tau=\frac{m}{K}$$

The solution of the equation (17) is given by

$$f = B_1 \cos Ay + C_1 \sin Ay \tag{18}$$

where B_1 and C_1 are constants to be determined subject to the following boundary conditions

$$\left.\begin{aligned} f &= \frac{\alpha}{\nu} \quad \text{when} \quad y = 0 \\ f &= -UA^2 + \frac{B}{\nu} \text{ when } y = h \end{aligned}\right\} \tag{19}$$

Using the boundary conditions (19) in (18) we can obtain the values of the constants B_1 and C_1 as

$$\left.\begin{aligned} B_1 &= \frac{\alpha}{\nu} \\ C_1 &= \frac{\alpha(1-\cos Ah) - UA^2)}{\nu \sin Ah} \end{aligned}\right\} \tag{20}$$

Substituting the values of the constants the velocity of the fluid particles and dust particles can be expressed respectively as

$$u = \frac{-\alpha(1-\cos Ay)(1-\tau\lambda^2)\sin Ah + \{\alpha(1-\tau\lambda^2)(1-\cos Ah) - U\lambda^2(1+1-\tau\lambda^2\}\sin Ay}{(1+1-\tau\lambda^2)\sin Ah} \tag{21}$$

$$u_1 = \frac{u}{1-\tau\lambda^2} \tag{22}$$

The discharge of the flux per second for unit width of the plane is

$$q = \int_0^h \pi yu \, dy$$

$$= \frac{\pi\alpha(1-\tau\lambda^2)(2Ah \sin Ah + 2A \cos Ah - h^2 A^2 - 2)}{A^2\lambda^2(1+l-\tau\lambda^2)}$$

$$+ \frac{2\pi\{\alpha(1-\tau\lambda^2)(1-\cos Ah - Uh^2)(1+l-\tau\lambda^2)}{A^2\lambda^2(1+l-\tau\lambda^2)\sin Ah}(\sin Ah - Ah \cos Ah) \tag{23}$$

The velocities of the fluid and dust particles at a height h above the plane are respectively given by

$$U = \frac{-\alpha(1-\cos Ah)(1-\tau\lambda^2) + \alpha(1-\tau\lambda^2)(1-\cos Ah) - Uh^2(1+l-\tau\lambda^2)}{\lambda^2(1+l-\tau\lambda^2)} \tag{24}$$

$$U_1 = \frac{U}{1-\tau\lambda^2} \tag{25}$$

Temperature Distribution. Assuming that there is no external heat addition the energy equation as given by Pai [5] is

$$\frac{\partial T}{\partial t}=K'\frac{\partial^2 T}{\partial y^2}+\mu'\left(\frac{\partial u}{\partial y}\right)^2 \tag{26}$$

where $K'=\frac{K}{\rho c_v}$, $\mu'=\frac{\mu}{\rho c_v}$

using the velocity expression (21) the above relations becomes

$$\frac{\partial T}{\partial t}=K'\frac{\partial^2 T}{\partial y^2}+\mu'\left(\frac{a^2+b^2}{2}+ab\sin 2Ay+\frac{b^2-a^2}{2}\cos 2Ay\right) \tag{27}$$

where $a=\frac{-\alpha A(1-\tau\lambda^2)\sin Ah}{\lambda^2(1+l-\tau\lambda^2)\sin Ah}$

$$b=\frac{A[\alpha(1-\tau\lambda^2)(1-\cos Ah)-Uh^2(1+l-\tau\lambda^2)]}{\lambda^2(1+l-\tau\lambda^2)\sin Ah}$$

In order to obtain the solution of the equation (27), we shall assume that

$$T(y,t)=\theta_1(t)\sin 2Ay+\theta_2(t)\cos 2Ay+\theta_3(t) \tag{28}$$

where θ_1, θ_2, θ_3 are functions of t. Using equation (25) in (24), and equating the powers of sin $2Ay$, cos $2Ay$ and constant terms we get the following differential equations

$$\left.\begin{aligned}\theta_1'(t)&=-\frac{K'}{4A^2}\theta_1+\mu' ab\\ \theta_2'(t)&=-\frac{K'}{4A^2}\theta_2+\mu'\frac{(b^2-a^2)}{2}\\ \theta_3'(t)&=\mu'\frac{(a^2+b^2)}{2}\end{aligned}\right\} \tag{29}$$

Solving the differential equation (29) with the boundary conditions

$$\left.\begin{aligned}t=0,\ T=0\\ \theta_1(0)=\theta_2(0)=\theta_3(0)=0\end{aligned}\right\} \tag{30}$$

One obtains the following relations

$$\theta_1(t)=\frac{4\mu' A^2}{K'}\left[1-e^{-\frac{K'}{4A^2}t}\right]$$

$$\theta_2(t)=\frac{2A^2\mu'(b^2-a^2)}{K'}\left[1-e^{-\frac{K'}{4A^2}t}\right]$$

$$\theta_3(t)=\mu'(a^2+b^2)\frac{t}{2}$$

From the above relations the temperature distribution is given by

$$T=\left[\frac{4\mu' A^2}{K'}\sin 2Ay+\frac{2A^2\mu'(b^2-a^2)}{K'}\cos 2Ay\right]\left[1-e^{-\frac{K't}{4A^2}}\right]+\frac{\mu'}{2}(a^2+b^2)\,t$$

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