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THE LAPLACE DIFFERENCE EQUATION OF HIGHER ORDER AND LAURICELLA'S FUNCTION $F_D^{(n)}$

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1. Introduction. If the substitution

$$u(x)=\int_C t^{x-1}\, V(t)\, dt, \tag{1}$$

where the contour of integration is to be determined, is made in the homogeneous Laplace difference equation of order n

$$\sum_{i=0}^{n} (a_i x+b_i)\, u(x+i)=O, \tag{2}$$

where, for the purposes of this note, the constants $\{a_i\}$ and $\{b_i\}$ are all taken to be non-zero, it has been shown ([6], p. 479) that the function $v(t)$ is determined by a linear ordinary differential equation of the first order.

Milne-Thompson ([6], sect. 15·5) considers (2) when $n=2$ in some detail. This treatment is extended here to cover some aspects of the case when n is unrestricted. For convenience, Milne-Thompson's notation is largely retained.

One family of solutions of (2) of the form $F_D^{(n)}$ will be specially discussed. This type of function has recently attracted some interest in connection with various applied problems ; cf., for example, [5].

2. Determination of v(t) and the contour of integration C. Write (2) in the form

$$\sum_{i=0}^{n} [\{A_i (x+i)+B_i\} u(x+i)=O, \tag{3}$$

when Laplace's transformation (1) gives the following differential equation for $v(t)$:

$$\sum_{i=0}^{n} A_i t^{i+1} V'(t)-\sum_{i=0}^{n} B_i t^i V(t)=O ; \tag{4}$$

it is assumed that the roots of the equation

$$\sum_{i=0}^{n} A_i t^i=O \tag{5}$$

are distinct, so that, we may write

$$\frac{V'(t)}{V(t)}=-\frac{\alpha}{t}+\sum_{i=1}^{n}\frac{B_i}{t-a_i} ; A_0\alpha+B_0=O.$$

Hence, $$V^*(t)=t^{-\alpha}\prod_{i=1}^{n}(t-a_i)^{\beta_i}.$$

In order that $u(x)$, as given by (1), is a solution of (2), the contour of integration C, must be such that

$$\left[t^{x-\alpha}\prod_{i=1}^{n}(t-a_i)^{\beta_i+1}\right]_C \tag{6}$$

returns to its initial value after t has described the contour.

For general purposes, the most convenient form of C which satisfies the condition (6) is a Pochhammer double loop ([7], p. 256) with at least one singularity of the integrand in each loop, the remaining singularities being outside the contour.

3. Solutions of (2) in the form $F_D^{(n-1)}$. Let P, Q, R denote the number of singularities of the integrand of (1) inside one loop of the contour, C, inside the other loop and outside the contour, respectively, and let $c_1,\ldots, c_p, c_{p+1}, \ldots, c_q, c_{q+1}, \ldots, c_{n+1}$ denote any permutation of the singularities of the integrand of

$$u(x)=\int_C t^{x-n-1} \prod_{i=1}^{n} (t-a_i)^{\beta_i}\, dt, \tag{7}$$

cf. [1].

In what follows, the symbol [0 ; 1] denotes a Pochhammer double loop with the origin inside one loop and the point 1 inside the other loop. The simplest form of C occurs when two of the numbers P, Q, R are both equal to unity. Let $P=Q=1$, when by means of a simple bi-linear transformation of the variable of integration may be selected to take the singularity inside one loop of C to the origin, and the singularity inside the other loop to 1.

The corresponding solution may be written as

$$R\int_{[0\,;\,1]} S^{-\xi}(S-1)^{-\beta_1'} \prod_{i=2}^{n} (S-a_i')^{-\beta_i'}\, ds, \tag{8}$$

where R consists of a product of such terms as

$$\prod_{i=1}^{n} a_i^{\rho_i}\,,\ \prod_{\substack{i=1\\ i\neq j}}^{n} (a_i-a_j)^{\sigma_i}$$

and where $\beta_1', \ldots, \beta_n', \rho_1, \ldots, \rho_n, \xi$ etc. depend upon the parameters $\beta_1, \ldots, \beta_n$ and the variable x, and the quantities $a_2', \ldots, a_n'$ are rational functions of $a_1, \ldots, a_n$.

If $a'_2, \ldots, a'_n$ are all of modulus less than unity, the integrand of (8) may be expanded as a multiple series which converges uniformly on the contour of integration, and so the operations of integration and summation may be interchanged, when the integral of (8) is proportional to

$$\sum_{m_2, \ldots, m_n=0}^{\infty} \frac{(\beta_2')_{m_2}\ldots (\beta_n')_{m_n}\, a_2'^{m_2}\ldots a_n'^{m_n}}{m_2! \ldots m_n!}\,.$$

$$\cdot \int_{[0\,;\,1]} S^{-\xi-m_2-\ldots-m_n}(S-1)^{-\beta_1'}\, ds \tag{9}$$

The inner integral of (9) may be evaluated by means of the formula

$$\int_{[0\,;\,1]} (-u)^{\alpha-1} \quad (u-1)^{\beta-1} = \frac{(2\pi i)^2}{\Gamma(1-\alpha)\Gamma(1-\beta)\Gamma(\alpha+\beta)}, \qquad (10)$$

so that, apart from a constant factor, the solution (8) becomes

$$R.\, F_D^{(n-1)} \;(\xi+\beta_1'-1,\, \beta_2',\, \ldots,\, \beta_n'\,;\, \xi\,;\, -a_2,\, \ldots,\, -a_n) \qquad (11)$$

where
$$F_D^{(n)} \;(a,\, b_1,\, \ldots,\, b_n\,;\, c\,;\, x_1,\, \ldots.\, x_n)$$

$$= \sum_{m_1,\, \ldots,\, m_n=0}^{\infty} \frac{(a)_{m_1+\ldots+m_n} (b_1)_{m_1} \ldots (b_n)_{m_n}}{(c)_{m_1+\ldots+m_n}} \frac{x_1^{m_1}}{m_1!} \cdots \frac{x_n^{m_n}}{m_n!},\ [4].$$

There are $\binom{n}{2} = \frac{1}{2}n\,(n-1)$ solutions of the type (11).

4. **Other hypergeometric solutions of** (2). If contours of integration other than that discussed in the previous section are considered, further families of solutions of (2) arise. Let only one of the three numbers P, Q, R be equal to unity. If, also, $Q \neq R$, $Q\ R$, then $n \binom{n-1}{Q}$ solutions of the type $R'C_{n-1}^{(k)}$ are obtainable for each set of values of P, Q, R. If $Q=R$, we have $\frac{1}{2}n \binom{n-1}{Q}$ solutions of similar type, where the generalised hypergeometric function $C_n^{(k)}$ was recently defined by the author in [1] as

$$C_n^{(k)} \;(a_1,\, \ldots,\, a_n\,;\, b_1,\, b_2\,;\, x_1,\, \ldots,\, x_n)$$

$$= \sum_{m_1,\, \ldots,\, m_n=0}^{\infty} \frac{(a_1)_{m_1} \ldots (a_n)_{m_n} (b_1)_{m_{k+1}+\ldots+m_n-m_1-\ldots-m_k}}{m_1! \ldots\ldots m_n!}.$$

$$(b_2)_{m_1+\ldots+m_k-m_{k+1}-\ldots-m_n}\, x_1^{m_1} \ldots\ldots x_n^{m_n}, \qquad (12)$$

in an investigation of the solutions of the partial differential system associated with the function $F_D^{(n)}$. $C_n^{(k)}$ generalises Horn's function G_2([3], p. 383).

If none of the numbers P, Q, R is equal to unity, then more complicated families of solutions of (2) ensue which may be expressed in terms of the function

$$D_{(n)}^{p,q}(a, b_1, \ldots, b_n ; c, c' ; x_1, \ldots, x_n)$$

$$= \sum_{m_1, \ldots, m_n=0}^{\infty} \frac{(a)_{m_{p+1}+\ldots+m_n-m_1-\ldots-m_p}(b_1)_{m_1}\ldots(b_n)_{m_n}}{(c)_{m_{q+1}+\ldots+m_n-m_1-\ldots-m_p}(c')_{m_{p+1}+\ldots+m_q}} \frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!}, \qquad (13)$$

also recently defined by the author [2].

This last possibility does not arise if the order of (2) is less than 4.

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