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SOME BILINEAR AND BILATERAL GENERATING RELATIONS INVOLVING HYPERGEOMETRIC FUNCTIONS OF THREE VARIABLES

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Abstract. The aim of this paper is to obtain two bilinear and one bilateral generating relations involving certain hypergeometric functions in three arguments. A few specializations, relevant to the present discussion, are also derived.

1. Introduction. If we use the notation

$$(a, n)=a(a+1)\,(a+2)\ldots(a+n-1);\ (a, 0)=1,$$

where a is arbitrary and n a positive integer, then the hypergeometric functions of three variables, which we require here, are defined as follows [8, p. 105], [1, p. 43] and [7, p. 356].

$$G_C(\alpha, \beta, \beta_1 ; \gamma ; x, y, z)$$

$$=\sum_{m,n,p=0}^{\infty} \frac{(\alpha, m+p)\,(\beta, m+n)\,(\beta_1, n-p)}{(1, m)\,(1, n)\,(1, p)\,(\gamma, m+n-p)}\, x^m y^n z^p, \tag{1·1}$$

$$G_D(\alpha, \alpha_1, \beta_1, \beta_2, \beta_3 ; \gamma ; x, y, z)$$

$$=\sum_{m,n,p=0}^{\infty} \frac{(\alpha, p-m)\,(\alpha_1, n)\,(\beta_1, m)\,(\beta_2, n)\,(\beta_3, p)}{(1, m)\,(1, n)\,(1, p)\,(\gamma, n+p-m)}\, x^m y^n z^p, \tag{1·2}$$

$$F_G(\alpha, \alpha, \alpha, \beta_1, \beta_2, \beta_3 ; \gamma_1, \gamma_2, \gamma_2 ; x, y, z)$$

$$= \sum_{m, n, p=0}^{\infty} \frac{(\alpha, m+n+p)\,(\beta_1, m)\,(\beta_2, n)\,(\beta_3, p)}{(1, m)\,(1, n)\,(1, p)\,(\gamma_1, m)\,(\gamma_2, n+p)}\, x^m\, y^n\, z^p \,; \qquad (1\cdot3)$$

the convergence conditions for the triple series in (1·1) and (1·2) are $|x|<1$, $|y|<1$, $|z|<1$ and in (1·3) $|x|<r$, $|y|<s$ and $|z|<t$ such that $r+s=1=r+t$.

Also, $$G_2(\alpha, \alpha_1, \beta, \beta_1 ; x, y)$$

$$= \sum_{m, n=0}^{\infty} \frac{(\alpha, m)\,(\alpha_1, n)\,(\beta, n-m)\,(\beta_1, m-n)}{(1, m)\,(1, n)}\, x^m\, y^n \,; \qquad (1\cdot4)$$

$$P_n^{(\alpha, \beta)}(x) = \frac{(1+\alpha, n)}{(1, n)}\; {}_2F_1(-n, 1+\alpha+\beta+n \,;\, 1+\alpha \,;\, (1-x)/2), \qquad (1\cdot5)$$

where G_2 is one of Horn's hypergeometric functions of two variables [2, p. 224] and $P_n^{(\alpha, \beta)}(x)$ are the familiar Jacobi polynomials [4, p. 254].

In the present investigation we require the following relations :

$$F_1(\alpha, \beta, \beta_1 ; \gamma ; x, x) = {}_2F_1(\alpha, \beta+\beta_1 ; \gamma ; x) \,; \qquad (1\cdot6)$$

$$F_2(\gamma+\gamma_1-1, \beta, \beta_1 ; \gamma, \gamma_1 ; x, y)$$

$$= (1-x)^{-\beta} (1-y)^{-\beta_1}\, G_2\left(\beta, \beta_1, 1-\gamma, 1-\gamma_1 ; \frac{x}{1-x}, \frac{y}{1-y}\right); \qquad (1\cdot7)$$

$$G_D\left(\alpha, \alpha_1, \beta_1, \beta_2, \beta_3 ; \alpha+\alpha_1 ; x, y, \frac{y}{y-1}\right)$$

$$= (1-y)^{-\beta_2}\, G_2\left(\beta_1, \beta_2+\beta_3, \alpha, 1-\alpha-\alpha_1 ; -x, \frac{y}{1-y}\right) ; \qquad (1\cdot8)$$

$$\sum_{n=0}^{\infty} (u-1)^n\, P_n^{(\alpha-n, \beta+n)}\left(\frac{1+u}{1-u}\right) {}_2F_1(-n, \delta-\gamma ; \delta ; t)$$

$$= \frac{\Gamma(\delta)\, \Gamma(\delta+\alpha-\gamma)}{\Gamma(\delta+\alpha)\, \Gamma(\delta-\gamma)}\, t^{\alpha} (1-u+ut)^{-\beta}\, {}_2F_1\left(\beta, \gamma ; \delta+\alpha ; -\frac{ut}{1-u+ut}\right); \qquad (1\cdot9)$$

$$\sum_{n=0}^{\infty} \binom{m+n}{n} P_{m+n}^{(\alpha-m-n,\ \beta-m-n)}(x)\ t^n$$

$$= \left(1+\frac{x+1}{2}\ t\right)^{\alpha-m} \left(1+\frac{x-1}{2}\ t\right)^{\beta-m} P_m^{(\alpha-m,\ \beta-m)}\left(x+\frac{x^2-1}{2}\ t\right); \quad (1{\cdot}10)$$

$$\sum_{n=0}^{\infty} \frac{(\lambda,\ n)}{(\delta,\ n)} P_n^{(\alpha-n,\ \beta-n)}(x)\ t^n$$

$$= F_1\left(\lambda,\ -\alpha,\ -\beta\ ;\ \delta\ ;\ -\frac{x+1}{2}\ t,\ \frac{1-x}{2}\ t\right); \quad (1{\cdot}11)$$

where F_1, F_2 are Appell's double hypergeometric functions [2, p. 224, (6) and (7)] and (1·6) is a known identity [2, p. 239, (11)]. Formula (1·7) is due to Erdélyi [3, p. 381], (1·8) is a transformation given by Dhawan [1, p. 46, (3·3)], equation (1·9) is a result discussed by Srivastava and Joshi [6, p. 28, (6·7)], (1·10) is a known relation given by Singhal and Srivastava [5, p. 759, (23)], and (1·11) is a generating function obtained by Srivastava [9, p. 65, 22)].

2. The generating relations to be derived here are

$$(1-z)^{1-\gamma}\, G_C\left(1-\beta_1,\ \beta,\ \beta_1 : \gamma\ ;\ -\frac{xy}{(1-x+xy)\,(1-z)},\ -\frac{xy}{(1-x+xy)\,(1-z)},\ z\right)$$

$$= (1-x+xy)^{\beta} \sum_{n=0}^{\infty} \frac{(1,\ n)}{(\gamma,\ n)} (x-1)^n P_n^{(-n,\ \beta+n)}\left(\frac{1+x}{1-x}\right) P_n^{(\gamma-1,\ -1-n)}(1-2y)\ ; \quad (2{\cdot}1)$$

$$\left(1+\frac{xy(1-y)}{1-x+xy}\right)^{-(\alpha+\alpha_1)}$$

$$G_D\left(\alpha,\ \alpha_1,\ \alpha+\alpha_1,\ \alpha,\ \alpha_1\ ;\ \alpha+\alpha_1\ ;\ \frac{xy(1-y)}{1-x+xy(2-y)},\ y,\ \frac{y}{y-1}\right)$$

$$=\frac{\Gamma(1-\alpha)\,\Gamma(-2\alpha-\alpha_1)}{\Gamma(-\alpha)\,\Gamma(1-2\alpha-\alpha_1)}\, y^{-1}\,(1-x+xy)^{\alpha_1}$$

$$\sum_{n=0}^{\infty}\frac{(1,n)}{(-\alpha,n)}(x-1)^n\, P_n^{(1-n,\,\alpha_1+n)}\left(\frac{1+x}{1-x}\right) P_n^{(-\alpha-1,\,-\alpha-\alpha_1-n)}(1-2y) \tag{2·2}$$

and
$$\left(1+\frac{x+1}{2}t\right)^{\alpha}\left(1+\frac{x-1}{2}t\right)^{\beta}\times$$

$$F_G\left(\lambda,\lambda,\lambda,\gamma,-\beta,-\alpha\,;\,\lambda-\delta+1,\delta,\delta\,;\,z,\frac{yt(x-1)}{2-(1-x)t},\frac{yt(1+x)}{2+(1+x)\,t}\right)$$

$$=(1-z)^{-\gamma}\sum_{n=0}^{\infty}((1-y)\,t)^n\, P_n^{(\alpha-n,\,\beta-n)}(x)$$

$$G_2\left(-n,\gamma,1-\delta,\delta-\lambda\,;\,\frac{y}{1-y},\frac{z}{1-z}\right). \tag{2·3}$$

Derivation of (2·1). Consider

$$T=(1-z)^{1-\gamma}\,G_C\left(1-\beta_1,\beta,\beta_1\,;\,\gamma\,;\,-\frac{xy}{(1-x+xy)(1-z)},\right.$$
$$\left.-\frac{xy}{(1-x+xy)(1-z)},z\right),$$

express G_C in series form as given by (1·1), which on using the formula [2, p. 224, (6)], yields

$$T=(1-z)^{1-\gamma}\sum_{p=0}^{\infty}\frac{(1-\beta_1,p)\,(\beta_1,-p)}{(1,p)\,(\gamma-p)}\,z^p$$

$$F_1\left(\beta,1-\beta_1+p,\beta_1-p\,;\,\gamma-p\,;\,-\frac{xy}{(1-x+xy)(1-z)},\right.$$
$$\left.-\frac{xy}{(1-x+xy)(1-z)}\right.$$

Again from the known identity (1·6),

$$T=(1-z)^{1-\gamma}\sum_{p=0}^{\infty}\frac{(1-\beta_1,p)(\beta_1-p)}{(1,p)(\gamma,-p)}z^p$$

$$ {}_2F_1\left(\beta,1;\gamma-p;-\frac{xy}{(1-x+xy)(1-z)}\right)$$

$$=(1-z)^{1-\gamma}\sum_{p,q=0}^{\infty}\frac{(\beta,q)(1,q)(1-\gamma,p-q)}{(1,p)(1,q)}z^p$$

$$\left(\frac{xy}{(1-x+xy)(1-z)}\right)^q$$

$$=(1-z)^{1-\gamma}\sum_{q=0}^{\infty}\left[\frac{(\beta,q)(1,q)}{(1,q)(\gamma,q)}\left(-\frac{xy}{(1-x+xy)(1-z)}\right)^q\right.$$

$$\left.\sum_{p=0}^{\infty}\frac{(1-\gamma-q,p)}{(1,p)}z^p\right]$$

$$={}_2F_1\left(\beta,1;\gamma;-\frac{xy}{1-x+xy}\right)$$

which in the light of (1·9) results in (2·1).

Derivation of 2·2. If we let

$$\Delta=\left(1+\frac{xy(1-y)}{1-x+xy}\right)^{-(\alpha+\alpha_1)}$$

$$G_D\left(\alpha,\alpha_1,\alpha+\alpha_1,\alpha,\alpha_1;\alpha+\alpha_1;\frac{xy(1-y)}{1-x+xy(2-y)},y,\frac{y}{y-1}\right),$$

and apply the relation (1·8), we get

$$\Delta=(1-y)^{-\alpha}\left(1+\frac{xy(1-y)}{1-x+xy}\right)^{-(\alpha+\alpha_1)}$$

$$G_2\left(\alpha+\alpha_1,\alpha+\alpha_1,\alpha,1-\alpha-\alpha_1;-\frac{xy(1-y)}{1-x+xy(2-y)},\frac{y}{1-y}\right).$$

Use the transformation (1·7) and simplify ; (2·4) provides

$$\Delta={}_2F_1\left(\alpha_1,\alpha+\alpha_1;1-\alpha;-\frac{xy}{1-x+xy}\right),$$

Finally, in view of (1·9), (2·2) is obtained

Derivation of 2·3. If we put

$$\Omega=\left(1+\frac{x+1}{2}\,t\right)^{\alpha}\left(1+\frac{x-1}{2}\,t\right)^{\beta}$$

$$F_G\left(\lambda,\lambda,\lambda,\gamma,-\beta,-\alpha;\lambda-\delta+1,\delta,\delta;z,\frac{yt(x-1)}{2-(1-x)t},\frac{yt(1+x)}{2+(1+x)t}\right)$$

$$=\left(1+\frac{x+1}{2}\,t\right)^{\alpha}\left(1+\frac{x-1}{2}\,t\right)^{\beta}\sum_{k=0}^{\infty}\frac{(\lambda,k)(\gamma,k)}{(1,k)(\lambda-\delta+1,k)}\,z^{k}$$

$$\times F_1\left(\lambda+k,-\alpha,-\beta;\delta;\frac{yt(1+x)}{2+(1+x)t},\frac{yt(x-1)}{2-(1-x)t}\right),$$

and express F_1 in terms of Jacobi polynomials with the help of (1·11), we find

$$\Omega=\left(1+\frac{x+1}{2}\,t\right)^{\alpha}\left(1+\frac{x-1}{2}\,t\right)^{\beta}\sum_{k=0}^{\infty}\frac{(\lambda,k)(\gamma,k)}{(1,k)(\lambda-\delta+1,k)}\,z^{k}$$

$$\times\sum_{r=0}^{\infty}\frac{(\lambda+k,r)}{(\delta,r)}\left(-\frac{yt}{\left(1+\frac{x+1}{2}\,t\right)\left(1+\frac{x-1}{2}\,t\right)}\right)^{r}$$

$$P_{r}^{(\alpha-r,\,\beta-r)}\left(x+\frac{x^2-1}{2}\,t\right)$$

which, in the presence of (1·10), provides

$$\Omega=\sum_{r,k=0}^{\infty}\frac{(\lambda,k+r)(\gamma,k)}{(1,k)(\delta,r)(\lambda-\delta+1,k)}(-yt)^{r}\,z^{k}$$

$$\sum_{n=0}^{\infty}\binom{n+r}{n}P_{n+r}^{(\alpha-r-n,\,\beta-r-n)}(x)\,t^{n}$$

$$=\sum_{n=0}^{\infty}P_{n}^{(\alpha-n,\,\beta-n)}(x)\,F_2(\lambda,-n,\gamma;\delta,\lambda-\delta+1;y,z)\,t^{n},$$

from which follows (2·3) after using the relation (1·7).

3. Particular Cases. In this section we discuss a few interesting specialized forms of the results obtained in section 2.

(i) In (2·1), putting $z=0$ and simplifying in view of (1·6), we get

$$
{}_2F_1\left(\beta,\ 1\ ;\ \gamma;\ -\frac{xy}{1-x+xy}\right)
$$

$$
=(1-x+xy)^{\beta}\sum_{n=0}^{\infty}\frac{(1,\ n)}{(\gamma,\ n)}\,(x-1)^n\ P_n^{(-n,\ \beta+n)}\left(\frac{1+x}{1-x}\right)
$$

$$
P_n^{(\gamma-1,\ -1-n)}(1-2y) \qquad (3\cdot 1)
$$

(ii) If we choose $x=0$, (2·2) provides

$$
F_3\left(\alpha,\ \alpha_1,\ \alpha_1,\ \alpha\ ;\ \alpha+\alpha_1\ ;\ \frac{y}{y-1},\ y\right)
$$

$$
=\frac{\Gamma(1-\alpha)\ \Gamma(-2\alpha-\alpha_1)}{\Gamma(-\alpha)\ \Gamma(1-2\alpha-\alpha_1)}\ y^{-1}
$$

$$
\sum_{n=0}^{\infty}\frac{(-1)^n\ (2-n,\ n)}{(-\alpha,\ n)}\ {}_2F_1\left(-n,\ 2+\alpha_1+n\ ;\ 2-n\ ;\ \tfrac{1}{2}\right)
$$

$$
P_n^{(-1-\alpha,\ -\alpha-\alpha_1-n)}(1-2y), \qquad (3\cdot 2)
$$

where F_3 is one of Appell's double hypergeometric functions [2, p. 224, (8)].

(iii) On setting $z=0$ and solving, (2·3) yields

$$
\left(1+\frac{x+1}{2}\ t\right)^{\alpha}\left(1+\frac{x-1}{2}\ t\right)^{\beta}
$$

$$
F_1\left(\lambda,\ -\beta,\ -\alpha\ ;\ \delta\ ;\ \frac{yt(x-1)}{2-(1-x)t},\ \frac{yt(1+x)}{2+(1+x)t}\right)
$$

$$
=\sum_{n=0}^{\infty}\frac{(1,\ n)}{(\delta,\ n)}\left((1-y)\ t\right)^n P_n^{(\alpha-n,\ \beta-n)}(x)
$$

$$
P_n^{(\delta-1,\ -\lambda-n)}\left(\frac{1+y}{1-y}\right). \qquad (3\cdot 3)
$$

Further on taking $\alpha=0$, (3·3) gives

$$\left(1+\frac{x-1}{2}\,t\right)^{\beta}\;{}_2F_1\left(-\beta,\lambda\,;\,\delta\,;\,\frac{yt(x-1)}{2-(1-x)t}\right)$$

$$=\sum_{n=0}^{\infty}\frac{(1,n)}{(\delta,n)}\left((1-y)\;t\right)^n\;P_n^{(-n,\,\beta-n)}(x)\;P_n^{(\delta-1,\,-\lambda-n)}\left(\frac{1+y}{1-y}\right). \qquad (3·4)$$

We conclude this paper with the remark that a number of results similar to those given in section 3 can be derived from results of section 2.

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