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SOME MULTIPLE SERIES TRANSFORMATION*

By

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SUMMARY

The present note gives a simple proof by induction of a theorem on multiple series transformation and applies this result to derive certain reduction formulas for the generalized Lauricella functions.

1. **Introduction**. Recently, Srivastava [5] gave a number of cases of reducibility of certain double series with arbitrary terms. Of our concern here is one of his results, which we state as :

Theorem 1. (Srivastava [5, p 297]). Let $\{c_n\}$ be any sequence of complex numbers. Then, for arbitrary ν and σ,

$$\sum_{m,\, n=0}^{\infty} c_{m+n}\, (\nu)_m\, (\sigma)_n\, \frac{x^{m+n}}{m!\, n!} = \sum_{n=0}^{\infty} c_n\, (\nu+\sigma)_n\, \frac{x^n}{n!}, \tag{1}$$

provided that the series involved converge absolutely.

In particular, if we let

$$c_n = (\mu)_n/(\rho)_n,\ n=0,\ 1,\ 2, \ldots, \tag{2}$$

then, as Srivastava observed [loc. cit.], (1) would yield a well-known case of reducibility of the Appell function F_1, viz.

$$F_1[\mu,\ \nu,\ \sigma;\ \rho;\ x,\ x] = {}_2F_1[\mu,\ \nu+\sigma;\ \rho;\ x].\ |x|<1. \tag{3}$$

A generalization of (3) is due to Lauricella [1], who indeed gave the reduction formula [op. cit., p. 150]

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(4) $$F_D^{(n)}[a, b_1, \ldots, b_n; c; x, \ldots, x]$$

$$= {}_2F_1[a, b_1+\ldots+b_n; c; x], \ |x|<1,$$

where $F_D^{(n)}$ denotes the fourth type of Lauricella's hypergeometric functions of n variables defined by [1, p. 113]

(5) $$F_D^{(n)}[a, b_1, \ldots, b_n; c; x_1, \ldots, x_n]$$

$$= \sum_{m_1, \ldots, m_n=0}^{\infty} \frac{(a)_{m_1+\ldots+m_n} (b_1)_{m_1} \cdots (b_n)_{m_n}}{(c)_{m_1+\ldots+m_n}} \frac{x_1^{m_1}}{m_1!} \cdots \frac{x_n^{m_n}}{m_n!},$$

$$\max\{|x_1|, \ldots, |x_n|\}<1.$$

In view of the known result (4) it would seem natural to look for a multidimensional generalization of the series transformation (1). As a matter of fact, such a generalization of Theorem 1 is given by (cf. [2]) :

Theorem 2. Let the coefficients $C(m)$ be defined for all values of the non-negative integer m, and let $\alpha_1, \ldots, \alpha_n$ be arbitrary complex numbers. Then

(6) $$\sum_{m_1, \ldots, m_n=0}^{\infty} C(m_1+\ldots+m_n)(\alpha_1)_{m_1} \cdots (\alpha_n)_{m_n} \frac{x^{m_1+\ldots+m_n}}{m_1! \ldots m_n!}$$

$$= \sum_{N=0}^{\infty} C(N)(\alpha_1+\ldots+\alpha_n)_N \frac{x^N}{N!},$$

provided that the series involved are absolutely convergent.

2. **Proof of Theorem** 2. At the outset we remark that the proof of Theorem 2 by the earlier writer [2] makes use of certain operators of finite differences. Our proof of the multiple series transformation (6) is by induction on the positive integer n. Indeed, (6) with $n=2$ is essentially the same as Srivastava's identity (1). Thus, if we assume (6) to hold true for some integer $n>1$ and denote the first member of (6) by Δ_n, we shall observe that, for arbitrary α_{n+1},

$$(7) \qquad \Delta_{n+1} = \sum_{m_{n+1}=0}^{\infty} (\alpha_{n+1})_{m_{n+1}} \frac{x^{m_{n+1}}}{m_{n+1}!}$$

$$\cdot \sum_{m_1, \ldots, m_n=0}^{\infty} C(m_1+\ldots+m_n+m_{n+1}) (\alpha_1)_{m_1} \cdots (\alpha_n)_{m_n}$$

$$\cdot \frac{x^{m_1+\ldots+m_n}}{m_1! \ldots m_n!}$$

$$= \sum_{M, m_{n+1}=0}^{\infty} C(M+m_{n+1}) (\alpha_1+\ldots+\alpha_n)_M (\alpha_{n+1})_{m_{n+1}} \frac{x^{M+m_{n+1}}}{M! \, m_{n+1}!}$$

$$= \sum_{N=0}^{\infty} C(N) (\alpha_1+\ldots+\alpha_{n+1})_N \frac{x^N}{N!},$$

by using (6) and (1), successively, it being understood that the various series involved are absolutely convergent.

The last member of (7) exhibits the fact that (6) is true also for $n+1$ if it is true for some positive integer n.

This evidently completes the proof of Theorem 2 by induction.

3. Applications. It is easily verified that the reduction formula (4) is contained in the multiple series transformation (6) with

$$(8) \qquad C(m) = (a)_m/(c)_m, \; m=0, 1, 2, \ldots .$$

More generally, if we set

$$(9) \qquad C(m) = \frac{\prod_{j=1}^{p} (a_j)_{m\lambda_j}}{\prod_{j=1}^{q} (b_j)_{m\mu_j}}, \; m=0, 1, 2, \ldots,$$

then (6) will yield a reduction formula for a certain class of the generalized Lauricella functions introduced and studied earlier by Srivastava and Daoust ([3], p. 454 et seq.). Following their notations [loc. cit.], we thus obtain

$$(10) \qquad F^{p:1;\ldots;1}_{q:0;\ldots;0}\left(\begin{matrix}[a_1:\lambda_1,\ldots,\lambda_1], \ldots, [a_p:\lambda_p,\ldots,\lambda_p]: \\ [b_1:\mu_1,\ldots,\mu_1], \ldots, [b_q:\mu_q,\ldots,\mu_q]:\end{matrix}\right.$$

$$\left.\begin{matrix}[\alpha_1:1]; \ldots; [\alpha_n:1]; \\ \text{——}; \ldots; \text{——};\end{matrix} \; x, \ldots, x\right)$$

$$= {}_{p+1}\Psi_q\left[\begin{matrix}(a_1, \lambda_1), \ldots, (a_p, \lambda_p), (\alpha_1+\ldots+\alpha_n, 1); \\ (b_1, \mu_1), \ldots, (b_q, \mu_q);\end{matrix} \; x\right],$$

where ${}_p\Psi_q$ denotes Wright's generalized hypergeometric function and, for convergence, $\lambda_j>0, j=1, \ldots, p$; $\mu_j>0, j=1, \ldots, q$, and

$$(11) \qquad \sum_{j=1}^{q} \mu_j - \sum_{j=1}^{p} \lambda_j \geqslant 0,$$

wherein the equality holds when $|x|$ is suitably restricted (cf. [4], pp. 157-1 ·8).

In particular, if $\lambda_j=1$, $j=1, \ldots, p$, and $\mu_j=1, j=1, \ldots, q$, the reduction formula (10) is simplified to the form

$$(12) \qquad F^{p:1;\ldots;1}_{q:0;\ldots;0}\left(\begin{matrix} a_1, \ldots, a_p : \alpha_1 ; \ldots ; \alpha_n ; \\ b_1, \ldots, b_q : — ; \ldots ; — ; \end{matrix} x, \ldots, x\right)$$

$$= {}_{p+1}F_q\left[\begin{matrix} a_1, \ldots, a_p, \alpha_1+\ldots+\alpha_n ; \\ b_1, \ldots, b_q ; \end{matrix} x\right],$$

provided $p<q$, or $p=q$ and $|x|<1$.

Evidently, for $p=q=1$, this last reduction formula (12) would correspond to the known result (4) above.

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