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ON SOME FINITE SUMMATIONS. II

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1. The object of this paper is to obtain some finite summation formulas involving the G-function of one and two variables. The method applied uses the known result [1, Eq. (6·2)]

$$(d-a)\ {}_2F_1\left[\begin{matrix} b,\ a-1 \\ d \end{matrix}\right]_{n+1} -(d-a-b)\ {}_2F_1\left[\begin{matrix} b,\ a \\ d \end{matrix}\right]_{n+1} = \frac{b+n}{n\,!}\,\frac{(a)_n\,(b)_n}{(d)_n} \tag{1·1}$$

where the subscript $n+1$ on the left indicates that only the first $n+1$ terms of the F series are to be included in the expansion.

2. We establish the following results :

$$\sum_{r=0}^{n} \frac{1}{r!}\,\frac{\Gamma(b+r)}{\Gamma(d+r)}\left[G^{l+1,\,u+1}_{p+2,\,q+2}\left(x \,\middle|\, \begin{matrix} 2-a-r,\ a_p,\ d-a \\ d-a+1,\ b_q,\ 2-a \end{matrix}\right) - G^{l+1,\,u+1}_{p+2,\,q+2}\left(x \,\middle|\, \begin{matrix} 1-a-r,\ a_p,\ d-a-b \\ d-a-b+1,\ b_q,\ 1-a \end{matrix}\right)\right]$$

$$=\frac{1}{n!}\,\frac{\Gamma(b+n+1)}{\Gamma(d+n)}\,G^{l,\,u+1}_{p+1,\,q+1}\left(x \,\middle|\, \begin{matrix} 1-a-n,\ a_p \\ b_q,\ 1-a \end{matrix}\right), \tag{2·1}$$

$$\sum_{r=0}^{n} \frac{1}{r!\,(d)_r}\left[G^{l+1,\, u+2}_{p+3,\, q+2}\left(x \,\middle|\, \begin{matrix} 1-b-r,\, 2-a-r,\, a_p,\, d-a \\ d-a+1,\, b_q,\, 2-a \end{matrix}\right)\right.$$

$$\left. -2G^{l+1,\, u+2}_{p+3,\, q+2}\left(x \,\middle|\, \begin{matrix} 1-b-r,\, 1-a-r,\, a_p\, \frac{d-a-b}{2} \\ \frac{d-a-b}{2}+1,\, b_q,\, 1-a \end{matrix}\right)\right]$$

$$= \frac{1}{n!\,(d)_n}\, G^{l,\, u+2}_{p+2,\, q+1}\left(x \,\middle|\, \begin{matrix} -b-n,\, 1-a-n,\, a_p \\ b_q,\, 1-a \end{matrix}\right), \qquad (2\cdot2)$$

$$\sum_{r=0}^{n} \frac{1}{r!\,(d)_r}\left[G^{l+2,\, u+1}_{p+3,\, q+3}\left(x \,\middle|\, \begin{matrix} 2-a-r,\, a_p,\, d-a,\, b \\ d-a+1,\, b+r,\, b_q,\, 2-a \end{matrix}\right)\right.$$

$$\left. -(d-a-b)\, G^{l+1,\, u+1}_{p+2,\, q+2}\left(x \,\middle|\, \begin{matrix} 1-a-r,\, a_p,\, b \\ b+r,\, b_q,\, 1-a \end{matrix}\right)\right]$$

$$= \frac{1}{n!\,(d)_n}\, G^{l+1,\, u+1}_{p+2,\, q+2}\left(x \,\middle|\, \begin{matrix} 1-a-n,\, a_p,\, b \\ b+n+1,\, b_q,\, 1-a \end{matrix}\right) \qquad (2\cdot3)$$

and

$$\sum_{r=0}^{n} \frac{\Gamma(b+r)}{r!}\left[(a-1)_r\, G^{l,\, u+2}_{p+2,\, q+2}\left(x \,\middle|\, \begin{matrix} a-d,\, 1-d,\, a_p \\ b_q,\, 1+a-d,\, 1-d-r \end{matrix}\right)\right.$$

$$\left. -(a)_r\, G^{l,\, u+2}_{p+2,\, q+2}\left(x \,\middle|\, \begin{matrix} a+b-d,\, 1-d,\, a_p \\ b_q,\, 1+a+b-d,\, 1-d-r \end{matrix}\right)\right]$$

$$= \frac{\Gamma(b+n+1)}{n!}\, (a)_n\, G^{l,\, u+1}_{p+1,\, q+1}\left(x \,\middle|\, \begin{matrix} 1-d,\, a_p \\ b_q,\, 1-d-n \end{matrix}\right). \qquad (2\cdot4)$$

Similar other results can be obtained from (1·1) which we are not giving here.

For the difinition of the G-function refer to [2, p. 207].

We shall denote it, for the sake of brevity, by

$$G^{l,\,u}_{p,\,q}\left(x\,\middle|\,\begin{matrix} a_p \\ b_q \end{matrix}\right) = \frac{1}{2\pi i}\int_L f(s)\, x^s\, ds, \tag{2·5}$$

where $$(fs) = \frac{\prod_{j=1}^{l}\Gamma(b_j - s)\prod_{j=1}^{u}\Gamma(1-a_j+s)}{\prod_{j=l+1}^{q}\Gamma(1-b_j+s)\prod_{j=u+1}^{p}\Gamma(a_j-s)}.$$

Proof. Considering (2·1) and employing (2·5) on its left side, we get

$$\sum_{r=0}^{n}\frac{1}{r!}\frac{1}{2\pi i}\int_L f(s)\left\{\frac{\Gamma(b+r)}{\Gamma(d+r)}\left[\frac{\Gamma(a-1+s+r)\,\Gamma(d-a-s+1)}{\Gamma(a-1+s)\,\Gamma(d-a-s)}\right.\right.$$

$$\left.\left.-\frac{\Gamma(a+s+r)\,\Gamma(d-a-b-s+1)}{\Gamma(a+s)\,\Gamma(d-a-b-s)}\right]\right\}x^s\,ds.$$

Changing the order of summation and integration, we get

$$\frac{1}{2\pi i}\int_L f(s)\sum_{r=0}^{n}\frac{1}{r!}\frac{\Gamma(b+r)}{\Gamma(d+r)}\,[(a-1+s)_r\,(d-a-s)$$

$$-(a+s)_r\,(d-a-b-s)]\,x^s\,ds.$$

Now using (1·1) we get the right-hand side of (2·1), which proves the result. Similarly, the other results can be proved.

3. Using the identity (1·1) we can also obtain a number of interesting results for the G-function of two variables defined [3, p. 537, and with slight variation, 4, p. 471] as follows :

$$G^{p,\,q,\,k,\,s,\,l}_{A,\,[C,\,E],\,B,\,[D,\,F]}\left[\begin{matrix} x \\ y \end{matrix}\,\middle|\,\begin{matrix} (a):(c);\,(e) \\ (b):(d);\,(f) \end{matrix}\right] \equiv G\left[\begin{matrix} x \\ y \end{matrix}\right]$$

$$= \frac{1}{(2\pi i)^2}\int_{-i\infty}^{i}\int_{-i\infty}^{i\infty}\Phi(\xi+\eta)\,\Psi(\xi,\eta)\,x^{\xi}\,y^{\eta}\,d\xi\,d\eta, \tag{3·1}$$

where $$\Phi(\xi+\eta)=\frac{\prod_{j=1}^{p}\Gamma[1-a_j+\xi+\eta]}{\prod_{j=p+1}^{A}\Gamma[a_j-\xi-\eta]\prod_{j=1}^{B}\Gamma[b_j+\xi+\eta]},$$

$$\Psi(\xi,\eta)$$

$$=\frac{\prod_{j=1}^{q}\Gamma[c_j+\xi]\prod_{j=1}^{s}\Gamma[d_j-\xi]\prod_{j=1}^{k}\Gamma[e_j+\eta]\prod_{j=1}^{l}\Gamma[f_j-\eta]}{\prod_{j=q+1}^{C}\Gamma[1-c_j-\xi]\prod_{j=s+1}^{D}\Gamma[1-d_j+\xi]\prod_{j=k+1}^{E}\Gamma[1-e_j-\eta]\prod_{j=l+1}^{F}\Gamma[1-f_j+\eta]}$$

with appropriate conditions for convergence.

Only two of the results are mentioned in this section, which are

$$\sum_{r=0}^{n}\frac{1}{r!}\Bigg\{G^{p+1,\,q+1,\,k,\,s,\,l+1}_{A+1,\,[C+1,\,E+1],\,B+1,\,[D+1,\,F+1]}\left[\begin{matrix}x\\y\end{matrix}\middle|\begin{matrix}a-d,\,(a):\,b+r, & (c) & ; & (e), & 2-a\\(b),\,d-a: & (d),\,1-d-r; & a-1+r, & (f)\end{matrix}\right]$$

$$-G^{p,\,q+1,\,k+1,\,s,\,l+1}_{A,\,[C+1,\,E+2],\,B,\,[D+1,\,F+2]}\left[\begin{matrix}x\\y\end{matrix}\middle|\begin{matrix}(a):b+r, & (c) & ; & 1+d-a-b, & (e), & 1-a\\(b):(d), & 1-d-r & ; & a+r, & (f), & 1-d+a+b\end{matrix}\right]\Bigg\}$$

$$=\frac{1}{n!}\;G^{p,\,q+1,\,k,\,s,\,l+1}_{A,\,[C+1,\,E+1],\,B,\,[D+1,\,F+1]}\left[\begin{matrix}x\\y\end{matrix}\middle|\begin{matrix}(a):1+b+n,\,(c);\,(e),\,1-a\\(b):(d),\,1-d-n;\,a+n,\,(f)\end{matrix}\right]\tag{3·2}$$

and

$$\sum_{r=0}^{n}\frac{\Gamma(b+r)}{r!}\Bigg\{G^{p+1,\,q,\,k,\,s,\,l+1}_{A+1,\,[C,\,E+1],\,B+1,\,[D+1,\,F+1]}\left[\begin{matrix}x\\y\end{matrix}\middle|\begin{matrix}a-d,\,(a):(c) & ;\,(e),\,2-a\\(b),\,d-a:(d),\,1-d-r & ;\,a-1+r,\,(f)\end{matrix}\right]$$

$$-G^{p+1,\, q,\, k,\, s,\, l+1}_{A+1,\, [C,\, E+1],\, B+1,\, [D+1,\, F+1]}\left[\begin{matrix} x \\ y \end{matrix}\middle|\begin{matrix} a-d-b,\, (a) : \quad (c) \quad ;\, (e),\, 1-a \\ (b),\, d+b-a : (d),\, 1-d-r;\, a+r,\, (f) \end{matrix}\right]\Bigg\}$$

$$=\frac{\Gamma(b+n+1)}{n!}\, G^{p,\, q,\, k,\, s,\, l+1}_{A,\, [C,\, E+1],\, B,\, [D+1,\, F+1]}\left[\begin{matrix} x \\ y \end{matrix}\middle|\begin{matrix} (a) : \quad (c) \quad ;\, (e),\, 1-a \\ (b) : (d),\, 1-d-n\,;\, a+n,\, (f) \end{matrix}\right] \tag{3·3}$$

To prove (3·2), apply (3·1) on its left side and we get

$$\sum_{r=0}^{n} \frac{1}{r!}\frac{1}{(2\pi i)^2}\int_{-i\infty}^{i\infty}\int_{-i\infty}^{i\infty} \Phi(\xi+\eta)\,\Psi(\xi,\eta)$$

$$\left[\frac{\Gamma(d-a+1+\xi+\eta)\,\Gamma(b+r+\xi)\,\Gamma(a-1+r-\eta)}{\Gamma(d-a+\xi+\eta)\,\Gamma(d+r+\xi)\,\Gamma(a-1-\eta)}\right.$$

$$\left.-\frac{\Gamma(d-a-b+1+\eta)\,\Gamma(b+r+\xi)\,\Gamma(a+r-\eta)}{\Gamma(d-a-b+\eta)\,\Gamma(d+r+\xi)\,\Gamma(a-\eta)}\right] x^{\xi}\, y^{\eta}\, d\xi\, d\eta.$$

On changing the order of summation and integration, we get

$$\frac{1}{(2\pi i)^2}\int_{-i\infty}^{i\infty}\int_{-i\infty}^{i\infty} \Phi(\xi+\eta)\,\Psi(\xi,\eta)$$

$$\sum_{r=0}^{n}\left[\frac{(d-a+\xi+\eta)\,(a-1-\eta)_r\,(b+\xi)_r}{(d+\xi)_r}\right.$$

$$\left.-\frac{(d-a-b+\eta)\,(a-\eta)_r\,(b+\xi)_r}{(d+\xi)_r}\right]\frac{\Gamma(b+\xi)}{\Gamma(d+\xi)}\, x^{\xi}\, y^{\eta}\, d\xi\, d\eta.$$

Now using (1·1) we get the right-hand side of (3·2), which proves the result. Similarly we can prove (3·3).

On specializing the parameters of the functions involved in the summations given in the preceding sections, one can obtain many results as their special cases.

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