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LINEAR FLOW OF HEAT IN AN ANISOTROPIC FINITE SOLID MOVING IN A CONDUCTING MEDIA

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1. Introduction. The problem of conduction of heat in anisotropic materials have gained much interest in the recent years. These problems occur mainly in wood technology, soil mechanics and mechanics of solids of fibrous structure.

The analytic approach of such problems and their exact solutions are hardly available in the present literature. Carslaw and Jaeger [1] have touched this topic and suggested some problems. They have discussed few of them dealing with linear flow of heat in solids having simple laws of conductivities, i.e., $K=K_0(1+ax)$: where K denotes the conductivity at a distance x from the original boundary and K_0 is constant. These problems are solved with the help of Laplace transform and other integral transforms involving modified Bessel's functions.

Recently exact solutions of some problems of anisotropic solids [2], [3] and [4] have appeared. In solving these problems we have made use of Jacobi transform of two variables introduced earlier [5]. In these problems the law of variation of conductivity was given by $K=K_0(1-i^2)$ where $i=x, y, z$.

In this paper we shall consider linear heat flow in a finite solid with the above conductivity and moving in a conducting media with constant velocity. With the help of Jacobi polynomials, we shall find an exact solution of the above problem.

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2. Formulation and Solution of Problem. We consider a solid rod $-1 \leqslant x \leqslant 1$ moving along the direction of its length. We know in ordinary case the flux vector is given by

$$\bar{f} = -K \text{ grad } u,$$

where u is temperature and is a function of the position and time. For one dimensional flow, the single component of the flux vector along any plane at a distance x from the origin is given by

$$f = -K \frac{\partial u}{\partial x} + pc\, uv$$

Here we have assumed that the solid is moving with a constant velocity v along the direction of x-axis. Also p is density of the solid and c is the specific heat (here we have taken both constants). Therefore, with the help of law of continuity and the fundamental laws of heat transfer we arrive at the following differential equation of heat conduction.

$$\frac{\partial}{\partial x}\left\{(1-x^2)\frac{\partial u}{\partial x}\right\} - \frac{pcv}{\lambda}\frac{\partial u}{\partial x} + \frac{Q(x)}{\lambda} = \frac{pc}{\lambda}\frac{\partial u}{\partial t} \tag{1}$$

with the law of conductivity $K = \lambda(1-x^2)$. $Q(x)$ is intensity of a continuous source of heat situated inside it.

Let the initial temperature of the rod be given by

$$u(x, 0) = g(x) \tag{2}$$

The equation (1) is easily comparable with the Jacobi equation.

$$(1-x^2)\frac{d^2y}{dx^2} + \{(\beta-\alpha)-(\alpha+\beta+2)x\}\frac{dy}{dx} + n(n+\alpha+\beta+1)y = 0 \tag{3}$$

which has Jacobi polynomials $P_n^{(\alpha, \beta)}(x)$ as its solution. Hence if we take

$$v = \frac{\alpha-\beta}{q}, \; Q(x) = -(\alpha+\beta)\lambda x \frac{\partial u}{\partial x}, \text{ where } q = \frac{cp}{\lambda}$$

then the solution of (1) can be taken in the form

$$u = \sum_{n=0}^{\infty} A_n e^{-B_n t} P_n^{(\alpha,\beta)}(x) \tag{4}$$

Substituting this in the equation (1), we get

$$B_n = \frac{n}{q}(n+\alpha+\beta+1) \tag{5}$$

To find the value of A_n we make use of the condition (2). This gives us

$$g(x)=\sum_{n=0}^{\infty} A_n P_n^{(\alpha, \beta)}(x) \tag{6}$$

Multiplying both the sides of (6) by $(1-x)^{\alpha} (1+x)^{\beta} P_n^{(\alpha, \beta)}(x)$, integrating with respect to x from -1 to 1 and using the orthogonal property,

$$\int_{-1}^{1} (1-x)^{\alpha} (1+x)^{\beta} P_m^{(\alpha, \beta)}(x) P_n^{(\alpha, \beta)}(x)\, dx=0,\ m \neq n$$

$$=\delta_n^{(\alpha, \beta)},\ m=n \tag{7}$$

where

$$\delta_n^{(\alpha,\beta)}=\frac{2^{\alpha+\beta+1}\,\Gamma(n+\alpha+1)\,\Gamma(n+\beta+1)}{n!\,(\alpha+\beta+2n+1)\,\Gamma(n+\alpha+\beta+1)} \tag{8}$$

we obtain

$$A_n=G_n/\delta_n^{(\alpha, \beta)} \tag{9}$$

where

$$G_n=\int_{-1}^{1} (1-x)^{\alpha} (1+x)^{\beta} P_n^{(\alpha, \beta)}(x)\, g(x)\, dx \tag{10}$$

Substituting the values of A_n and B_n from (9) and (5) respectively in (4), we get

$$u(x, t)=\sum_{n=0}^{\infty} G_n\left(\delta_n^{\alpha, \beta}\right)^{-1} \exp\left\{\frac{-nt}{q}(n+\alpha+\beta+1)\right\} P_n^{(\alpha, \beta)}(x) \tag{11}$$

This gives exact solution of the problem. To ensure the convergence of the infinite series on the right hand side of (11), we must have $\alpha,\beta<-1$.

Special Case. Now as a special case we assume that the initial temperature is given by $g(x)=1+x$ and the velocity $v=1$. For this we take $\alpha=\frac{1}{2}$, $\beta=-\frac{1}{2}$.

Substituting these values in (10) and using the result [6, (p. 261)].

$$\int_{-1}^{1} (1-x)^{\alpha} (1+x)^{\beta} x^k P_n^{(\alpha, \beta)}(x)=0 \text{ for } 0\leqslant k<n$$

$$=\mu_n^{\alpha, \beta} \text{ for } k=n \tag{12}$$

where

$$\mu_n^{\alpha,\beta}=\frac{2^{1+\alpha+\beta+n}\,\Gamma(1+\alpha+n)\,\Gamma(1+\beta+n)}{\Gamma(2+\alpha+\beta+2n)}$$

We get

$$\left.\begin{aligned} G_n &= \pi/2 \quad \text{for} \quad n=0 \\ &= \pi/4 \quad \text{for} \quad n=1 \\ &= 0 \quad\;\; \text{for} \quad n\geqslant 2 \end{aligned}\right\} \tag{13}$$

Hence, after substituting the values of $P_0^{(\alpha,\beta)}(x)$ and $P_n^{(\alpha,\beta)}(x)$* we get from (11)

$$u(x,t)=\tfrac{1}{2}+\tfrac{1}{2}(2x+1)\,e^{-2t/q} \tag{14}$$

where q is given in (3).

Table-1 and Figure-1 describe temperature variation with respect to time at different points on the rod. $\bar{x}$, $\bar{u}$ and $\bar{t}$ express the quantities $2x+1$, $2u(x,t)-1$ and $2t/q$ respectively.

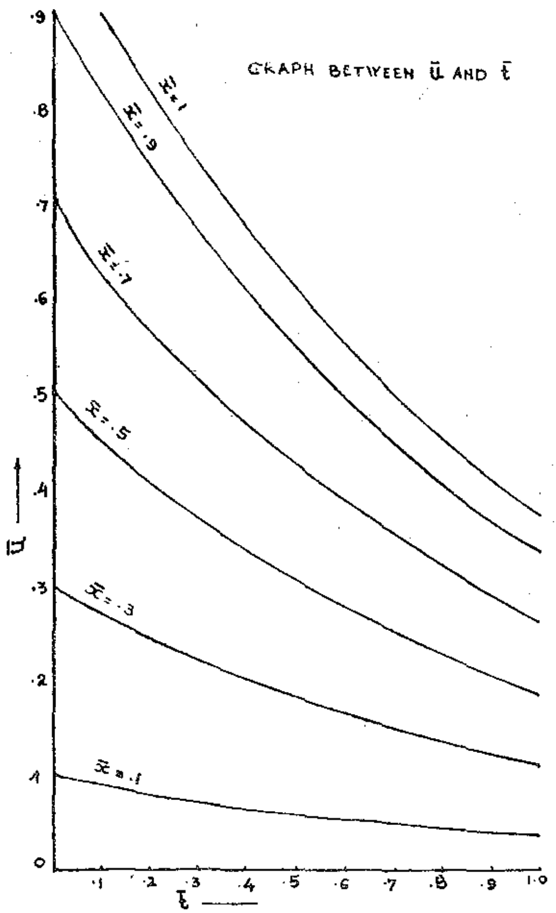

FIGURE. 1

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* $P_0^{(\alpha,\beta)}(x)=1$, $P_n^{(\alpha,\beta)}(x)=\tfrac{1}{2}\{(\alpha+\beta+2)x+(\alpha-\beta)\}$.

TABLE 1

Values of $\overline{u}(x, t)$

$\bar{x}$	$\bar{t}$										
	0	·1	·2	·3	·4	·5	·6	·7	·8	·9	1·0
0·1	0·100	0·090	0·820	0·074	0·067	0·061	0·055	0·050	0·045	0·041	0·037
0·3	0·300	0·270	0·246	0·222	0·201	0·183	0·163	0·150	0·135	0·123	0·111
0·5	0·500	0·450	0·410	0·370	0·335	0·305	0·275	0·250	0·225	0·205	0·185
0·7	0·700	0·630	0·514	0·518	0·469	0·427	0·385	0·350	0·315	0·287	0·259
0·9	0·900	0·810	0·738	0·666	0·603	0·549	0·495	0·450	0·405	0·369	03·33
1·0	1·000	0·900	0·820	0·742	0·671	0·612	0·550	0·504	0·451	0·411	0·370

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REFERENCE

1. H. S. Carslaw and J. C. Jaeger : '*Conduction of Heat in Solids*', Oxford (1959)
2. P. M. Gupta and V. P. Saxena : 'Unsteady Heat Distribution in an Orthotropic Rectangular Rod Moving Along the Direction of Its Length, *Indian Journal of Physics*, Vol. 44, (1970), p. 48.
3. P. M. Gupta and V. P. Saxena : 'Unsteady Temperature Distribution in an Moving Anisotropic Slab', *Indian Jour. Pure and Appl. Math.*, Vol. 2 (1971) p. 290.
4. P. M. Gupta and V. P. Saxena : 'Unsteady Temperature Distribution in an Orthogotropic Rectangular Slab', presented to P.R.C. Convention of C.S.I.R. (Nainital, 1970).
5. V. P. Saxena : 'A Study of Transform Calculus with Technical Applications,' Ph.D. Thesis (Vikram Univ. 1970).
6. E. D. Rainville : '*Special Functions*', McMillan (1960).