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ABSOLUTE RIESZ SUMMABILITY OF FACTORED FOURIER SERIES

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1·1 Definitions and notations. Let $\lambda=\lambda(w)$ be a differentiable monotonic increasing function of w and tending to infinity with it. For a given infinite series[1] $\Sigma\, a_n$, we write

$$A_r(w)=\sum_{n\leqslant w}\{\lambda(w)-\lambda(n)\}^r a_n \quad (r\geqslant 0).$$

The series $\Sigma\, a_n$ is said to be summable $|R, \lambda, r|$ $(r>0)$

or $\quad \Sigma\, a_n \in |R, \lambda, r|$ $(r>0)$ if

$$\int_A^\infty |d\,[A_r(w)/\{\lambda(w)\}^r]\,|<\infty,$$

where A is a positive number (Obrechkoff [1], [2]).

Now, for $r>0$ and $m<w<m+1$, we have

$$\frac{d}{dw}[A_r(w)/\{\lambda(w)\}^r]=\frac{r\,\lambda'(w)}{\{\lambda(w)\}^{r+1}}\sum_{n\leqslant w}\{\lambda(w)-\lambda(n)\}^{r-1}\lambda(n)\,a_n.$$

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[1] Summations are over 1, 2, 3, ... ∞ when there is no indication to the contrary.

Thus the series $\Sigma\, a_n \in |R, \lambda, r|$ $(r>0)$ if

$$\int_A^\infty \left| \frac{r\,\lambda'(w)}{\{\lambda(w)\}^{r+1}} \sum_{n\leqslant w} \{\lambda(w)-\lambda(n)\}^{r-1}\,\lambda(n)\,a_n \right| dw$$

is convergent.

We define the summability $|R, \lambda, 0|$ equivalent to the absolute convergence.

1.2. Let $f(t)$ be a 2π-periodic function and L-integrable over $(-\pi, \pi)$. Without any loss of generality the constant term of the Fourier series of $f(t)$ can be taken to be zero, that is

$$\int_{-\pi}^{\pi} f(t)\,dt=0,$$

so that the Fourier series of $f(t)$, at $t=x$, may be written as

$$\Sigma\,(a_n \cos nx+b_n \sin nx)=\Sigma\,A_n(x).$$

1.3. Throughout this paper $\beta>0$ and we use the following notations :

$$\phi(t)=\tfrac{1}{2}\,\{f(x+t)+f(x-t)\}. \tag{1}$$

$$\phi_1(t)=\frac{1}{t}\int_0^t \phi(u)\,du. \tag{2}$$

$$\log_2=\log\log. \tag{3}$$

$$y(n)=\{\log(n+1)\}^{-2}\,\{\log_2(n+2)\}^{-\beta}. \tag{4}$$

$$e(w)=\exp\{\log w\,(\log_2 w)^{\beta}\}\ (w\geqslant \exp(e^{-\beta})). \tag{5}$$

$$E(w, t)=\sum_{n\leqslant w} e(n)\,y(n)\cos nt. \tag{6}$$

$$G(w, t)=\int_0^t \frac{u}{(\log_2 k/u)^{\beta}}\,\frac{\partial}{\partial u}\,E(w, u)\,du. \tag{7}$$

$$H(w, t)=\int_t^{\pi} \frac{u}{(\log_2 k/u)^{\beta}}\,\frac{\partial}{\partial u}\,E(w, u)\,du). \tag{8}$$

2. Introduction. Concerning the absolute Riesz summability of Fourier series of the type "exp $\{\log w \log_2 w\}$" and order $1+\delta$ $(\delta>0)$, the following theorem is known (Chandra [3]).

Theorem A. If $\phi_1(t) \log_2 k/t \in BV(0, \pi)$, where $k \geqslant \pi e^2$, then

$$\Sigma A_n(x) \in |R, \exp\{\log w \log_2 w\}, 1+\delta| \ (\delta > 0).$$

In 1971, Chandra [4] investigated the summability factors for the absolute Riesz summability of the type $\exp\{\log w \log_2 w\}$ and order unity by imposing the conditon of theorem A on the generating function of the Fourier series. Precisely, he proved the following theorem:

Theorem B. Let, for $k \geqslant \pi e^2$, $\phi_1(t) \log_2 k/t \in BV(0, \pi)$.

Then $\displaystyle\sum \frac{A_n(x)}{(\log(n+1))^2 \log_2(n+2)} \in |R, \exp\{\log w \log_2 w\}, 1|$.

The object of this paper is to generalise theorem B by proving the following :

Theorem. *Let* $\phi_1(t) (\log_2 k/t)^\beta \in BV(0, \pi)$, *where* $k \geqslant \pi e^{4+\beta}$ *and* $\beta > 0$, *then*

$$\Sigma A_n(x)\, y(x) \in |R, \exp\{\log w \log_2 w)^\beta\}, 1|.$$

For $\beta = 1$, we get theorem B of Chandra [4].

3. We shall use the following order-estimates.

$$E(w, t) = O\left\{\frac{w\, e(w)\, y(w)\, (\log_2 w)^{1-\beta}}{(\beta + \log_2 w)}\right\}. \tag{9}$$

$$E(w, t) = O\{t^{-1} e(w)\, y(w)\}. \tag{10}$$

$$G(w, t) = O\left\{\frac{t\, w\, e(w)\, y(w)\, \log_2 w)^{1-\beta}}{\log_2 k/t)^\beta\, (\beta + \log_2 w)}\right\}. \tag{11}$$

$$H(w, t) = O\{(\log k/t)\, e(w)\, y(w)\}. \tag{12}$$

Proof of (9). Let p denote the integral part of $exp\{exp(-\beta)\}$, for $\beta > 0$, and $e(n) = O(1)$ for $1 \leqslant n \leqslant p$ and let $m \leqslant w < m+1$. Then

$$E(w, t) = \sum_{n=1}^{m} e(n)\, y(n) \cos nt$$

$$= O(1) + \sum_{n=p+1}^{m} e(n)\, y(n) \cos nt$$

$$= O(1) + O\left\{\sum_{n=p+1}^{m} e(n)\, y(n)\right\}.$$

Now,

$$\sum_{n=p+1}^{m} e(n)\, y(n) = \sum_{n=p+1}^{q} e(n)\, y(n) + \sum_{n=q+1}^{m} e(n)\, y(n)$$

$$= O(1) + \sum_{n=q+1}^{m} e(n)\, y(n),$$

where q is so chosen that $q \geqslant exp\, \{e^{\beta+2}\}$ and

$$\frac{s\, y(s)\, (\log_2 s)^{1-\beta}}{(\beta + \log_2 s)}$$

is monotonic increasing ($\uparrow$) with s for $s \geqslant q$. Thus, since $e(s)\, y(s) \uparrow$ with s, for $s \geqslant q$, we have

$$\sum_{n=q+1}^{m} e(n)\, y(n) < \int_{q+1}^{w} e(s)\, y(s)\, ds + e(m)\, y(m)$$

$$\leqslant \int_{q+1}^{w} e^{(1)}(s)\, \frac{s\, y(s)\, (\log_2 s)^{1-\beta}}{(\beta + \log_2 s)}\, ds + e(w)\, y(w)$$

$$\leqslant \frac{w\, y(w)\, (\log_2 w)^{1-\beta}}{(\beta + \log_2 w)} \int_{q}^{w} e^{(1)}(s)\, ds + e(w)\, y(w)$$

$$= O\left\{\frac{w\, e(w)\, y(w)\, (\log_2 w)^{1-\beta}}{(\beta + \log_2 w)}\right\}.$$

Proof of (10). Proceeding as in (9), we write, for $q \geqslant \exp\, \{e^{2+\beta}\}$

$$E(w, t) = O(1) + \sum_{n=q+1}^{m} e(n)\, y(n) \cos nt.$$

By applying Abel's Lemma, we have

$$\sum_{n=q+1}^{m} e(n)\, y(n) \cos nt \leqslant e(m)\, y(m) \max_{1+q \leqslant q' \leqslant m} \left| \sum_{n=q'}^{m} \cos nt \right|$$

$$= O\, \{t^{-1}\, e(w)\, y(w)\},$$

uniformly in $0 < t \leqslant \pi$.

Proof of (11).

$$G(w, t) = \int_{0}^{t} \frac{u}{(\log_2 k/u)^{\beta}} \frac{\partial}{\partial u} E(w, u)\, du.$$

Integrating by parts, we have

$$G(w, t)=\left[\frac{u}{(\log_2 k/u)^{\beta}} E(w, u)\right]_0^t$$

$$-\int_0^t E(w, u)\left[\frac{1}{(\log_2 k/u)^{\beta}}+\frac{\beta(\log k/u)^{-1}}{(\log_2 k/u)^{1+\beta}}\right] du$$

$$=t(\log_2 k/t)^{-\beta} E(w, t)$$

$$+O\left\{\frac{w\, e(w)\, y(w)\, (\log_2 w)^{1-\beta}}{(\beta+\log_2 w)}\int_0^t \frac{1}{(\log_2 k/u)^{\beta}} du\right\}$$

$$+O\left\{\frac{w\, e(w)\, y(w) \log_2 w)^{1-\beta}}{(\beta+\log_2 w)}\int_0^t \frac{\beta(\log_2 k/u)^{-1-\beta}}{\log k/u} du\right\}$$

(by (9))

$$=t\,(\log_2 k/t)^{-\beta} E(w, t)$$

$$+O\left\{\frac{w\, e(w)\, y(w)\, (\log_2 w)^{1-\beta}}{(\beta+\log_2 w)} \frac{t}{(\log_2 k/t)^{\beta}}\right\}$$

$$=O\left\{t(\log_2 k/t)^{-} \frac{w\, e(w)\, y(w)\, (\log_2 w)^{1-\beta}}{(\beta+\log_2 w)}\right\},$$

by (9)

Proof of (12).

$$H(w, t)=\int_t^{\pi} \frac{u}{(\log_2 k/u)^{\beta}} \frac{\partial}{\partial u} E(w, u)\, du.$$

Integrating by parts, we have

$$H(w, t)=\left[\frac{u}{(\log_2 k/u)^{\beta}} E(w, u)\right]_t^{\pi}$$

$$-\int_t^{\pi} E(w, u)\left\{(\log_2 k/u)^{-\beta}+\frac{\beta(\log_2 k/u)^{-1-\beta}}{(\log k/u)}\right\} du$$

$$=O\,\{(\log_2 k/t)^{-\beta} e(w)\, y(w)\}$$

(by (10))

$$+O\left\{\int_t^{\pi} |E(w, u)|\, (\log_2 k/u)^{-\beta}\, du\right\}$$

$$=O\,\{(\log_2 k/t)^{-\beta} e(w)\, y(w)\}$$

$$+O\,\{e(w)\, y(w) \log k/t\}$$ (by (10))

$$=O\{(\log k/t)\, e(w)\, y(w)\}.$$

4. For the proof of the theorem we shall require the following Lemmas.

Lemma 1. (Obrechkoff [1], [2]). *If a series* $\Sigma a_n \varepsilon |R, \lambda, r|$ $(r \geqslant 0)$, *then* $\Sigma a_n \varepsilon |R, \lambda, r'|$ $(r' > r)$.

Lemma 2. *Let, for* $\beta > 0$ *and* $k \geqslant \pi e^{3+\beta}$,

$$\alpha_n = \int_0^{\pi} \frac{\cos nt}{(\log_2 k/t)^{\beta}} dt.$$

Then, for large n,

$$\alpha_n = O\left\{\frac{1}{n(\log n)\ (\log_2 n)^{1+\beta}}\right\}.$$

See Chandra [3], Lemma 2.

Lemma 3. *Let, for* $\beta > 0$ *and* $k \geqslant \pi e^{4+\beta}$,

$$\beta_n = \int_0^{\pi} \frac{\cos nt}{(\log k/t)\ (\log_2 k/t)^{1+\beta}} dt.$$

Then, for large n,

$$\beta_n = O\left\{\frac{1}{n(\log n)^2\ (\log_2 n)^{1+\beta}}\right\}.$$

Proof. Integrating by parts,

$$\beta_n = \left[(\log k/t)^{-1} (\log_2 k/t)^{-1-\beta} \frac{\sin nt}{n} \right]_0^{\pi}$$

$$- \int_0^{\pi} \left\{ \frac{(\log k/t)^{-2}\ (\log_2 k/t)^{-1-\beta}}{t} + \frac{(1+\beta)\ (\log k/t)^{-2}\ (\log_2 k/t)^{-2-\beta}}{t} \right\} \frac{\sin nt}{n}\ dt$$

$$= - \int_0^{\pi} \frac{(\log k/t)^{-2}\ (\log_2 k/t)^{-1-\beta}}{t} \frac{\sin nt}{n}\ dt$$

$$-(1+\beta) \int_0^{\pi} \frac{(\log k/t)^{-2}\ (\log_2 k/t)^{-2-\beta}}{t} \frac{\sin nt}{n}\ dt.$$

$= \beta_{n,1} + \beta_{n,2}$, say.

Now, $$\beta_{n,1}=-\frac{1}{n}\int_0^{\pi}\frac{(\log_2 k/t)^{-1-\beta}}{t\,(\log k/t)^2}\sin nt\,dt$$

$$=-\frac{1}{n}\left(\int_0^{\frac{1}{n}}+\int_{\frac{1}{n}}^{\pi}\right)\frac{(\log_2 k/t)^{-1-\beta}}{t\,(\log k/t)^2}\sin nt\,dt$$

$$=-\frac{1}{n}\,(I_1+I_2),\text{ say.}$$

Since $(\log k/t)^{-2}\,(\log_2 k/t)^{-1-\beta}$ increases in $(0,\ 1/n)$, we have

$$I_1=O\left\{(\log n)^{-2}\,(\log_2 n)^{-1-\beta}\int_0^{\frac{1}{n}}\frac{\sin nt}{t}\,dt\right\}$$

$$=O\left\{(\log n)^{-2}\,(\log_2 n)^{-1-\beta}\right\}.$$

For $k\geqslant\pi e^{4+\beta}$, $t^{-1}\,(\log k/t)^{-2}\,(\log_2 k/t)^{-1-\beta}$ is decreasing in $(1/n,\ \pi)$. Therefore, by using the Second Mean Value theorem, we have

$$I_2=O\left\{n\,(\log n)^{-2}\,(\log_2 n)^{-1-\beta}\left|\int_{1/n}^{\xi}\sin nt\,dt\right|\right\}\qquad(1/n<\xi<\pi)$$

$$=O\left\{\frac{1}{(\log n)^2\,(\log_2 n)^{1+\beta}}\right\}.$$

Thus combining I_1 and I_2, we get

$$\beta_{n,1}=O\left\{\frac{1}{n(\log n)^2\,(\log_2 n)^{1+\beta}}\right\}.$$

Proceeding as above, in case of $\beta_{n,2}$, we shall get

$$\beta_{n,2}=O\left\{\frac{1}{n(\log n)^2\,(\log_2 n)^{2+\beta}}\right\}.$$

Collecting the estimates of $\beta_{n,1}$ and $\beta_{n,2}$ we follow the proof of the lemma.

Lemma 4. *The integral*

$$I=\int_{e^2}^{\infty} \frac{(\beta+\log_2 w)}{w\, e(w)\, (\log_2 w)^{1-\beta}}\, |G(w,\pi)|\, dw <\infty .$$

Proof. $$G(w\ \pi)=\int_0^{\pi} \frac{u}{(\log_2 k/u)^{\beta}}\, \frac{\partial}{\partial u}\, E(w, u)\, du.$$

Integrating by parts, we have

$$G(w, \pi)=\left[\frac{u}{(\log_2 k/u)^{\beta}}\, E(w, u)\right]_0^{\pi}$$

$$-\int_0^{\pi} E(w, u)\, \frac{\partial}{\partial u}\, \{u(\log_2 k/u)^{-\beta}\}\, du$$

$$=\pi\,(\log_2 k/\pi)^{-\beta}\, E(w, \pi)$$

$$-\int_0^{\pi} \sum_{n\leqslant w} e(n)\, y(n)\, \cos nu \frac{\partial}{\partial u}\left(u\left(\log_2 \frac{k}{u}\right)^{-\beta}\right) du$$

$$=O\,\{e(w)\, y(w)\}+\{\sum_{n\leqslant w} e(n)\, y(n)\, |\beta_n'|\}$$

(by (10))

where $$\beta'_n=\int_0^{\pi} \frac{\partial}{\partial u}\left\{u\,(\log_2 k/u)^{-\beta}\right\} \cos nu\, du$$

$$=\{(\log_2 k/u)^{-\beta}+\beta(\log k/u)^{-1}\, (\log_2 k/u)^{-1-\beta}\}\cos nu\, du$$

$$=\alpha_n+\beta_n,\ \text{say}.$$

Now, by Lemma 2 and 3, we have

$$\beta'_n=O\left\{\frac{1}{n\log(n+1)\,(\log_2 (n+2))^{1+\beta}}\right\}$$

for $\beta>0$ and large n.

Therefore, on substituting the estimate for $G(w, \pi)$ in I, we get

$$I=O\left\{\int_{e^2}^{\infty}\frac{(\beta+\log_2 w)}{w\, e(w)\,(\log_2 w)^{1-\beta}}\, e(w)\, y(w)\, dw\right\}$$

$$+O\left\{\int_{e^2}^{\infty}\frac{(\beta+\log_2 w)}{w\, e(w)\,(\log_2 w)^{1-\beta}}\left|\sum_{n\leqslant w} e(n)\, y(n)\, \beta'_n\right| dw\right\}$$

$$=O(1)+O\left\{\int_{e^2}^{\infty}\frac{(\beta+\log_2 w)}{w\, e(w)\,(\log_2 w)^{1-\beta}}\left|\sum_{n\leqslant w} e(n)\, y(n)\, \beta'_n\right| dw\right\}$$

$$=O\,(1),$$

by Lemma 1, since $\Sigma\, \beta'_n \varepsilon\, |R, \lambda, 0|$.

This completes the proof of Lemma 4.

5. Proof of the theorem. We have

$$A_n(x)=\frac{2}{\pi}\int_0^{\pi} \phi(t)\cos nt\, dt.$$

Integrating by parts and using the fact that $\phi_1(\pi)=0$, we have

$$A_n(x)=\frac{2}{\pi}\int_0^{\pi} \phi_r(t)\, nt \sin nt\, dt$$

$$=\frac{2}{\pi}\int_0^{\pi} \phi_1(t)\,(\log_2 k/t)^{\beta}\left(\frac{nt\sin nt}{(\log_2 k/t)^{\beta}}\right) dt$$

$$=-\frac{2}{\pi}\int_0^{\pi} d\,\{\phi_1(t)\,(\log_2 k/t)^{\beta}\}\int_0^{t}\frac{nu\sin nu}{(\log_2 k/u)^{\beta}}\, du,$$

integrating by parts.

The series $\Sigma\, A_n(x)\, y(x)\, \varepsilon\, |R, e(w), 1|$, if

$$I=\frac{2}{\pi}\int^{\infty}\frac{(\beta+\log_2 w)}{{}_{9}w\, e(w)\,(\log_2 w)^{1-\beta}}\left|\sum_{n\leqslant w} e(n)\, y(n).\right.$$

$$\cdot\ \int_0^{\pi} d\,\{\phi_1(t)\,(\log_2 k/t)^{\beta}\}\int_0^{t}\frac{nu\sin nu}{(\log_2 k/u)^{\beta}}\, du\Big|\, dw$$

is convergent.

But, by using (6) and (7), we have

$$I \leqslant \frac{2}{\pi} \int_0^{\pi} | d \{\phi_1(t) (\log_2 k/t)^{\beta} \} |.$$

$$\cdot \int_{e^2}^{\infty} \frac{(\beta+\log_2 w)}{w\, e(w)\, (\log_2 w)^{1-\beta}} |G(w, t)| \, dw.$$

Now, since

$$\int_0^{\pi} | d \{\phi_1 (t) (\log_2 k/t)^{\beta}\} | < \infty,$$

by the hypothesis of the theorem, therefore, it is sufficient for the proof of the theorem to show that

$$J = \int_{e^2}^{\infty} \frac{(\beta+\log_2 w)}{w\, e(w)\, (\log_2 w)^{1-\beta}} |G(w, t)| \, dw$$

$$= O(1),$$

uniformly in $0 < t < \pi$.

On taking $e^2 < \tau = \frac{k}{t} - (\log_2 k/t)^{\beta}$, we write

$$J = \int_{e^2}^{\tau} + \int_{\tau}^{\infty} = J_1 + J_2, \text{ say,}$$

so that, by (11), we have

$$J_1 = O \left\{ t (\log_2 k/t)^{-\beta} \int_{e^2}^{\tau} \gamma(w) \, dw \right\}$$

$$= O(1).$$

And by using the fact

$$G(w, t) = G(w, \pi) - H(w, t),$$

we have

$$J_2 \leqslant \int_{\tau}^{\infty} \frac{(\beta+\log_2 w)}{w\, e(w)\, (\log_2 w)^{1-\beta}} \,|G(w,\pi)|\, dw$$

$$+\int_{\tau}^{\infty} \frac{(\beta+\log_2 w)}{w\, e(w)\, (\log_2 w)^{1-\beta}} \,|H(w,t)|\, dw$$

$$< \int_{e^2}^{\infty} \frac{(\beta+\log_2 w)}{w\, e(w)\, (\log_2 w)^{1-\beta}} \,|G(w,\pi)|\, dw$$

$$+\int_{\tau}^{\infty} \frac{(\beta+\log_2 w)}{w\, e(w)\, (\log_2 w)^{1-\beta}} \,|H(w,t)|\, dw$$

$$=J_{2,1}+J_{2,2}, \text{ say.}$$

Now, by Lemma 4

$$J_{2,1}=O\,(1),$$

and, by (12), we have

$$J_{2,2}=O\left\{(\log k/t) \int_{\tau}^{\infty} \frac{(\beta+\log_2 w)}{w\, e(w)\, (\log_2 w)^{1-\beta}}\, e(w)\, y(w)\, dw\right\}$$

$$=O\left\{(\log k/t) \int_{\tau}^{\infty} \frac{(\beta+\log_2 w)}{w\, (\log_2 w)^{1-\beta}}\, \frac{1}{(\log w)^2\, (\log_2 w)^{\beta}}\, dw\right\}$$

$$=O\left\{(\log k/t) \int_{\tau}^{\infty} \frac{(1+\beta\,(\log_2 w)^{-1})}{w\,(\log w)^2}\, dw\right\}$$

$$=O\,(1),$$

uniformly in $0<t<\pi$.

This terminates the proof of the theorem.

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