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A THEOREM ON THE TRANSFORMATION OF KAMPÉ DE FÉRIET FUNCTIONS

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In this paper, a proof of an extension of Bailey's theorem ([2], sect. 1,) to two-dimensional series is given. The basis of this was considered by Abel more than one hundred years earlier.

Theorem 1. If the values of the parameters and the variables are such that the following series either converge or terminate, then

$$\sum_{m,\,n=0}^{\infty} \frac{((a))_{m+n}((d))_{m+n}((v))_{2m+2n}((c))_m((l))_m((z))_{2m}((c'))_n((l'))_n((z'))_{2n}}{((h))_{m+n}((g))_{m+n}((f))_{2m+2n}((p))_m((b))_m((j))_{2m}((p'))_n((b'))_n((j'))_{2n}\ m!\ n!}$$

$$.\quad x^m y^m s^n t^n v^{2m} z^{2n}$$

$$F\left(\begin{array}{l|c|l} U+D+V & (u), \quad (d)+m+n, \quad (v)+2m+2n & \\ W+L+Z & w), (l)+m,\ (z)+2m : (w'),\ (l')+n,\ (z')+2n & xuv \\ E+G+F & (e), \quad (g)+m+n, \quad (f)+2m+2n & swz \\ K+B+J & (k), (b)+m,\ (j)+2m : (k'), (b')+n,\ (j')+2n & \end{array}\right)$$

$$= \sum_{m,\,n=0}^{\infty} \frac{((d)_{m+n}((u))_{m+n}((v))_{m+n}((l))_m((w))_m((z))_m((l'))_n((w'))_n((z'))_n}{((g))_{m+n}((e))_{m+n}((f))_{m+n}((b))_m((k))_m((j))_m((b'))_n((k'))_n((j'))_n\ m!\ n!}$$

$$.\quad x^m u^m y^m w^n s^n z^n\ F$$

$$F\left(\begin{array}{l|c|c} A+E+V & 1-m-n-(e), \quad (a), \quad (v)+m+n & \\ K+C+Z+1 & 1-m-(k),(c),(z)+m,\ -m : 1-n-(k'),(c'),(z')+n,\ -n & X \\ U+H+F & 1-m-n--(u), \quad (h), \quad (f)+m+n & Y \\ W+P+J & 1-m-(w),(p),(j)+m : 1-n-(w'),(p'),(j')+n & \end{array}\right)$$

where $X=(-1)^{1-U-W+E+K}\ yv/u$, and $Y=(-1)^{1-U-W+E+K}\ tz/w$

and where $(a)_m=\Gamma(a+m)/\Gamma(a)$, $((a))_m=(a_1)_m\ (a_2)_m \ldots (a_A)_m$, etc.,

$$\text{and } F\left(\begin{array}{c|ccc|c} \mu & & (\alpha) & & \\ \nu & (\beta) & : & (\beta') & x \\ \rho & & (\gamma) & & y \\ \sigma & (\delta) & : & (\delta') & \end{array}\right) \text{(the Kampé de Fériet function)}$$

$$=\sum_{m,\, n=0}^{\infty} \frac{((\alpha))_{m+n}\,((\beta))_m\,((\beta'))_n\, x^m y^n}{((\gamma))_{m+n}\,((\delta))_m\,((\delta'))_n\, m!\, n\,!} \text{ (cf. [1]),}$$

where $$((\alpha))_{m+n}=\prod_{j=1}^{A} (\alpha)_{m+n}, \text{ etc.}$$

The author feels that the above notation is more suited to the present needs than a contracted notation for the Kampé de Fériet function.

Before proceeding to the main proof, the following lemma is established.

Lemma. If $$\beta_{m,\,n}=\sum_{p=0}^{m}\sum_{q=0}^{n} \alpha_{p,\,q}\, u_{m-p,\, n-q}\, v_{m+p,\, n+q}$$

and $$\gamma_{m,\,n}=\sum_{p=m}^{\infty}\sum_{q=n}^{\infty} \delta_{p,\,q}\, u_{p-m,\, q-n}\, v_{p+m,\, q+n}$$

where α, δ, u, v are functions of p and q only, then

$$\sum_{m,\, n=0}^{\infty} \alpha_{m,\,n}\,\gamma_{m,\,n} = \sum_{m,\, n=0}^{\infty} \beta_{m,\,n}\,\delta_{m,\,n}$$

provided that the series are convergent.

Proof. Consider a quadruply infinite array of objects which can be represented as points in four-dimensional space. We may form the sum

$$\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} f(p,\, q,\, r,\, s)$$

by summing parallel to the coordinate axes. Also, we may form the sum

$$\sum_{p=0}^{\infty} \sum_{q=0}^{p} \sum_{r=0}^{\infty} \sum_{s=0}^{r} f(p, q, r, s)$$

by summing first along two pairs of diagonals and then outwardly perpendicular to each set of diagonals, respectively.

By hypothesis, the quadruple series is assumed to be convergent, so that it may be written

$$\sum_{p=0}^{\infty} \sum_{r=0}^{\infty} \sum_{q=0}^{p} \sum_{s=0}^{q} f(p, q, r, s).$$

Hence,

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \alpha_{m, n} \gamma_{m, n} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{p=m}^{\infty} \sum_{q=n}^{\infty} \alpha_{m, n} \delta_{p, q} u_{p-m, q-n} v_{p+m, q+n}$$

$$= \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \sum_{m=0}^{p} \sum_{n=0}^{q} \alpha_{m, n} \delta_{p, q} u_{p-m, q-n} v_{p+m, q+n}$$

$$= \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \beta_{p, q} \delta_{p, q}$$

so that the lemma is established.

The main proof of Theorem I is as follows.

Let $$u_{m, n} = \frac{((u))_{m+n} ((w))_m ((w'))_n u^m w^n}{((e))_{m+n} ((k))_m ((k'))_n m! n!},$$

$$v_{m, n} = \frac{((v))_{m+n} ((z))_m ((z'))_n v^m z^n}{((f))_{m+n} ((j))_m ((j'))_n},$$

$$\delta_{m, n} = \frac{((d))_{m+n} ((l))_m ((l'))_n x^m s^n}{((g))_{m+n} ((b))_m ((b'))_n}$$

and

$$\alpha_{m+n} = \frac{((a))_{m+n} ((c))_m ((c'))_n y^m t^n}{((h))_{m+n} ((p))_m ((p'))_n m! n!}.$$

If we write s for $p-m$ and t for $q-n$,

$$\gamma_{m, n} = \sum_{s=0}^{\infty} \sum_{t=0}^{\infty} \delta_{s+m, t+n} u_{s, t} v_{s+2m, t+2n},$$

so it is found that

$$\gamma_{m,\,n}=\frac{((d))_{m+n}\,((l))_{m}\,((l'))_{n}\,((v))_{2m+2n}\,((z))_{2m}\,((z'))_{2n}\,x^{m}s^{n}v^{2m}z^{2n}}{((g))_{m+n}\,((b))_{m}\,((b'))_{n}\,((f))_{2m+2n}\,((j))_{2m}\,((j'))_{2n}}$$

$$\times F\left(\begin{array}{l|c|c} U+D+V & (u),\quad (d)+m+n,\quad (v)+2m+2n & \\ W+L+Z & (w),\,(l)+m,\,(z)+2m:(w'),\,(l')+n,\,(z')+2n & xuv \\ E+G+F & (e),\quad (g)+m+n,\quad (f)+2m+2n & swz \\ K+B+J & (k),\,(b)+m,\,(j)+2m:(k'),\,(b')+n,\,(j')+2n & \end{array}\right)$$

and

$$\beta_{m,\,n}=\frac{((u))_{m+n}\,((v))_{m+n}\,((w))_{m}\,((w'))_{n}\,((z))_{m}\,((z'))_{n}\,u^{m}v^{m}w^{n}z^{n}}{((e))_{m+n}\,((f))_{m+n}\,((k))_{m}\,((k'))_{n}\,((j))_{m}\,((j'))_{n}\,m!\,n!}$$

$$\times F\left(\begin{array}{l|c|c} A+E+V & 1-m-n-(e),\quad (a),\quad (v)+m+n & \\ K+C+Z+1 & 1-m-(k),\,(c),\,(z)+m,\,-m:1-n-(k'),\,(c'),\,(z')+n,\,-n & X \\ U+H+F & 1-m-n-(u),\quad (h),\quad (f)+m+n & Y \\ W+P+J & 1-m-(w),\,(p),\,(j)+m:1-n-(w'),\,(p'),\,(j')+n & \end{array}\right).$$

Theorem I now follows by combining these results with the lemma, it is an extension of Bailey's Theorem as applied by L. J. Slater ([3], p. 60). When the inner series are summable, more straightforward, direct transformations involving Kampé de Fériet functions are obtainable which it is hoped to discuss in further papers.

REFERENCES

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[2] W. N. Bailey, *Identities of Rogers-Ramanujan type*, Proc. Lond. Math. Soc. (2) 50 (1949), 1-10.

[3] L. J. Slater, *Generalized hypergeometric functions*, Cambridge University Press, 1966.