ISSN 0304-9892 (Print) ISSN 2455-7463 (Online)

Jñānābha ज्ञानाभ

(HALF YEARLY JOURNAL OF MATHEMATICS)
[Included : UGC-CARE List]

VOLUME 49

NUMBER 2

DECEMBER 2019

Published by :

The Vijñāna Parishad of India

[Society for Applications of Mathematics]

DAYANAND VEDIC POSTGRADUATE COLLEGE

(Bundelkhand University)

ORAI-285001, U.P., INDIA

www.vijnanaparishadofindia.org/jnanabha

ISSN 0304-9892 (Print) ISSN 2455-7463 (Online)

Jñānābha

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Jñānābha, Vol. 49(2) (2019), 1-5

FIXED POINT THEOREMS FOR GENERALIZED NON-EXPANSIVE MAPPINGS

By
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(*Received : June* 06, 2018 *; Revised: October* 20, 2019)

Abstract

In this paper, we obtain a fixed point theorem for the mappings satisfying non-expansive type conditions.
2010 Mathematics Subject Classifications: 47H10, 54H25.
Keywords and phrases: Generalized non-expansive mappings and Fixed points.

1 Introduction and Preliminaries

A mapping T be on a metric space (X, d) is said to be non-expansive if $d(Tx, Ty) \leq d(x, y)$ for all $x, y \in X$, while mapping T is called contraction if there exists a non-negative real number $k < 1$ such that $d(Tx, Ty) \leq k\, d(x, y)$ for all $x, y \in X$. It is well known that every contraction on a complete metric space has a unique fixed point (Banach Contraction Principle). But, this is not true for non-expansive mappings on complete metric space, e.g., $Tx = x + 1$ in usual metric space $X = [0, \infty)$. It is well known that for the existence of fixed points for non-expansive mappings one needs the convex structure of the space X. Many researchers tried to explore the existence of fixed points for non-expansive type mappings and have done a good work in this direction for Banach spaces as well as metric spaces (see, [1, 4, 6, 2, 3, 5, 7, 8, 9] and references therein).

Bogin [1] proved the following result.

Theorem 1.1. *Let (X, d) be a complete metric space and $T : X \to X$ a mapping satisfying*

$$d(Tx, Ty) \leq a\, d(x, y) + b\, [d(x, Tx) + d(y, Ty)] + c\, [d(x, Ty) + d(y, Tx)], \tag{1.1}$$

where $a \geq 0$, $b > 0$, $c > 0$ and $a + 2b + 2c = 1$. Then T has a unique fixed point.

Greguš [10] considered a class of self mappings on X which satisfy (1.1) with $c = 0$. In fact, he proved the following theorem.

Theorem 1.2. *Let C be a non-empty closed convex subset of a Banach space X and $T : C \to C$ a mapping satisfying*

$$||Tx - Ty|| \leq a\, ||x - y|| + b\, [||x - Tx|| + ||y - Ty||], \tag{1.2}$$

for all $x, y \in C$, where $a > 0$, $b > 0$ and $a + 2b = 1$. Then T has a unique fixed point.

Greguš's result has inspired many authors for further investigations in this direction, (see Abdelijawad et al.[11], Ćirić [3, 5, 7, 8, 6], Jungck[13] and references therein). In 1993, Ćirić [4] proved the the following result which is a proper generalization of the above theorems.

Theorem 1.3. *Let (X,d) be a complete metric space and $T : X \to X$ a mapping satisfying*

$$\begin{aligned} d(Tx,Ty) &\leq a \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{1}{2}[d(x,Ty)+d(y,Tx)]\right\} \\ &\quad +b \max\{d(x,Tx), d(y,Ty)\} + c\,[d(x,Ty)+d(y,Tx)], \qquad (1.3) \end{aligned}$$

where $a \geq 0$, $b > 0$, $c > 0$ and $a+b+2c = 1$. Then T has a unique fixed point and T is continuous at that fixed point.

In 2008, Suzuki [14] introduced a weaker notion of contractions and proved the following theorem.

Theorem 1.4 ([14])**.** *Let (X,d) be a complete metric space, T be a mapping on X. Define a non-increasing function $\theta : [0,1) \to (\frac{1}{2}, 1]$ by*

$$\theta(r) = \begin{cases} 1 & \text{if } 0 \leq r \leq \frac{\sqrt{5}-1}{2}, \\ \frac{1-r}{r^2} & \text{if } \frac{\sqrt{5}-1}{2} \leq r \leq \frac{1}{\sqrt{2}}, \\ \frac{1}{1+r} & \text{if } \frac{1}{\sqrt{2}} \leq r < 1. \end{cases}$$

Assume that there exists $r \in [0,1)$ such that

$$\theta(r)d(x,Tx) \leq d(x,y) \text{ implies } d(Tx,Ty) \leq r\,d(x,y), \qquad (1.4)$$

for each $x,y \in X$. Then there exists a unique fixed point z of T. Moreover, $\lim_{n\to\infty} T^n x = z$ for all $x \in X$.

Since $\lim_{r\to 1-0} \theta(r) = \frac{1}{2}$, it is very natural to consider the following condition.

Definition 1.1 ([15])**.** *Let T be a mapping on a subset A of a Banach Space X. Then T is said to satisfy condition (C) if*

$$\frac{1}{2}||x - Tx|| \leq ||x-y|| \quad \text{implies} \quad ||Tx - Ty|| \leq ||x-y||, \qquad (1.5)$$

for all $x,y \in A$.

The condition (C) is weaker than non-expansive (see, Proposition 1 of [15]).

Using above condition (C), many results have been obtained by researchers for the existence of fixed points, see [12, 16] and references therein. Recently, Popescu [16] generalized the result due to Bogin [1] for non-expansive mappings in the setting of condition (C). In fact, he proved the following.

Theorem 1.5. *Let (X,d) be a nonempty complete metric space and $T : X \to X$ a mapping satisfying*

$$\begin{aligned} \frac{1}{2}d(x,Tx) &\leq d(x,y) \text{ implies} \\ d(Tx,Ty) &\leq a\,d(x,y) + b\,[d(x,Tx)+d(y,Ty)] + c\,[d(x,Ty)+d(y,Tx)], \end{aligned}$$

where $a \geq 0, b > 0, c > 0$ and $a + 2b + 2c = 1$. Then T has a unique fixed point.

Let (X,d) be a metric space and $T : X \to X$ a self-mapping of X. For $x,y \in X$, we use the following notation:

$$\begin{aligned} M(Tx,Ty) &= a \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{1}{2}[d(x,Ty)+d(y,Tx)]\right\} \\ &\quad +b \max\{d(x,Tx), d(y,Ty)\} + c\,[d(x,Ty)+d(y,Tx)]. \end{aligned}$$

Now, we investigate a new generalized class of self-mappings T on metric space X which satisfy the following generalized non-expansive type condition:

$$\frac{1}{2}d(x,Tx) \leq d(x,y) \text{ implies } d(Tx,Ty) \leq M(Tx,Ty), \qquad (1.6)$$

for all $x,y \in X$, where a,b and c are non-negative real numbers such that

$$a + b + 2c = 1. \qquad (1.7)$$

2 Main Results

Theorem 2.1. *Let (X, d) be a complete metric space and $T : X \to X$ satisfying* (1.6), *where $a \geq 0, b > 0, c > 0$ and such that* (1.7) *holds. Then T has a unique fixed point, i.e., there is a unique $z \in X$ such that $Tz = z$.*

Proof. Let $x \in X$ be an arbitrary point. Since $\frac{1}{2}d(x,Tx) \leq d(x,Tx)$, by condition (1.6) and (1.7), we obtain

$$\begin{aligned} d(Tx,T^2x) &\leq M(Tx,T^2x) \\ &\leq (a+b)\max\{d(x,Tx),d(Tx,T^2x)\} + c\,d(x,T^2x) \\ &\leq (a+b+2c)\max\{d(x,Tx),d(Tx,T^2x)\} \\ &= \max\{d(x,Tx),d(Tx,T^2x)\}. \end{aligned}$$

Hence, $d(Tx,T^2x) \leq d(x,Tx) \quad \forall x \in X$.

Thus, if we define a sequence $\{x_n\}$ in X such that $x_n = T^n x$, where $x \in X$ is arbitrary. Then

$$d(x_n,x_{n+1}) \leq d(x_{n-1},x_n) \quad \forall n \geq 1. \tag{2.1}$$

That is, the sequence $\{d(x_n,x_{n+1})\}_{n=0}^{\infty}$ is a non-increasing, where $x_0 = T^0x = x$.

Now, we show that there is a non-negative real number $m < 2$ such that $d(Tx,T^3x) \leq m\,d(x,Tx)$. If $d(x,Tx) \leq d(x,T^2x)$, then $\frac{1}{2}d(x,Tx) \leq d(x,T^2x)$ and using (2.1), we have

$$\begin{aligned} d(Tx,T^3x) &\leq M(Tx,T^3x) \\ &\leq a\,[d(x,Tx)+d(Tx,T^2x)] + b\,d(x,Tx) \\ &\quad +c\,[d(x,Tx)+d(Tx,T^2x)+d(Tx,T^3x)] \\ &\leq (2a+b+2c)d(x,Tx) + c\,d(Tx,T^3x) \\ \Rightarrow \qquad d(Tx,T^3x) &\leq \frac{2a+b+2c}{1-c}\,d(x,Tx) = \frac{1+a}{1-c}\,d(x,Tx). \end{aligned}$$

Setting $m_1 = \frac{1+a}{1-c} < 2$, we get $d(Tx,T^3x) \leq m_1\,d(x,Tx)$.

Now suppose that $d(x,Tx) > d(x,T^2x)$. Since $\frac{1}{2}d(x,Tx) \leq d(x,Tx)$,

$$d(Tx,T^2x) \leq M(Tx,T^2x) < (a+b+c)\,d(x,Tx).$$

Then,

$$\begin{aligned} d(Tx,T^3x) &\leq d(Tx,T^2x) + d(T^2x,T^3x) \leq 2\,d(Tx,T^2x) \\ &< (1+a+b)\,d(x,Tx). \end{aligned}$$

Setting $m_2 = (1+a+b) < 2$, we get $d(Tx,T^3x) < m_2\,d(x,Tx)$.

Thus taking $m = \max\{m_1,m_2\}$, we obtain $0 < m < 2$ and

$$d(Tx,T^3x) \leq m\,d(x,Tx) \quad \forall\, x \in X. \tag{2.2}$$

Since $\frac{1}{2}d(Tx,T^2x) \leq d(Tx,T^2x)$, using the condition (2.2) we have

$$\begin{aligned} d(T^2x,T^3x) &\leq M(T^2x,T^3x) \\ &\leq (a+b)\,d(Tx,T^2x) + c\,d(Tx,T^3x) \\ &\leq (a+b+mc)\,d(x,Tx). \end{aligned}$$

Taking $k = (a+b+mc) < 1$, we get

$$d(T^2x,T^3x) \leq k\,d(x,Tx) \quad \forall x \in X.$$

Hence, by induction we have

$$d(x_n, x_{n+1}) \leq k^{[n/2]} d(x, Tx) \quad \forall\, n \geq 0, \tag{2.3}$$

where $[n/2]$ means the greatest integer not exceeding $n/2$. Since $0 < k < 1$, condition (2.3) implies that $\{x_n\}$ is a Cauchy sequence and by completeness of X, there exist $z \in X$ such that $x_n \to z$ as $n \to \infty$.

Next, we will prove that $z \in X$ is a fixed point of T. Assuming that there is some n such that

$$\frac{1}{2} d(x_n, x_{n+1}) > d(z, x_n) \text{ and } \frac{1}{2} d(x_{n+1}, x_{n+2}) > d(z, x_{n+1}).$$

Then,

$$d(x_n, x_{n+1}) \leq d(z, x_n) + d(z, x_{n+1}) < \frac{1}{2}[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})] \leq d(x_n, x_{n+1}),$$

which is a contradiction, so for all $n \geq 0$, we get

$$\text{either } \frac{1}{2} d(x_n, x_{n+1}) \leq d(z, x_n) \text{ or } \frac{1}{2} d(x_{n+1}, x_{n+2}) \leq d(z, x_{n+1}).$$

Thus, there exist a subsequence $\{n_j\}$ of $\{n\}$ such that $\frac{1}{2} d(x_{n_j}, x_{n_j+1}) \leq d(z, x_{n_j})$ for all $j \geq 0$. Then, we have

$$d(Tz, x_{n_j+1}) \leq M(Tz, Tx_{n_j}).$$

Now, taking $n \to \infty$, we get

$$d(z, Tz) \leq (a + b + c)\, d(z, Tz) \quad \Rightarrow \quad d(z, Tz) = 0 \quad \Rightarrow \quad Tz = z.$$

For uniqueness of fixed point, let z' be another fixed point of T. Then, $\frac{1}{2} d(z, Tz) = 0 \leq d(z, z')$ and hence

$$d(Tz, Tz') \leq M(Tz, Tz') \quad \Rightarrow \quad d(z, z') \leq (a + 2c)\, d(z, z') \quad \Rightarrow \quad z = z'.$$

□

Remark 2.1. *It is to be noted that, Theorem* 2.1 *is a weaker version of the result due to Ćirić* [4] *(Theorem* 2.1*, page* 148*) for non-expansive mappings.*

From our main result, following corollaries which are generalizations of results due to Popescu [16] (Theorem 2.1, page 3913) and Suzuki [15] (Theorem 4, page 1094) for non-expansive mappings in metric sense respectively.

Corollary 2.1. *Let (X, d) be a complete metric space and $T : X \to X$ a mapping satisfying $\frac{1}{2} d(x, Tx) \leq d(x, y)$ implies*

$$d(Tx, Ty) \leq a\, d(x, y) + b\,[d(x, Tx) + d(y, Ty)] + c\,[d(x, Ty) + d(y, Tx)],$$

where $a \geq 0$ and $b > 0, c > 0$ such that $a + b + 2c = 1$. Then T has a unique fixed point.

Corollary 2.2. *Let (X, d) be a complete metric space and $T : X \to X$ a mapping satisfying $\frac{1}{2} d(x, Tx) \leq d(x, y)$ implies*

$$d(Tx, Ty) \leq a\, d(x, y) + b \max\{d(x, Tx), d(y, Ty)\} + c\,[d(x, Ty) + d(y, Tx)],$$

where $a \geq 0$ and $b > 0, c > 0$ such that $a + b + 2c = 1$. Then T has a unique fixed point.

Taking $a = 0$ in Theorem 2.1, we get the following corollary.

Corollary 2.3. *Let (X, d) be a complete metric space and $T : X \to X$ a mapping satisfying $\frac{1}{2} d(x, Tx) \leq d(x, y)$ implies*

$$d(Tx, Ty) \leq b \max\{d(x, Tx), d(y, Ty)\} + c\,[d(x, Ty) + d(y, Tx)],$$

where $b > 0, c > 0$ such that $b + 2c = 1$. Then T has a unique fixed point.

Acknowledgment
First author thanks to UGC, New Delhi for support under RGNF. We are also thankful to the referee for his valuable suggestions to improve the paper in its present form.

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Jñānābha, Vol. 49(2) (2019), 6-14

AIR QUALITY PREDICTION USING TIME SPACE ANALYSIS

By

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(*Received : August* 02, 2019 *; Revised: October* 18, 2019)

Abstract

Air pollution is a serious threat to the environment and ecology. Monitoring and prediction of air quality is an important aspect, as it helps to issue early warnings and adopt suitable control measures in time. Particulate matter of size less than and equal to 2.5 microns is the prominent air pollutant. It easily penetrates through lungs affecting human health. This paper investigates the performance of the empirical mode decomposition and the wavelet transform in non linear non stationary PM2.5 time series prediction problem. The prediction is carried out by applying adaptive neuro-fuzzy inference system (ANFIS). It is found that the wavelet transform outperforms empirical mode decomposition for non linear PM2.5 time series.

2010 Mathematics Subject Classification: 62P12
Keywords and phrases: Particulate matter, Air quality index, Empirical mode decomposition, Wavelet transform, Adaptive neuro-fuzzy inference system.

1 Introduction

Air pollutants are growing rapidly, which is one of the most serious health issues. Among pollutants, particulate matter is gaining more attention because of its ability to penetrate deep into lungs. Many studies have been carried out showing significant association between particulate matter and chronic diseases and even resulting in death [2,7]. The WHO has called it second tobacco epidemic. The PM2.5 can hamper brain development by 10-20 per cent and we are going to make our next generation retarded. Evidence is now emerging that heart attacks, a brain attacks are linked to air pollution. A recent study found strong relation between PM2.5 exposure and neurological disorders [9]. It has aroused worldwide concerns over the last few years. The study of particulate matter is of great relevance from worlds economy and health point of view. The spatial and temporal distribution of PM2.5 concentration which has strong nonlinear characteristics is influenced by the meteorological field, the emission source, the complex underlying surface, and the coupling of the physical and chemical processes, which makes it difficult to predict. At present, the commonly used prediction methods are mechanism analysis and statistical prediction method. Many studies related to prediction of PM2.5 has been done using stepwise regression keeping in mind its chemical composition and physical properties, and the other pollutants (such as SO_2 ,NO_x, O_3) and meteorological factors [3,4,8]. PM10 and PM2.5 in urban areas using a chemical transport model is predicted [11]. In this paper, PM2.5 prediction is carried out by breaking down time series into sub series. The study examines Wavelet-ANFIS (WANFIS) and Empirical mode decomposition-ANFIS (EMD-ANFIS) models for predicting PM2.5 using the embedded series.

The proposed model is trained for PM2.5 concentration monitored at Shadipur, Delhi. Shadipur is one of the pollution hotspots in Delhi. It is residential cum industrial area. Delhi Transport Corporation depot is also located in Shadipur connecting many parts of Delhi. Its neighboring areas are Mayapuri and Naraina. Naraina is divided into industrial, residential and rural areas. It is the location of the headquarters of the Steel Authority of India Limited. Mayapuri is combination of residential flats, light metal factories and automobile service stations. Air quality in this area lies mainly in very poor or severe category. Apart from meteorological parameters and vehicular emissions, industries plays major role in pollution levels making study of particulate matter more complex. The main cause of air pollution is garbage burning. The garbage predominantly consists of rubber and plastic waste which is carcinogenic. PM2.5 (24-hourly) concentrations are obtained from Central Pollution Control Board (CPCB) from the period from March, 2015 to March, 2019.

2 Time space analysis method

The aim is to forecast PM2.5 daily concentration using past values. The uncertainty of PM2.5 makes precise prediction a challenging task. Alternate way is to break the original series into components with lower variability and then training each of them using ANFIS. Summing individual results gives the final output [12,14]. The time space analysis methods- wavelet transformation and empirical mode decomposition are used for breaking down high variability time series into subseries. Further, performance analysis of both methods is carried out by comparing PM2.5 actual and forecasted values.

2.1 Empirical mode Decomposition (EMD)

The EMD decomposes a signal into intrinsic membership functions (IMF). IMF has only one extreme between zero crossings, and has an average of zero [6]. The process used for decomposing is iterative and is stopped when the standard deviation between two successive shifting is smaller than 0.2 or 0.3. Given a signal $y(t)$ shifting process works in following steps:

1. From cube-spline interpolation of local extremes determine upper and lower envelope respectively. Let μ_1 be the average of upper and lower envelope.
2. First component, $A_1(t) = y(t) - \mu_1(t)$.
3. If μ_1 and A_1 satisfy stopping criteria, then first IMF $d_1(t) = \mu_1(t)$ and residue $R_1(t) = A_1(t)$.
4. Otherwise, steps 1-3 is repeated for $A_1(t)$.
5. For $R_1(t)$, steps 1-4 are repeated until all IMFs and residue are obtained say J is IMF count

$$y(t) = \Sigma_{i=1}^{J} D_i(t) + R(t). \tag{2.1}$$

2.2 Wavelet Transform

Fourier transforms limitation of uniform frequency resolution at all frequencies lead to wavelet transforms. With Fourier transforms time-frequency grid is uniform whereas wavelet transform is collection of different windows thus descrying low and high frequencies. Wavelet transforms are further divided into continuous and discrete [1]. The present study deals with discrete wavelet transform (DWT) for multi resolution of signal. Multi resolution analysis (finer to coarser in time domain) is based on high and low pass filter. High pass filter referred to as mother wavelet captures high frequency (details) and low pass filter are father wavelet which captures low frequency (approximations) [10]

$$\phi(t) = \sqrt(2)\sum g_a\phi(2t-a), g_a = \frac{1}{\sqrt(2)}\int \phi(t)\phi(2t-a)dt, \tag{2.2}$$

$$\psi(t) = \sqrt(2)\sum f_a\psi(2t-a), f_a = \frac{1}{\sqrt(2)}\int \psi(t)\psi(2t-a)dt. \tag{2.3}$$

Using mother and father wavelet, signal x(t) can be disintegrated as:

$$x(t) = \Sigma_c X_{l,c}\psi_{l,c}(t) + \Sigma_c d_{l,c}\psi_{l,c}(t) + \Sigma_c d_{l-1,c}\psi_{l-1,c}(t) + ... + \Sigma_c d_{1,k}\psi_{1,c}, \tag{2.4}$$

$$X_{l,c} = \int \phi_{l,c}x(t)dt, \phi_{l,c} = 2^{\frac{-l}{2}}\phi(2^{-l}t-c), \tag{2.5}$$

$$d_{l,c} = \int \psi_{l,c}x(t)dt, \psi_{l,c} = 2^{\frac{-l}{2}}\psi(2^{-l}t-c). \tag{2.6}$$

Here, l ranges from 1 to number of number of coefficients and c is number of levels. High and low pass filter vary for different wavelets. For present study, daubechies wavelet due to its decay pattern in time and frequency unlike Haar and Shannon wavelets which are compact in time and in frequency respectively.

3 ANFIS

ANFIS is based on five adaptive and fixed layers comprising of premise and consequent parameters. The premise parameters are the input parameters defined by the membership functions of data. The consequent parameters are tuned. The learning ability of neural networks helps in tuning the parameters. ANFIS is basically used for handling non linear behavior of time series. The algorithm of the process learns in accordance with the back propagation method. Many prediction studies based on data is carried out using ANFIS [5]. The performance of the system in learning from data helps in predicting non stationary non linear time series. More insights to the system architecture are discussed [13].

4 Methodology

PM2.5 concentration monitored at Shadipur is undertaken for the study for the period from 22/03/2015 to 28/03/2019. The original series is decomposed using time space analysis methods. Consider time series of PM2.5 concentration as $x(t)$, $t = 1, 2, ..., m$ where m is the count of data points under consideration. The optimal lag undertaken in for the prediction is four keeping in mind the learning rate which will directly affect the computational time. The first step is to decompose series than apply ANFIS on decomposed components for prediction as depicted in Fig. 5.1.

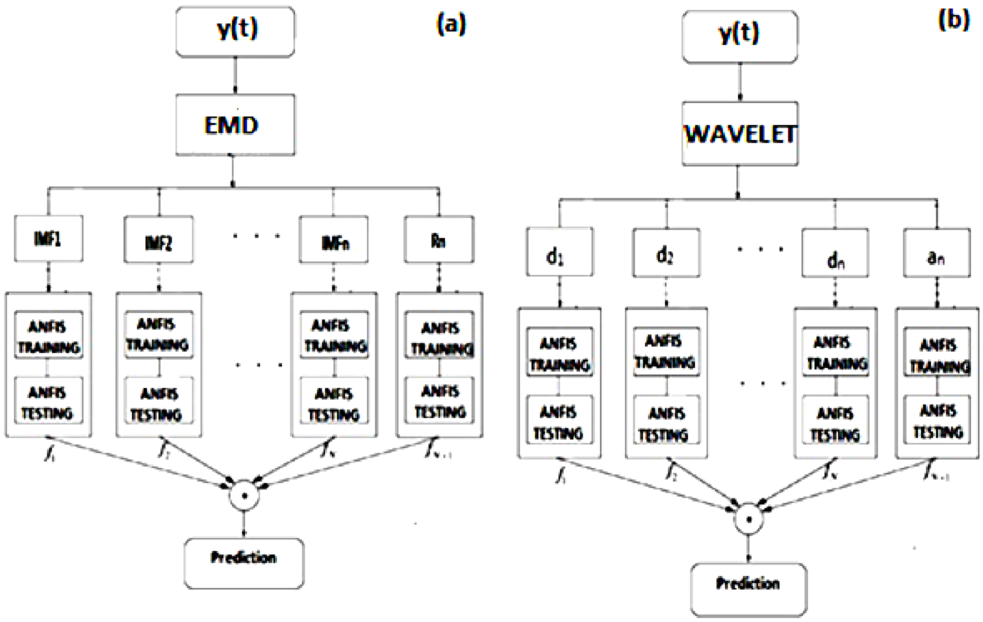

Figure 4.1: (a) EMD-ANFIS (b) Wavelet ANFIS (WANFIS)

5 Performance Testing

The section comprises of performance analysis of proposed EMD-ANFIS and WANFIS algorithm is tested for PM2.5 series. The data is split into training (70 percent), testing (15 percent) and validation (15 percent) sets. The performance has been compared between the observed and predicted value using mean absolute error (MAE) and coefficient of determination (R^2). R^2 close to 1 and smaller value of MAE proves good prediction of the methods. where, a_i is actual data, pi predicted data, m data points and a is mean of the actual data. Further to investigate the validated data, air quality index (AQI) is evaluated and compared with the actual as depicted on CPCB site on daily basis. AQI is not just a number describing the quality of air but also explains what we are breathing in. The purpose was to acquaintance public about decaying air quality. AQI is calculated using the standards defined by CPCB. AQI is broadly categorized as follows:

Category	Value	24-hr Average $PM_{2.5}$ Concentration
Good	0-50	0-30
Satisfactory	51-100	31-60
Moderately polluted	101-150	61-90
Poor	151-200	91-120
Very poor	201-300	121-250
Severe	301-500	250+

The time series of PM2.5 for the period under consideration is depicted in Figure 5.1 and plot area describes PM2.5 categories. The series clearly depicts few cases of less PM2.5

concentration and more of poor and very poor categories. Firstly the series is decomposed using EMD. IMF is shown in Figure 5.2.

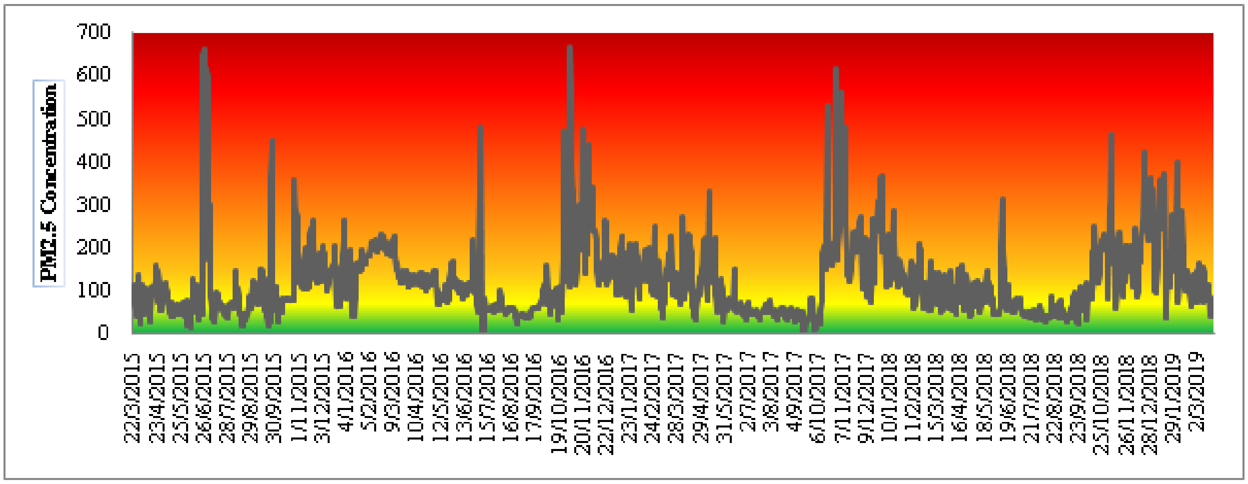

Figure 5.1: PM2.5 time series for the period from 22/03/2015 to 28/03/2019.

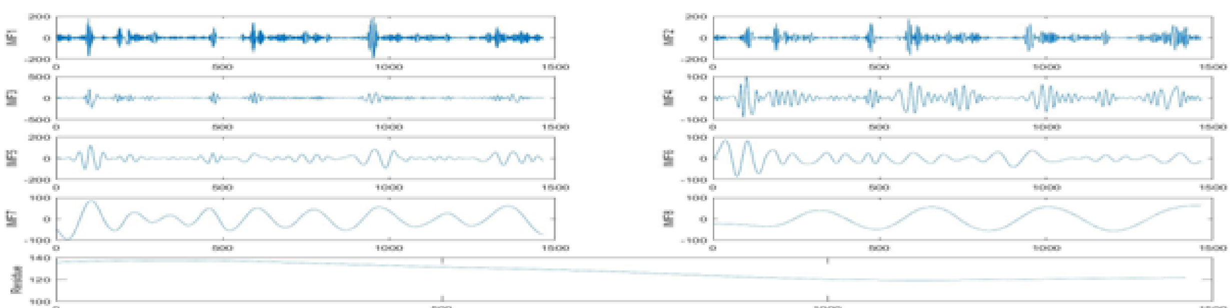

Figure 5.2: IMFs/ Residue series of PM2.5 data.

The decomposed IMFs and residue are predicted using ANFIS. Next PM2.5 series undergo wavelet transformation and is decomposed into approximations and details which follow the algorithm depicted in Figure 4.1b. Finally the predicted data is obtained by summing up predicted value of components. The performance of models can be seen in Figure 5.3 and Figure 5.4 for observed and predicted value of testing and validation data.

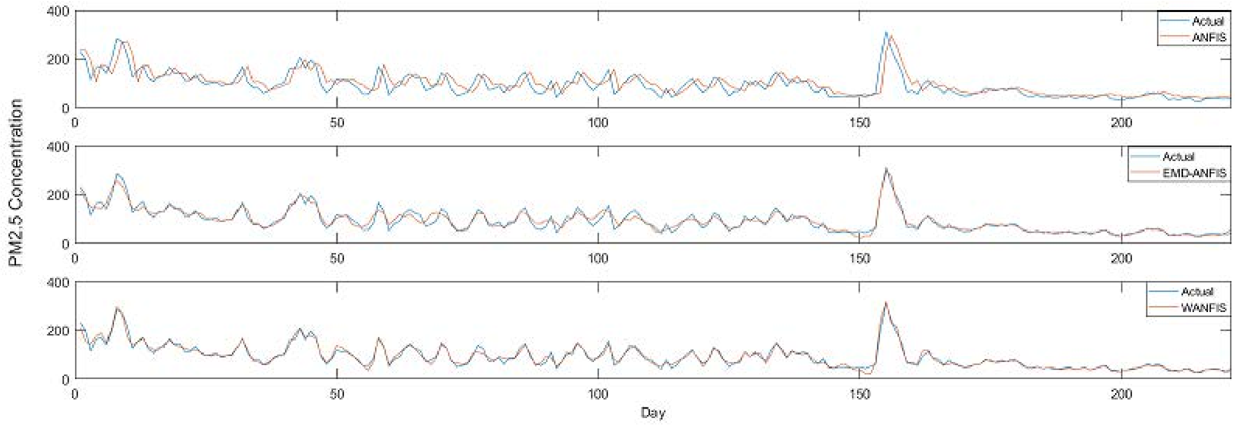

Figure 5.3: Observed and predicted PM2.5 Testing data.

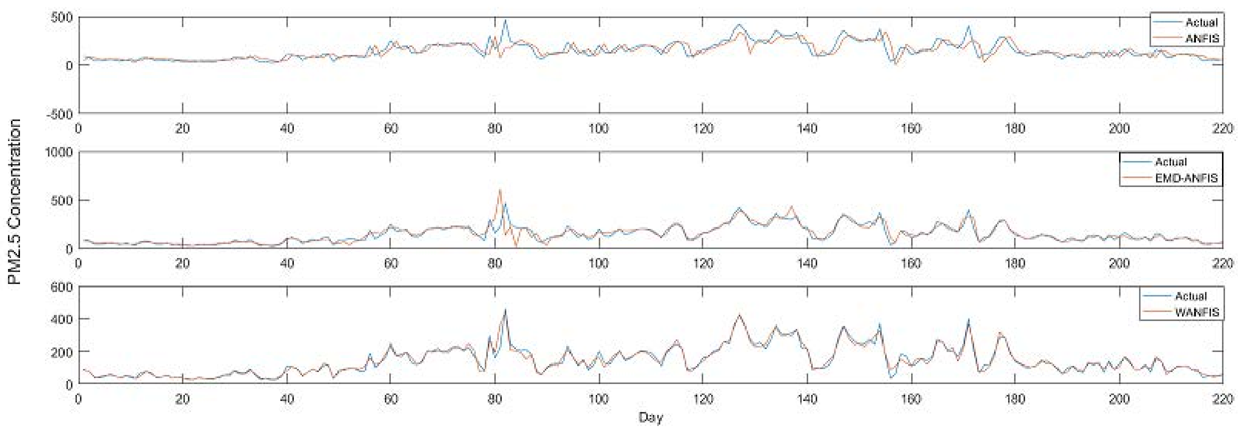

Figure 5.4: Observed and predicted PM2.5 Validation data.

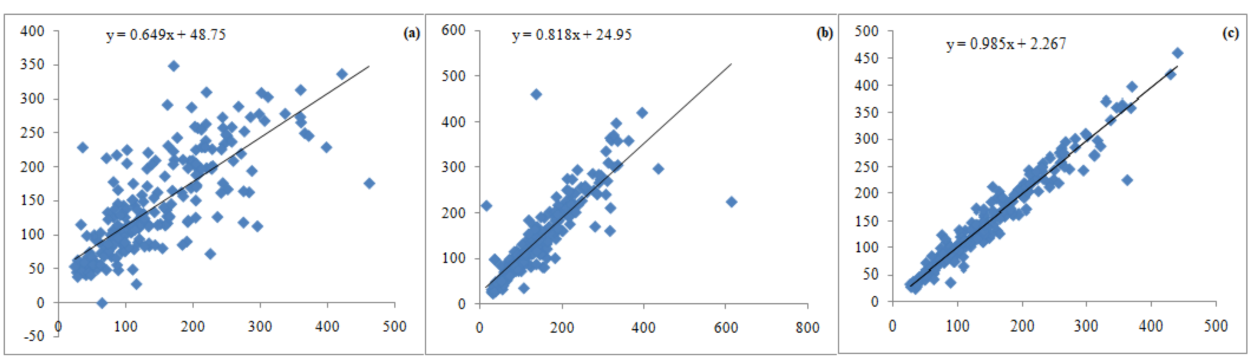

Figure 5.5: Observed vs. predicted PM2.5 using(a)ANFIS;(b)EMDANFIS;(c)WaveletANFIS

Further, the efficiency of models is statistically compared using MAE and R^2 as depicted in Table 5.1.

Table 5.1: Performance Analysis

Model	R^2			**MAE**		
	Training	Testing	Validation	Training	Testing	Validation
ANFIS	0.7324	0.6251	0.5814	31.19	27.9552	33.125
EMD ANFIS	0.9491	0.9079	0.7163	14.4091	11.6029	25.7133
Wavelet ANFIS	**0.9762**	**0.9507**	**0.9505**	**9.5249**	**8.0686**	**13.0044**

It is evident that WANFIS performs much better than other models. Using the forecasted and observed validation data AQI is calculated. Approximately 96 percent of forecasted and observed AQIs lie in the same category as in Figure 5.6. Thus the model can be used by pollution controlling and monitoring agencies to predict AQI which can contribute in controlling its severe affects and taking precaution beforehand by the government and people.

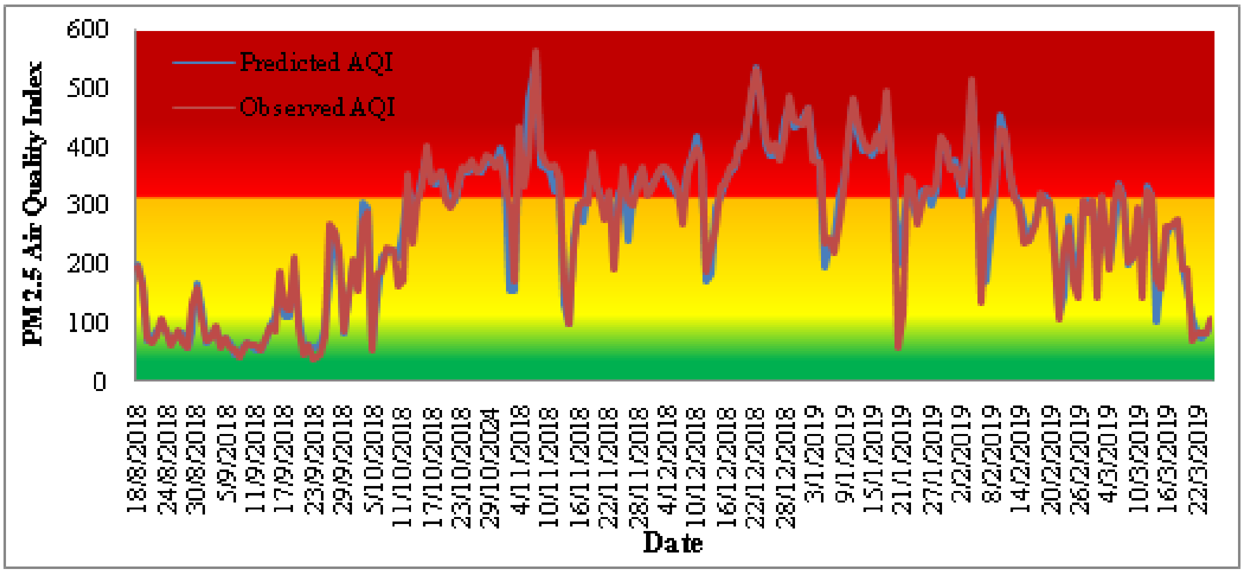

Figure 5.6: Predicted and observed air quality index, plot area is divided into categories.

6 Conclusion

A hybrid model consisting of time space analysis methods and adaptive neuro-fuzzy inference system is proposed for accurate prediction. The performance of the model has been verified by calculating AQI of the predicted values and comparing with AQIs for PM2.5 as shown on www.cpcb.nic.in. It is found that the wavelet transformation is competent to improve the PM2.5 forecasting accuracy. Simulation results have shown that Wavelet-ANFIS outperformed EMD-ANFIS and ANFIS. Air quality index falls in poor and severe categories indicating need of action plan to control air pollution. Wavelet-ANFIS lead to accurate prediction of AQI. WANFIS can be used in order to forecast air quality index and issuing health advisories according to the air quality categories. The proposed model can be considered for prediction of any non linear non stationary time series using only the lagged values of the concerned series. The model reduced dependency complexity on other factors. Although the work reported in this paper improves prediction accuracy, there is still ample space for improvement in learning rate of the proposed model. For future studies authors will consider other optimization techniques to optimize learning rate and computational time for more convenient model in forecasting air quality.

Acknowlegements
The authors are thankful to Guru Gobind Singh Indraprastha University, Delhi (India) for providing research facilities and financial support. They are also thankful to the referee for his valuable suggestions.

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Jñānābha, Vol. 49(2) (2019), 15-21

RESULTS FOR ZEROS OF POLYNOMIAL AND ANALYTIC FUNCTION

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(*Received : August* 02, 2019 ; *Revised: September* 21, 2019)

Abstract

Let $p(z)$ be a polynomial of degree n. In this paper, we have obtained ring shaped regions containing zeros of complex valued polynomial as well as analytic functions in terms of coefficient of function. Our results improve upon the results earlier proved.

2010 Mathematics Subject Classifications: 30C10, 30C15.

Keywords and phrases: Polynomials, analytic function, zeros, region.

1 Introduction and Statement of Results

The following elegant result is commonly known as Eneström-Kakeya Theorem, firstly proved by Eneström [3] and later independently by Kakeya [7] and Hurwitz [5].

Theorem A. *Let* $p(z)=\sum_{\nu=0}^{n} a_\nu z^\nu$ *be a polynomial of degree* n *such that*

$$a_n \geq a_{n-1} \geq \cdots \geq a_1 \geq a_0 > 0. \tag{1.1}$$

Then $p(z)$ *does not vanish in* $|z|>1$.

Joyal, Labelle and Rahman [6] extended Theorem A to the polynomials with coefficients not necessarily non-negative. More precisely, they proved the following

Theorem B. *Let* $p(z)=\sum_{\nu=0}^{n} a_\nu z^\nu$ *be a polynomial of degree* n *such that*

$$a_n \geq a_{n-1} \geq \cdots \geq a_1 \geq a_0. \tag{1.2}$$

Then $p(z)$ *has all its zeros in*

$$|z| \leq \frac{a_n - a_0 + |a_0|}{|a_n|}. \tag{1.3}$$

The following result was proved by Rather and Ahmad [8].

Theorem C. *Let* $p(z)=\sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree* n *with complex coefficients. Let* $a_j=\alpha_j+i\beta_j$, *for* $j=0,1,2,\ldots,n$ *and for some* $K\geq 1$,

$$K\alpha_n \geq \alpha_{n-1} \geq \cdots \geq \alpha_1 \geq \alpha_0,$$

and

$$K\beta_n \geq \beta_{n-1} \geq \cdots \geq \beta_1 \geq \beta_0, \tag{1.4}$$

then $p(z)$ *has all its zeros in*

$$|z+(K-1)| \leq \frac{K(\alpha_n+\beta_n)-(\alpha_0+\beta_0)+|a_0|}{|a_n|}. \tag{1.5}$$

Aziz and Mohammad [1] extended Eneström-Kakeya Theorem to the class of analytic function $f(z) = \sum_{j=0}^{\infty} a_j z^j$(not identically zero), with its coefficients a_ν satisfying a relation analogous to (1.1) and proved the following theorem.

Theorem D. *Let $f(z) = \sum_{j=0}^{\infty} a_j z^j$ (not identically zero) be analytic in $|z| \leq t$. If $a_j > 0$ and $a_{j-1} - ta_j \geq 0, j = 1, 2, 3, \ldots,$ then $f(z)$ does not vanish in $|z| < t$.*

Aziz and Shah [2] relaxed the hypothesis of Theorem D and proved the following theorem.

Theorem E. *Let $f(z) = \sum_{j=0}^{\infty} a_j z^j$(not identically zero) be analytic in $|z| \leq t$, such that for some $K \geq 1$,*

$$Ka_0 \geq ta_1 \geq t^2 a_2 \geq \cdots . \tag{1.6}$$

Then $f(z)$ does not vanish in

$$|z - \left(\frac{K-1}{2K-1}\right) t| \leq \frac{Kt}{2K-1}. \tag{1.7}$$

Shah and Liman [9] proved the following result concerning the location of zeros of analytic function.

Theorem F. *Let $f(z) = \sum_{j=0}^{\infty} a_j z^j$(not identically zero) be analytic in $|z| < t$. If for someK ≥ 1,*

$$K|a_0| \geq t|a_1| \geq t^2|a_2| \geq \cdots \tag{1.8}$$

and for some real β

$$|\arg a_j - \beta| \leq \alpha \leq \frac{\pi}{2}, j = 0, 1, 2, \ldots .$$

Then $f(z)$ does not vanish in

$$|z - \frac{(K-1)t}{M^2 - (K-1)^2}| < \frac{Mt}{M^2 - (K-1)^2}, \tag{1.9}$$

where

$$M = K(\cos\alpha + \sin\alpha) + 2\frac{\sin\alpha}{|a_0|} \sum_{j=1}^{\infty} |a_j| t^j. \tag{1.10}$$

In this paper, firstly we prove the following result, which gives maximum number of zeros that can lie in a prescribed region and also a zero-free region thereby improving Theorem C.

Theorem 1.1. *Let $p(z) = \sum_{j=0}^{n} a_j z^j$be a polynomial of degree n with complex coefficients. Also let $a_j = \alpha_j + i\beta_j$, and for some $K, L \geq 1$,*

$$K\alpha_n \geq \alpha_{n-1} \geq \cdots \geq \alpha_\lambda \leq \alpha_{\lambda-1} \leq \cdots \leq \alpha_1 \leq \alpha_0$$

and

$$L\beta_n \geq \beta_{n-1} \geq \cdots \geq \beta_\mu \leq \beta_{\mu-1} \leq \cdots \leq \beta_1 \leq \beta_0, \tag{1.11}$$

where α_0 and β_0 are not zero simultaneously. Then the maximum number of zeros in$\frac{|a_0|}{M_1} < |z| \leq \delta, 0 < \delta < 1$, does not exceed

$$\frac{1}{\log(1/\delta)} \log\left[\frac{(\alpha_0 + |\alpha_0|) + (\beta_0 + |\beta_0|) - 2(\alpha_\lambda + \beta_\mu) + K(\alpha_n + |\alpha_n|) + L(\beta_n + |\beta_n|)}{|a_0|}\right],$$

where

$$M_1 = (\alpha_0 + \beta_0) - 2(\alpha_\lambda + \beta_\mu) + K(\alpha_n + |\alpha_n|) + L(\beta_n + |\beta_n|). \tag{1.12}$$

For $\lambda = \mu = 0$, Theorem 1.1 reduces to following

Corollary 1.1. *Let $p(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with complex coefficients. Also let $a_j = \alpha_j + i\beta_j$, for $j = 0, 1, 2, \ldots, n$ and for some $K, L \geq 1$,*

$$K\alpha_n \geq \alpha_{n-1} \geq \cdots \geq \alpha_1 \geq \alpha_0,$$

and

$$L\beta_n \geq \beta_{n-1} \geq \cdots \geq \beta_1 \geq \beta_0, \tag{1.13}$$

with α_0 and β_0 are not simultaneously zero. Then the maximum number of zeros in $\frac{|a_0|}{M_2} < |z| \leq \delta, 0 < \delta < 1$, does not exceed

$$\frac{1}{\log(1/\delta)} \log\left[\frac{(|\alpha_0| + |\beta_0|) - (\alpha_0 + \beta_0) + K(|\alpha_n| + \alpha_n) + L(|\beta_n| + \beta_n)}{|a_0|}\right]$$

where

$$M_2 = K(|\alpha_n| + \alpha_n) + L(|\beta_n| + \beta_n) - (\alpha_0 + \beta_0). \tag{1.14}$$

If $\alpha_j > 0, \beta_j > 0$, for $j = 0, 1, 2, \ldots, n$; in (1.13) then we get the following

Corollary 1.2. *Let $p(z) = \sum_{j=0}^{n} a_j z^j$; $a_j = \alpha_j + i\beta_j, j = 0, 1, 2, \ldots, n$; be a polynomial of degree n such that for some $K, L \geq 1$,*

$$\begin{aligned} K\alpha_n &\geq \alpha_{n-1} \geq \cdots \geq \alpha_1 \geq \alpha_0 > 0 \\ L\beta_n &\geq \beta_{n-1} \geq \cdots \geq \beta_1 \geq \beta_0 > 0. \end{aligned} \tag{1.15}$$

Then maximum number of zeros in $\frac{|a_0|}{M_3} < |z| \leq \delta, 0 < \delta < 1$ does not exceed

$$\frac{1}{\log(1/\delta)} \log\left[\frac{2(K\alpha_n + L\beta_n)}{|a_0|}\right],$$

where

$$M_3 = 2(K\alpha_n + L\beta_n) - (\alpha_0 + \beta_0). \tag{1.16}$$

For the location of zeros of analytic functions we prove the following result which not only generalizes Theorem F, but in particular cases reduces to Theorem E and Theorem D also. More precisely we prove

Theorem 1.2. *Let $f(z) = \sum_{j=0}^{\infty} a_j z^j$ (not identically zero) be analytic in $|z| < t$. If for some $K \geq 1$,*

$$K|a_0| \geq t|a_1| \geq t^2|a_2| \geq \cdots \geq t^\lambda|a_\lambda| \leq t^{\lambda+1}|a_{\lambda+1}| \leq \cdots \tag{1.17}$$

and for some β,

$$|\arg a_j - \beta| \leq \alpha \leq \frac{\pi}{2}, j = 0, 1, 2, \ldots,$$

then $f(z)$ does not vanish in

$$\left|z - \frac{(K-1)t}{M_4^2 - (K-1)^2}\right| < \frac{M_4 t}{M_4^2 - (K-1)^2}, \tag{1.18}$$

where

$$M_4 = \left(K - 2t^\lambda\left|\frac{a_\lambda}{a_0}\right|\right)\cos\alpha + K\sin\alpha + 2\frac{\sin\alpha}{|a_0|}\sum_{j=1}^{\infty}|a_j|t^j. \tag{1.19}$$

Remark 1.1. *If we let $\lambda \to \infty$ in Theorem 1.2, then we get Theorem F. For $\lambda \to \infty$, $\alpha = \beta = 0$, Theorem 1.2 reduces to Theorem E and for the case $\lambda \to \infty$, $\alpha = \beta = 0, K = 1$, Theorem 1.2 reduces to Theorem D.*

2 Lemma

We need the following lemma for the proof of the above theorems.

Lemma 2.1. *Let $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ be a polynomial of degree n such that for some real β,*
$$|\arg a_j - \beta| \leq \alpha \leq \frac{\pi}{2}, j = 0, 1, 2, \ldots, n.$$
Then for some $t > 0$,
$$|ta_j - a_{j-1}| \leq |t|a_j| - |a_{j-1}|| \cos\alpha + (t|a_j| + |a_{j-1}|) \sin\alpha. \tag{2.1}$$

This Lemma follows from inequality (6) in [4].

3 Proof of the Theorems

Proof of Theorem 1.1. Consider the polynomial
$$\begin{aligned}
F(z) &= (1-z)p(z) \\
&= (1-z)(a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n) \\
&= a_0 + (a_1 - a_0)z + (a_2 - a_1)z^2 + \cdots + (a_n - a_{n-1})z^n - a_n z^{n+1} \\
&= a_0 + \sum_{j=1}^{n} (a_j - a_{j-1})z^j - a_n z^{n+1}.
\end{aligned}$$

Now for $|z| = 1$,
$$\begin{aligned}
|F(z)| &\leq |a_0| + \sum_{j=1}^{n} |a_j - a_{j-1}| + |a_n| \\
&\leq |\alpha_0| + |\beta_0| + |\alpha_n| + |\beta_n| + \sum_{j=1}^{n} |\alpha_j - \alpha_{j-1}| + \sum_{j=1}^{n} |\beta_j - \beta_{j-1}| \\
&= |\alpha_0| + |\beta_0| + |\alpha_n| + |\beta_n| + |\alpha_n - \alpha_{n-1}| + \sum_{j=1}^{n-1} |\alpha_j - \alpha_{j-1}| + |\beta_n - \beta_{n-1}| \\
&\quad + \sum_{j=1}^{n-1} |\beta_j - \beta_{j-1}| \\
&= |\alpha_0| + |\beta_0| + |\alpha_n| + |\beta_n| + |K\alpha_n - \alpha_{n-1} - K\alpha_n + \alpha_n| + \sum_{j=1}^{\lambda} |\alpha_j - \alpha_{j-1}| \\
&\quad + \sum_{j=\lambda+1}^{n-1} |\alpha_j - \alpha_{j-1}| + |L\beta_n - \beta_{n-1} - L\beta_n + \beta_n| + \sum_{j=1}^{\mu} (\beta_{j-1} - \beta_j) \\
&\quad + \sum_{j=\mu+1}^{n-1} (\beta_j - \beta_{j-1}).
\end{aligned}$$

or
$$\begin{aligned}
|F(z)| &\leq |\alpha_0| + |\beta_0| + |\alpha_n| + |\beta_n| + |K\alpha_n - \alpha_{n-1}| + (K-1)|\alpha_n| + \sum_{j=1}^{\lambda} (\alpha_{j-1} - \alpha_j) \\
&\quad + \sum_{j=\lambda+1}^{n-1} (\alpha_j - \alpha_{j-1}) + |L\beta_n - \beta_{n-1}| + (L-1)|\beta_n| + \sum_{j=1}^{\mu} (\beta_{j-1} - \beta_j)
\end{aligned}$$

$$+\sum_{j=\mu+1}^{n-1}(\beta_j-\beta_{j-1})$$
$$=|\alpha_0|+|\beta_0|+|\alpha_n|+|\beta_n|+K\alpha_n-\alpha_{n-1}+(K-1)|\alpha_n|+\alpha_0-2\alpha_\lambda+\alpha_{n-1}$$
$$+L\beta_n-\beta_{n-1}+(L-1)|\beta_n|+\beta_0-2\beta_\mu+\beta_{n-1}$$
$$=(\alpha_0+|\alpha_0|)+(\beta_0+|\beta_0|)-2(\alpha_\lambda+\beta_\mu)+K(\alpha_n+|\alpha_n|)+L(\beta_n+|\beta_n|)$$
$$=M \text{ (say)}.$$

Thus $|F(z)|\le M$, for $|z|=1$.

Also $|F(0)|=|p(0)|=|a_0|\neq 0$, as α_0 and β_0 are not zero simultaneously.

Now it is known (see [10; page 171] that if $f(z)$ is regular, $f(0)\neq 0$ and $|F(z)|\le M$ in $|z|\le 1$; then the number of zeros of $f(z)$ in $|z|\le\delta<1$ does not exceed $\frac{1}{\log(1/\delta)}\log(\frac{M}{|f(0)|})$. Applying this result to $F(z)$, we get the number of zeros of $F(z)$ and hence of $p(z)$ in $|z|\le\delta$ does not exceed

$$\frac{1}{\log(1/\delta)}\log\left[\frac{(\alpha_0+|\alpha_0|)+(\beta_0+|\beta_0|)-2(\alpha_\lambda+\beta_\mu)+K(\alpha_n+|\alpha_n|)+L(\beta_n+|\beta_n|)}{|a_0|}\right].$$

This proves one part of the theorem.

Now to show that no zeros lie in

$$|z|<\frac{|a_0|}{M_1}.$$

For this we have

$$F(z)=a_0+(a_1-a_0)z+(a_2-a_1)z^2+\cdots+(a_n-a_{n-1})z^n-a_nz^{n+1} \tag{3.1}$$

i.e. $\quad F(z)=a_0+h(z),$

where

$$h(z)=(a_1-a_0)z+(a_2-a_1)z^2+\cdots+(a_n-a_{n-1})z^n-a_nz^{n+1},$$
$$h(z)=-a_nz^{n+1}+(a_n-a_{n-1})z^n+\sum_{j=1}^{n-1}(a_j-a_{j-1})z^j.$$

For $|z|=1$,

$$\max_{|z|=1}|h(z)|\le|a_n|+|a_n-a_{n-1}|+\sum_{j=1}^{n-1}|a_j-a_{j-1}|$$
$$\le(|\alpha_n|+|\beta_n|)+|\alpha_n-\alpha_{n-1}|+|\beta_n-\beta_{n-1}|+\sum_{j=1}^{n-1}|\alpha_j-\alpha_{j-1}|+\sum_{j=1}^{n-1}|\beta_j-\beta_{j-1}|$$
$$\le(|\alpha_n|+|\beta_n|)+|K\alpha_n-\alpha_{n-1}-K\alpha_n+\alpha_n|+|L\beta_n-\beta_{n-1}-L\beta_n+\beta_n|$$
$$+\sum_{j=1}^{\lambda}|\alpha_j-\alpha_{j-1}|+\sum_{j=\lambda+1}^{n-1}|\alpha_j-\alpha_{j-1}|+\sum_{j=1}^{\mu}|\beta_j-\beta_{j-1}|+\sum_{j=\mu+1}^{n-1}|\beta_j-\beta_{j-1}|$$
$$\le(|\alpha_n|+|\beta_n|)+K\alpha_n-\alpha_{n-1}+(K-1)|\alpha_n|+\alpha_0-2\alpha_\lambda+\alpha_{n-1}$$
$$+L\beta_n-\beta_{n-1}+(L-1)|\beta_n|+\beta_0-2\beta_\mu+\beta_{n-1}$$
$$=(\alpha_0+\beta_0)-2(\alpha_\lambda+\beta_\mu)+K(\alpha_n+|\alpha_n|)+L(\beta_n+|\beta_n|)$$
$$=M_1. \text{ (Let)}$$

Thus $\max\limits_{|z|=1} |h(z)| \leq M_1$.

Therefore by Schwarz's lemma $|h(z)| \leq M_1|z|$ for $|z| \leq 1$.

Now from (3.1)

$$\begin{aligned} F(z) &= a_0 + h(z) \\ |F(z)| &\geq |a_0| - |h(z)| \\ &\geq |a_0| - M_1|z| \qquad\qquad \text{for } |z| \leq 1 \\ &> 0 \text{ if } |z| < \frac{|a_0|}{M_1}. \end{aligned}$$

Therefore no zeros of $F(z)$ and hence of $p(z)$ lie in

$$|z| < \frac{|a_0|}{M_1}.$$

This completes the proof of Theorem 1.1.

Proof of Theorem 1.2. Since $f(z)$ is analytic in $|z| < t$, therefore $\lim\limits_{j\to\infty} a_j z^j = 0$.

Now consider the function

$$\begin{aligned} F(z) &= (z-t)f(z) \\ &= (z-t)(a_0 + a_1 z + a_2 z^2 + \cdots + a_\lambda z^\lambda + \cdots) \\ &= -ta_0 + (a_0 - ta_1)z + (a_1 - ta_2)z^2 + \cdots \\ &= -ta_0 + a_0 z - Ka_0 z + (Ka_0 - ta_1)z + \sum_{j=2}^{\infty}(a_{j-1} - ta_j)z^j \end{aligned}$$

or

$$F(z) = -ta_0 + a_0 z - Ka_0 z + G(z), \tag{3.2}$$

where

$$G(z) = (Ka_0 - ta_1)z + z\sum_{j=2}^{\infty}(a_{j-1} - ta_j)z^{j-1}.$$

Clearly $G(z)$ is analytic; $G(0) = 0$ and for $|z| = t$

$$\begin{aligned} |G(z)| &\leq t|Ka_0 - ta_1| + t\sum_{j=2}^{\infty}|a_{j-1} - ta_j|t^{j-1} \\ &\leq t[(K|a_0| - t|a_1|)\cos\alpha + (K|a_0| + t|a_1|)\sin\alpha \\ &+ (t|a_1| - t^2|a_2|)\cos\alpha + (t|a_1| + t^2|a_2|)\sin\alpha \\ &+ (t^2|a_2| - t^3|a_3|)\cos\alpha + (t^2|a_2| + t^3|a_3|)\sin\alpha + \cdots + \\ &+ (t^{\lambda-1}|a_\lambda| - t^\lambda|a_\lambda|)\cos\alpha + (t^{\lambda-1}|a_{\lambda-1}| + t^\lambda|a_\lambda|)\sin\alpha \\ &+ (t^{\lambda+1}|a_{\lambda+1}| - t^\lambda|a_\lambda|)\cos\alpha + (t^{\lambda-1}|a_{\lambda+1}| + t^\lambda|a_\lambda|)\sin\alpha \\ &+ \cdots) \\ &= t[(K|a_0| - 2t^\lambda|a_\lambda|)\cos\alpha + K|a_0|\sin\alpha + 2\sin\alpha\sum_{j=1}^{\infty}|a_j|t^j] \\ &= t|a_0|\left[\left(K - 2t^\lambda\frac{|a_\lambda|}{|a_0|}\right)\cos\alpha + K\sin\alpha + \frac{2\sin\alpha}{|a_0|}\sum_{j=1}^{\infty}|a_j|t^j\right] \end{aligned}$$

$$= t|a_0|M_4, \text{ (say).}$$

where

$$M_4 = \left(K - 2t^{\lambda}|\frac{a_{\lambda}}{a_0}|\right)\cos\alpha + K\sin\alpha + 2\frac{\sin\alpha}{|a_0|}\sum_{j=1}^{\infty}|a_j|t^j. \tag{3.3}$$

This implies $|G(z)| \leq |a_0|M_4|z|$ for $|z| \leq t$ by Schwarz's lemma.

Hence from (3.2), we have

$$\begin{aligned}|F(z)| &\geq |ta_0 - a_0 z + Ka_0 z| - |G(z)| \\ &\geq |a_0||(K-1)z + t| - |a_0|M_4|z| \\ &> 0\end{aligned}$$

if

$$M_4|z| < |(K-1)z + t|. \tag{3.4}$$

Since it is easy to verify that the region defined by (3.4) is precisely the disk

$$\left[z : |z - \frac{(K-1)t}{M_4^2 - (K-1)^2}| < \frac{M_4 t}{M_4^2 - (K-1)^2}\right]. \tag{3.5}$$

It follows from (3.4) that $F(z)$ and hence $f(z)$ does not vanish in the disk defined by (3.5). This completes the proof of Theorem 1.2.

Acknowledgement

Author is highly thankful to anonymous referee for his valuable suggestions.

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Jñānābha, Vol. 49(2) (2019), 22-27

CHAOS CONTROL DYNAMICS IN COMPETITIVE HERBIVORE SPECIES NETWORK

By

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(*Received : August* 02, 2019 *; Revised: September* 25, 2019)

Abstract

This paper studies the mathematical modelling of a competitive ecological system in which the interactions between different species are being studied in the framework of ecological systems. Both linear and non-linear interactions have been accounted in the model. Through fixed point analysis, the critical value of parameter has been evaluated after which the system enters critical phase from phase of stability and then to chaos. Bifurcation plot for variation in coefficient of indirect dependency is plotted and used to verify the different phases of evolution of the interspecies relation. The system dynamics is observed to transit from stable to chaotic state through state of critical stability. To control chaos in the competitive ecological system under master slave scheme, it is synchronized to another stable identical ecosystem. Using Lyapunov stability theorem controller are devised. The active controller is observed to completely control the chaos in the system and restore stability of the ecological system.
2010 Mathematics Subject Classification: 93D15
Keywords and phrases: Competitive Species interaction, Bifurcation, Lyapunov function, Active Controller, Chaos.

1 Introduction

Ecosystems are fundamental units of nature in which species are connected in networks of food chain. These food chains are governed by the interactions between the species for the procurement of resources in nature and sustenance. Limited resources often lead to resource crunch which leads to competition between the species. Through competition and encounters nature strikes a balance between species growth and resource abundance. Mathematically fluctuations in case of species abundance were studied first by Volterra in 1926 [11]. Fluctuations in population evolution gradually lead to chaotic states in food chain in which species population randomly evolve with time.

Chaos was first studied in deterministic non fluidic flow by Lorenz in 1963 [6]. Chaos was first time mathematically observed in his study. Chaos has attracted the attention of several researchers since then which has led to further realization and understanding of random states in different kind of systems in various fields including ecology. The study on random impacts of complex damped systems gives an insight on significance behind observation and study of chaos [9]. The observation of chaos in different fields and their systems led to the rise of research interest of controlling it and restore stability of the system. The synchronization of chaotic Lorenz systems by using the active control mechanism discussed [1].

Active control mechanisms are one of the most widely used scheme of synchronizing the chaotic systems for controlling of chaos since then. For physical systems like nanofluid convection models [2], energy systems [7], and complex duffing system [10] chaos and its synchronization have been studied in detail. Similarly, in ecology chaos and its control in ecosystem models have been studied. Dynamics of cooperation in competition interaction models was discussed [5]. The controlling of chaos in food chain models was demonstrated [8]. Ecological food chains can be both linear and nonlinear kind with direct and indirect species dependence. The transition to chaos in three species nonlinear model of competitive ecosystem with an intermediate competitive herbivore species is studied [3].

The problem of study comprises of modelling a competitive herbivore ecosystem where one species indirectly effects the growth of other while competing for resources in situation of lack of resource abundance. The dynamics of the system have been analysed through stability analysis from which the parametric conditions have been derived that govern the system stability and its transition to random states. The bifurcation plots and Lyapunov exponents are determined to validate the observations and derived parametric conditions. Synchronization of chaos in supply chain systems using a Lyapunov function based single controller a multistate controller has been derived [4] for the system which on activation synchronizes two such identical chaotic systems in different dynamic phases and controls chaos. The role of migration is observed to be crucial for controlling the chaos in such competitive ecosystems.

2 Mathematical Modelling

Let us consider x_1, x_2 and x_3 represent the three population levels of three herbivore species residing in ecosystem where there is a resource shortage. The shortage of resources leads to higher mortality among these species primarily due to starvation and infightings. The species indirectly affect each others population level in a positive way as more is the population of one herbivore species higher are its chances to be getting preyed by the predators which provides indirect refuge to the other herbivore prey species. The encounters between the two species might possibly affect the population levels of a third species in either a positive or negative manner depending on the interactions between them. The model is described by following equations:

$$\begin{aligned} \dot{x_1} &= -a_2x_1 + b_2x_2, \\ \dot{x_2} &= -a_1x_2 + b_1x_1 - c_2x_1x_3, \\ \dot{x_3} &= -a_3x_3 + c_1x_1x_2, \end{aligned} \tag{2.1}$$

where a_1=coefficient of decay of species 1; a_2= coefficient of decay of species 2; a_3= coefficient of decay of species 3; b_1=coefficient of indirect dependency of species 2 on species 1; b_2=coefficient of indirect dependency of species 1 on species 2; c_1=coefficient of encounter between species 1 and species 3; c_2=coefficient of encounter between species 1 and species 2.

3 Stability Analysis

Three fixed point for system (1) are $(0,0,0)$, $\left(\sqrt{\frac{(b_1b_2-a_1a_2)a_3}{a_2c_1c_2}}, \frac{a_2}{b_2}\sqrt{\frac{(b_1b_2-a_1a_2)a_3}{a_2c_1c_2}}, \frac{(b_1b_2-a_1a_2)a_3}{a_2c_1c_2}\right)$ and $\left(-\sqrt{\frac{(b_1b_2-a_1a_2)a_3}{a_2c_1c_2}}, -\frac{a_2}{b_2}\sqrt{\frac{(b_1b_2-a_1a_2)a_3}{a_2c_1c_2}}, \frac{(b_1b_2-a_1a_2)a_3}{a_2c_1c_2}\right)$.

The Jacobian(J) of the system is given as follows:

$$J = \begin{bmatrix} -a_2 & b_2 & 0 \\ b_1 - c_2x_3 & -a_1 & -c_2x_1 \\ c_1x_2 & c_1x_1 & -a_3 \end{bmatrix} \tag{3.1}$$

3.1 <u>Case I</u>: For fixed point $(0,0,0)$ the characteristic equation is as follows:

$$\lambda^3 + e_1\lambda^2 + e_2\lambda + e_3 = 0,$$

where $e_1 = (a_1 + a_2 + a_3)$, $e_2 = (a_1a_2 + a_2a_3 + a_1a_3 - b_1b_2)$and $e_3 = (-a_3)(b_1b_2 - a_1a_2)$. From Routh-Hurwitz criteria for stability $e_1 > 0$, $e_3 > 0$ and $e_1e_2 - e_3 > 0$. Thus for stability at $(0,0,0)$ it is required that $b_1 < \left(\frac{a_1a_2}{b_2}\right) = b_0$. The system is asymptotically stable when $b_1 < b_0$, critically stable when $b_1 = b_0$ and unstable when $b_1 > b_0$.

3.2 <u>Case II</u>: For fixed point $\left(-\sqrt{\frac{(b_1b_2-a_1a_2)a_3}{a_2c_1c_2}}, -\frac{a_2}{b_2}\sqrt{\frac{(b_1b_2-a_1a_2)a_3}{a_2c_1c_2}}, \frac{(b_1b_2-a_1a_2)a_3}{a_2c_1c_2}\right)$ and $\left(\sqrt{\frac{(b_1b_2-a_1a_2)a_3}{a_2c_1c_2}}, \frac{a_2}{b_2}\sqrt{\frac{(b_1b_2-a_1a_2)a_3}{a_2c_1c_2}}, \frac{(b_1b_2-a_1a_2)a_3}{a_2c_1c_2}\right)$ show invariance under the transformation $(x_1, x_2, x_3) \to (-x_1, -x_2, -x_3)$. The characteristic equation is as follows:

$$\lambda^3 + e_1\lambda^2 + e_2\lambda + e_3 = 0,$$

where $e_1 = (a_1 + a_2 + a_3)$, $e_2 = (a_2a_3 + \frac{b_1b_2a_3}{a_2})$ and $e_3 = 2a_3(b_1b_2 - a_1a_2)$. Thus for stability it is required that $b_0 < b_1 < b_c$, where $b_c = (\frac{(a_1+a_2+a_3)(a_2a_3)+2(a_1a_2a_3)}{a_3b_2-(a_1+a_3)(\frac{b_2a_3}{a_2})})$ is the critical value of $b_1$and $b_0 = (\frac{a_1a_2}{b_2})$. The system is asymptotically stable when $b_0 < b_1 < b_c$, critically stable when $b_1 = b_c$ and unstable at $b_1 > b_c$ which leads to chaos finally.

4 Synchronization

For synchronization two identical system one in stable state and another in chaotic state are considered as master and slave system which are mentioned as follows:

Master system

$$\begin{aligned} \dot{x_1} &= -a_2x_1 + b_2x_2, \\ \dot{x_2} &= -a_1x_2 + b_1x_1 - c_2x_1x_3, \\ \dot{x_3} &= -a_3x_3 + c_1x_1x_2. \end{aligned}$$

Slave system

$$\begin{aligned} \dot{y_1} &= -a_2y_1 + b_2x_2, \\ \dot{y_2} &= -A_1y_2 + B_1x_1 - c_2x_1x_3, \\ \dot{y_3} &= -a_3y_3 + c_1x_1x_2. \end{aligned}$$

This leads to the following error system :

$$\begin{aligned} \dot{e_1} &= -a_2e_1 + b_2e_2, \\ \dot{e_2} &= -(A_1 - a_1)y_2 + (B_1 - b_1)y_1 - a_1e_2 + b_1e_1 - c_2y_3e_1 - c_2x_1e_3, \\ \dot{e_3} &= -a_3e_3 + c_1(y_2e_1 - x_1e_2). \end{aligned}$$

The Lyapunov function ϕ is given as follows:

$$\phi(e_1, e_2, e_3) = \frac{1}{2}(e_1^2 + e_2^2 + e_3^2)$$

$$\implies \dot{\phi} = e_1\dot{e_1} + e_2\dot{e_2} + e_3\dot{e_3}.$$

The synchronization between the master and slave is obtained from the following conditions:

$$Condition\ 1 \ : \quad \lim_{t\to\infty} \|e(t)\| = 0,$$
$$Condition\ 2 \ : \quad \dot{\theta} \leq 0.$$

From Condition 1 and Condition 2 two controllers Controller 1 and Controller 2 are derived which are given as follows:

Controller 1*:*

$$\begin{aligned} u_1 &= a_2e_1 - b_2e_2 - e_1, \\ u_2 &= (A_1 - a_1)y_2 - (B_1 - b_1)y_1 + a_1e_2 - b_1e_1 + c_2y_3e_1 + c_2x_1e_3 - e_2, \\ u_3 &= a_3e_3 - c_1(y_2e_1 - x_1e_2) - e_3. \end{aligned} \tag{4.1}$$

and *Controller* 2*:*

$$\begin{aligned} h_1 &= -b_2e_2, \\ h_2 &= (A_1 - a_1)y_2 - (B_1 - b_1)y_1 + a_1e_2 - b_1e_1 + c_2(y_3 - x_3), \\ h_3 &= -c_1(y_1 - x_1). \end{aligned}$$

Using controller 1 and controller 2 the master and slave system are synchronized for controlling of chaos.

5 Results and Discussion

Numerical simulation of the above system is carried out for different values of the fixed parameters, a_1= 1,a_2=5, a_3=1, b_2=6, c_1=2, c_2=1 and varying parameter b_1. For the fixed parameter values, one gets $b_0 = (\frac{a_1a_2}{b_2}) = 0.83$ while $b_c = (\frac{(a_1+a_2+a_3)(a_2a_3)+2(a_1a_2a_3)}{a_3b_2-(a_1+a_3)(\frac{b_2a_3}{a_2})}) = 11.5$. For different value of b_1 the following observation are made:

- when b_1=6, the condition $b_1 > b_0$ and $b_1 < b_c$ is satisfied and the system is observe to be in stable state.
- when b_1=11, the condition $b_1 > b_0$ and $b_1 \approx b_c$ is satisfied and so critically stable state is observed.
- when b_1=16, $b_1 > b_0$ and $b_1 > b_c$ the system is in chaotic state.

In Figure 5.1, the transition of dynamic state from stable to chaotic phase can be observed through the bifurcation diagram for variation in b_1.

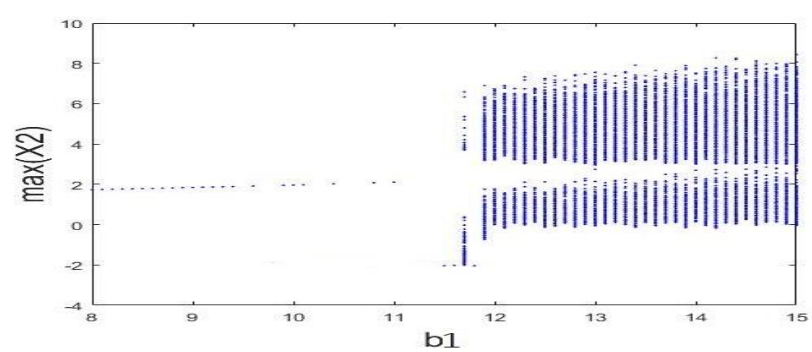

Figure 5.1: Bifurcation diagram for variation in b1 parameter

The Lyapunov exponent analysis is further carried out to ensure the presence of chaos after . The details of the different states and the value of the Lyapunov exponents which validate the dynamics of the system are given in Table 5.1.

Table 5.1: Lyapunov exponent values determined for different stages of system dynamics

Value of b_1	λ_1	λ_2	λ_3	Observed Dynamic State
6	-0.209	-0.213	-6.578	Stable Spiral State
11	-0.04192	-0.04583	-6.912	Critical Two Torus State
16	0.4007	-0.002381	-7.398	Chaotic Strange Attractor State

From Figure 5.2 it is evident that the Controller 1 and Controller 2 derived from Condition 1 and Condition 2 of synchronization between master and slave system are perfectly synchronizing and controlling chaos.

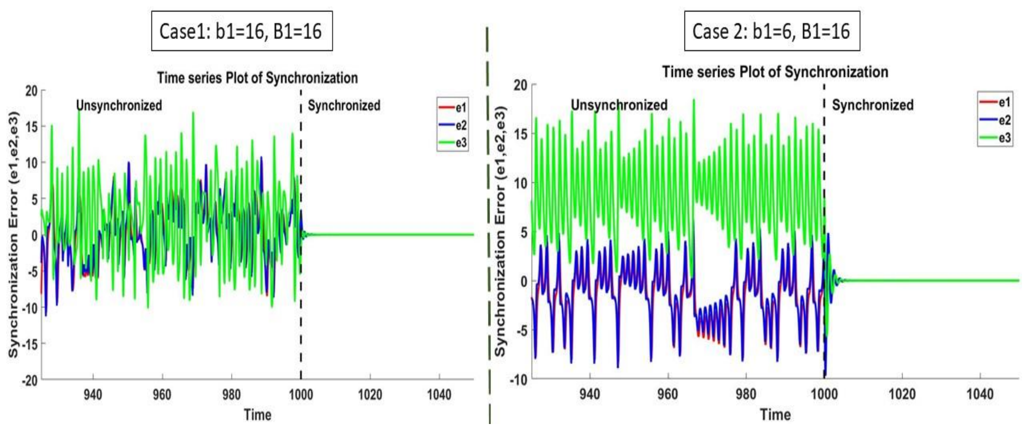

Figure 5.2: Synchronization in Case 1 and Synchronization with chaos control of chaos in Case 2 on activation of controller at t=1000

It is further observed that Controller 1 is faster than Controller 2 by three-unit time due to presence of more migration terms, details of which are mentioned in Table 5.1. This highlights the importance of migration in stabilizing the population levels ensuring stability of the system.

Table 5.2: Time taken by Controllers to synchronize and control chaos after activation at t=1000

Controller	Time for Case1: Synchronization	Time for Case2: Synchronization+ Control
Controller 1	1005	1008
Controller 2	1008	1011

6 Conclusion

In this paper, the herbivore competitive ecosystem in a resource crunch environment is studied. From stability analysis parametric condition governing the transition of dynamic state of system is derived. All the three phases: stable, intermittence and chaotic states are observed. Using Lyapunov function, the controllers are designed which synchronize the chaotic system successfully for control of chaos. Controller 1 is observed to be faster than Controller 2 in controlling chaos due to presence of more population interaction terms. It can be concluded that interaction plays a crucial role in stabilizing the population levels and restoring the stability of the system which susceptible to fluctuations to ecological disturbances.

Acknowledgements
For providing research facilities and financial support authors are thankful to Guru Gobind Singh Indraprastha University, Delhi (India). The authors are also thankful to the referee for his suggestions to bring the paper in its present form.

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Jñānābha, Vol. 49(2) (2019), 28-33

SOME CHARACTERIZATIONS OF CHAUDHRY AND AHMAD PROBABILITY DISTRIBUTION USEFUL IN SIZE MODELING

By

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(*Received : August* 11, 2019 ; *Revised: September* 22, 2019)

Abstract

The objective of this paper is to give some new characterizations by truncated moment, order statistics and upper record values for a probability distribution introduced by Chaudhry and Ahmad useful in size modeling [8].

2010 Mathematics Subject Classification: 60E05, 62E10, 62E15, 62G30

Keywords and phrases: Characterizations; Order Statistics; Record Values; Truncated Moment.

1 Introduction

Before a particular probability distribution is applied to real world data, it is important to characterize it subject to certain conditions. As pointed out by Nagaraja [12], A characterization is a certain distributional or statistical property of a statistic or statistics that uniquely determines the associated stochastic model. For example, all the Pearson distributions can be characterized by their first four moments, provided these exist; see Stuart and Ord [16]. Similarly, as pointed out by Patel and Read [14], If a normally distributed rv, or if a random sample from a normal distribution, has some property P, then it may be of interest to know whether such a property characterizes the normal law. Many authors and researchers have investigated the problems of characterizations of probability distributions at different times. For some recent research work, the interested readers are referred to Ahsanullah [2], Ahsanullah, et al. [4, 5, 6], Galambos and Kotz [10], Kotz and Shanbhag [11], and Nagaraja [12], among others. In this paper, motivated by the importance of the characterizations of probability distributions in practical problems, we establish some new characterization results by truncated moment, order statistics and record values for a probability distribution introduced by Chaudhry and Ahmad useful in size modeling [8], with the probability density function (pdf) and cumulative distribution function (cdf) respectively given by:

$$f_X(x) = 2\sqrt{\frac{\alpha}{\pi}} \exp\left[-\left\{\left(\sqrt{\alpha}\right)x - \left(\sqrt{\beta}\right)x^{-1}\right\}^2\right], \quad \alpha > 0, \beta > 0, x > 0 \quad (1.1)$$

and

$$F(x) = \frac{1}{\sqrt{\pi}} \exp\left(2\sqrt{\alpha\beta}\right) \gamma\left(\frac{1}{2}, \alpha x^2; \alpha\beta\right) \quad (1.2)$$

$$= 1 - \frac{1}{\sqrt{\pi}} \exp\left(2\sqrt{\alpha\beta}\right) \Gamma\left(\frac{1}{2}, \alpha x^2; \alpha\beta\right), \alpha > 0, \beta > 0, \quad (1.3)$$

where $\gamma(a, z; b) = \int\limits_0^z t^{a-1} e^{\left(-t-bt^{-1}\right)} dt$ and $\Gamma(a, z; b) = \int\limits_z^\infty t^{a-1} e^{\left(-t-bt^{-1}\right)} dt$ denote the generalized incomplete gamma functions. The moment is given by

$$E(X) = 2\sqrt{\frac{\beta}{\pi}} e^{2\sqrt{\alpha\beta}} K_1\left(2\sqrt{\alpha\beta}\right) \tag{1.4}$$

where $K_\nu(z)$ denotes the modified Bessel function of the second kind (or Macdonald function). For applications and more distributional properties, please refer to Chaudhry and Ahmad [8].

2 Characterization Results

In this section, we give our proposed characterizations of Chaudhry and Ahmad's probability distribution by truncated moment, order statistics and record values.

2.1 Characterization by Truncated Moment

For this, we will need the following assumption and lemmas.

Assumption 2.1. *Suppose the random variable X is absolutely continuous with the cumulative distribution function $F(x)$ and the probability density function $f(x)$. We assume that $\omega = \inf\{x|F(x) > 0\}$, and $\delta = \sup\{x|F(x) < 1\}$. We also assume that $f(x)$ is a differentiable for all x, and $E(X)$ exists.*

Lemma 2.1. *Under the Assumption 2.1, if $E(X|X \le x) = g(x)\tau(x)$, where $\tau(x) = \frac{f(x)}{F(x)}$ and $g(x)$ is a continuous differentiable function of x with the condition that $\int_0^x \frac{u-g'(u)}{g(u)} du$is finite for $x > 0$, then $f(x) = ce^{\int_0^x \frac{u-g'(u)}{g(u)} du}$, where c is a constant determined by the condition $\int_0^\infty f(x)dx = 1$.*

Proof. For proof, see Shakil, et al. [15]. □

Lemma 2.2. *Under the Assumption 2.1, if $E(X|X \ge x) = \tilde{g}(x)r(x)$, where $r(x) = \frac{f(x)}{1-F(x)}$ and $\tilde{g}(x)$ is a continuous differentiable function of x with the condition that $\int_x^\infty \frac{u+[\tilde{g}(u)]'}{\tilde{g}(u)}$ is finite for $x > 0$, then $f(x) = ce^{-\int_0^x \frac{u+[\tilde{g}(u)]'}{\tilde{g}(u)} du}$, where c is a constant determined by the condition $\int_0^\infty f(x)dx = 1$.*

Proof. For proof, see Shakil, et al. [15]. □

Theorem 2.1. *If the random variable X satisfies the Assumption 2.1 with $\omega = 0$ and $\delta = \infty$, then $E(X|X \le x.) = g(x)\frac{f(x)}{F(x)}$, where*

$$g(x) = \frac{\gamma(1, \alpha x^2; \alpha\beta)}{2\alpha \exp(-\alpha x^2 - \beta x^{-2})}, \tag{2.1}$$

if and only if X has the pdf $f_X(x) = 2\sqrt{\frac{\alpha}{\pi}} \exp\left[-\left\{\left(\sqrt{\alpha}\right)x - \left(\sqrt{\beta}\right)x^{-1}\right\}^2\right]$.

Proof. Suppose that $E(X|X \le x) = g(x)\dfrac{f(x)}{F(x)}$. Then, since $E(X|X \le x) = \dfrac{\int_0^x uf(u)du}{F(x)}$, we have $g(x) = \dfrac{\int_0^x uf(u)du}{f(x)}$. Now, if the random variable X satisfies the Assumption 2.1 and has the distribution with the pdf (1), then we have

$$g(x) = \frac{\int_0^x uf(u)du}{f(x)} = \frac{\int_0^x u\exp(-\alpha u^2 - \beta u^{-2})du}{\exp(-\alpha x^2 - \beta x^{-2})}. \tag{2.2}$$

Let $\alpha u^2 = t$ in Eq. (1.6). Then, using the definition of the generalized incomplete gamma function, we have

$$\begin{aligned} g(x) &= \frac{\int_0^{\alpha x^2} \exp(-t - \alpha\beta t^{-1})dt}{2\alpha\exp(-\alpha x^2 - \beta x^{-2})} \\ &= \frac{\gamma(1, \alpha x^2; \alpha\beta)}{2\alpha\exp(-\alpha x^2 - \beta x^{-2})}. \end{aligned}$$

Conversely, suppose that

$$g(x) = \frac{\gamma(1, \alpha x^2; \alpha\beta)}{2\alpha\exp(-\alpha x^2 - \beta x^{-2})}.$$

Then, differentiating $g(x)$ with respect to x and using Lemma 2.1, we have

$$g^{/}(x) = x - g(x)(\frac{2\beta}{x^3} - 2\alpha x),$$

from which we obtain

$$\frac{x - g^{/}(x)}{g(x)} = \frac{2\beta}{x^3} - 2\alpha x.$$

Since, by Lemma 2.1, we have

$$\frac{x - g^{/}(x)}{g(x)} = \frac{f^{/}(x)}{f(x)},$$

it follows that

$$\frac{f^{/}(x)}{f(x)} = \frac{2\beta}{x^3} - 2\alpha x.$$

On integrating the above expression with respect to x and simplifying, we obtain

$$\ln f(x) = \ln(ce^{-\alpha x^2 - \beta x^{-2}}),$$

or,

$$f(x) = ce^{-\alpha x^2 - \beta x^{-2}},$$

where c is the normalizing constant to be determined. Thus, on integrating the above equation with respect to x from $x = 0$ to $x = \infty$, and using the condition $\int_0^\infty f(x)dx = 1$, we obtain $c = 2\sqrt{\frac{\alpha}{\pi}}e^{2\sqrt{\alpha\beta}}$. This completes the proof of Theorem 2.1. □

Theorem 2.2. *If the random variable X satisfies the Assumption 2.1 with $\omega = 0$ and $\delta = \infty$, then $E(X|X \geq x) = \tilde{g}(x)\frac{f(x)}{1-F(x)}$, where*

$$\tilde{g}(x) = \frac{(E(X) - g(x)f(x))(\sqrt{\frac{\pi}{\alpha}})}{2\exp[-\{(\sqrt{\alpha})x - (\sqrt{\beta})x^{-1}\}^2]},$$

where $g(x)$ is given by Eq. (1.5) and $E(X)$ is given by Eq. (1.4), if and only if

$$f_X(x) = 2\sqrt{\frac{\alpha}{\pi}}\exp[-\{(\sqrt{\alpha})x - (\sqrt{\beta})x^{-1}\}^2].$$

Proof. Suppose that $E(X|X \geq x) = \tilde{g}(x)\frac{f(x)}{1-F(x)}$. Then, since $E(X|X \geq x) = \frac{\int_x^\infty uf(u)du}{1-F(x)}$, we have $\tilde{g}(x) = \frac{\int_x^\infty uf(u)du}{f(x)}$. Now, if the random variable X satisfies the Assumptions 2.1 and has the distribution with the pdf (1), then we have

$$\begin{aligned}\tilde{g}(x) &= \frac{\int_x^\infty uf(u)du}{f(x)} = \frac{\int_0^\infty uf(u)du - \int_0^x uf(u)du}{f(x)} \\ &= \frac{(E(X) - g(x)f(x))(\sqrt{\frac{\pi}{\alpha}})}{2\exp[-\{(\sqrt{\alpha})x - (\sqrt{\beta})x^{-1}\}^2]}.\end{aligned}$$

Conversely, suppose that $\tilde{g}(x) = \frac{(E(X)-g(x)f(x))(\sqrt{\frac{\pi}{\alpha}})}{2\exp[-\{(\sqrt{\alpha})x-(\sqrt{\beta})x^{-1}\}^2]}$. Then, using Lemma 2.2, differentiating $\tilde{g}(x)$ with respect to x, and simplifying, we have

$$(\tilde{g}(x)) = -x - \tilde{g}(x)\left(\frac{2\beta}{x^3} - 2\alpha x\right),$$

from which we obtain

$$\frac{x + (\tilde{g}(x'))}{\tilde{g}(x)} = -\left(\frac{2\beta}{x^3} - 2\alpha x\right).$$

Since, by Lemma 2.2, we have

$$\frac{f'(x)}{f(x)} = -\frac{x+[\tilde{g}(x)]'}{\tilde{g}(x)},$$

it follows that

$$\frac{f'(x)}{f(x)} = \frac{2\beta}{x^3} - 2\alpha x.$$

On integrating the above expression with respect to x and simplifying, we obtain

$$\ln f(x) = \ln(ce^{-\alpha x^2 - \beta x^{-2}}),$$

or,

$$f(x) = ce^{-\alpha x^2 - \beta x^{-2}},$$

where c is the normalizing constant to be determined. Thus, on integrating the above equation with respect to x from $x = 0$ to $x = \infty$, and using the condition $\int_0^\infty f(x)dx = 1$, we obtain $c = 2\sqrt{\frac{\alpha}{\pi}}e^{2\sqrt{\alpha\beta}}$. This completes the proof of Theorem 2.2. □

2.2 Characterizations by Order Statistics

If $X_1, X_2, \ldots, X_n$ be the n independent copies of the random variable X with absolutely continuous distribution function $F(x)$ and pdf $f(x)$, and if $X_{1,n} \leq X_{2,n} \leq \ldots \leq X_{n,n}$ be the corresponding order statistics, then it is known from Ahsanullah, et al. [3], chapter 5, or Arnold, et al. [7], Chapter 2, that $X_{j,n}|X_{k,n} = x$, for $1 \leq k < j \leq n$, is distributed as the $(j-k)th$ order statistics from $(n-k)$ independent observations from the random variable V having the pdf $f_V(v|x)$ where $f_V(v|x) = \frac{f(v)}{1-F(x)}$, $0 \leq v < x$, and $X_{i.,n}|X_{k,n} = x$, $1 \leq i < k \leq n$, is distributed as ith order statistics from k independent observations from the random variable W having the pdf $f_W(w|x)$where $f_W(w|x) = \frac{f(w)}{F(x)}$, $w < x$. Let $S_{k-1} = \frac{1}{k-1}(X_{1,n} + X_{2,n} + \ldots + X_{k-1,n})$, and $T_{k,n} = \frac{1}{n-k}(X_{k+1,n} + X_{k+2,n} + \ldots + X_{n.n})$.

Theorem 2.3. *Suppose the random variable X satisfies the Assumption* 2.1 *with* $\omega = 0$ *and* $\delta = \infty$*, then* $E(S_{k-1}|X_{k,n} = x) = g(x)\tau(x)$*, where* $\tau(x) = \frac{f(x)}{F(x)}$ *and*

$$g(x) = \frac{\gamma(1, \alpha x^2; \alpha\beta)}{2\alpha \exp(-\alpha x^2 - \beta x^{-2})},$$

if and only if $f_X(x) = 2\sqrt{\frac{\alpha}{\pi}} \exp[-\{(\sqrt{\alpha})x - (\sqrt{\beta})x^{-1}\}^2]$.

Proof. It is known from Ahsanullah, et al. [3], and David and Nagaraja [9] that $E(S_{k-1}|X_{k,n} = x) = E(X|X \leq x)$. Hence, by Theorem 2.1, the result follows. □

Theorem 2.4. *Suppose the random variable X satisfies the Assumption* 2.1 *with* $\omega = 0$ *and* $\delta = \infty$*, then* $E(T_{k,n}|X_{k,n} = x) = \tilde{g}(x)\frac{f(x)}{1-F(x)}$*, where*

$$\tilde{g}(x) = \frac{(E(X) - g(x)f(x))(\sqrt{\frac{\pi}{\alpha}})}{2\exp[-\{(\sqrt{\alpha})x - (\sqrt{\beta})x^{-1}\}^2]},$$

if and only if $f_X(x) = 2\sqrt{\frac{\alpha}{\pi}} \exp[-\{(\sqrt{\alpha})x - (\sqrt{\beta})x^{-1}\}^2]$.

Proof. It is known from Ahsanullah, et al. [3], and David and Nagaraja [9] that $E(T_{k,n}|X_{k,n} = x) = E(X|X \geq x)$. Hence the result follows from Theorem 2.2. □

2.3 Characterization by Upper Record Values

For details on record values, see Ahsanullah [1]. Let $X_1, X_2, ...$ be a sequence of independent and identically distributed absolutely continuous random variables with distribution function $F(x)$ and pdf $f(x)$. If $Y_n = \max(X_1, X_2, ..., X_n)$ for $n \geq 1$ and $Y_j > Y_{j-1}, j > 1$, then X_j is called an upper record value of $\{X_n, n \geq 1\}$. The indices at which the upper records occur are given by the record times $\{U(n) > \min(j|j > U(n+1), X_j > X_{U(n-1)}, n > 1)\}$ and $U(1) = 1$. Let the *nth* upper record value be denoted by $X(n) = X_{U(n)}$.

Theorem 2.5. *Suppose the random variable X satisfies the Assumption* 2.1 *with* $\omega = 0$ *and* $\delta = \infty$*, then* $E(X(n+1)|X(n) = x) = \tilde{g}(x)\frac{f(x)}{1-F(x)}$*, where* $\tilde{g}(x) = \frac{(E(X)-g(x)f(x))(\sqrt{\frac{\pi}{\alpha}})}{2\exp[-\{(\sqrt{\alpha})x-(\sqrt{\beta})x^{-1}\}^2]}$*, if and only if* $f_X(x) = 2\sqrt{\frac{\alpha}{\pi}} \exp[-\{(\sqrt{\alpha})x - (\sqrt{\beta})x^{-1}\}^2]$.

Proof. It is known from Ahsanullah et al. [3], and Nevzorov [13] that $E(X(n+1)|X(n) = x) = E(X|X \geq x)$. Then, the result is evident from Theorem 2.2. □

3 Concluding Remarks

In this paper, we have considered a two-parameter probability distribution introduced by Chaudhry and Ahmad useful in size modeling [8]. We have discussed some of its basic distributional properties. Based on these properties, we have established some new characterization results for Chaudhry and Ahmad's distribution by truncated moment, order statistics and upper record values. We hope the findings of the paper will be quite useful for the practitioners in various fields of sciences.

Acknowledgement

We are very much thankful to the referee and Executive Editor for their suggestions to bring the paper in its present form.

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Jñānābha, Vol. 49(2) (2019), 34-44

COMMON FIXED POINTS FOR GENERALIZED MULTIVALUED CONTRACTION MAPPINGS ON WEAK PARTIAL METRIC SPACES

By

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(*Received : August* 16, 2019 *; Revised: October* 25, 2019)

Abstract

In this paper, we define multivalued generalized contraction map and establish a common fixed point theorem for this map in the setting of newly introduced weak partial metric space which generalize some well known results of partial metric space in the literature.

2010 Mathematics Subject Classifications: 47H10, 54H25.
Keywords and phrases: Weak Partial metric space, Partial Hausdorff metric, Common fixed point.

1 Introduction and Preliminaries

The famous Banach Contraction Principle is an important tool in the metric fixed point theory because of its constructive method to find fixed point of certain self maps of metric spaces. It has been widely used principle. Many researchers generalize this theorem in different directions.

Following the Banach Contraction Principle Nadler[7] first initiated the study of fixed point theorems for multi-valued contraction self-mappings and established a remarkable result for multivalued contractions in metric spaces.

Let $CB(X)$ be a collection of all nonempty closed and bounded subsets of X. For $E, F \in CB(X)$, define

$$H(E,F) = \max\{\sup_{a\in E} d(a,F), \sup_{b\in F} d(b,E)\},$$

where $d(x,E) = \inf\{d(x,a) : a \in E\}$ is the distance between a point x and a set E. It is known that H is a metric on $CB(X)$, called the Hausdorff metric induced by d.

Definition 1.1. [7]. *Let $T : X \to 2^X$ be a multi-valued map. An element $x \in X$ is called a fixed point of T if $x \in Tx$.*

The Nadler's fixed point theorem is the following.

Theorem 1.1. [7]. *Let (X,d) be a complete metric space, and $T : X \to CB(X)$ be a multi-valued map satisfying*

$$H(Tx,Ty) \le kd(x,y),$$

for all $x, y \in X$, where $k \in [0,1)$. Then T has a fixed point.

Afterwards a rapid progress in the fixed point theory for multivalued mappings has been observed.

In 1992 *Matthews* [6] *introduced a new type of metric called partial metric and corresponding space is called the partial metric space as the part of the study of denotational semantics of dataflow networks. Matthews proved a version of Banach fixed point theorem in partial metric space. Much work has been done in this direction (see, for instance* [1], [3], [5] *and references therein).*

Following are some definitions and results needed in the sequel.

Definition 1.2. [6]*. Let X be a non empty set. Then a mapping $p : X \times X \to \mathbb{R}^+$ is said to be a partial metric on X if for all $x, y, z \in X$,*
$(P1)$ $x = y \Leftrightarrow p(x, x) = p(x, y) = p(y, y)$;
$(P2)$ $p(x, x) \leq p(x, y)$;
$(P3)$ $p(x, y) = p(y, x)$;
$(P4)$ $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.

The pair (X, p) is called a partial metric space.

In 2012*, Aydi et al.* [2] *proved the Banach-type fixed point result for set-valued mappings in complete partial metric space.*

Theorem 1.2. [2]*. Let (X, p) be a complete partial metric space. If $T : X \to CB^p(X)$ is a multi-valued mapping such that for all $x, y \in X$, we have*

$$H_p(Tx, Ty) \leq kp(x, y),$$

where $k \in (0, 1)$. Then T has a fixed point.

The above contraction condition called as H_p-contraction.

Recently Ismat Beg and H.K.Pathak [4] *introduced a weaker form of partial metric called Weak Partial Metric Space which is defined as follows:*

Definition 1.3. [4]*. Let X be a nonempty set. A function $q : X \times X \to \mathbb{R}^+$ is called a weak partial metric on X if for all $x, y, z \in X$, the following conditions hold:*
$(WP1)$ $q(x, x) = q(x, y) \Leftrightarrow x = y$;
$(WP2)$ $q(x, x) \leq q(x, y)$;
$(WP3)$ $q(x, y) = q(y, x)$;
$(WP4)$ $q(x, y) \leq q(x, z) + q(z, y)$.
The pair (X, q) is a weak partial metric space.
Examples:
(1) $(\mathbb{R}^+, q)$*, where $q : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is defined as*

$$q(x, y) = |x - y| + 1 \text{ for all } x, y \in \mathbb{R}^+.$$

(2) $(\mathbb{R}^+, q)$*, where $q : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is defined as*

$$q(x, y) = \frac{1}{4}|x - y| + \max\{x, y\} \text{ for all } x, y \in \mathbb{R}^+.$$

Notice that
- *If $q(x, y) = 0$, then $(WP1)$ and $(WP2) \Rightarrow x = y$. But the converse need not be true.*
- $(P1) \Rightarrow (WP1)$*, but the converse need not be true.*
- $(P4) \Rightarrow (WP4)$*, but the converse need not be true.*

Each weak partial metric q on X generates a T_0 topology τ_q on X which has as a base the family of open q-balls $\{B_q(x, \epsilon) : x \in X,\ \epsilon > 0\}$, where $B_q(x, \epsilon) = \{y \in X : q(x, y) < q(x, x) + \epsilon\}$ for all $x \in X$ and $\epsilon > 0$.

If q is a weak partial metric on X, then the function $q^s : X \times X \to \mathbb{R}^+$ given by

$$q^s(x,y) = q(x,y) - \frac{1}{2}\big[q(x,x) + q(y,y)\big], \text{ defines a metric on } X.$$

A sequence $\{x_n\}$ in (X,q) converges to a point $x \in X$ with respect to τ_q if and only if $q(x,x) = \lim_{n\to\infty} q(x,x_n)$. Furthermore, a sequence $\{x_n\}$ converges in (X,q^s) to a point $x \in X$ if and only if

$$q(x,x) = \lim_{n\to\infty} q(x_n,x) = \lim_{n,m\to\infty} q(x_n,x_m).$$

Let (X,q) be a weak partial metric space. Let $CB^q(X)$ be the family of all nonempty, closed and bounded subsets of (X,q). Here boundedness is given as follows: A is a bounded subset in (X,q) if there exist $x_0 \in X$ and $M \geq 0$ such that for all $a \in A$, we have $a \in B_q(x_0,M)$, that is,

$$q(x_0,a) < q(a,a) + M.$$

For $E,F \in CB^q(X)$ and $x \in X$, define $q(x,E) = \inf\{q(x,a) : a \in E\}$, $\delta_q(E,F) = \sup\{q(a,F) : a \in E\}$, and $\delta_q(F,E) = \sup\{q(b,E) : b \in F\}$.

Clearly $q(x,E) = 0 \Longrightarrow q^s(x,E) = 0$ where $q^s(x,E) = \inf\{q^s(x,a) : a \in E\}$.

Remark 1.1. [4]. *Let (X,q) be a weak partial metric space and E any nonempty set in (X,p), then*

$$a \in \bar{E} \Leftrightarrow q(a,E) = q(a,a),$$

where $\bar{E}$ denotes the closure of E with respect to the weak partial metric q. First we observe that $q(a,\bar{E}) = q(a,E)$. If $a \in \bar{E}$ then $q(a,E) = q(a,\bar{E}) = \inf_{b\in\bar{E}} d(a,b) = d(a,a)$. Now for any a in X there is a minimizing sequence $\{a_n\}$ in E such that $q(a,E) = \lim q(a,a_n)$. So if $q(a,E) = q(a,a), \lim q(a,a_n) = q(a,a)$; that is, $a = \lim a_n$ and so $a \in \bar{E}$.

Note that E is closed in $(X,q) \Leftrightarrow E = \bar{E}$. The followings are some properties of mapping $\delta_q : CB^q(X) \times CB^q(X) \to [0,\infty)$.

Proposition 1.1. [4]. *Let (X,q) be a weak partial metric space. For any $E,F,H \in CB^q(X)$, we have the following:*
(i) $\delta_q(E,E) = \sup\{q(a,a) : a \in E\}$;
(ii) $\delta_q(E,E) \leq \delta_q(E,F)$;
(iii) $\delta_q(E,F) = 0 \Rightarrow E \subseteq F$;
(iv) $\delta_q(E,F) \leq \delta_q(E,H) + \delta_q(H,F)$.

Proposition 1.2. [4]. *Let (X,q) be a weak partial metric space. For all $E,F,H \in CB^q(X)$, we have*
$(wh1)$ $H_q^+(E,E) \leq H_q^+(E,F)$;
$(wh2)$ $H_q^+(E,F) = H_q^+(F,E)$;
$(wh3)$ $H_q^+(E,F) \leq H_q^+(E,H) + H_q^+(H,F)$.

Definition 1.4. [4]. *Let (X,q) be a weak partial metric space. For $E,F \in CB^q(X)$, define*

$$H_q^+(E,F) = \frac{1}{2}\{\delta_q(E,F) + \delta_q(F,E)\}.$$

The mapping $H_q^+ : CB^q(X) \times CB^q(X) \to [0,+\infty)$, is called H_q^+-type Hausdorff metric induced by q.

Definition 1.5. [4]. *Let (X,q) be a complete weak partial metric space. A multi-valued map $T : X \to CB^q(X)$ is called H_q^+ -contraction if for every $x, y \in X$,*
(i) there exists α in $(0,1)$ such that
$$H_q^+(T(x) \setminus \{x\}, T(y) \setminus \{y\}) \leq \alpha \; q(x,y),$$
(ii) for every x in X, y in $T(x)$ and $\epsilon > 0$, there exists z in $T(y)$ such that
$$q(y,z) \leq H_q^+(T(y), T(x)) + \epsilon.$$

Remark 1.2. [4]. *Since $\max\{a,b\} \geq \frac{1}{2}(a+b)$ for all $a, b \geq 0$, it follows that H_q - contraction always implies H_q^+ - contraction, but the converse implication need not be true.*

Beg and Pathak[4] gave the following variant of Nadler's fixed point theorem.

Theorem 1.3. [4]. *Every H_q^+- type multivalued contraction on a complete metric space (X,q) has a fixed point.*

We define H_q^+ - Generalized contraction mapping in the following way.

Definition 1.6. *Let (X,q) be a complete weak partial metric space. A multivalued mapping $T : X \to CB^q(X)$ is called H_q^+ - generalized contraction if for every $x, y \in X$,*
(I) there exists α in $(0,1)$ such that
$$H_q^+(T(x) \setminus \{x\}, T(y) \setminus \{y\}) \leq \alpha \; M(x,y),$$
where $\quad M(x,y) = \max\{q(x,y), q(x,Tx), q(y,Ty)\}$,
(II) *for every x in X, y in $T(x)$ and $\epsilon > 0$, there exists z in $T(y)$ such that*
$$q(y,z) \leq H_q^+(T(y), T(x)) + \epsilon.$$

Definition 1.7. *An element $x \in X$ is called a common fixed point of two multivalued mappings $T, S : X \to CB^q(X)$ if $x \in Tx \cap Sx$.*

2 Main Result

Now we state our main result.

Theorem 2.1. *Let (X,q) be a complete weak partial metric space and $T, S : X \to CB^q(X)$ be two multivalued mappings satisfying, for all $x, y \in X$, the following condition:*
$$H_q^+(Tx \setminus \{x\}, Sy \setminus \{y\}) \leq \alpha \; M(x,y), \tag{2.1}$$
where $\alpha \in (0,1)$ and $M(x,y) = \max\{q(x,y), q(x,Tx), q(y,Sy)\}$.

Suppose also that, for all x in X, y in $T(x)$ and $\epsilon > 0$, there exists z in $S(y)$ such that
$$q(z,y) \leq \; H_q^+(S(y), T(x)) + \epsilon, \tag{2.2}$$
and for all x in X, y in $S(x)$ and $\epsilon > 0$, there exists z in $T(y)$ such that
$$q(z,y) \leq \; H_q^+(T(y), S(x)) + \epsilon. \tag{2.3}$$

Then, T and S have a common fixed point.

Proof. Let $x_0 \in X$ be arbitrary and $x_1 \in Sx_0$ such that $x_0 \notin Sx_0$. If $M(x_1, x_0) = 0$ then x_0 is common fixed point of S and T but $x_0 \notin Sx_0$, therefore we take $M(x_1, x_0) > 0$. By (2.3), there exists $x_2 (\neq x_1) \in Tx_1$ such that $x_1 \notin Tx_1$ and $q(x_2, x_1) \leq H_q^+(Tx_1, Sx_0) + \epsilon$. Similarly, assume $M(x_2, x_1) > 0$. Again by (2.2), there exists $x_3 (\neq x_2) \in Sx_2$ such that $x_2 \notin Sx_2$ and

$q(x_3, x_2) \leq H_q^+(Sx_2, Tx_1) + \epsilon$. Continuing this process, we can construct a sequence $\{x_n\}$ in X such that $x_{2n+2}(\neq x_{2n+1}) \in Tx_{2n+1}$ with $x_{2n+1} \notin Tx_{2n+1}$ and $x_{2n+1}(\neq x_{2n}) \in Sx_{2n}$ with $x_{2n} \notin Sx_{2n}$, and $M(x_{n+1}, x_n) > 0$ satisfying

$$q(x_{2n+1}, x_{2n}) \leq H_q^+(Sx_{2n}, Tx_{2n-1}) + \epsilon,$$

and

$$q(x_{2n+2}, x_{2n+1}) \leq H_q^+(Tx_{2n+1}, Sx_{2n}) + \epsilon.$$

By (2.1) and choosing $\epsilon = (\frac{1}{\sqrt{\alpha}} - 1)H_q^+(Tx_{2n-1}, Sx_{2n})$, we have

$$\begin{aligned} q(x_{2n}, x_{2n+1}) &\leq H_q^+(Tx_{2n-1}, Sx_{2n}) + \epsilon \\ &= H_q^+(Tx_{2n-1}, Sx_{2n}) + (\frac{1}{\sqrt{\alpha}} - 1)H_q^+(Tx_{2n-1}, Sx_{2n}) \\ &= \frac{1}{\sqrt{\alpha}} H_q^+(Tx_{2n-1}, Sx_{2n}) \\ &= \frac{1}{\sqrt{\alpha}} H_q^+(Tx_{2n-1} \setminus \{x_{2n-1}\}, Sx_{2n} \setminus \{x_{2n}\}) \\ &\leq \sqrt{\alpha}\ M(x_{2n-1}, x_{2n}) \\ &= \sqrt{\alpha}\ \ \max\{q(x_{2n-1}, x_{2n}), q(x_{2n-1}, Tx_{2n-1}), q(x_{2n}, Sx_{2n})\} \\ &\leq \sqrt{\alpha}\ \max\{q(x_{2n-1}, x_{2n}), q(x_{2n-1}, x_{2n}), q(x_{2n}, x_{2n+1})\} \\ &= \sqrt{\alpha} \max\{q(x_{2n-1}, x_{2n}), q(x_{2n}, x_{2n+1}))\}. \end{aligned} \tag{2.4}$$

Now, if $q(x_{2n}, x_{2n+1}) > q(x_{2n-1}, x_{2n})$, then by (2.4) we have

$$q(x_{2n}, x_{2n+1}) < \sqrt{\alpha}\ q(x_{2n}, x_{2n+1}),$$

because $\sqrt{\alpha} < 1$, then above inequality implies that $q(x_{2n}, x_{2n+1}) = 0 \Rightarrow x_{2n} = x_{2n+1}$ but $x_{2n} \neq x_{2n+1}$.

So, a contradiction occurs. Hence

$$q(x_{2n}, x_{2n+1}) \leq \sqrt{\alpha}\ q(x_{2n-1}, x_{2n}). \tag{2.5}$$

Adopting similar process, we obtain

$$q(x_{2n+1}, x_{2n+2}) \leq \sqrt{\alpha}\ q(x_{2n}, x_{2n+1}). \tag{2.6}$$

From (2.5) and (2.6) we conclude that

$$q(x_{n+1}, x_n) \leq \sqrt{\alpha}\ q(x_{n-1}, x_n).$$

Now by induction on n, we get

$$q(x_{n+1}, x_n) \leq (\sqrt{\alpha})^n q(x_0, x_1).$$

For any $m \in \mathbb{N}$, we have

$$\begin{aligned} q^s(x_n, x_{n+m}) &\leq q(x_n, x_{n+m}) \\ &\leq q(x_n, x_{n+1}) + q(x_{n+1}, x_{n+2}) + ... + q(x_{n+m-1}, x_{n+m}) \\ &\leq (\sqrt{\alpha})^n q(x_0, x_1) + (\sqrt{\alpha})^{n+1} q(x_0, x_1) + ... + (\sqrt{\alpha})^{n+m-1} q(x_0, x_1) \\ &= (\sqrt{\alpha})^n [1 + (\sqrt{\alpha}) + ... + (\sqrt{\alpha})^{m-1})]\ q(x_0, x_1) \\ &\leq \frac{(\sqrt{\alpha})^n}{1 - (\sqrt{\alpha})} q(x_0, x_1) \longrightarrow 0 \ \ as \ \ n \longrightarrow +\infty. \end{aligned}$$

It further implies that $\{x_n\}$ is a Cauchy sequence in (X, q^s). Since (X, q) is complete, therefore (X, q^s) is also complete metric space. Then the sequence $\{x_n\}$ converges to some $x^* \in X$ with respect to the metric q^s, that is

$$\lim_{n\to+\infty} q^s(x_n, x^*) = 0.$$

Moreover, we have

$$q(x^*, x^*) = \lim_{n\to+\infty} q(x_n, x^*) = \lim_{n\to+\infty} q(x_n, x_n) = 0. \tag{2.7}$$

As we know that

$$q(x_n, Sx^*) \leq q(x_n, x^*) + q(x^*, Sx^*)$$

and

$$q(x^*, Sx^*) \leq q(x^*, x_n) + q(x_n, Sx^*).$$

Taking limit and using (2.7) in the above two inequalities, we get

$$\lim_{n\to+\infty} q(x_n, Sx^*) = q(x^*, Sx^*). \tag{2.8}$$

Similarly,

$$\lim_{n\to+\infty} q(Sx^*, x_n) = q(Sx^*, x^*). \tag{2.9}$$

Now as, $\quad q(x_{2n+2}, Sx^*) \leq \delta_q(Tx_{2n+1}, Sx^*)$. Taking limit and using (2.8), we get

$$q(x^*, Sx^*) \leq \lim_{n\longrightarrow+\infty} \delta_q(Tx_{2n+1}, Sx^*). \tag{2.10}$$

Now let $a \in Sx^*$ then,

$$q(a, Tx_{2n+2}) \leq \delta_q(Sx^*, Tx_{2n+1}).$$

Also we know that

$$q(a, x_{2n+2}) \leq q(a, Tx_{2n+1}) + q(Tx_{2n+1}, x_{2n+2}).$$

Then we have

$$q(Sx^*, x_{2n+2}) \leq q(a, x_{2n+2}) \leq q(a, Tx_{2n+1}) + q(Tx_{2n+1}, x_{2n+2}),$$

which further implies

$$q(Sx^*, x_{2n+2}) \leq q(a, x_{2n+2}) \leq \delta_q(Sx^*, Tx_{2n+1}) + q(x_{2n+2}, x_{2n+2}).$$

Letting $n \to \infty$ and using (2.7) and (2.9) we get

$$q(Sx^*, x^*) \leq \lim_{n\to\infty} q(a, x_{2n+2}) \leq \lim_{n\to\infty} \delta_q(Sx^*, Tx_{2n+1}) + 0,$$

that is

$$q(Sx^*, x^*) \leq \lim_{n\to\infty} \delta_q(Sx^*, Tx_{2n+1}). \tag{2.11}$$

Since by definition, we have

$$\frac{1}{2}\big\{\delta_q(Tx_{2n+1}, Sx^*) + \delta_q(Sx^*, Tx_{2n+1})\big\} = H_q^+(Tx_{2n+1}, Sx^*).$$

Taking limit in the above expression, we get

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{2}\big\{\delta_q(Tx_{2n+1}, Sx^*) + \delta_q(Sx^*, Tx_{2n+1})\big\} &= \lim_{n\to\infty} H_q^+(Tx_{2n+1}, Sx^*) \\ &= \lim_{n\to\infty} H_q^+(Tx_{2n+1} \setminus \{x_{2n+1}\}, Sx^* \setminus \{x^*\})\end{aligned}$$

$$\begin{aligned}
&\le \alpha \lim_{n\to\infty} M(x_{2n+1}, x^*)\\
&= \alpha \lim_{n\to\infty} \max\{q(x_{2n+1}, x^*), q(x_{2n+1}, Tx_{2n+1}), q(x^*, Sx^*)\}\\
&= \alpha\, q(x^*, Sx^*).
\end{aligned}$$

Using (2.10) and (2.11) in the last inequality, we get

$$\frac{1}{2}\{q(x^*, Sx^*) + q(Sx^*, x^*)\} \le \lim_{n\to\infty} \frac{1}{2}\{\delta_q(Tx_{2n+1}, Sx^*) + \delta_q(Sx^*, Tx_{2n+1})\} \le \alpha\, q(x^*, Sx^*).$$

This implies

$$q(x^*, Sx^*) \le \alpha\, q(x^*, Sx^*).$$

As $\alpha \in (0,1)$, therefore $q(x^*, Sx^*) = 0 = q(x^*, x^*)$. This implies that $x^* \in Sx^*$, since Sx^* is closed.

Analogously, we get $x^* \in Tx^*$.

Hence T and S have a common fixed point. □

Now we present a simple example to validate our result.

Example 2.1. *Let $X = \left\{0, \frac{1}{4}, 1\right\}$ be endowed with the weak partial metric $q : X \times X \to \mathbb{R}^+$ defined by*

$q(x,y) = \frac{1}{4}|x-y| + \frac{1}{2}\max\{x,y\}$ *for all $x, y \in X$.*

Define the mappings $T, S : X \to CB^q(X)$ by

$$Tx = \begin{cases} \{0\} & \text{if } x = \{0,1\} \\ \left\{0, \frac{1}{4}\right\} & \text{if } x = \frac{1}{4} \end{cases} \quad \text{and} \quad Sx = \begin{cases} \{0\} & \text{if } x = \{0, \frac{1}{4}\} \\ \left\{1, \frac{1}{4}\right\} & \text{if } x = 1. \end{cases}$$

Since $q(1,1) = \frac{1}{2} \neq 0$. So q is not a metric on X.

As $q^s(x,y) = |x-y|$. So, (X,q) is a complete weak partial metric space.

Also,

$$\begin{aligned}
x \in \overline{\{0\}} &\Leftrightarrow q(x, \{0\}) = q(x,x)\\
&\Leftrightarrow \frac{3}{4}x = \frac{1}{2}x \Leftrightarrow x = 0\\
&\Leftrightarrow x \in \{0\}.
\end{aligned}$$

Hence, $\{0\}$ is closed with respect to the weak partial metric q.

$$\begin{aligned}
x \in \overline{\left\{0, \frac{1}{4}\right\}} &\Leftrightarrow q\Big(x, \left\{0, \frac{1}{4}\right\}\Big) = q(x,x)\\
&\Leftrightarrow \min\left\{\frac{3x}{4}, \frac{1}{4}\Big|x - \frac{1}{4}\Big| + \frac{1}{2}\max\left\{x, \frac{1}{4}\right\}\right\} = \frac{x}{2}\\
&\Leftrightarrow x \in \left\{0, \frac{1}{4}\right\}.
\end{aligned}$$

Hence, $\left\{0, \frac{1}{4}\right\}$ is closed with respect to the weak partial metric q.

$$\begin{aligned}
x \in \overline{\left\{1, \frac{1}{4}\right\}} &\Leftrightarrow q\Big(x, \left\{1, \frac{1}{4}\right\}\Big) = q(x,x)\\
&\Leftrightarrow \min\left\{\frac{1}{4}\Big|x-1\Big| + \frac{1}{2}\max\{x,1\}, \frac{1}{4}\Big|x - \frac{1}{4}\Big| + \frac{1}{2}\max\left\{x, \frac{1}{4}\right\}\right\} = \frac{x}{2}\\
&\Leftrightarrow x \in \left\{1, \frac{1}{4}\right\}.
\end{aligned}$$

Hence, $\left\{1, \frac{1}{4}\right\}$ is closed with respect to the weak partial metric q. Now, we shall show that for all $x, y \in X$, the contractive condition (2.1) is satisfied. For this, we consider the following cases:
(i) $x = y = 0$. We have

$$H_q^+(T(0) \setminus \{0\}, S(0) \setminus \{0\}) = H_q^+(\phi, \phi) = 0,$$

and (2.1) is satisfied.
(ii) $x = 0,\ y = \frac{1}{4}$. We have

$$H_q^+\Big(T(0) \setminus \{0\}, S(\frac{1}{4}) \setminus \{\frac{1}{4}\}\Big) = H_q^+(\phi, \{0\}) = 0,$$

and (2.1) is satisfied.
(iii) $x = \frac{1}{4},\ y = 0$. We have

$$H_q^+\Big(T(\frac{1}{4}) \setminus \{\frac{1}{4}\}, S(0) \setminus \{0\}\Big) = H_q^+(\{0\}, \phi) = 0,$$

and (2.1) is satisfied.
(iv) $x = 0,\ y = 1$. We have

$$H_q^+(T(0) \setminus \{0\}, S(1) \setminus \{1\}) = H_q^+\Big(\phi, \{\frac{1}{4}\}\Big) = 0,$$

and (2.1) is satisfied.
(v) $x = 1,\ y = 0$. We have

$$H_q^+(T(1) \setminus \{1\}, S(0) \setminus \{0\}) = H_q^+(\{0\}, \phi) = 0,$$

and (2.1) is satisfied.
(vi) $x = y = \frac{1}{4}$. We have

$$H_q^+\Big(T(\frac{1}{4}) \setminus \{\frac{1}{4}\}, S(\frac{1}{4}) \setminus \{\frac{1}{4}\}\Big) = H_q^+(\{0\}, \{0\}) = 0,$$

and (2.1) is satisfied.
(vii) $x = \frac{1}{4},\ y = 1$. We have

$$H_q^+\Big(T(\frac{1}{4}) \setminus \{\frac{1}{4}\}, S(1) \setminus \{1\}\Big) = H_q^+\Big(\{0\}, \{\frac{1}{4}\}\Big) = q\Big(0, \frac{1}{4}\Big) = \frac{3}{16} \leq \alpha \frac{11}{16} = \alpha\ M\Big(\frac{1}{4}, 1\Big),$$

and (2.1) is satisfied.
(viii) $x = 1,\ y = \frac{1}{4}$. We have

$$H_q^+\Big(T(1) \setminus \{1\}, S(\frac{1}{4}) \setminus \{\frac{1}{4}\}\Big) = H_q^+(\{0\}, \{0\}) = q(0, 0) = 0,$$

and (2.1) is satisfied.
(ix) $x = y = 1$. We have

$$H_q^+(T(1) \setminus \{1\}, S(1) \setminus \{1\}) = H_q^+\Big(\{0\}, \{\frac{1}{4}\}\Big) = q\Big(0, \frac{1}{4}\Big) = \frac{3}{16} \leq \alpha \frac{3}{4} = \alpha\ M(1, 1),$$

and (2.1) is satisfied.

Further, we will show that for every x in X, y in $T(x)$ and $\epsilon > 0$, $\exists$ z in $S(y)$ such that $q(y, z) \leq H_q^+(S(y), T(x)) + \epsilon$. Indeed,
(a) if $x = 0$, $y \in T(0) = \{0\}$, and $\epsilon > 0$, $\exists$ $z \in S(y) = \{0\}$ such that

$$0 = q(y, z) < H_q^+(S(y), T(x)) + \epsilon,$$

(b_1) *if* $x = \frac{1}{4},\ y \in T(\frac{1}{4}) = \{0, \frac{1}{4}\}$, *say* $y = 0$ *and* $\epsilon > 0,\ \exists\ z \in S(y) = \{0\}$ *such that*

$$0 = q(y, z) < \frac{3}{32} + \epsilon = H_q^+(S(y), T(x)) + \epsilon,$$

(b_2) *if* $x = \frac{1}{4},\ y \in T(\frac{1}{4}) = \{0, \frac{1}{4}\}$, *say* $y = \frac{1}{4}$ *and* $\epsilon > 0,\ \exists\ z \in S(y) = \{0\}$ *such that*

$$\frac{3}{16} = q(y, z) \leq \frac{3}{32} + \epsilon = H_q^+(S(y), T(x)) + \epsilon,$$

(c) *if* $x = 1,\ y \in T(1) = \{0\}$, *and* $\epsilon > 0,\ \exists\ z \in S(0) = \{0\}$ *such that*

$$0 = q(y, z) < H_q^+(S(y), T(x)) + \epsilon,$$

and also we will show that for every x *in* X, y *in* $S(x)$ *and* $\epsilon > 0,\ \exists\ z$ *in* $T(y)$ *such that* $q(y, z) \leq H_q^+(T(y), S(x)) + \epsilon$. *Indeed,*

(á) if $x = 0,\ y \in S(0) = \{0\}$ *and* $\epsilon > 0,\ \exists\ z \in T(0) = \{0\}$ *such that*

$$0 = q(y, z) < H_q^+(T(y), S(x)) + \epsilon,$$

(b́) if $x = \frac{1}{4},\ y \in S(\frac{1}{4}) = \{0\}$ *and* $\epsilon > 0,\ \exists\ z \in T(0) = \{0\}$ *such that*

$$0 = q(y, z) < H_q^+(T(y), S(x)) + \epsilon,$$

(ć₁) if $x = 1,\ y \in S(1) = \{1, \frac{1}{4}\}$, *say* $y = 1$ *and* $\epsilon > 0,\ \exists\ z \in T(1) = \{0\}$ *such that*

$$\frac{3}{4} = q(y, z) \leq \frac{15}{32} + \epsilon = H_q^+(T(y), S(x)) + \epsilon,$$

(ć₂) if $x = 1,\ y \in S(1) = \{1, \frac{1}{4}\}$, *say* $y = \frac{1}{4}$ *and* $\epsilon > 0,\ \exists\ z$*(say* $z = \frac{1}{4}$*)* $\in T(\frac{1}{4}) = \{0, \frac{1}{4}\}$ *such that*

$$\frac{1}{8} = q(y, z) < \frac{14}{32} + \epsilon = H_q^+(T(y), S(x)) + \epsilon.$$

Hence all conditions of Theorem 2.1 are satisfied with $\alpha = 0.3$ *and* $\epsilon = 1$. *Here* $\boldsymbol{x = 0}$, *is the common fixed point of* S *and* T.

Corollary 2.1. *Every* H_q^+ *- type generalized multivalued contraction on a complete weak partial metric space* (X, q) *with Lipschitz constant* $\alpha < 1$ *has a fixed point.*

Proof. The proof follows from Theorem 2.1 by taking $T = S$. □

The following example satisfy the condition of Corollary 2.1 but not of the Theorem 3 of [4].

Example 2.2. *Let* $X = \{0, \frac{1}{3}, 1\}$ *and define a weak partial metric* $q : X \times X \to [0, \infty)$ *as follows:* $q(0, 0) = 0,\ q(\frac{1}{3}, \frac{1}{3}) = \frac{1}{5},\ q(1, 1) = \frac{1}{3},\ q(1, \frac{1}{3}) = q(\frac{1}{3}, 1) = \frac{3}{5},\ q(0, \frac{1}{3}) = q(\frac{1}{3}, 0) = \frac{1}{4},\ q(0, 1) = q(1, 0) = \frac{2}{5}$. *Define the mapping* $T : X \to CB^q(X)$ *by*

$$Tx = \begin{cases} \{0\} & \text{if } x = 0 \\ \{1\} & \text{if } x = \frac{1}{3} \\ \{0, \frac{1}{3}\} & \text{if } x = 1. \end{cases}$$

Clearly, (X, q) *is a weak partial metric space. Now we check that for all* $x, y \in X$, *contractive condition* (I) *is satisfied. For this, we consider the following cases:*

(i) $x = y = 0$. *We have*

$$H_q^+(T(0) \setminus \{0\}, T(0) \setminus \{0\}) = H_q^+(\phi, \phi) = 0,$$

and (I) *is satisfied.*
(ii) $x = 0,\ y = \frac{1}{3}$. *We have*

$$H_q^+(T(0) \setminus \{0\}, T(\frac{1}{3}) \setminus \{\frac{1}{3}\}) = H_q^+(\phi, \{1\}) = 0,$$

and (I) *is satisfied.*
(iii) $x = 0,\ y = 1$. *We have*

$$H_q^+(T(0) \setminus \{0\}, T(1) \setminus \{1\}) = H_q^+(\phi, \{0, \frac{1}{3}\}) = 0,$$

and (I) *is satisfied.*
(iv) $x = y = \frac{1}{3}$. *We have*

$$H_q^+\Big(T(\frac{1}{3}) \setminus \{\frac{1}{3}\}, T(\frac{1}{3}) \setminus \{\frac{1}{3}\}\Big) = H_q^+(\{1\}, \{1\}) = q(1,1) = \frac{1}{3} \le \alpha\frac{3}{5} = \alpha\ M\Big(\frac{1}{3}, \frac{1}{3}\Big),$$

and (I) *is satisfied.*
(v) $x = \frac{1}{3},\ y = 1$. *We have*

$$H_q^+\Big(T(\frac{1}{3}) \setminus \{\frac{1}{3}\}, T(1) \setminus \{1\}\Big) = H_q^+\Big(\{1\}, \{0, \frac{1}{3}\}\Big) = \frac{1}{2} \le \alpha\frac{3}{5} = \alpha\ M\Big(\frac{1}{3}, 1\Big),$$

and (I) *is satisfied.*
(vi) $x = y = 1$. *We have*

$$H_q^+(T(1) \setminus \{1\}, T(1) \setminus \{1\}) = H_q^+\Big(\{0, \frac{1}{3}\}, \{0, \frac{1}{3}\}\Big) = \frac{1}{5} \le \alpha\frac{2}{5} = \alpha\ M(1,1),$$

and (I) *is satisfied.*

Further, we will show that for every x *in* X, y *in* $T(x)$ *and* $\epsilon > 0$, $\exists$ z *in* $T(y)$ *such that* $q(y,z) \le H_q^+(T(y), T(x)) + \epsilon$. *Indeed,*
(a_1) *if* $x = 0,\ y \in T(0) = \{0\}$, *and* $\epsilon > 0$, $\exists$ $z \in T(y) = \{0\}$ *such that*

$$0 = q(y,z) \le H_q^+(T(y), T(x)) + \epsilon,$$

(b_1) *if* $x = \frac{1}{3},\ y \in T(\frac{1}{3}) = \{1\}$, *and* $\epsilon > 0$, $\exists$ $z \in T(y) = \big\{0, \frac{1}{3}\big\}$ *such that*

$$\frac{2}{5} = q(y,z) < \frac{1}{2} + \epsilon = H_q^+(T(y), T(x)) + \epsilon,$$

(c_{11}) *if* $x = 1,\ y \in T(1) = \big\{0, \frac{1}{3}\big\}$, *say* $y = 0$ *and* $\epsilon > 0$, $\exists\ z \in T(y) = \{0\}$ *such that*

$$0 = q(y,z) < H_q^+(T(y), T(x)) + \epsilon,$$

(c_{12}) *if* $x = 1,\ y \in T(1) = \big\{0, \frac{1}{3}\big\}$, *say* $y = \frac{1}{3}$ *and* $\epsilon > 0$, $\exists\ z \in T(\frac{1}{3}) = \{1\}$ *such that*

$$\frac{3}{5} = q(y,z) \le \frac{1}{2} + \epsilon = H_q^+(T(y), T(x)) + \epsilon.$$

Hence all the conditions of Corollary 2.1 *are satisfied with* $\alpha = 0.9$ *and* $\epsilon = \frac{1}{10}$. *Here* $\boldsymbol{x = 0}$ *is the fixed point of* T.
On the other hand we see that the result of Beg and Pathak[4] is not applicable. As,

$$H_q^+\Big(T\Big(\frac{1}{3}\Big) \setminus \Big\{\frac{1}{3}\Big\}, T\Big(\frac{1}{3}\Big) \setminus \Big\{\frac{1}{3}\Big\}\Big) = H_q^+(\{1\}, \{1\}) = \frac{1}{3} \not\le \alpha\ q\Big(\frac{1}{3}, \frac{1}{3}\Big) = \alpha\frac{1}{5}$$

for any $\alpha \in (0,1)$.

Acknowledgement
Authors are indebted to the referee for his careful reading of the manuscript and valuable suggestions.

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Jñānābha, Vol. 49(2) (2019), 45-63

EXISTENCE AND ATTRACTIVITY THEOREMS FOR NONLINEAR FIRST ORDER HYBRID DIFFERENTIAL EQUATIONS WITH ANTICIPATION AND RETARDATION

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(*Received : October* 13, 2019 *; Revised: October* 23, 2019)

Abstract

In this paper, we introduce the notion of a pulling function and establish an existence and global attractivity results for a nonlinear hybrid differential equation of quadratic type on the unbounded intervals of real line with mixed arguments of anticipations and retardations. A positivity result is also obtained under some usual natural conditions. The hybrid fixed point theoretic technology of Dhage (2004) is used while establishing our main results of the paper. Our hypotheses and claims have also been explained with the help of a natural realization.

2010 Mathematics Subject Classifications: 34K10, 47H10.

Keywords and phrases: Hybrid differential equation; Hybrid fixed point theorem; Existence theorem; Attractivity of solution.

1 Statement of the Problem

Let $t_0 \in \mathbb{R}$ be a fixed real number and let $J_\infty = [t_0, \infty)$ be a closed but unbounded interval in $\mathbb{R}$. Then Dhage *et. al* [15] introduced the class $\mathcal{CRB}(J_\infty)$ of functions $a : J_\infty \to (0, \infty)$ satisfying the following properties:

(i) a is continuous, and

(ii) $\lim\limits_{t\to\infty} a(t) = \infty$.

The members of the class $\mathcal{CRB}(J_\infty)$ are the weight functions and may sometimes be called the **pulling functions** on J_∞. There do exist several pulling functions $a : J_\infty \to (0, \infty)$ satisfying the above two conditions. In fact, $|t| + 1$, $e^{|t|}$, $t^2 + 1$, $\log(2 + |t|)$, $\cosh t$ etc. are such pulling functions on J_∞ and commonly used pulling functions on $\mathbb{R}_+ = [0, \infty)$ are $a_1(t) = e^{ct}$, $c > 0$ and $a_2(t) = t^2 + 1$ (see Banas and Dhage [1], Dhage [7, 8, 9, 10] etc. and references therein). Again, functions from the class of continuous and strictly monotone positive functions $a : J_\infty \to (0, \infty)$ going increasingly to ∞ satisfy the above criteria. Note that if $a \in \mathcal{CRB}(J_\infty)$, then the reciprocal function $\overline{a} : J_\infty \to \mathbb{R}_+$ defined by $\overline{a}(t) = \frac{1}{a(t)}$ is continuous and bounded on J_∞ with $\lim\limits_{t\to\infty} \overline{a}(t) = 0$.

Given a pulling function $a \in \mathcal{CRB}(J_\infty)$, we consider the following hybrid functional differential equation (in short HFDE),

$$\left.\begin{array}{c}\dfrac{d}{dt}\left[\dfrac{a(t)x(t) - k(t, x(t)), x(\gamma(t))}{f(t, x(t), x(\alpha(t)))}\right] = g(t, x(t), x(\eta(t))), \;\; t \in J_\infty, \\ x(t_0) = x_0 \in \mathbb{R},\end{array}\right\} \tag{1.1}$$

where $f : J_\infty \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \setminus \{0\}$, $k, g : J_\infty \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $\alpha, \gamma, \eta : J_\infty \to J_\infty$ are continuous functions satisfying

(i) the map $(x, y, z) \mapsto \dfrac{a(t)x - k(t, x, z)}{f(t, x, y)}$ is well defined for each $t \in J_\infty$; and

(ii) the functions α, γ and η are respectively anticipatory and retardatory, that is $\alpha(t) \geq t$, $\gamma(t) \geq t$ and $t_0 \leq \eta(t) \leq t$ for all $t \in J_\infty$ respectively with $\alpha(t_0) = t_0 = \gamma(t_0)$.

Definition 1.1. *By a solution for the functional differential equation* (1.1) *we mean a function $x \in BC(J_\infty, \mathbb{R})$ such that*

(i) the function $t \mapsto \dfrac{a(t)x(t) - k(t, x(t), x(\gamma(t)}{f(t, x(t), x(\alpha(t)))}$ is continuous on J_∞, and

(ii) x satisfies the equations in (1.1) *on J_∞,*

where $BC(J_\infty, \mathbb{R})$ is the space of bounded and continuous real-valued functions defined on J_∞.

The HFDE (1.1) is a mixed quadratic and linear perturbations of second type for the initial value problem of linear first order ordinary differential equations. The details of different types of perturbations of a linear differential equation appear in Dhage [11]. It is also new to the literature and includes a good number of known hybrid differential equations as special cases.

1.1 Special cases

1. If $f(t, x, y) = 1$ and $k(t, x, y) = 0$ for all $(t, x, y) \in J_\infty \times \mathbb{R} \times \mathbb{R}$ in (1.1), we get the second type quadratically perturbed nonlinear differential equation,

$$\left.\begin{aligned} \frac{d}{dt}\big[a(t)x(t)\big] &= g(t, x(t), x(\eta(t))), \ \ t \in J_\infty, \\ x(t_0) &= x_0 \in \mathbb{R}. \end{aligned}\right\} \tag{1.2}$$

2. When $f(t, x, y) = 1$ for all $(t, x, y) \in J_\infty \times \mathbb{R} \times \mathbb{R}$ in (1.1), we obtain the second type linearly perturbed nonlinear differential equation,

$$\left.\begin{aligned} \frac{d}{dt}\big[a(t)x(t) - k(t, x(t)), x(\gamma(t)))\big] &= g(t, x(t), x(\eta(t))), \ \ t \in J_\infty, \\ x(t_0) &= x_0 \in \mathbb{R}, \end{aligned}\right\} \tag{1.3}$$

which is again new to the literature for nonlinear differential equations with anticipation and retardation.

3. Similarly, if $k \equiv 0$ on $J_\infty \times \mathbb{R} \times \mathbb{R}$, then the HFDE (1.1) reduces to the quadratically perturbed nonlinear differential equation

$$\left.\begin{aligned} \frac{d}{dt}\left[\frac{a(t)x(t)}{f(t, x(t), x(\alpha(t)))}\right] &= g(t, x(t), x(\eta(t))), \ \ t \in J_\infty, \\ x(t_0) &= x_0 \in \mathbb{R}, \end{aligned}\right\} \tag{1.4}$$

which is studied in Dhage *et al* [15] for existence and asymptotically attractivity of solutions on J_∞. It is known that the HFDE (1.4) is interesting and includes the following perturbed nonlinear differential equations as special cases.

4. Let $k : J_\infty \to \mathbb{R}_+$ be a continuous function such that $\lim\limits_{t\to\infty} e^{K(t)} = \infty$, where $K(t) = \int_{t_0}^{t} k(s)\,ds > 0$. Then the hybrid functional differential equation,

$$\left.\begin{aligned} \frac{d}{dt}\left[\frac{x(t)}{f(t,x(t),x(\alpha(t)))}\right] &+ k(t)\left[\frac{x(t)}{f(t,x(t),x(\alpha(t)))}\right] \\ &= g(t,x(t),x(\gamma(t))),\ \ t \in J_\infty, \\ x(t_0) = x_0 \in \mathbb{R}, & \end{aligned}\right\} \tag{1.5}$$

is of the type of HFDE (1.4) on the interval J_∞ with $a(t) = e^{K(t)}$.

5. The special case when $f(t,x,y) = f(t,x)$ and $g(t,x,y) = g(t,x)$, the HFDE (1.5) reduces to quadratic HFDE

$$\left.\begin{aligned} \frac{d}{dt}\left[\frac{x(t)}{f(t,x(t))}\right] + k(t)\left[\frac{x(t)}{f(t,x(t))}\right] &= g(t,x(t)),\ \ t \in J_\infty \\ x(t_0) = x_0 \in \mathbb{R}. & \end{aligned}\right\} \tag{1.6}$$

The HFDE (1.6) with $J_\infty = \mathbb{R}_+$ has been discussed in Dhage [8] for the local attractivity results under mixed Lipschitz and compactness type conditions.

6. Again, a special case of (1.6) with $f(t,x) = 1$ in the form

$$\left.\begin{aligned} x'(t) + k(t)x(t) = g(t,x(t)),\ t \in J_\infty \\ x(0) = x_0 \in \mathbb{R}, \end{aligned}\right\} \tag{1.7}$$

has been treated in Burton and Furumochi [3] for asymptotic stability of solutions.

7. Finally, when $g(t,x) = q(t)x^n$ on $J_\infty \times \mathbb{R}$, the DE (1.7) includes as a special case the well-known Bernoulli's equation,

$$\left.\begin{aligned} x'(t) + k(t)x(t) = q(t)x^n(t),\ t \in J_\infty \\ x(0) = x_0 \in \mathbb{R}, \end{aligned}\right\} \tag{1.8}$$

where $q : J_\infty \to \mathbb{R}$ and n is a nonnegative real number.

Thus, in a nutshell, our HFDE (1.1) is more general and therefore, the global attractivity and ultimate positivity results proved in this paper are of great interest and include the existence as well as attractivity results for the above mentioned HFDEs (1.2)-(1.8) on the unbounded interval J_∞ as special cases.

2 Auxiliary Results

Let X be a non-empty set and let $\mathcal{T} : X \to X$. An invariant point under $\mathcal{T}$ in X is called a fixed point of $\mathcal{T}$, that is, the fixed points are the solutions of the functional equation $\mathcal{T}x = x$. Any statement asserting the existence of fixed point of the mapping $\mathcal{T}$ is called a fixed point theorem for the mapping $\mathcal{T}$ in X. The fixed point theorems are obtained by imposing the conditions on T or on X or on both $\mathcal{T}$ and X. By experience, better the mapping $\mathcal{T}$ or X, we have better fixed point principles. As we go on adding richer structure to the non-empty set X, we derive richer fixed point theorem useful for applications to different areas of mathematics and particularly to nonlinear differential and integral equations. Below we give some fixed point theorems useful in establishing the attractivity and ultimate positivity of the solutions for HFDE (1.1) on unbounded intervals. Before stating these results we give some preliminaries.

Let X be an infinite dimensional Banach space with the norm $\|\cdot\|$. A mapping $\mathcal{T} : X \to X$ is called **$\mathcal{D}$-Lipschitz** if there is an upper semi-continuous and nondecreasing function $\psi_{\mathcal{T}} : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying

$$\|\mathcal{T}x - \mathcal{T}y\| \le \psi_{\mathcal{T}}(\|x - y\|) \tag{2.1}$$

for all $x, y \in X$, where $\psi_{\mathcal{T}}(0) = 0$. If $\psi_{\mathcal{T}}(r) = k\,r$, $k > 0$, then $\mathcal{T}$ is called **Lipschitz** with the Lipschitz constant k. In particular, if $k < 1$, then $\mathcal{T}$ is called a **contraction** on X with the contraction constant k. Further, if $\psi_{\mathcal{T}}(r) < r$ for $r > 0$, then $\mathcal{T}$ is called **nonlinear $\mathcal{D}$-contraction** and the function $\psi_{\mathcal{T}}$ is called $\mathcal{D}$-function of $\mathcal{T}$ on X. There do exist $\mathcal{D}$-functions and the commonly used $\mathcal{D}$-functions are $\psi_{\mathcal{T}}(r) = k\,r$, $\psi_{\mathcal{T}}(r) = \ln(1 + r)$ and $\psi_{\mathcal{T}}(r) = \dfrac{r}{1+r}$, etc. (see Banas and Dhage [1] and the references therein).

Definition 2.1. *An operator $\mathcal{T}$ on a Banach space X into itself is called **totally bounded** if for any bounded subset S of X, $\mathcal{T}(S)$ is a relatively compact subset of X. If $\mathcal{T}$ is continuous and totally bounded, then it is called **completely continuous** on X.*

Our essential tool used in the paper is the following fixed point theorem of Dhage [5, 6] for a quadratic operator equation involving three operators in a Banach algebra X.

Theorem 2.1 (Dhage [6])**.** *Let S be a non-empty, closed convex and bounded subset of the Banach algebra X and let $\mathcal{A}, \mathcal{C} : X \to X$ and $\mathcal{B} : S \to X$ be three operators such that*

(a) $\mathcal{A}$ and $\mathcal{C}$ are $\mathcal{D}$-Lipschitz with $\mathcal{D}$-functions $\psi_{\mathcal{A}}$ and $\psi_{\mathcal{C}}$ respectively,
(b) $\mathcal{B}$ is completely continuous,
(c) $x = \mathcal{A}x\,\mathcal{B}y + \mathcal{C}x \Longrightarrow x \in S$ for all $y \in S$, and
(d) $M\,\psi_{\mathcal{A}}(r) + \psi_{\mathcal{C}}(r) < r$, $r > 0$, where $M = \|\mathcal{B}(S)\| = \sup\{\|\mathcal{B}x\| : x \in S\}$.

Then the operator equation

$$\mathcal{A}x\,\mathcal{B}x + \mathcal{C}x = x \tag{2.1}$$

has a solution in S.

Corollary 2.1 (Dhage [5])**.** *Let S be a non-empty, closed convex and bounded subset of the Banach algebra X and let $\mathcal{A} : X \to X$ and $\mathcal{B} : S \to X$ be two operators such that*

(a) $\mathcal{A}$ and $\mathcal{C}$ are Lipschitz with Lipschitz constant L_1 and L_2 respectively,
(b) $\mathcal{B}$ is completely continuous,
(c) $x = \mathcal{A}x\,\mathcal{B}y + \mathcal{C}x \Longrightarrow x \in S$ for all $y \in S$, and
(d) $L_1\,M +\, L_2 < 1$,where $M = \|\mathcal{B}(S)\| = \sup\{\|\mathcal{B}x\| : x \in S\}$.

Then the operator equation

$$\mathcal{A}x\,\mathcal{B}x + \mathcal{C}x = x \tag{2.2}$$

has a solution in S.

A collection of a good number of applicable fixed point theorems may be found in the monographs of Granas and Dugundji [16], Deimling [4], Zeidler [20] and the references therein. In the following section we give different types of characterizations of the solutions for nonlinear functional differential equations on unbounded intervals of real line.

3 Characterizations of Solutions

We seek solutions of the HFDE (1.1) in the space $BC(J_\infty, \mathbb{R})$ of continuous and bounded real-valued functions defined on J_∞. Define a standard supremum norm $\|\cdot\|$ and a multiplication "$\cdot$" in $BC(J_\infty, \mathbb{R})$ by

$$\|x\| = \sup_{t \in J_\infty} |x(t)|$$

and

$$(x \cdot y)(t) = (xy)(t) = x(t)y(t), \ t \in J_\infty.$$

Clearly, $BC(J_\infty, \mathbb{R})$ becomes a Banach algebra w.r.t. the above norm and the multiplication in it. Let $\mathcal{A}, \mathcal{B} : BC(J_\infty, \mathbb{R}) \to BC(J_\infty, \mathbb{R})$ be two continuous operators and consider the following operator equation in the Banach algebra $BC(J_\infty, \mathbb{R})$,

$$\mathcal{A}x(t)\,\mathcal{B}x(t) + \mathcal{C}x(t) = x(t) \tag{3.1}$$

for all $t \in J_\infty$. Below we give different characterizations of the solutions for the operator equation (3.1) in the space $BC(J_\infty, \mathbb{R})$.

Definition 3.1. *We say that solutions of the operator equation* (3.1) *are* **locally attractive** *if there exists a closed ball $\overline{B}_r(x_0)$ in the space $BC(J_\infty, \mathbb{R})$ for some $x_0 \in BC(J_\infty, \mathbb{R})$ such that for arbitrary solutions $x = x(t)$ and $y = y(t)$ of equation* (3.1) *belonging to $\overline{B}_r(x_0)$ we have that*

$$\lim_{t \to \infty} (x(t) - y(t)) = 0. \tag{3.2}$$

In the case when the limit (3.2) *is uniform with respect to the set $\overline{B}_r(x_0)$, i.e., when for each $\varepsilon > 0$ there exists $T > 0$ such that*

$$|x(t) - y(t)| \le \epsilon \tag{3.3}$$

for all $x, y \in \overline{B}_r(x_0)$ being solutions of (3.1) *and for $t \ge T$, we will say that solutions of equation* (3.1) *are* **uniformly locally attractive** *on J_∞.*

Definition 3.2. *A solution $x = x(t)$ of equation* (3.1) *is said to be* **globally attractive** *if (3.2) holds for each solution $y = y(t)$ of* (3.1) *in $BC(J_\infty, \mathbb{R})$. In other words, we may say that solutions of the equation* (3.1) *are globally attractive if for arbitrary solutions $x(t)$ and $y(t)$ of* (3.1) *in $BC(J_\infty, \mathbb{R})$, the condition* (3.2) *is satisfied. In the case when the condition* (3.2) *is satisfied uniformly with respect to the space $BC(J_\infty, \mathbb{R})$, i.e., if for every $\epsilon > 0$ there exists $T > 0$ such that the inequality* (3.2) *is satisfied for all $x, y \in BC(J_\infty, \mathbb{R})$ being the solutions of* (3.1) *and for $t \ge T$, we will say that solutions of the equation* (3.1) *are* **uniformly globally attractive** *on J_∞.*

Remark 3.1. Let us mention that the details of the global attractivity of solutions may be found in a recent paper of Hu and Yan [19] while the concepts of uniform local and global attractivity (in the above sense) may be found in Banas and Dhage [1].

Now we introduce the new concept of local and global ultimate positivity of the solutions for the operator equation (3.1) in the space $BC(J_\infty, \mathbb{R})$.

Definition 3.3 (Dhage [8]). *A solution x of the equation* (3.1) *is called* **locally ultimately positive** *if there exists a closed ball $\overline{B}_r(x_0)$ in the space $BC(J_\infty, \mathbb{R})$ for some $x_0 \in BC(J_\infty, \mathbb{R})$ such that $x \in \overline{B}_r(x_0)$ and*

$$\lim_{t\to\infty} \big[|x(t)| - x(t)\big] = 0. \tag{3.4}$$

In the case when the limit (3.4) *is uniform with respect to the solution set of the operator equation* (3.1) *in $BC(J_\infty, \mathbb{R})$, i.e., when for each $\varepsilon > 0$ there exists $T > 0$ such that*

$$\big|\,|x(t)| - x(t)\big| \le \epsilon \tag{3.5}$$

for all x being solutions of (3.1) *in $BC(J_\infty, \mathbb{R})$ and for $t \ge T$, we will say that solutions of equation* (3.1) *are* **uniformly locally ultimately positive** *on J_∞.*

Definition 3.4 (Dhage [9]). *A solution $x \in BC(J_\infty, \mathbb{R})$ of the equation* (3.1) *is called* **globally ultimately positive** *if* (3.4) *is satisfied. In the case when the limit* (3.5) *is uniform with respect to the solution set of the operator equation* (3.1) *in $BC(J_\infty, \mathbb{R})$, i.e., when for each $\varepsilon > 0$ there exists $T > 0$ such that* (3.5) *is satisfied for all x being solutions of* (3.1) *in in $BC(J_\infty, \mathbb{R})$ and for $t \ge T$, we will say that solutions of equation* (3.1) *are* **uniformly globally ultimately positive** *on J_∞.*

Remark 3.2. *We note that global attractivity implies the local attractivity and uniform global attractivity implies the uniform local attractivity of the solutions for the operator equation* (3.1) *on J_∞. Similarly, global ultimate positivity implies local ultimate positivity of the solutions for the operator equation* (3.1) *on unbounded intervals. However, the converse of the above two statements may not be true.*

4 Attractivity and Positivity Results

Now, in this section, we discuss the existence, attractivity and positivity results for the first order ordinary differential equation (1.1) on J_∞.

Definition 4.1. *A function $\beta : J_\infty \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is called Carathéodory if*

(i) the map $t \mapsto \beta(t, x, y)$ is measurable for all $x, y \in \mathbb{R}$ and
(ii) the map $(x, y) \mapsto \beta(t, x, y)$ is jointly continuous for all $t \in J_\infty$.

The following lemma is often times used in the study of nonlinear discontinuous and specially Carathéodory theory of nonlinear differential equations.

Lemma 4.1 (Carathéodory). *Let $\beta : J_\infty \times \mathbb{R} \to \mathbb{R}$ be a mapping such that $\beta(\cdot, x)$ is measurable for all $x \in \mathbb{R}$ and $\beta(t, \cdot)$ is continuous for all $t \in J_\infty$. Then the map $(t, x) \mapsto \beta(t, x)$ is jointly measurable.*

We need the following hypotheses in the sequel.

(A_0) The function $x \mapsto \dfrac{a(t_0)x - k(t_0, x, x)}{f(t_0, x, x)}$ is injective in $\mathbb{R}$.

(A_1) The function f is continuous and there exists a function $\ell_1 \in BC(J_\infty, \mathbb{R}_+)$ and a constant $K_1 > 0$ such that

$$\big|f(t, x_1, x_2) - f(t, y_1, y_2)\big| \le \frac{1}{2} \cdot \frac{\ell_1(t) \max\{|x_1 - y_1|, |x_2 - y_2|\}}{K_1 + \max\{|x_1 - y_1|, |x_2 - y_2|\}}$$

for all $t \in J_\infty$ and $x_1, x_2, y_1, y_2 \in \mathbb{R}$. Moreover, $\sup\limits_{t\in J_\infty} \ell_1(t) = L_1$.

(A_2) The function $t \mapsto F(t) = f(t,0,0)$ is continuous and bounded on J_∞ with bound F_0.

(A_3) The function f is continuous and there exists a function $\ell_2 \in BC(J_\infty, \mathbb{R}_+)$ and a constant $K_2 > 0$ such that

$$\big|k(t,x_1,x_2) - k(t,y_1,y_2)\big| \le \frac{1}{2} \cdot \frac{\ell_2(t)\max\{|x_1-y_1|,|x_2-y_2|\}}{K_2 + \max\{|x_1-y_1|,|x_2-y_2|\}}$$

for all $t \in J_\infty$ and $x_1,x_2,y_1,y_2 \in \mathbb{R}$. Moreover, $\sup\limits_{t\in J_\infty} \ell_2(t) = L_2$.

(A_4) The function $t \mapsto K(t) = k(t,0,0)$ is continuous and bounded on J_∞ with bound K_0.

(B_1) The function g is Carathéodory.

(B_2) There exists a function $b \in BC(J_\infty, \mathbb{R}_+)$ such that

$$|g(t,x,y)| \le b(t)$$

for all $t \in J_\infty$ and $x, y \in \mathbb{R}$. Moreover, we assume that

$$\lim_{t\to\infty} \overline{a}(t) \int_{t_0}^{t} b(s)\, ds = 0.$$

Remark 4.1. *If $a \in \mathcal{CRB}(J_\infty)$, then $\overline{a} \in BC(J_\infty, \mathbb{R}_+)$ and so the number $\|\overline{a}\| = \sup_{t\in J_\infty} \overline{a}(t)$ exists. Again, since the hypothesis (B_2) holds, the function $w : \mathbb{R}_+ \to \mathbb{R}_+$ defined by the expression $w(t) = \overline{a}(t) \int_{t_0}^{t} b(s)\, ds$ is continuous and satisfies the condition $\lim_{t\to\infty} w(t) = 0$. So the number $W = \sup_{t\ge t_0} w(t)$ exists.*

The following lemma is useful in the sequel.

Lemma 4.2. *Assume that hypothesis (A_0) holds. Then, for any function $h \in L^1(J_\infty, \mathbb{R}_+)$, the function $x \in BC(J_\infty, \mathbb{R}_+)$ is a solution of the HFDE*

$$\frac{d}{dt}\left[\frac{a(t)x(t) - k(t,x(t),x(\gamma(t)))}{f(t,x(t),x(\alpha(t)))}\right] = h(t), \quad t \in J_\infty, \tag{4.1}$$

and

$$x(0) = x_0 \tag{4.2}$$

if and only if x satisfies the hybrid integral equation (HIE)

$$\begin{aligned} x(t) = \big[f(t,x(t),x(\alpha(t)))\big]&\left(C_0\, \overline{a}(t) + \overline{a}(t) \int_{t_0}^{t} h(s)\, ds\right) \\ &+ \overline{a}(t)\, k(t,x(t),x(\gamma(t))) \end{aligned} \tag{4.3}$$

for all $t \in J_\infty$, where $C_0 = \dfrac{a(t_0)x_0 - k(t_0,x_0,x_0)}{f(t_0,x_0,x_0)}$.

Proof. Let $h \in L^1(J_\infty, \mathbb{R}_+)$. Assume first that x is a solution of the HFDE (4.1)-(4.2). By dfn, the map

$$t \mapsto \frac{a(t)x(t) - k(t,x(t),x(\gamma(t)))}{f(t,x(t),x(\alpha(t)))}$$

is continuous, whence $\dfrac{d}{dt}\left[\dfrac{a(t)x(t) - k(t,x(t),x(\gamma(t)))}{f(t,x(t),x(\alpha(t)))}\right]$ is integrable on J_∞. Applying integration to (4.1) from t_0 to t, we obtain the HFDE (4.3) on J_∞.

Conversely, assume that the function x satisfies the HFIE (4.3) on J_∞. Since $h \in L^1(J_\infty, \mathbb{R}_+)$, it can be proved that the function

$$t \mapsto \frac{a(t)x(t) - k(t, x(t), x(\gamma(t)))}{f(t, x(t), x(\alpha(t)))}$$

is continuous for each $x \in BC(J_\infty, \mathbb{R}_+)$ and hence differential on J_∞. By direct differentiation of the HFIE (4.3), we obtain the HFDE (4.1). Again, substituting $t = t_0$ in the HFIE (4.3) yields

$$\frac{a(t_0)x(t_0) - k(t_0, x(t_0), x(t_0))}{f(t_0, x(t_0), x(t_0))} = \frac{a(t_0)x_0 - k(t_0, x_0, x_0)}{f(t_0, x_0, x_0)}.$$

Since the mapping $x \mapsto \dfrac{a(t_0)x - k(t_0, x, x)}{f(t_0, x, x)}$ is injective in $\mathbb{R}$, we obtain $x(t_0) = x_0$. Hence the proof of the lemma is complete. $\square$

Our main existence and global attractivity result is as follows.

Theorem 4.1. *Assume that the hypotheses* (A_1) *through* (A_3) *and* (B_1) *through* (B_2) *hold. Further, assume that the condition*

$$\max\left\{L_1\left(\left|\frac{x_0 - k(t_0, x_0, x_0)}{f(t_0, x_0, x_0)}\right| \|\overline{a}\| + W\right)\,,\ L_2\right\} \le \min\{K_1\,,\ K_2\} \tag{4.4}$$

holds. Then the HFDE (1.1) *has a solution and solutions are uniformly globally attractive defined on* J_∞.

Proof. Now, by an application of Lemma 4.2, the HFDE (1.1) is equivalent to the following hybrid functional integral equation (in short HFIE)

$$\begin{aligned} x(t) = \big[f(t, x(t), x(\alpha(t)))\big]\left(C_0\,\overline{a}(t) + \overline{a}(t)\int_{t_0}^{t} g(s, x(s), x(\gamma(s)))\,ds\right) \\ + \overline{a}(t)\,k(t, x(t), x(\gamma(t))) \end{aligned} \tag{4.5}$$

for all $t \in J_\infty$. Set $X = BC(J_\infty, \mathbb{R})$ and define a closed ball $\overline{B}_r(0)$ in X centered at origin 0 of radius r given by

$$r = \big(L_1 + F_0\big)\ \big(|C_0|\,\|\overline{a}\| + W\big) + L_2 + K_0$$

where, C_0 is defined as in Lemma 4.2.

Define three operators $\mathcal{A}$, $\mathcal{C}$ on X and $\mathcal{B}$ on $\overline{B}_r(0)$ by

$$\mathcal{A}x(t) = f(t, x(t), x(\alpha(t))),\ \ t \in J_\infty, \tag{4.6}$$

$$\mathcal{B}x(t) = C_0\,\overline{a}(t) + \overline{a}(t)\int_{t_0}^{t} g(s, x(s), x(\gamma(s)))\,ds,\ \ t \in J_\infty, \tag{4.7}$$

and

$$\mathcal{C}x(t) = k(t, x(t), x(\gamma(t))),\ \ t \in J_\infty. \tag{4.8}$$

Then the HFIE (4.5) is transformed into the operator equation as

$$\mathcal{A}x(t)\,\mathcal{B}x(t) + \mathcal{C}x(t) = x(t),\ \ t \in J_\infty. \tag{4.9}$$

We show that the operators $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ satisfy all the conditions of Theorem 2.1 on $BC(J_\infty, \mathbb{R})$. First we we show that the operators $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ define the mappings

$\mathcal{A}, \mathcal{C} : X \to X$ and $\mathcal{B} : \overline{B}_r(0) \to X$. Let $x \in X$ be arbitrary. Obviously, $\mathcal{A}x$ is a continuous function on J_∞. We show that $\mathcal{A}x$ is bounded on J_∞. Thus, if $t \in J_\infty$, then we obtain:

$$\begin{aligned}
|\mathcal{A}x(t)| &= |f(t, x(t), x(\alpha(t)))| \\
&\leq |f(t, x(t), x(\alpha(t))) - f(t, 0, 0)| + |f(t, 0, 0)| \\
&\leq \frac{1}{2} \cdot \frac{\ell_1(t) \max\{|x(t)|, |x(\alpha(t))|\}}{K_1 + \max\{|x(t)|, |x(\alpha(t))|\}} + F_0 \\
&\leq L_1 + F_0.
\end{aligned}$$

Therefore, taking the supremum over t,

$$\|\mathcal{A}x\| \leq L_1 + F_0 = N_1.$$

Thus $\mathcal{A}x$ is continuous and bounded on J_∞. As a result $\mathcal{A}x \in X$. Similarly, it can be shown that $\mathcal{C}x \in X$ and in particular, $\mathcal{A}, \mathcal{C} : X \to X$. Next, the function $\overline{a}$ as well as the indefinite integral is continuous on J_∞ and so the function $\mathcal{B}x$ is continuous on J_∞. We show that $\mathcal{B}x$ is bounded on J_∞. Let $x \in \overline{B}_r(0)$ be arbitrary. Then we have

$$\begin{aligned}
\left|\mathcal{B}x(t)\right| &\leq \left|C_0\, \overline{a}(t)\right| + \overline{a}(t) \int_{t_0}^{t} \left|g(s, x(s), x(\gamma(s)))\right| ds \\
&\leq |C_0|\, \overline{a}(t) + \overline{a}(t) \int_{t_0}^{t} b(s)\, ds \\
&\leq |C_0|\, \|\overline{a}\} + W
\end{aligned}$$

for all $t \in t_\infty$. Now, taking the supremum over t, we obtain $\|\mathcal{B}x\| \leq |C_0|\, \|\overline{a}\} + W$ for all $x \in \overline{B}_r(0) \to X$. As a result, $\mathcal{B} : \overline{B}_r(0) \to X$.

Next, we show that $\mathcal{A}$ is a Lipschitz on X. Let $x, y \in X$ be arbitrary. Then, by hypothesis (H_3),

$$\begin{aligned}
\|\mathcal{A}x - \mathcal{A}y\| &= \sup_{t \in J_\infty} |\mathcal{A}x(t) - \mathcal{A}y(t)| \\
&\leq \sup_{t \in J_\infty} \frac{1}{2} \cdot \frac{\ell_1(t) \max\{|x(t) - y(t)|,\, |x(\alpha(t)) - y(\alpha(t))|\}}{K_1 + \max\{|x(t) - y(t)|,\, |x(\alpha(t)) - y(\alpha(t))|\}} \\
&\leq \frac{1}{2} \cdot \frac{L_1 \|x - y\|}{K_1 + \|x - y\|} \\
&= \psi_{\mathcal{A}}(\|x - y\|)
\end{aligned}$$

for all $x, y \in X$. This shows that $\mathcal{A}$ is a $\mathcal{D}$-Lipschitz on X with the $\mathcal{D}$-function $\psi_{\mathcal{A}}(r) = \frac{1}{2} \cdot \frac{L_1\, r}{K_1 + r}$, $r > 0$. Similarly, it can be shown that $\mathcal{C}$ is a $\mathcal{D}$-Lipschitz on X with the $\mathcal{D}$-function $\psi_{\mathcal{C}}(r) = \frac{1}{2} \cdot \frac{L_2\, r}{K_2 + r}$, $r > 0$.

Next we shows that $\mathcal{B}$ is a completely continu0ous operator on $\overline{B}_r(0)$. First, we show that $\mathcal{B}$ is continuous on $\overline{B}_r(0)$. To do this, let us fix arbitrarily $\epsilon > 0$ and let $\{x_n\}$ be a sequence of points in $\overline{B}_r(0)$ converging to a point $x \in \overline{B}_r(0)$. Then we get:

$$\begin{aligned}
&|(\mathcal{B}x_n)(t) - (\mathcal{B}x)(t)| \\
&\quad \leq |\overline{a}(t)| \int_{t_0}^{t} |g(s, x_n(s), x_n(\gamma(s))) - g(s, x(s), x(\gamma(s)))| ds
\end{aligned}$$

$$
\begin{aligned}
&\le \overline{a}(t) \int_{t_0}^{t} \Big[|g(s, x_n(s), x_n(\gamma(s)))| + |g(s, x(s), x(\gamma(s)))| \Big]\, ds \\
&\le 2\,\overline{a}(t) \int_{t_0}^{t} b(s)\, ds \\
&= 2\, w(t). \qquad (4.10)
\end{aligned}
$$

Hence, by virtue of hypothesis (B$_2$), we infer that there exists a $T > 0$ such that $w(t) \le \epsilon$ for $t \ge T$. Thus, for $t \ge T$, from the estimate (4.10) we derive that

$$
|(\mathcal{B}x_n)(t) - (\mathcal{B}x)(t)| \le 2\epsilon \quad \text{as} \quad n \to \infty.
$$

Furthermore, let us assume that $t \in [t_0, T]$. Then, by dominated convergence theorem, we obtain the estimate:

$$
\begin{aligned}
\lim_{n\to\infty} \mathcal{B}x_n(t) &= \lim_{n\to\infty} \left[C_0 \overline{a}(t) + \overline{a}(t) \int_{t_0}^{t} g(s, x_n(s), x_n(\gamma(s)))\, ds \right] \\
&= C_0 \overline{a}(t) + \overline{a}(t) \int_{t_0}^{t} \Big[\lim_{n\to\infty} g(s, x(s), x_n(\gamma(s))) \Big]\, ds \\
&= \mathcal{B}x(t) \qquad (4.11)
\end{aligned}
$$

for all $t \in [t_0, T]$. Moreover, it can be shown as below that $\{\mathcal{B}x_n\}$ is an equicontinuous sequence of functions in X. Now, following the arguments similar to that given in Granas *et al.* [17], it is proved that $\mathcal{B}$ is a continuous operator on $\overline{B}_r(0)$ into X.

Next, we show that $\mathcal{B}$ is a compact operator on $\overline{B}_r(0)$. To finish, it is enough to show that every sequence $\{\mathcal{B}x_n\}$ in $\mathcal{B}(\overline{B}_r(0))$ has a Cauchy subsequence. Now, by hypothesis (B$_2$),

$$
\begin{aligned}
|\mathcal{B}x_n(t)| &\le \big|C_0\big|\, |\overline{a}(t)| + |\overline{a}(t)| \int_{t_0}^{t} |g(s, x_n(s), x_n(\gamma(s)))|\, ds \\
&\le |C_0|\ \|\overline{a}\| + w(t) \\
&\le |C_0|\ \|\overline{a}\| + W \qquad (4.12)
\end{aligned}
$$

for all $t \in J_\infty$. Taking the supremum over t, we obtain

$$
\|\mathcal{B}x_n\| \le |C_0|\ \|\overline{a}\| + W
$$

for all $n \in \mathbb{N}$. This shows that $\{\mathcal{B}x_n\}$ is a uniformly bounded sequence in $\mathcal{B}(\overline{B}_r(0))$.

Next, we show that $\{\mathcal{B}x_n\}$ is also a equicontinuous sequence in $\mathcal{B}(\overline{B}_r(0))$. Let $\epsilon > 0$ be given. Since $\lim_{t\to\infty} w(t) = 0$, there is a constant $T_1 > 0$ such that $|w(t)| < \frac{\epsilon}{8}$ for all $t \ge T_1$. Similarly, since $\lim\limits_{t\to\infty} \overline{a}(t) = 0$, for above $\epsilon > 0$, there is a $T_2 > 0$ such that

$$
|\overline{a}(t)| < \frac{\epsilon}{8\,|C_0|}
$$

for all $t \ge T_2$. Thus, if $T = \max\{T_1, T_2\}$, then

$$
|w(t)| < \frac{\epsilon}{8}
$$

and

$$
|\overline{a}(t)| < \frac{\epsilon}{8\,|C_0|}
$$

for all $t \geq T$. Let $t, \tau \in J_\infty$ be arbitrary. If $t, \tau \in [t_0, T]$, then we have

$$\begin{aligned}
|\mathcal{B}x_n(t) - \mathcal{B}x_n(\tau)| &\leq |C_0| \; |\overline{a}(t) - \overline{a}(\tau)| \\
&\quad + \Big|\overline{a}(t) \int_{t_0}^{t} g(s, x_n(s), x_n(\gamma(s)))\, ds \\
&\qquad - \overline{a}(\tau) \int_{t_0}^{\tau} g(s, x_n(s), x_n(\gamma(s)))\, ds\Big| \\
&\leq |C_0| \; |\overline{a}(t) - \overline{a}(\tau)| \\
&\quad + \Big|\overline{a}(t) \int_{t_0}^{t} g(s, x_n(s), x_n(\gamma(s)))\, ds \\
&\qquad - \overline{a}(\tau) \int_{t_0}^{t} g(s, x_n(s), x_n(\gamma(s)))\, ds\Big| \\
&\quad + \Big|\overline{a}(\tau) \int_{t_0}^{t} g(s, x_n(s), x_n(\gamma(s)))\, ds \\
&\qquad - \overline{a}(\tau) \int_{t_0}^{\tau} g(s, x_n(s), x_n(\gamma(s)))\, ds\Big| \\
&\leq |C_0| \; |\overline{a}(t) - \overline{a}(\tau)| \\
&\quad + |\overline{a}(t) - \overline{a}(\tau)| \Big|\int_{t_0}^{t} g(s, x_n(s), x_n(\gamma(s)))\, ds\Big| \\
&\quad + |\overline{a}(\tau)| \Big|\int_{\tau}^{t} g(s, x_n(s), x_n(\gamma(s)))\, ds\Big| \\
&\leq |C_0| \; |\overline{a}(t) - \overline{a}(\tau)| \\
&\quad + |\overline{a}(t) - \overline{a}(\tau)| \int_{t_0}^{T} b(s)\, ds + \overline{a} \Big|\int_{\tau}^{t} b(s)\, ds\Big| \\
&\leq |C_0| \; |\overline{a}(t) - \overline{a}(\tau)| \\
&\quad + |\overline{a}(t) - \overline{a}(\tau)| \int_{t_0}^{T} b(s)\, ds + |w(t) - w(\tau)| \\
&\leq \big[\big|C_0\big| + \|b\|_{L^1}\big] \; |\overline{a}(t) - \overline{a}(\tau)| + |w(t) - w(\tau)|
\end{aligned}$$

where, $w(t) = \overline{a}(t) \displaystyle\int_{t_0}^{t} b(s)\, ds$ and $\|b\|_{L^1} = \displaystyle\int_{t_0}^{T} b(s)\, ds$.

By the uniform continuity of the function $\overline{a}$ and w on $[t_0, T]$, for above ϵ we have the numbers $\delta_1 > 0$ and $\delta_2 > 0$ depending only on ϵ such that

$$|t - \tau| < \delta_1 \Longrightarrow |\overline{a}(t) - \overline{a}(\tau)| < \frac{\epsilon}{8\,\big[\big|C_0\big| + \|b\|_{L^1}\big]}$$

and

$$|t - \tau| < \delta_2 \Longrightarrow |w(t) - w(\tau)| < \frac{\epsilon}{8}$$

Let $\delta_3 = \min\{\delta_1, \delta_2\}$. Then

$$|t-\tau| < \delta_3 \Longrightarrow |\mathcal{B}x_n(t) - \mathcal{B}x_n(\tau)| < \frac{\epsilon}{4}$$

for all $n \in \mathbb{N}$.

Again, if $t, \tau > T$, then we have a $\delta_4 > 0$ depending only on ϵ such that

$$\begin{aligned}
&|\mathcal{B}x_n(t) - \mathcal{B}x_n(\tau)| \\
&\quad \le |C_0|\ |a(t) - a(\tau)| \\
&\qquad + \left|\overline{a}(t)\int_{t_0}^{t} g(s, x_n(s))\, ds - \overline{a}(t)\int_{t_0}^{\tau} g(s, x_n(s), x_n(\gamma(s)))\, ds\right| \\
&\quad \le |C_0|\ |\overline{a}(t)| + |\overline{a}(\tau)| + w(t) + w(\tau) \\
&\quad < \frac{\epsilon}{2} < \epsilon
\end{aligned}$$

for all $n \in \mathbb{N}$ whenever $|t-\tau| < \delta_4$. Similarly, if $t, \tau \in \mathbb{R}_+$ with $t < T < \tau$, then we have

$$|\mathcal{B}x_n(t) - \mathcal{B}x_n(\tau)| \le |\mathcal{B}x_n(t) - \mathcal{B}x_n(T)| + |\mathcal{B}x_n(T) - \mathcal{B}x_n(\tau)|.$$

Take $\delta = \min\{\delta_3, \delta_4\} > 0$ depending only on ϵ. Therefore, from the above obtained estimates, it follows that

$$|\mathcal{B}x_n(t) - \mathcal{B}x_n(T)| < \frac{\epsilon}{2} \quad \text{and} \quad |\mathcal{B}x_n(T) - \mathcal{B}x_n(\tau)| < \frac{\epsilon}{2}$$

for all $n \in \mathbb{N}$ whenever $|t-\tau| < \delta$. As a result, $|\mathcal{B}x_n(t) - \mathcal{B}x_n(\tau)| < \epsilon$ for all $t, \tau \in J_\infty$ and for all $n \in \mathbb{N}$ whenever $|t-\tau| < \delta$. This shows that $\{\mathcal{B}x_n\}$ is a equicontinuous sequence in X. Now an application of Arzelà-Ascoli theorem yields that $\{\mathcal{B}x_n\}$ has a uniformly convergent subsequence on the compact subset $[t_0, T]$ of J_∞. Without loss of generality, call the subsequence to be the sequence itself. We show that $\{\mathcal{B}x_n\}$ is Cauchy in X. Now $|\mathcal{B}x_n(t) - \mathcal{B}x(t)| \to 0$ as $n \to \infty$ for all $t \in [t_0, T]$. Then for given $\epsilon > 0$ there exists an $n_0 \in \mathbb{N}$ such that

$$\sup_{t_0 \le p \le T} \overline{a}(p)\int_{t_0}^{p} |g(s, x_m(s), x_m(\gamma(s))) - g(s, x_n(s), x_n(\gamma(s)))|\, ds < \frac{\epsilon}{2}$$

for all $m, n \ge n_0$. Therefore, if $m, n \ge n_0$, then we have

$$\begin{aligned}
&\|\mathcal{B}x_m - \mathcal{B}x_n\| \\
&\quad = \sup_{t_0 \le p < \infty}\left|\overline{a}(p)\int_{t_0}^{p} |g(s, x_m(s), x_m(\gamma(s))) - g(s, x_n(s), x_n(\gamma(s)))|\, ds\right| \\
&\quad \le \sup_{t_0 \le p \le T}\left|\overline{a}(p)\int_{t_0}^{p} |g(s, x_m(s), x_m(\gamma(s))) - g(s, x_n(s), x_m(\gamma(s)))|\, ds\right| \\
&\qquad + \sup_{p \ge T}\overline{a}(p)\int_{t_0}^{p}\Big[|g(s, x_m(s), x_m(\gamma(s)))| + |g(s, x_n(s), x_m(\gamma(s)))|\Big]\, ds \\
&\quad < \epsilon.
\end{aligned}$$

This shows that $\{\mathcal{B}x_n\} \subset \mathcal{B}(\overline{B}_r(0)) \subset X$ is Cauchy. Since X is complete, $\{\mathcal{B}x_n\}$ converges to a point in X. As $\mathcal{B}(\overline{B}_r(0))$ is closed, we have that $\{\mathcal{B}x_n\}$ converges to a point in $\mathcal{B}(\overline{B}_r(0))$. Hence $\mathcal{B}(\overline{B}_r(0))$ is relatively compact and consequently $\mathcal{B}$ is a continuous and compact operator on $\overline{B}_r(0)$ into X.

Next, we estimate the value of the constant M. By dfn of M, one has

$$\begin{aligned}\|\mathcal{B}(\overline{B}_r(0))\| &= \sup\{\|\mathcal{B}x\| \ : \ x \in \overline{B}_r(0)\}\\ &= \sup\left\{\sup_{t\in J_\infty} |\mathcal{B}x(t)| \ : \ x \in \overline{B}_r(0)\right\}\\ &\le \sup_{x\in\overline{B}_r(0)}\left\{\sup_{t\in J_\infty} \big|C_0\big|\,|\overline{a}(t)|\right.\\ &\qquad \left. + \sup_{t\in J_\infty}|\overline{a}(t)|\int_{t_0}^{t}|g(s,x(s),x(\gamma(s)))|\,ds\right\}\\ &\le \big|C_0\big|\,\|\overline{a}\| + W\\ &= M.\end{aligned}$$

Thus,

$$\|\mathcal{B}x\| \le \big|C_0\big|\,\|\overline{a}\| + W = M$$

for all $x \in \overline{B}_r(0)$. Next, let $x, y \in X$ be arbitrary. Then,

$$\begin{aligned}|x(t)| &\le |\mathcal{A}x(t)|\,|\mathcal{B}y(t)| + |\mathcal{C}x(t)|\\ &\le \|\mathcal{A}x\|\,\|\mathcal{B}y\| + \|\mathcal{C}x\|\\ &\le \|\mathcal{A}(X)\|\,\|\mathcal{B}(\overline{B}_r(0))\| + \|\mathcal{C}(X)\|\\ &\le \big(L_1 + F_0\big)\,M + L_2 + K_0\\ &\le \big(L_1 + F_0\big)\,\big(\big|C_0\big|\,\|\overline{a}\| + W\big) + L_2 + K_0\end{aligned}$$

for all $t \in J_\infty$. Therefore, we have:

$$\|x\| \le \big(L_1 + F_0\big)\,\big(\big|C_0\big|\,\|\overline{a}\| + W\big) + L_2 + K_0 = r.$$

This shows that $x \in \overline{B}_r(0)$ and hypothesis (c) of Theorem 2.1 is satisfied. Again,

$$\begin{aligned}M\psi_{\mathcal{A}}(r) + \psi_{\mathcal{C}}(r) &\le \frac{1}{2}\cdot\frac{L_1\left(\big|C_0\big|\,\|\overline{a}\| + W\right)r}{K_1 + r} + \frac{1}{2}\cdot\frac{L_2 r}{K_2 + r}\\ &\le \frac{\max\left\{L_1\big(\big|C_0\big|\,\|\overline{a}\| + W\big)\,,\,L_2\right\}r}{\min\{K_1\,,\,K_2\} + r}\\ &\le \frac{\max\left\{L_1\left(\left|\dfrac{a(t_0)x_0 - k(t_0,x_0,x_0)}{f(t_0,x_0,x_0)}\right|\,\|\overline{a}\| + W\right)\,,\,L_2\right\}r}{\min\{K_1\,,\,K_2\} + r} < r\end{aligned}$$

for all $r > 0$, because

$$\max\left\{L_1\left(\left|\frac{a(t_0)x_0 - k(t_0,x_0,x_0)}{f(t_0,x_0,x_0)}\right|\,\|\overline{a}\| + W\right)\,,\,L_2\right\} \le \min\{K_1\,,\,K_2\}.$$

Therefore, hypothesis (d) of Theorem 2.1 is satisfied. Now we apply Theorem 2.1 to the operator equation $\mathcal{A}x\,\mathcal{B}x + \mathcal{C}x = x$ to yield that the HFDE (1.1) has a solution on J_∞. Moreover, the solutions of the HFDE (1.1) are in $\overline{B}_r(0)$. Hence, solutions are global in nature.

Finally, let $x, y \in \overline{B}_r(0)$ be any two solutions of the HFDE (1.1) on J_∞. Then

$$|x(t) - y(t)|$$

$$
\begin{aligned}
&\leq \Bigg| \Big[\, f(t,x(t),x(\alpha(t)))\,\Big] \left(C_0\, \overline{a}(t) + \overline{a}(t) \int_{t_0}^{t} g(s,x(s),x(\gamma(s)))\, ds \right) \\
&\qquad - \Big[\, f(t,y(t),y(\alpha(t)))\,\Big] \left(C_0\, \overline{a}(t) + \overline{a}(t) \int_{t_0}^{t} g(s,y(s),y(\gamma(s)))\, ds \right) \Bigg| \\
&\qquad + \Big| k(t,x(t),x(\gamma(t))) - k(t,x(t),x(\gamma(t))) \Big|\, \overline{a}(t) \\
&\leq \big| f(t,x(t),x(\alpha(t))) - f(t,y(t),y(\alpha(t)))\ \big|\ \big(|C_0|\, \overline{a}(t) + w(t) \big) \\
&\qquad + \big| f(t,y(t),y(\gamma(t))) \big|\, \overline{a}(t) \int_{t_0}^{t} \big|\ g(s,x(s),x(\gamma(s))) - g(s,y(s),y(\gamma(s)))\ \big|\ ds \\
&\qquad + \big|\ k(t,x(t),x(\gamma(t))) - k(t,x(t),x(\gamma(t)))\ \big|\ \overline{a}(t) \\
&\leq \frac{L_1\, \max\{|x(t)-y(t)|,\, |x(\alpha(t)) - y(\alpha(t))|\}}{K_1 + \max\{|x(t)-y(t)|,\, |x(\alpha(t)) - y(\alpha(t))|\}} \big(|C_0|\, \|\overline{a}\| + W \big) \\
&\qquad + \frac{L_2\, \max\{|x(t)-y(t)|,\, |x(\alpha(t)) - y(\alpha(t))|\}}{K_2 + \max\{|x(t)-y(t)|,\, |x(\alpha(t)) - y(\alpha(t))|\}} \|\overline{a}\| \\
&\qquad + 2(F_0 + L_1)\, \overline{a}(t) \int_{t_0}^{t} b(s)\, ds \\
&\leq \frac{\max\{L_1 \big(|C_0|\, \|\overline{a}\| + W \big),\, L_2\}\, \max\{|x(t)-y(t)|,\, |x(\alpha(t)) - y(\alpha(t))|\}}{\min\{K_1,\, K_2\} + \max\{|x(t)-y(t)|,\, |x(\alpha(t)) - y(\alpha(t))|\}} \\
&\qquad + 2(F_0 + L_1)\, w(t)
\end{aligned}
\tag{4.13}
$$

Now, taking the limit superior as $t \to \infty$ in the above inequality yields,

$$
\begin{aligned}
&\limsup_{t\to\infty} |x(t) - y(t)| \\
&\leq \limsup_{t\to\infty} \frac{L\, \max\{|x(t)-y(t)|,\, |x(\alpha(t)) - y(\alpha(t))|\}}{K + \max\{|x(t)-y(t)|,\, |x(\alpha(t)) - y(\alpha(t))|\}} \\
&\qquad + \limsup_{t\to\infty} 2(F_0 + L_1)\, w(t) \\
&\leq \frac{L\, \max\left\{ \limsup\limits_{t\to\infty} |x(t)-y(t)|,\, \limsup\limits_{t\to\infty} |x(\alpha(t)) - y(\alpha(t))| \right\}}{K + \max\left\{ \limsup\limits_{t\to\infty} |x(t)-y(t)|,\, \limsup\limits_{t\to\infty} |x(\alpha(t)) - y(\alpha(t))| \right\}} \\
&\qquad + 2(F_0 + L_1) \limsup_{t\to\infty}\, w(t) \\
&\leq \frac{L\, \limsup\limits_{t\to\infty} |x(t)-y(t)|}{K + \limsup\limits_{t\to\infty} |x(t)-y(t)|}
\end{aligned}
$$

where $L = \max\{L_1 \big(|C_0|\, \|\overline{a}\| + W \big),\, L_2\} \leq \min\{K_1,\, K_2\} = K$. As a result, we obtain $\limsup\limits_{t\to\infty} |x(t) - y(t)| = 0$ and consequently $\lim\limits_{t\to\infty} |x(t) - y(t)| = 0$. Therefore, there is a real number $T > 0$ such that $|x(t) - y(t)| < \epsilon$ for all $t \geq T$. Consequently, the solutions of HFDE (1.1) are uniformly globally attractive on J_∞. This completes the proof. □

Remark 4.2. The conclusion of Theorem 4.1 also remains true under the following more general modified conditions:

(i) The hypothesis (A_2) is replaced with the following hypothesis:

(A_2') The function f is continuous and there exist a $\mathcal{D}$-function $\psi_f \in \mathfrak{D}$ such that
$$\big|f(t,x_1,x_2) - f(t,y_1,y_2)\big| \leq \psi_f(\max\{|x_1-y_1|,|x_2-y_2|\})$$
for all $t \in J_\infty$ and $x_1,x_2,y_1,y_2 \in \mathbb{R}$.

(ii) The hypothesis (A_3) is replaced with the following hypothesis:

(A_3') The function k is continuous and and there exist a $\mathcal{D}$-function $\psi_k \in \mathfrak{D}$ such that
$$\big|k(t,x_1,x_2) - k(t,y_1,y_2)\big| \leq \psi_k(\max\{|x_1-y_1|,|x_2-y_2|\})$$
for all $t \in J_\infty$ and $x_1,x_2,y_1,y_2 \in \mathbb{R}$.

(iii) The inequality (4.4) is replaced by
$$\left(\Big|\frac{a(t_0)x_0 - k(t_0,x_0,x_0)}{f(t_0,x_0,x_0)}\Big| \, \|\overline{a}\| + W\right)\psi_f(r) + \psi_k(r) < r,$$
for all $r > 0$.

Theorem 4.2. *Assume that all conditions Theorem* 4.1 *hold. Then the HFDE* (1.1) *has a solution and solutions are uniformly globally attractive and ultimately positive defined on* J_∞.

Proof. By Theorem 4.1, the HFDE (1.1) has a global solution in the closed ball $\overline{B}_r(0)$, where the radius r is given as in the proof of Theorem 4.1 and the solutions are uniformly globally attractive on J_∞. We know that for any $x,y \in \mathbb{R}$, one has the inequality,
$$|x|\,|y| = |xy| \geq xy,$$
and therefore,
$$\big||xy| - (xy)\big| \leq |x|\,\big||y| - y\big| + \big||x| - x\big|\,|y| \tag{4.14}$$
for all $x,y \in \mathbb{R}$. Now, for any solution $x \in \overline{B}_r(0)$, one has
$$\begin{aligned}
&\big||x(t)| - x(t)\big| \\
&= \Bigg|\Big|f(t,x(t),x(\alpha(t)))\Big|\,\Big|\left(C_0\overline{a}(t) + \overline{a}(t)\int_{t_0}^t g(s,x(s),x(\gamma(s)))\,ds\right)\Big| \\
&\quad - \big[f(t,x(t),x(\alpha(t)))\big]\left(C_0\overline{a}(t) + \overline{a}(t)\int_{t_0}^t g(s,x(s),x(\gamma(s)))\,ds\right)\Bigg| \\
&\quad + \overline{a}(t)\Big|\big|k(t,x(t),x(\alpha(t)))\big| - k(t,x(t),x(\alpha(t)))\Big| \\
&\leq \Bigg|\,\big|\,f(t,x(t),x(\alpha(t)))\,\big|\,\big(|C_0| - C_0\big)\,\overline{a}(t)\Bigg| \\
&\quad + \big|f(t,x(t),x(\alpha(t)))\big|\,\Bigg|\Big|\overline{a}(t)\int_{t_0}^t g(s,x(s),x(\gamma(s)))\,ds\Big| \\
&\qquad\qquad - \overline{a}(t)\int_{t_0}^t g(s,x(s),x(\gamma(s)))\,ds\Bigg|
\end{aligned}$$

$$
\begin{aligned}
&+ \Big| |f(t,x(t),x(\alpha(t)))| - f(t,x(t),x(\alpha(t)))\Big| \\
&\quad \times \left| C_0 \overline{a}(t) + \overline{a}(t) \int_{t_0}^{t} g(s,x(s),x(\gamma(s)))\,ds \right| \\
&+ \overline{a}(t) \Big| |k(t,x(t),x(\alpha(t)))| - k(t,x(t),x(\alpha(t)))\Big| \\
\leq{}& 2|C_0|(L_1+F_0)\overline{a}(t) + 2(L_1+F_0)\,w(t) \\
&+ 2(L_1+F_0)\big[|C_0|\,\overline{a}(t) + w(t)\big] + 2(L_2+K_0)\overline{a}(t)
\end{aligned} \tag{4.15}
$$

for all $t \in J_\infty$. Taking the limit superior as $t \to \infty$ in the above inequality (4.14), we obtain the estimate that $\lim_{t\to\infty} \big||x(t)| - x(t)\big| = 0$. Therefore, there is a real number $T > 0$ such that $|\,|x(t)| - x(t)| \leq \epsilon$ for all $t \geq T$. Hence, solutions of the HFDE (1.1) are uniformly globally attractive as well as ultimately positive defined on J_∞. This completes the proof. □

In the following we indicate an exm to illustrate the abstract ideas contained in Theorem 4.1.

Example 4.1. Let $J_\infty = \mathbb{R}_+ = [0,\infty) \subset \mathbb{R}$. Given a function $a(t) = e^t \in \mathcal{CRB}(\mathbb{R}_+)$, consider the following functional hybrid differential equation with the mixed arguments of anticipation and retardation,

$$
\left.
\begin{aligned}
\frac{d}{dt}\left[\frac{e^t x(t) - k_1(t,x(t),x(2t))}{f_1(t,x(t),x(2t))}\right] &= \frac{e^{-t}\log(1+|x(t)|+|x(t/2)|)}{1+|x(t)|+|x(t/2)|}, \\
x(0) &= 0,
\end{aligned}
\right\} \tag{4.16}
$$

for all $t \in \mathbb{R}_+$, where the functions $k_1 : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $f_1 : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}_+ \setminus \{0\}$ are defined by

$$
f_1(t,x,y) = \begin{cases} 2 & \text{if } -\infty < x \leq 0, \\ \dfrac{1}{4}\Big(\dfrac{x}{1+x} + \dfrac{y}{1+y}\Big) + 2 & \text{if } \ 0 < x < \infty, \end{cases}
$$

and

$$
k_1(t,x,y) = \begin{cases} 1 & \text{if } -\infty < x \leq 0, \\ \dfrac{1}{4}\Big(\dfrac{x}{1+x} + \dfrac{y}{1+y}\Big) + 1 & \text{if } \ 0 < x < \infty, \end{cases}
$$

for all $t \in [0,1]$. Define a function $g : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$
g(t,x,y) = \frac{e^{-t}\log(1+|x|+|y|)}{1+|x|+|y|}.
$$

Clearly, the functions f and k satisfies the hypothesis (A_1) and are bounded on $\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}$ with bounds $M_{f_1} = 3$ and $M_{k_1} = 2$ respectively. See Dhage [13] and references therein. Now, it can be shown as in Banas and Dhage [1] that the function f satisfies the hypothesis (A_2') with $\psi_f(r) = \dfrac{1}{2} \cdot \dfrac{r}{1+r}$ for $r > 0$. Again, the function k also satisfies the hypothesis (A_3') with $\psi_k(r) = \dfrac{1}{2} \cdot \dfrac{r}{1+r}$ for $r > 0$. Also the function g satisfies the hypotheses (B_1)-(B_2) with

$b(t) = e^{-t}$. Hence, $\lim\limits_{t\to\infty} w(t) = \lim\limits_{t\to\infty} e^{-t} \int_0^t e^{-s}\, ds = 0$. Hence by Theorem 4.1, the HFDE (4.16) has a solution and solutions are globally attractive defined on $\mathbb{R}_+$. Moereover, the solutions are globally ultimately positive defined on $\mathbb{R}_+$.

Remark 4.3. Finally, we remark that the ideas of this paper may be extended to a more general hybrid functional differential equations,

$$\left.\begin{aligned} \frac{d}{dt}\left[\frac{a(t)x(t) - k(t, x(\gamma_1(t)), \cdots, x(\gamma_n(t)))}{f(t, x(\alpha_1(t)), \cdots, x(\alpha_n(t)))}\right] & \\ = g(t, x(\eta_1(t)), \cdots, x(\eta_n(t))),\ \ t \in J_\infty, & \\ x(t_0) = x_0 \in \mathbb{R}, & \end{aligned}\right\} \tag{4.17}$$

for proving the similar results on unbounded intervals J_∞ of the real line $\mathbb{R}$ under appropriate modifications.

Remark 4.4. If g is assumed to be continuous function on $J_\infty \times \mathbb{R} \times \mathbb{R}$, then the attractivity and existence results for the HFDE (1.1) may be obtained via another fixed point theoretic approach of using measure of noncompactness. See the details of this procedure that appears in Banas and Dhage [1], Dhage [10, 12], Dhage *et. al* [14] and the references given therein. The advantage of this approach is that we automatically get different characterizations of the solutions in the long run of independent variable t on J_∞, however the difficulty lies in the construction of the handy tool related to the corresponding measure of noncompactness.

5 The Conclusion

From foregoing discussion, it is clear that the pulling functions are very much useful in the existence as well as qualitative study of asymptotic behavior of the solutions of nonlinear differential equations. Hence there is a importance of the class of pulling functions $\mathcal{CRB}(J_\infty)$ in the study of such nonlinear equations on the unbounded intervals of real line. The same fact is also true for asymptotic qualitative study of the nonlinear integral equations. The choice of pulling functions depends upon the nature of the differential equations. For different asymptotic qualitative behavior of the solution for nonlinear hybrid differential equations, the different pulling functions may be used. Similar is the case with nonlinear integral equations. See Banas and Dhage [1], Burton and Furumochi [3], Dhage [8, 9, 10], Dhage *et. al* [14] and references therein. Again, the hybrid fixed point theorems are useful in the study of different asymptotic characterizations of the solutions for different types of functional differential equations on the unbounded intervals of real line (see Dhage *et. al* [15]), Betea *et. al* [2] and Hallaci *et. al* [18]). The choice of the hybrid fixed point theorem depends upon the situations and the circumstances of the nonlinearities involved in the problems. The cleverer selection of the hybrid fixed point theorem yields very powerful existence results as well as different asymptotic characterizations of the nonlinear functional differential equations. In this article, we proved the existence as well as global attractivity and ultimate positivity of the solutions only, however other asymptotic characterizations may also be proved for nonlinear functional hybrid differential equations involving three nonlinearites under suitable hypotheses with appropriate modifications on the unbounded intervals of real line.

Acknowledgments
The author is very much thankful to **Professor J.N. Salunke** (India) for critically reading and pointing out some misprints in the earlier version of this paper. We are also thankful to the referee for his valuable suggestions to bring the paper in its present form.

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Jñānābha, Vol. 49(2) (2019), 64-81

STATISTICAL ANALYSIS OF THE ASYMMETRIC BEHAVIOR OF DIFFERENT SOLAR ACTIVITY FEATURES DURING SOLAR CYCLES 20-24

By

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(*Received : October* 16, 2019 ; *Revised: November* 11, 2019)

Abstract

The study of North-South (N-S) asymmetry phenomenon is quite important because of its applicability in understanding the nature of solar dynamo action. The aim of our present work is to study and analyze various solar active phenomena occurring in both north and south hemispheres of the sun during solar cycles 20-24 (up to December 2011). The data of X-ray solar flares from 1975 to 2011 and of $H\alpha$ flares, Sunspot Area and Solar Active Prominences (SAP) from 1964 to 2011 are first normalized and then analyzed statistically. The N-S asymmetry indices and correlation for several solar phenomena have been calculated and plotted. Our study shows that most of the activity features in solar cycles 20-23 starts from northern hemisphere and afterward dominated in the southern hemisphere but in the rise phase of the solar cycle 24, activity features start in the southern hemisphere and afterward dominated in the northern hemisphere. The study indicates that some of the solar activity features like Sunspot Area and $H\alpha$ Flares are highly correlated with one another where as in the northern hemisphere Sunspot Area and north dominated Soft X-rays are poorly correlated.

2010 Mathematics Subject Classifications: 03H10, 85A99
Keywords and phrases: North-South asymmetry, X-ray solar flares, $H\alpha$ flares, Sunspot Area and Solar Active Prominences (SAP).

1 Introduction

There is stunning variety of magnetic field related phenomena across a wide range of spatial, temporal and energy scales displayed by the Sun. These time dependent processes are controlled by the magnetic fields generated by the combined action of convection and differential rotation of a nonlinear dynamo in the solar interior. Collectively, these processes are called solar activity. The activity exhibits a regular variation on a time scale called solar cycle. All solar activity phenomena viz. Sunspot number, Solar active prominences (SAP), $H\alpha$ flares, Soft X- ray flux (SXR) etc, are related to sunspots and thus magnetic activity. Sunspots often appear in bipolar pairs whose polarity orientation is opposite in two hemisphere. Years of sunspot observations have now firmly established that the sunspot cycle has an average period of 11 years. The related magnetic phenomena on the Sun like Solar flares, Solar Prominences, Coronal Mass Ejections etc. are caused by the Sun's twisting and turning magnetic field. Solar activity features have a great effect not only on climatological parameters but also its study is important for other scientific developments like telecommunications, power lines, geophysical explorations and long term planning of space missions.

It is well known that solar active phenomena are distributed non-uniformly over the solar disk. Non-uniform occurrence of the solar activity events in one (northern or eastern) or the other (southern or western) part is known as asymmetry (N-S or E-W). The study of asymmetries may find applications in predicting the behavior of activity in coming solar cycle which is important for the prediction of the space weather and climate. The study of N-S asymmetry of solar active features through the study of nonlinear solar dynamo models has attracted the researchers during the last couple of years. The (N-S) distribution, including asymmetries, of several solar activity indices such as flares , filament, magnetic flux , relative sunspot number, sunspot areas have been discovered by various authors ([21, 10, 22, 25, 8, 26, 24, 11]). Recently the asymmetry has been also reported in solar energetic particle events ($SEPs$) ([9, 6]). These studies indicate that there exists an asymmetry in the N-S distribution of the solar activity. Several studies have been done on the asymmetric behavior of solar activities using different features, such as Sunspot number, Sunspot group number and Sunspot area ([30, 27, 16, 4]), $H\alpha$ flares (SF), Soft X-ray flares ([22, 23, 12, 13]), Solar Active Prominences (SAP), SAP at low latitude ($\leq$40) and SAP at high latitude ($\geq$50) ([28, 17, 14]). [17] studied the N-S asymmetry of the SAP (low latitude $\leq$40), SAP (high latitude $\geq$50) from 1957 to 1998 (Solar cycles 19-22). [14] presented a comparison of N-S asymmetry and distribution of Solar Active Prominence (SAP) during solar cycles 20-23. They have also presented the same analysis for different disk features of SAP in two groups of cycles 23. [27] studied various solar phenomena occurring in both northern and southern hemispheres of the sun for solar cycle 18-22. They also calculated the N-S asymmetry indices for these solar active phenomena and plotted them against the number of solar cycles. Similarly, [3] have studied N-S asymmetry using different phenomena of solar activity. Summaries of the studies of hemispherical asymmetries of solar activity have been included in the works of ([29, 18, 19]). Statistical analysis show that the N-S asymmetry is statistically significant meaning thereby that it is a real physical phenomena and not a random mathematical fluctuation ([19, 5, 7]). The asymmetry tells that the magnetic field systems originating in the two hemispheres are weakly coupled, which was also inferred from the observations.

The aim of the present work is to make a detailed study of N-S asymmetry of daily solar activity features (Sunspot Area, $H\alpha$ Flares, Soft X- rays, Solar Active Prominences (SAP), SAP(Low latitude), SAP(High latitude)) from 1964 to 2011(Solar cycle 20-24).

2 Data Sets

For the present study we used the data from following sources:
The data have been collected from National Geophysical Data centers (NGDC) anonymous ftp server. The monthly north and south number of SAP (1964-2011) obtained from ftp//ftp.ngdc.noaa.gov/STP/SOLAR_DATA/SOLAR_FILAMENT, with 149652 data points.
At low latitudes ($\leq$40) a total number of 135457 SAP events and at high latitude ($\geq$50) 5971 SAP events have been reported.

The monthly north and south soft X-rays flares (1975-2011), detected by the GOES satellites, and were downloaded from
ftp//ftp.ngdc.noaa.gov/STP/SOLAR_DATA/XRAY_FLARES/XRAYS_FLARES, with 33301 data points.
The monthly north and south H flares (1964-2011) obtained from
ftp//ftp.ngdc.noaa.gov/STP/SOLAR_DATA/SOLAR_FLARES.
The monthly north and south sunspot area (1964-2011) compiled by [26] from
http://solarscience.msfc.nasa.gov/greenwich.shtml.
This period covers solar cycle 20-24. The URL address of this website is
ftp//ftp.ngdc.noaa.gov/STP/SOLAR_DATA.

3 Analysis Techniques

For the present analysis, first we have taken events both in the northern and southern hemisphere for different solar activity features like Sunspot Area (SA), Solar Active Prominences(SAP), Solar Active Prominences (low), Solar Active Prominences (high), Soft X-rays (SXR) and $H\alpha$ Flares. The north and south data have been normalized using the technique given below. After normalization the asymmetry (A_{NS}) of north and south events has been calculated. Finally, we have calculated the dispersion of the N-S asymmetry and for analyzing the significance level of the asymmetry we have used the following criteria:
If $A_{NS} > \triangle A_{NS}$, then asymmetry is insignificant.
If $A_{NS} \geq \triangle A_{NS}, > 2\triangle A_{NS}$ then asymmetry is significant.
If $A_{NS} \leq 2\triangle A_{NS}$, then asymmetry is highly significant.

3.1 Normalization

The process of taking raw data and reducing it to a set of relations or a particular range $[a, b]$, while ensuring the integrity of the raw data with data redundancy eliminated, is said to be normalization of the data. Particularly, in mathematical analysis, the data set is normalized by taking the range as the interval $[0, 1]$ for accurate analysis and minimization of errors. We use following formula for normalization of the data:

$$Normalized(x) = \frac{a + (x - p)(b - a)}{(p - q)}, \tag{3.1}$$

where: a = minimum range,
b = maximum range,
p = Minimum North events, South events,
q = Maximum North events, South events.
In order to characterize the N-S asymmetry of solar activity, usually the N-S asymmetry index is defined by

$$A_{NS} = \frac{N\text{-}S}{N + S}. \tag{3.2}$$

N represents number of solar activity phenomena in northern hemisphere,
S represents number of solar activity phenomena in southern hemisphere.
Thus, if $A_{NS} > 0$, the dominant hemisphere of activity is northern one and if $A_{NS} < 0$, the dominant hemisphere of activity is southern one.
To investigate that up to what extent the asymmetry is real, we have followed the method

of [20] in which we can define the asymmetry of random distribution on the solar disk as

$$\triangle A_{NS} = \pm\sqrt{2(N+S)}, \tag{3.3}$$

where $\triangle A_{NS}$ is the dispersion of the N-S asymmetry and N and S are defined as above.

3.2 Correlation Analysis

There are several methods to calculate the correlation coefficient, but Pearsons correlation is commonly used in linear regression. It indicates the strength of a linear relationship between two variables. In the present analysis we follow general criteria to describe strong and weak correlations, i.e., a correlation higher than 0.8 is as strong, whereas, and a correlation less than 0.5 as weak. For calculating the correlation coefficient we use the following mathematical formula

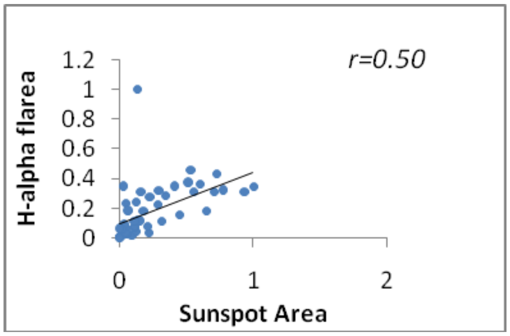
$$r = \frac{n\sum NS - (\sum N)(\sum S)}{\sqrt{\{n\sum N^2 - (\sum N)^2\}}\sqrt{\{n\sum S^2 - (\sum S)^2\}}}, \tag{3.4}$$

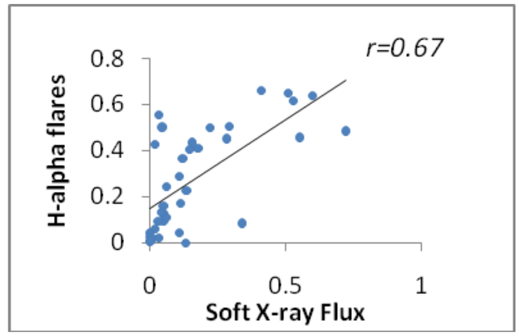
where r is correlation coefficient and

n : Total number of events.

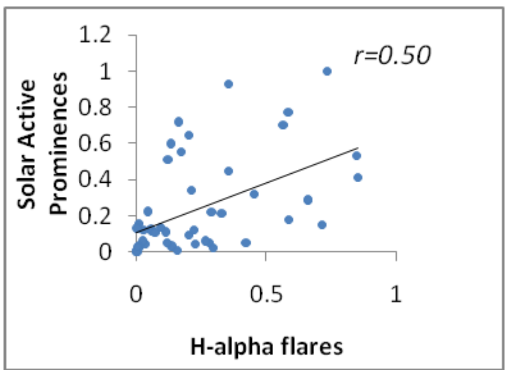
N : Normalized North events.
S : Normalized South events.
$\sum N$: Sum of the Normalized North events.
$\sum S$: Sum of the Normalized South events.
$\sum NS$: Sum of the product of Normalized North South events.
$\sum N^2$: Sum of square of Normalized North events.
$\sum S^2$: Sum of square of the Normalized South events.

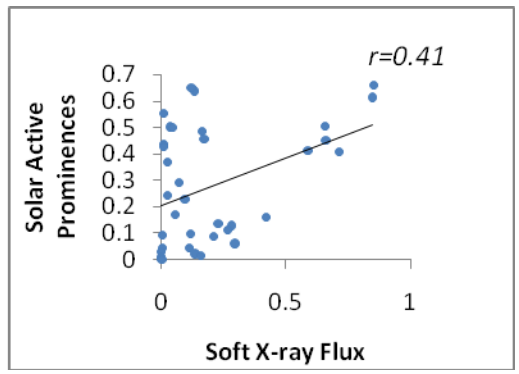
Further, the coefficient of determination (r^2) is calculated to discuss the proportion of the variance amongst the activity features.



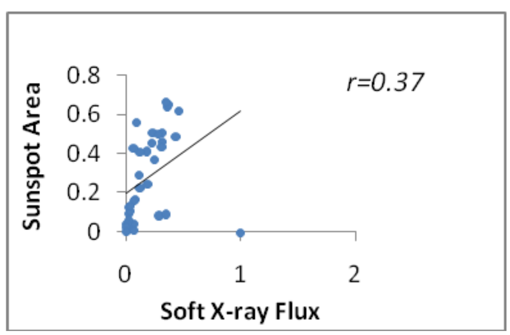


(a) **(b)**



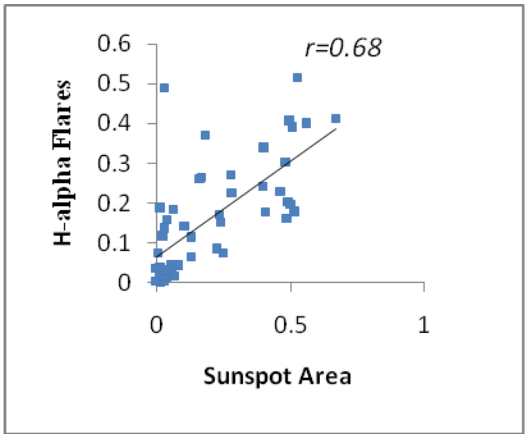


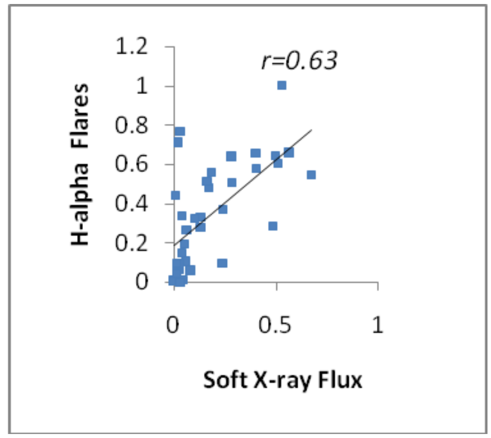
(c) **(d)**



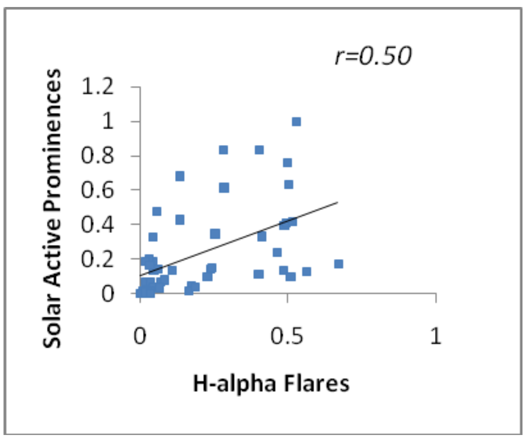
(e)

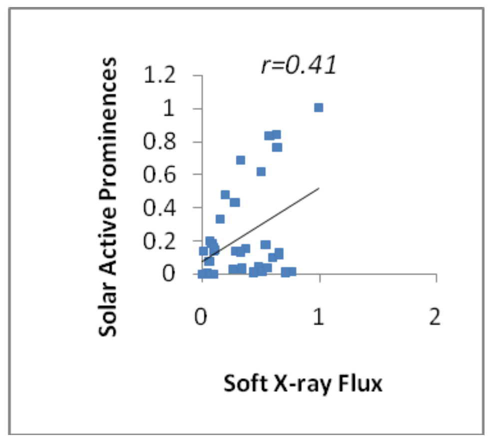
Figure 3.1: (a), (b), (c), (d), (e). Correlation Plots of Normalized North Events for different Solar Activity Features during Solar Cycle (SC) 20-24 (1964 to 2011).



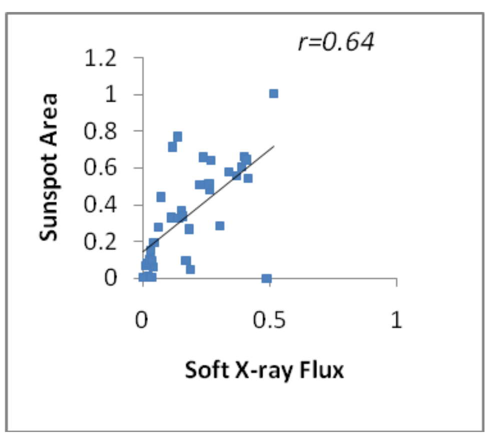


(a) **(b)**



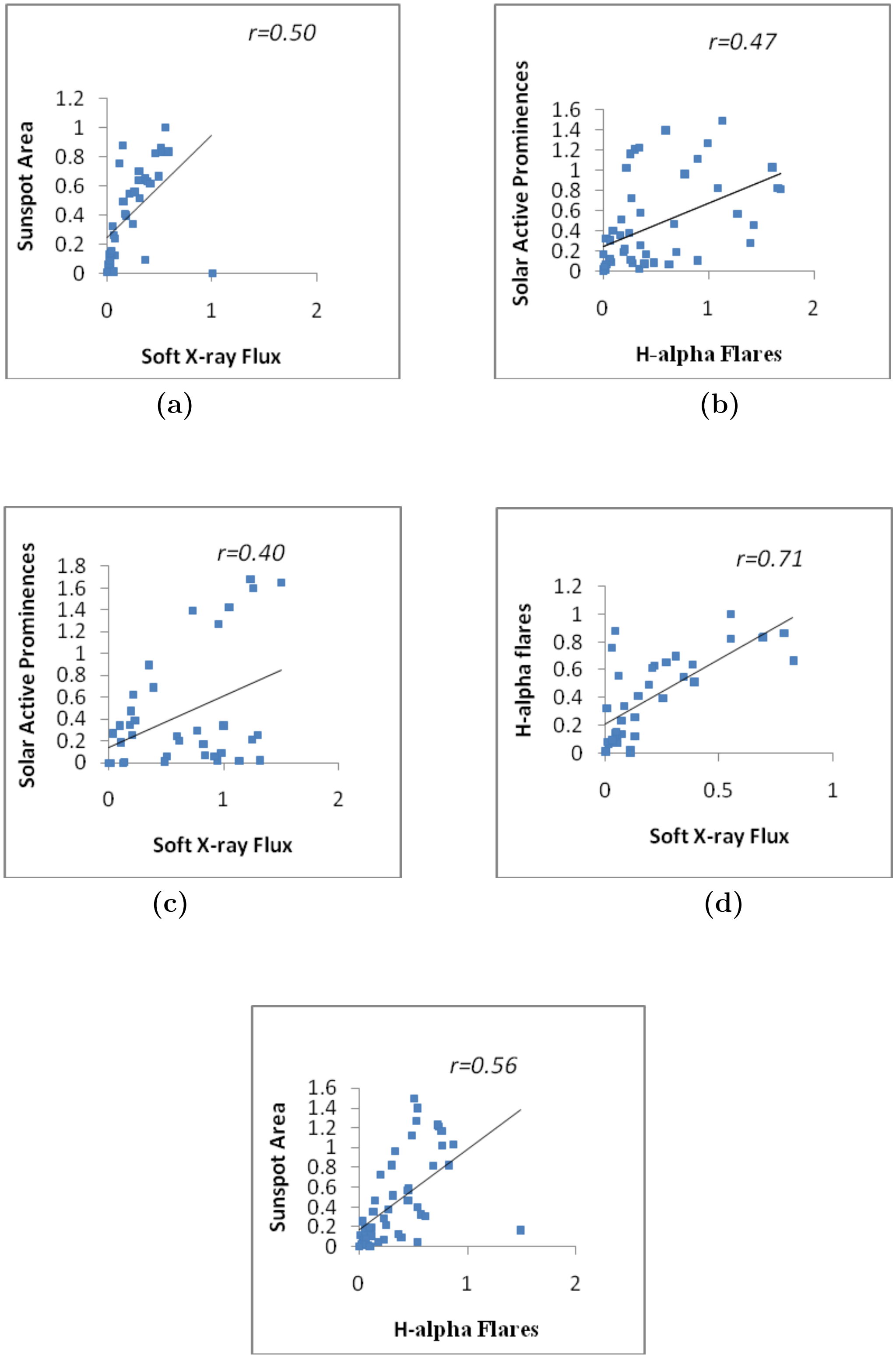


(c) **(d)**



(e)

Figure 3.2: (a), (b), (c), (d), (e). Correlation Plots of Normalized South Events for different Solar Activity Features during Solar Cycle (SC) 20-24 (1964 to 2011).



(a) (b) (c) (d) (e)

Figure 3.3: (a), (b), (c), (d), (e). Correlation Plots of Normalized Total (N+S) Events for different Solar Activity Features during Solar Cycle (SC) 20-24 (1964 to 2011).

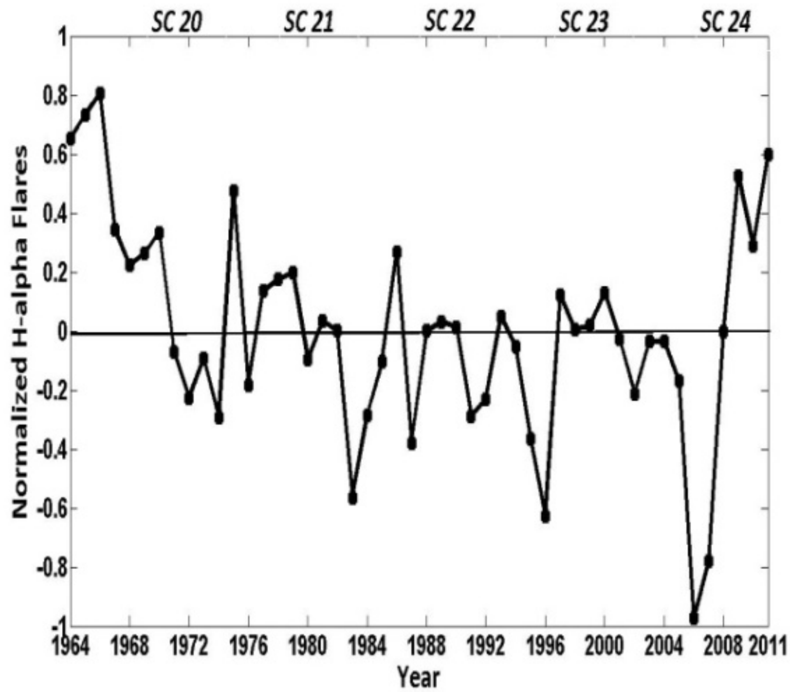


Figure 4(a). Asymmetry plot for normalized $H\alpha$ Flares for Solar Cycles (SC) 20-24 (1964 to 2011).

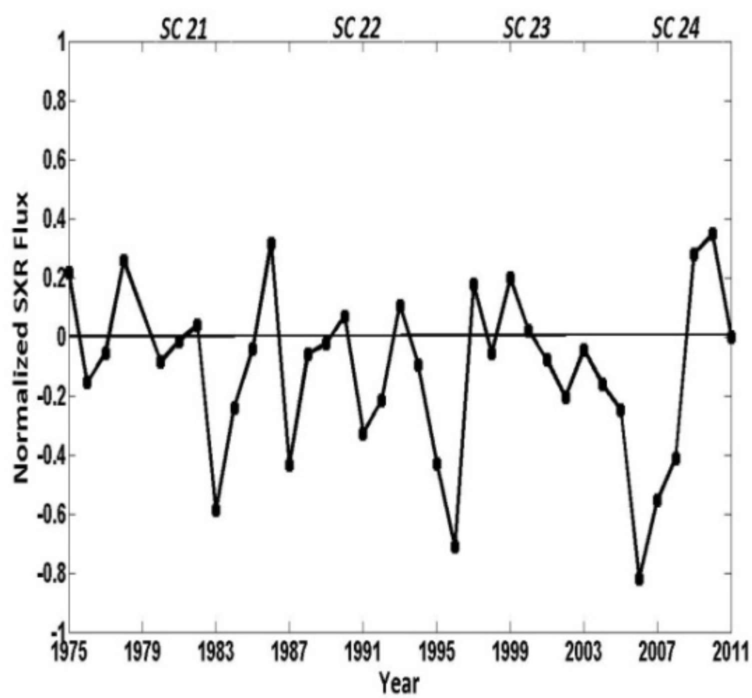


Figure 4(b). Asymmetry plot for normalized SXR Flux for Solar Cycles (SC) 21-24 (1975 to 2011).

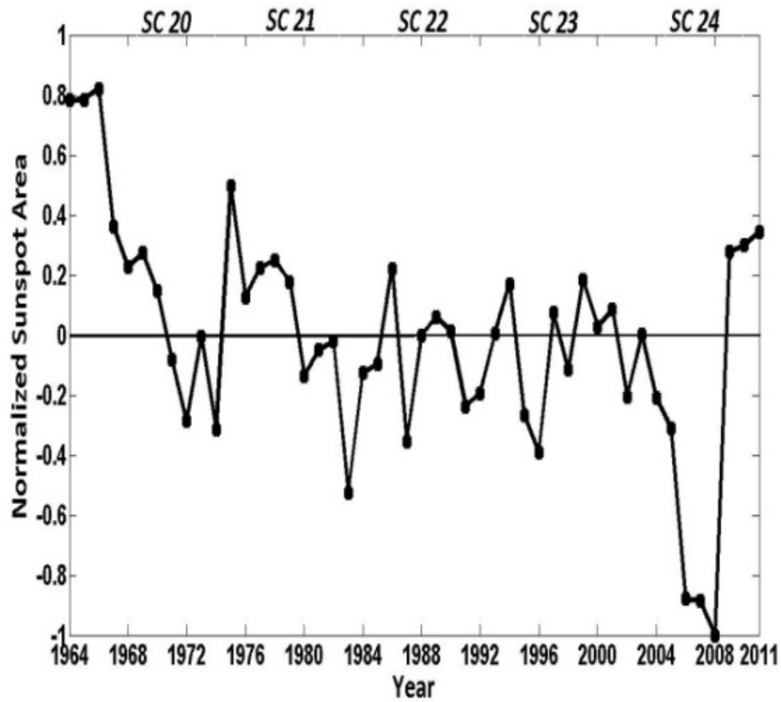


Figure 4(c). Asymmetry plot for normalized Sunspot Area for Solar Cycles (SC) 20-24 (1964 to 2011).

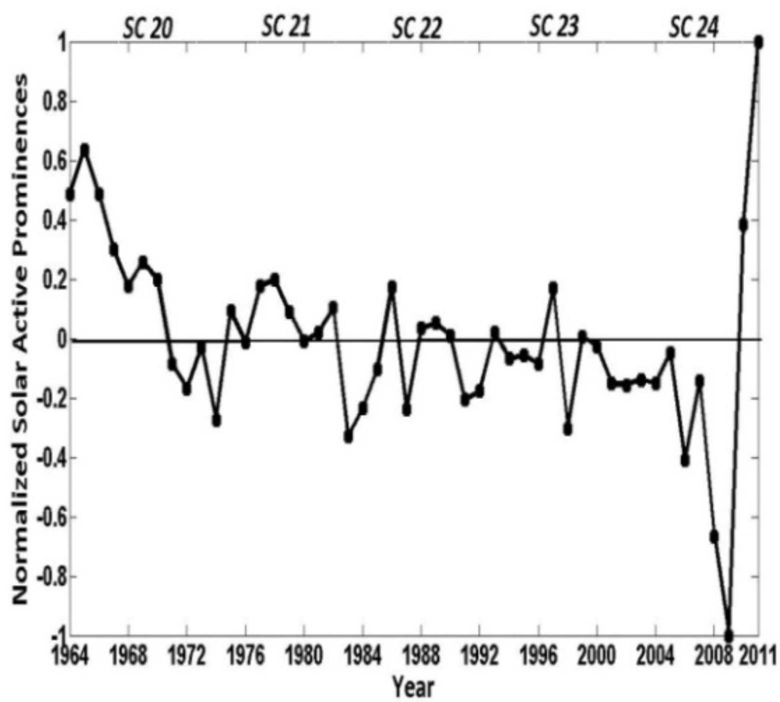


Figure 4(d). Asymmetry plot for normalized Solar Active Prominences for Solar Cycles (SC) 20-24 (1964 to 2011).

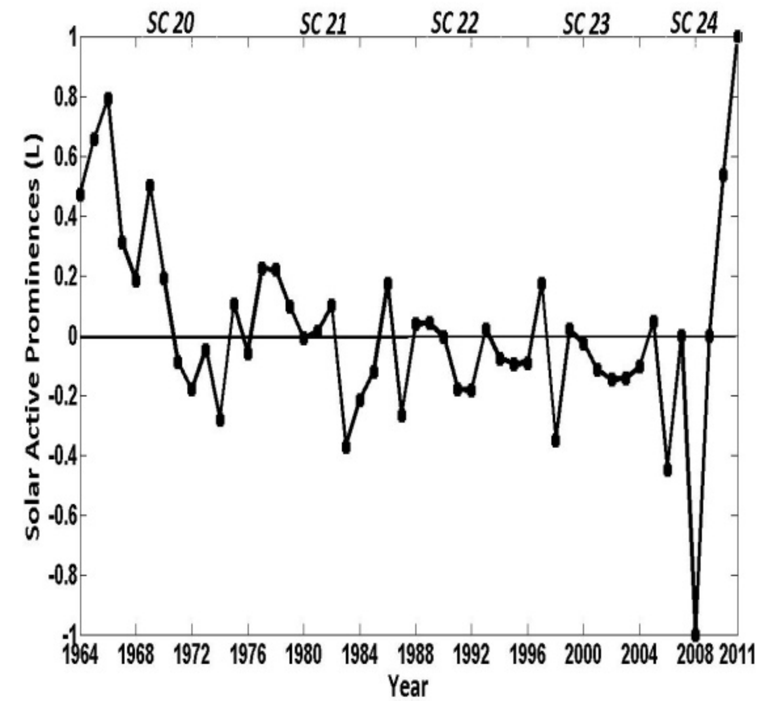


Figure 4(e). Asymmetry plot for normalized Solar Active Prominences (Low) for Solar Cycles (SC) 20-24 (1964 to 2011).

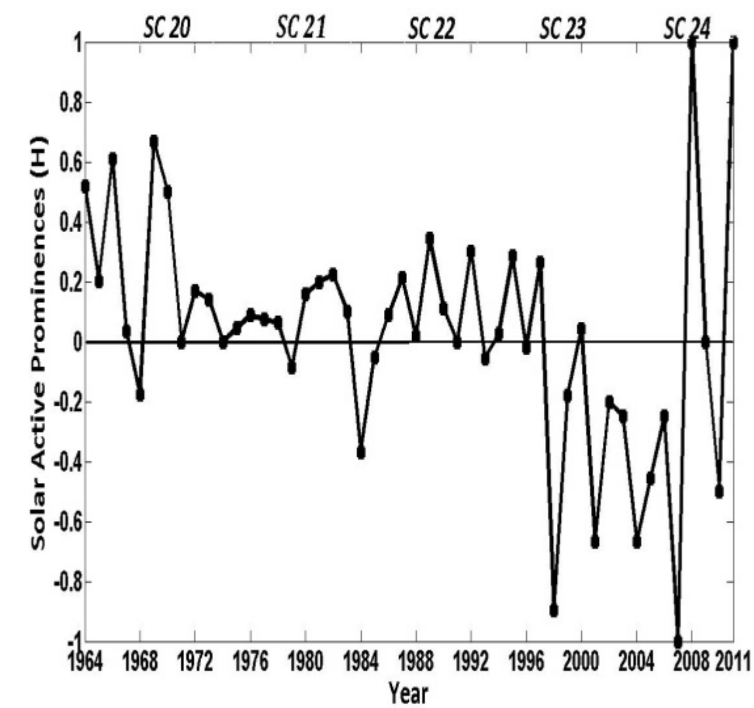


Figure 4(f). Asymmetry plot for normalized Solar Active Prominences (High) for Solar Cycles (SC) 20-24 (1964 to 2011).

4 Results and Discussions

4.1 N-S Asymmetry

The N-S asymmetry of different activity features i.e. $H\alpha$-flares, SXR flux, Sunspot area, Solar Active Prominences and Solar Active Prominences (both low and high latitude) have been investigated and shown in Figures 4(a), 4(b), 4(c), 4(d), 4(e) and 4(f) respectively. Figure 4(a) shows that occurrence of $H\alpha$-flares activity is dominated in the northern hemisphere in solar cycle 20, and during the decay phase of solar cycle 20 it moved towards the southern hemisphere. Again in solar cycle 21 it moved to northern hemisphere. In solar cycle 22, the activity was started from northern hemisphere and again it moved to southern hemisphere. Throughout the solar cycle 23, it was dominated in southern hemisphere. In the rise phase of solar cycle 24 the flares activity started in southern hemisphere and afterward got dominated in the northern hemisphere. Our results confirm the earlier results of ([1, 15]). In addition to previous studies, our study extends the results up to rising phase of solar cycle 24 (up to December 2011).

Further, the same trend of asymmetry as noted in case of $H\alpha$-flares activity have been found for SXR flares (Figure 4(b)), Sunspot Area (Figure 4(c)), Solar Active Prominences (Figure 4(d)) and Solar Active Prominences (low latitude) (Figure 4(e)). However, in the case of Solar Active Prominences (high latitude) (Figure 4(f)), the eruption of the activity was dominated to northern hemisphere up to solar cycle 22. In solar cycle 23, it was shifted to southern hemisphere. Again, in the rise phase of cycle 24, it was shifted into northern hemisphere as commonly noted in other activity features also.

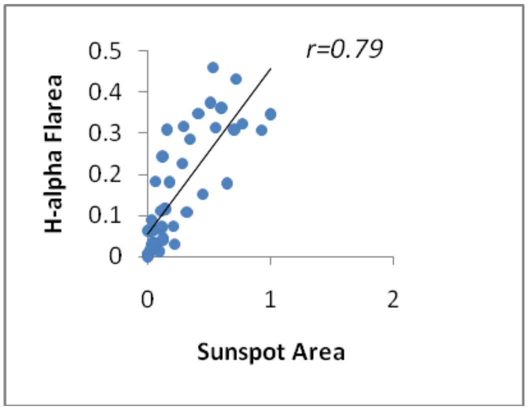
4.2 Correlation among different features

Apart from the N-S asymmetry, we have also presented the correlation study between different solar activity features. The correlation between different solar activity features have been ploted in Figure 3.1 (for northen hemishere), Figure 3.2 (for southern hemishere) and Figure 3.3 (for total number of events). In northern hemisphere the Sunspot Area and number of soft X-ray flares are poorly correlated (r=0.37) (Figure 3.1(e)). The poor correlation shows that for soft X- rays flares production, Sunspot Area is solely not responsible. There may be other reasons for the production of X-rays Flares. As reported in the literature the flares can be produced by the complexivity of the Sunspot group. Therefore, in the northern hemisphere, the sunspot may be less magnetically complex. Good correlation (r=0.68) is observed in the southern hemisphere between Sunspot Area and $H\alpha$ Flares (Figure 3.2(a)). However, it is not significantly high. One possibility for the good correlation may be that the active region in the southern hemisphere are more magnetically complex. It is interesting to look in the magnetic configuration of Sunspot group in details to understand the above correlation better. However this is not the scope of this paper.

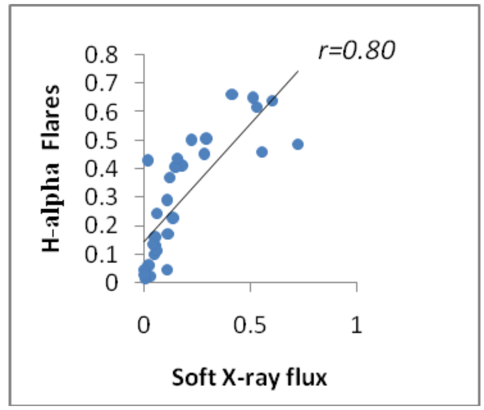
The correlations between the SAP and X-ray flares, and SAP and $H\alpha$ flares in north as well as south hemishere are 0.41 and 0.50 respectively (Figures 3.1(d), 3.2(d), and 3.1(c), 3.2(c)). One possible explanation may be that the prominances/filament eruption produces weak reconnection and hence the weak X-rays and $H\alpha$ flares are produced.

The good correlation (r=0.71) between total soft X-rays and $H\alpha$ flares(Figure 3.3(d)) shows that almost all the soft X-rays produce $H\alpha$ emission. Almost similar correlations have been obtained in case of northern as well as southern hemisphere (r=0.67 and r=0.63 resp.) for these events (Figure 3.1(b) and 3.2(b)). This study also confirms the earlier findings ([2, 15]). This shows that soft X-rays emissions are usually orignated in the higher solar atmosphere.The particle orignated during X-ray flares emission later produced the $H\alpha$ emission in the lower solar atmosphere. There is no firm conclusive statement for correlation between certain activity features like total SAP and $H\alpha$ flares, total Sunspot Area and X-rays, total sunspot area and $H\alpha$ flares, Sunspot Area and X-rays in southern hemisphere etc. as the correlation lies between 0.47 and 0.64.

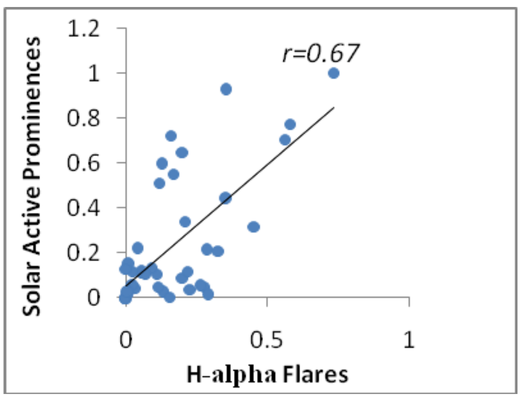
It is quite interesting to note that the correlation coefficient increases drastically between certain activity features if few of the outlying data points are removed. The respective revised correlation plots are given in Figures 5,6 and 7 below. It is found that the correlation between total soft X-rays and $H\alpha$ flares as well as in northern and southern hemisphere is strong as the correlation coefficient are 0.83, 0.80 and 0.86 respectively (Figures 7(e), 5(b) and 6(b)). This shows that regression line is quite close to the data points as the coefficient of determination is greater than 0.64 in all cases. Similarly, the Sunspot Area and Soft X-rays are highly correlated as the correlation coefficients are 0.86, .82 and 0.92 in total, northern and southern hemisphere respectively (Figures 7(a), 5(e), and 6(e)) and the coefficient of determination is greater than 0.67. Further, Sunspot area and $H\alpha$ flares are also highly correlated as shown in Figures 5(a), 6(a) and 7(d). It is also to be noted that rest of the activity features like SAP and $H\alpha$ flares, SAP and Soft X-rays show good correlation (Figures 5(c), 5(d), 6(c), 6(d), 7(b), and 7(c)).



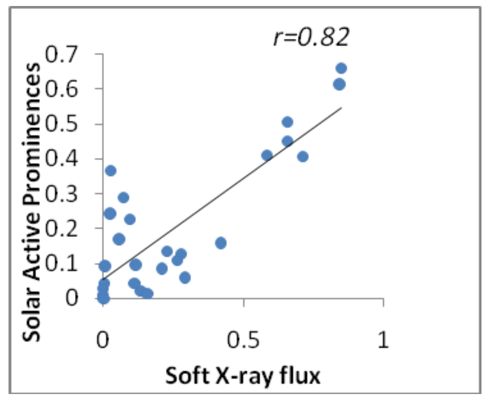
(a)



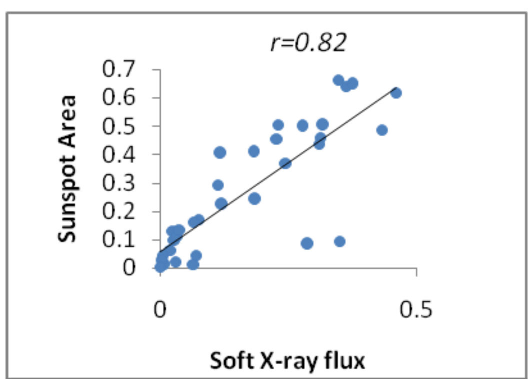
(b)



(c)

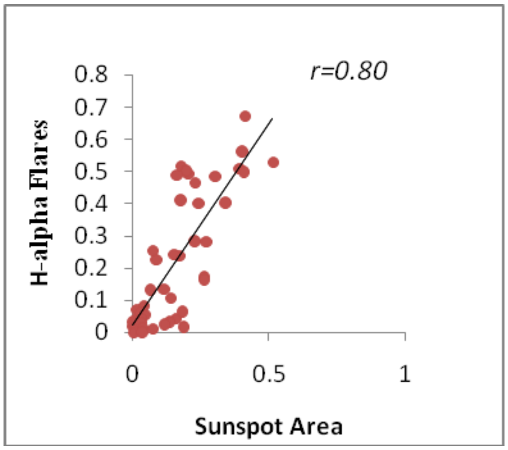


(d)

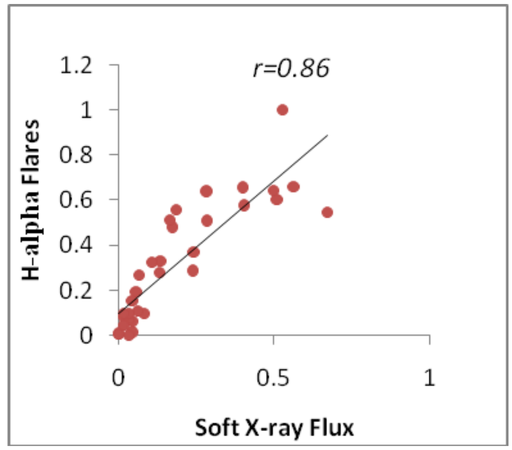


(e)

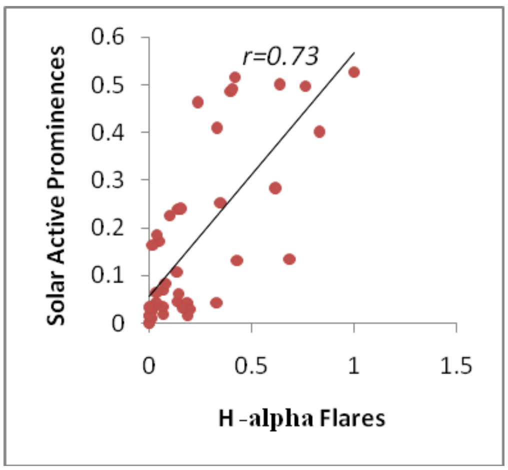
Figure 5 (a), (b), (c), (d), (e). Correlation Plots of Normalized North Events for different Solar Activity Features during Solar Cycle 20-24 after removing outliers.



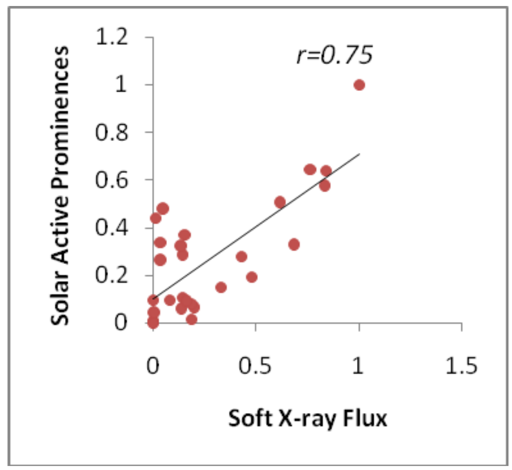
(a)



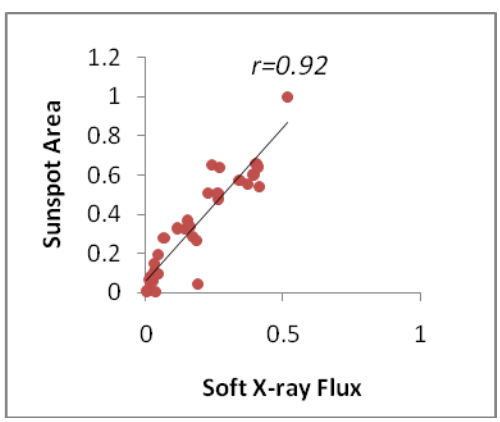
(b)



(c)

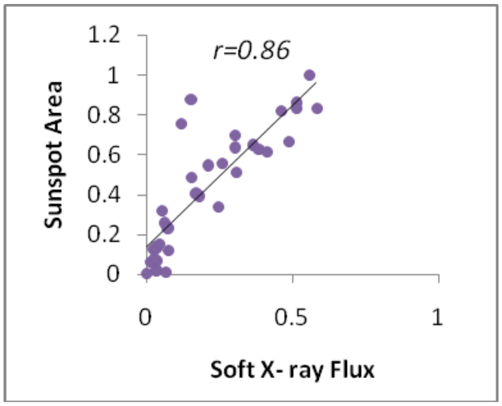


(d)

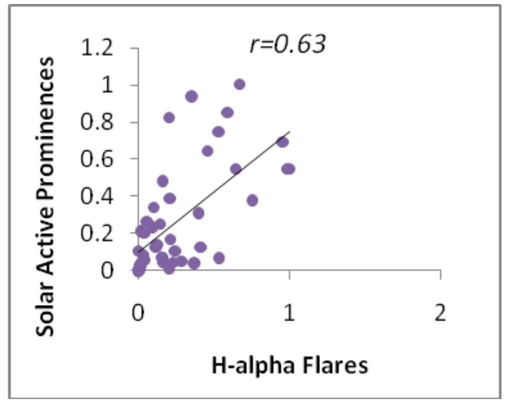


(e)

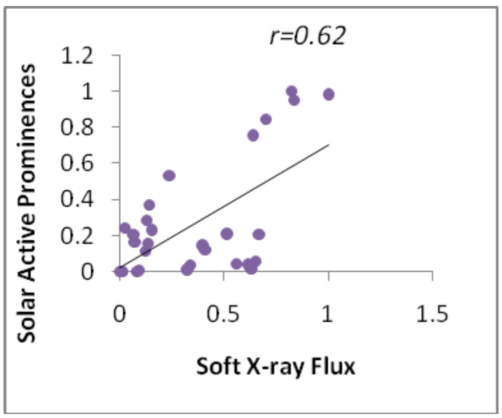
Figure 6. (a), (b), (c), (d), (e) Correlation Plots of Normalized South Events for different Solar Activity Features during Solar Cycle 20-24 after removing outliers.



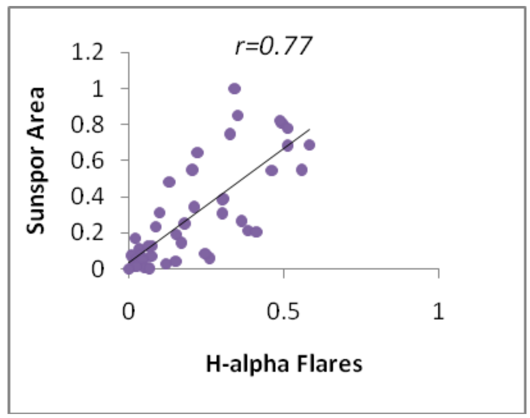
(a)



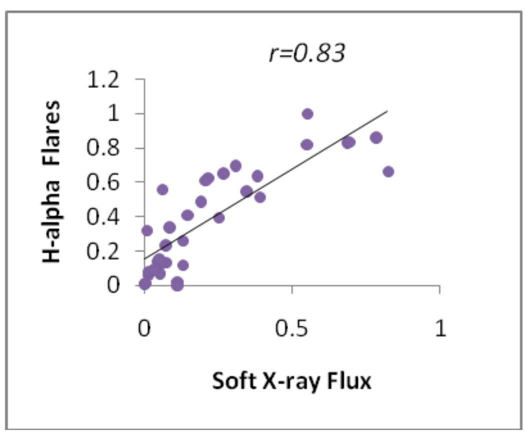
(b)



(c)



(d)



(e)

Figure 7 (a), (b), (c), (d), (e). Correlation Plots of Normalized Events for different Solar Activity Features during Solar Cycle 20-24 after removing outliers.

5 Conclusions:

To understand the nature of the solar dynamo system, it is important to analyze the N-S asymmetry phenomena of solar activity features. In this paper, the normalized data of Sunspot Area, SAP, SAP(low latitude), SAP (high latitude) and $H\alpha$ flares for solar cycles 20-24 (1964-2011) and soft X-rays (SXR) for solar cycles 21-24 (1975-2011) have been analyzed statistically to observe the asymmetric behavior of the solar active phenomena and correlation among them. It is concluded that the normalized N-S asymmetry of all solar activity features shows a drift from northern hemisphere to southern hemisphere during solar cycle 20. For the remaining cycles 21-23, all activities except $SAP(H)$ are dominant in southern hemisphere. However, the asymmetry of $SAP(H)$ varies dominantly in northern hemisphere for solar cycles 20-22 and shifts to southern hemisphere during solar cycle 23. In the beginning of the solar cycle 24, all the activities were started in southern hemisphere and afterward dominated in northern hemisphere. The different solar activity features are physically associated with each other. It becomes important to discuss the correlation among them. In this study it is found that there is a weak correlation between sunspot area and number of X-ray flares in northern hemisphere, where as the correlation is stronger in southern hemisphere. This result indicates that only the Sunspot Area is not responsible for the production of X-ray flares. Also, there is weak correlation between the prominence/filament eruption and X-ray and $H\alpha$ solar flares, which confirms the idea that all the filament eruptions are not associated with the flares. Further, there is a strong correlation between $H\alpha$ and X-ray flares, which supports the earlier studies. Here it is important to mention that the correlation amongst the activities increases tremendously as few of the outliers are removed from the data tables. In some cases, like sunspot area and X-ray flares, $H\alpha$ and X-ray flares, SAP and X-ray flares etc. strong correlation has been obtained. This may be because of some observational errors in data or any other hidden reasons, but the strong correlation supports the dependency of solar activity features on one another.

Acknowledgments

We are very much thankful for the open data policy of National Geophysical Data centers ($NGDC$). We are also thankful to the referee for his valuable suggestions to bring the paper in its present form.

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Jñānābha, Vol. 49(2) (2019), 82-93

ON EXISTENCE AND UNIQUENESS THEOREMS FOR ABSTRACT MEASURE INTEGRODIFFERENTIAL EQUATIONS

By

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(*Received : November* 02, 2019 ; *Revised: November* 11, 2019)

Abstract

In this paper, we prove the relevance, existence and uniqueness theorems for a nonlinear abstract measure integrodifferential equation via classical fixed point theorems of Schauder [14] and Dhage [10] under weaker Carathéodory condition. Our hypotheses and claims have also been illustrated with a couple of examples.

2010 Mathematics Subject Classifications: 34K10, 47H10.

Keywords and phrases: Abstract measure integrodifferential equation; Existence and uniqueness theorem.

1 Statement of the Problem

Let X be a real Banach space with a convenient norm $\|\cdot\|$ and let $x, y \in X$ be any two elements. Then the line segment $\overline{xy}$ in X is defined by

$$\overline{xy} = \{z \in X \mid z = x + r(y - x), 0 \leq r \leq 1\}. \tag{1.1}$$

Let $x_0 \in X$ be a fixed point and $z \in X$. Then for any $x \in \overline{x_0 z}$, we define the sets S_x and $\overline{S}_x$ in X by

$$S_x = \{rx \mid -\infty < r < 1\}, \tag{1.2}$$

and

$$\overline{S}_x = \{rx \mid -\infty < r \leq 1\}. \tag{1.3}$$

Let $x_1, x_2 \in \overline{xy}$ be arbitrary. We say $x_1 < x_2$ if $S_{x_1} \subset S_{x_2}$, or equivalently, $\overline{x_0 x_1} \subset \overline{x_0 x_2}$. In this case we also write $x_2 > x_1$.

Let M denote the σ-algebra of all subsets of X such that (X, M) is a measurable space. Let ca(X, M) be the space of all vector measures (real signed measures) and define a norm $|\cdot|$ on ca(X, M) by

$$\|p\| = |p|(X), \tag{1.4}$$

where $|p|$ is a total variation measure of p and is given by

$$|p|(X) = \sup_{\sigma} \sum_{i=1}^{\infty} |\, p(E_i)|,\ E_i \subset X, \tag{1.5}$$

where the supremum is taken over all possible partitions $\sigma = \{E_i : i \in \mathbb{N}\}$ of measurable subsets of X. It is known that ca(X, M) is a Banach space with respect to the norm $\|\cdot\|$ given by (1.4).

Let μ be a σ-finite positive measure on X, and let $p \in \text{ca}(X, M)$. We say p is absolutely continuous with respect to the measure μ if $\mu(E) = 0$ implies $p(E) = 0$ for some $E \in M$. In this case we also write $p << \mu$.

Let $x_0 \in X$ be fixed and let M_0 denote the σ- algebra on S_{x_0}. Let $z \in X$ be such that $z > x_0$ and let M_z denote the σ-algebra of all sets containing M_0 and the sets of the form S_x, $x \in \overline{x_0 z}$.

Given a vector measure $p \in \text{ca}(X, M)$ with $p << \mu$, consider the nonlinear abstract measure integrodifferential equation (in short AMIGDE) of the form

$$\frac{dp}{d\mu} = \int\limits_{\overline{S}_x - S_{x_0}} f\big(\tau, p(\overline{S}_\tau)\big)\, d\mu \quad \text{a.e. } [\mu] \text{ on } \overline{x_0 z}. \tag{1.6}$$

and

$$p(E) = q(E), \quad E \in M_0, \tag{1.7}$$

where q is a given known vector measure, $\dfrac{dp}{d\mu}$ is a Radon-Nikodym derivative of p with respect to μ, $f : S_z \times \mathbb{R} \to \mathbb{R}$ and $x \mapsto f\big(x, p(\overline{S}_x)\big)$ is μ-integrable for each $p \in \text{ca}(S_z, M_z)$.

Definition 1.1. Given an initial real measure q on M_0, a vector $p \in \text{ca}(S_z, M_z)$ $(z > x_0)$ is said to be a *solution* of AMIGDE (1.6)-(1.7) if

(i) $p(E) = q(E)$, $E \in M_0$,
(ii) $p << \mu$ on $\overline{x_0 z}$, and
(iii) p satisfies (1.6) a.e. $[\mu]$ on $\overline{x_0 z}$.

Remark 1.1. *The AMIGDE* (1.6)-(1.7) *is equivalent to the abstract measure integral equation (in short AMIGDE)*

$$p(E) = \int\limits_E \Bigg(\int\limits_{\overline{S}_x - S_{x_0}} f\big(\tau, p(\overline{S}_\tau)\big)\, d\mu \Bigg) d\mu, \tag{1.8}$$

if $E \in M_z$, $E \subset \overline{x_0 z}$. and

$$p(E) = q(E) \quad \textit{if } E \in M_0. \tag{1.9}$$

A solution p of the AMIGDE (1.6)-(1.7) *on $\overline{x_0 z}$ will be denoted by $p(\overline{S}_{x_0}, q)$.*

Sharma [16, 17] introduced the abstract measure differential equations (in short AMDEs) as the generalizations of the ordinary differential equations in which ordinary derivative is replaced with the Radon-Nykodym derivative of vector measures in an abstract space. Similarly, Dhage [4, 5, 6] initiated the study of nonlinear AMIGDEs as the generalization of the ordinary integrodifferential equations. Later, the study is continued in the papers of Bellale *et. al* [1], Bellale and Birajdar [2] and Bellale and Dapke [3], however since there is no relevance result of the considered AMIGDEs, the work contained in the above mentioned papers [1, 2, 3] is artificial and incorrect duplication of the previously known results. In the present paper we discuss the relevance, existence and uniqueness theorems to the AMIGDE (1.6)-(1.7) under suitable natural conditions via fixed point techniques from nonlinear functional analysis. In the following section we prove the relevance theorem for the AMIGDE (1.6)-(1.7) by relating it to an ordinary integrodifferential equations. Section 3 deals with the fixed point results needed in the subsequent sections of the paper. The main existence and uniqueness results along with a couple of examples are presented in Section 4.

2 Relevance Theorem

In this section we prove the relevance theorem for the AMIGDE (1.6)-(1.7) and it is shown that the AMIGDE (1.6)-(1.7) reduces to an ordinary integrodifferential equation, viz.,

$$\left.\begin{aligned} y'(x) &= \int_{x_0}^{x} f(\tau, y(\tau))\, d\tau, \quad x \geq x_0, \\ y(x_0) &= y_0, \end{aligned}\right\} \tag{2.1}$$

under certain suitable natural conditions, where f is Carathéodory real-valued function on $[x_0, x_0+T] \times \mathbb{R}$ into $\mathbb{R}$.

Let $X = \mathbb{R}$, $\mu = m$, the Lebesgue measure on $\mathbb{R}$, $\overline{S}_x = (-\infty, x]$, $x \in \mathbb{R}$, and q a given real Borel measure on M_0. Then equations (1.6)-(1.7) take the form

$$\left.\begin{aligned} &\frac{d}{dm} p((-\infty, x]) = \int\limits_{[x_0,x]} f\big(\tau, p(-\infty, \tau]\big)\, dm, \\ &p(E) = q(E), \quad E \in M_0. \end{aligned}\right\} \tag{2.2}$$

It will now be shown that the equations (2.1) and (2.2) are equivalent in the sense of the following theorem.

Theorem 2.1. *Let $q : M_0 \to \mathbb{R}$ be a given initial measure such that $q(E) = 0$ for all $E \in M_0$ and $q(\{x_0\}) = 0$. Then,*

(a) to each solution $p = p(\overline{S}_{x_0}, q)$ of (2.2) existing on $[x_0, x_1)$, there corresponds a solution y of (2.1) satisfying $y(x_0) = y_0$.
(b) Conversely, to every solution $y(x)$ of (2.1), there corresponds a solution $p(\overline{S}_{x_0}, q)$, of (2.2) existing on $[x_0, x_1)$ with a suitable initial measure q provided f satisfies the relation $f(x_0, 0) = 0$.

Proof. (a) Let $p = p(\overline{S}_{x_0}, q)$ be a solution of (2.2), existing on $[x_0, x_1)$. Define a real Borel measure p_1 on $\mathbb{R}$ as follows.

$$p_1((-\infty, x)) = \begin{cases} 0, & \text{if } x \leq x_0, \\ p((-\infty, x]) - p((-\infty, x_0]), & \text{if } x_0 < x < x_1 \\ p((-\infty, x_1)), & \text{if } x \geq x_1, \end{cases} \tag{2.3}$$

and

$$p_1(-\infty, x_0]) = p(-\infty, x_0]).$$

Define the functions $y_1(x)$ and $y(x)$ by

$$\begin{aligned} y_1(x) &= p_1((-\infty, x)), && x \in \mathbb{R} \\ y(x) &= y_1(x) + p((-\infty, x_0]), && x \in [x_0, x_1). \end{aligned} \tag{2.4}$$

The condition $q(\{x_0\}) = 0$, the definition of the solution p, and the definitions of $y(x)$ together imply that

$$p_1(\{x_0\}) = p(\{x_0\}) = 0.$$

Now for each $x \in [x_0, x_1)$ we obtain from (2.2) and the definition of $y(x)$

$$\begin{aligned} y(x) &= y_1(x) + p((-\infty, x_0]) \\ &= p_1((-\infty, x)) + p((-\infty, x_0]) \\ &= p(\overline{S}_x). \end{aligned} \tag{2.5}$$

Since p is a solution of (2.2) we have $p << m$ on $[x_0, x_1)$. Hence $y(x)$ is absolutely continuous on $[x_0, x_1)$. The details concerning these arguments appear in Rudin [15, pages 163-165]. This shows that $y'(x)$ exists a.e. on $[x_0, x_1)$. Now for each $x \in [x_0, x_1)$, we have, by virtue of (2.3) and (2.4)

$$p([x_0, x]) = \int\limits_{[x_0,x]} \frac{d}{dm} p((-\infty, t])\, dm.$$

Therefore,

$$p((-\infty, x]) - p((-\infty, x_0]) = \int\limits_{[x_0,x]} \frac{d}{dm} p((-\infty, t])\, dm.$$

This further implies that

$$p(\overline{S}_x) = p(\overline{S}_{x_0}) + \int_{x_0}^{x} \left(\int_{x_0}^{t} f\big(\tau, p(\overline{S}_\tau)\big)\, dm \right) dm.$$

That is,

$$y(x) = y(x_0) + \int_{x_0}^{x} \left(\int_{x_0}^{t} f\big(\tau, y(\tau)\big)\, d\tau \right) dt.$$

Hence,

$$y'(x) = \int_{x_0}^{x} f\big(\tau, y(\tau)\big)\, d\tau \quad \text{a.e on } [x_0, x_1).$$

This proves that $y(x)$ is a solution of (2.1) on $[x_0, x_1)$ satisfying

$$y(x_0) = y_0.$$

(b) Conversely, suppose that $y(x)$ be a solution of (2.1) existing on $[x_0, x_1]$. Then, y is absolutely continuous on $[x_0, x_1]$. Now, corresponding to the absolutely continuous function $y(x)$ which is a solution of (2.1) on $[x_0, x_1)$, we can construct a absolutely continuous real Borel measure p on M_{x_1} such that,

$$\begin{aligned} &p(E) = 0 \ \text{ for all } \ E \in M_0, \\ &p(\overline{S}_x) = y(x), \text{ if } x \in [x_0, x_1). \end{aligned} \tag{2.6}$$

The details concerning these arguments appear in Rudin [15, pages 163-165]. Since $y(x)$ is a solution of (2.1) we have for $x \in [x_0, x_1)$,

$$y(x) = y(x_0) + \int_{x_0}^{x} \left(\int_{x_0}^{t} f\big(\tau, y(\tau)\big)\, d\tau \right) dt.$$

Now, $y(x_0) = p\big(S_{x_0}\big) = 0$ and so, by (2.6) we obtain that

$$\big[p(\overline{S}_x) - p(\overline{S}_{x_0})\big] = \int\limits_{[x_0,x]} \left(\int\limits_{[x_0,t]} f\big(\tau, p(\overline{S}_\tau)\big)\, dm \right) dm.$$

That is,

$$p([x_0,x]) = \int\limits_{[x_0,x]} \left(\int\limits_{[x_0,t]} f\big(\tau, p(\overline{S}_\tau)\big)\, dm \right) dm.$$

In general, if $E \in M_{x_1}, E \subset \overline{x_0x_1}$, then

$$p(E) = \int\limits_{E} \left(\int\limits_{\overline{S}_x - S_{x_0}} f\big(\tau, p((-\infty,\tau]\big)\, dm \right) dm.$$

By definition of Radon-Nykodym derivative, we obtain

$$\frac{d}{dm}\Big[p((-\infty,x])\Big] = \left(\int\limits_{\overline{S}_x - S_{x_0}} f\big(\tau, p((-\infty,\tau]\big)\, dm \right) \ a.e.\ [\mu] \ \text{ on } \overline{x_0z},$$

$$p(E) = 0 \ \text{ for } \ E \in M_0.$$

This shows that p is a solution of (2.2) on $[x_0,x_1)$ and the proof of (b) is complete. □

3 Fixed Point Results

To state the required fixed point techniques that will be used in the proofs of main results, we need the following definitions in what follows.

Definition 3.1 (Dhage [8, 10])**.** *An upper semi-continuous and nondecreasing function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is called a $\mathcal{D}$-function if $\psi(0) = 0$. The class of all $\mathcal{D}$-functions on $\mathbb{R}+$ is denoted by $\mathfrak{D}$.*

Definition 3.2 (Dhage [8, 10])**.** *An operator $\mathcal{T} : \mathfrak{X} \to \mathfrak{X}$ is called $\mathcal{D}$-Lipschitz if there exists a $\mathcal{D}$-function $\psi_\mathcal{T} \in \mathfrak{D}$ such that*

$$\|\mathcal{T}x - \mathcal{T}y\| \le \psi_\mathcal{T}\big(\|x-y\|\big) \tag{3.1}$$

for all elements $x, y \in \mathfrak{X}$. If $\psi_\mathcal{T}(r) = k\,r$, $k > 0$, $\mathcal{T}$ is called a Lipschitz operator on $\mathfrak{X}$ with the Lipschitz constant k. Again, if $0 \le k < 1$, then $\mathcal{T}$ is called a contraction on $\mathfrak{X}$ with contraction constant k. Furthermore, if $\psi_\mathcal{T}(r) < r$ for $r > 0$, then $\mathcal{T}$ is called a nonlinear $\mathcal{D}$-contraction on $\mathfrak{X}$. The class of all $\mathcal{D}$-functions satisfying the condition of nonlinear $\mathcal{D}$-contraction is denoted by $\mathfrak{DN}$.

An operator $\mathcal{T} : \mathfrak{X} \to \mathfrak{X}$ is called compact if $\overline{\mathcal{T}(\mathfrak{X})}$ is a compact subset of $\mathfrak{X}$. $\mathcal{T}$ is called totally bounded if for any bounded subset S of $\mathfrak{X}$, $\mathcal{T}(S)$ is a totally bounded subset of $\mathfrak{X}$. T is called completely continuous if $\mathcal{T}$ is continuous and totally bounded on $\mathfrak{X}$. Every compact operator is totally bounded, but the converse may not be true, however, two notions are equivalent on bounded subsets of $\mathfrak{X}$. The details of different types of nonlinear contraction, compact and completely continuous operators appear in Granas and Dugundji [14].

To prove the main existence results of next section, we need the following classical topological and analytic fixed point principles.

Theorem 3.1 (Schauder [14])**.** *Let S be a closed convex and bounded subset of a Banach space $\mathfrak{X}$ and let $\mathcal{T} : S \to S$ be a completely continuous operator. Then the operator equation $\mathcal{T}x = x$ has a solution.*

Theorem 3.2 (Dhage [10])**.** *Let $\mathfrak{X}$ be a Banach space and let $\mathcal{T} : \mathfrak{X} \to \mathfrak{X}$ be a nonlinear $\mathcal{D}$-contraction. Then the operator equation $\mathcal{T}x = x$ has a unique solution.*

4 Existence and Uniqueness Theorems

We need the following definition in the sequel.

Definition 4.1. *A function $\beta : S_z \times \mathbb{R} \to \mathbb{R}$ is called Carathéodory if*

(i) $x \to \beta(x,u)$ *is μ-measurable for each $u \in \mathbb{R}$, and*
(ii) $(u) \to \beta(x,u)$ *is continuous almost everywhere $[\mu]$ on $\overline{x_0 z}$.*

Further a Carathéodory function $\beta(x,u)$ is called $L^{\mu}_{\mathbb{R}}$-Carathéodory if

(iii) *there exists a μ-integrable function $h : S_z \to \mathbb{R}$ such that*
$$|\beta(x,u)| \le h(x) \quad a.e.\ [\mu] \ \text{ on } x \in \overline{x_0 z},$$
for all $u \in \mathbb{R}$.

We consider the following set of assumptions.

(H_0) For any $z > x_0$, the σ-algebra M_z is compact with respect to the topology generated by the Pseudo-metric d defined on M_z by
$$d(E_1, E_2) = \mu(E_1 \Delta E_2)$$
for all $E_1, E_2 \in M_z$.
(H_1) $\mu(\{x_0\}) = 0$.
(H_2) q is continuous on M_0 with respect to the Pseudo-metric d defined in (H_0).
(H_3) The function $f(x,u)$ is $L^{\mu}_{\mathbb{R}}$-Carathéodory.
(H_4) There exists a $\mathcal{D}$-function $\psi_f \in \mathfrak{D}$ such that
$$|f(x,u) - f(x,v)| \le \psi_f\big(|u-v|\big) \ \text{ a.e. } [\mu] \text{ on } x \in \overline{x_0 z},$$
for all $u, v \in \mathbb{R}$. Moreover, $\psi_f(r) < r$ for each $r > 0$.

Theorem 4.1. *Suppose that the hypotheses (H_0)-(H_3) hold. Then the AMIGDE* (1.6)-(1.7) *has a solution.*

Proof. By expressions (1.2) and (1.3), we have a real number $r(>1)$ such that $r \to 1$ and $S_{rx_0} \supset S_{x_0}$. Then, from hypothesis (H_1), it follows that
$$\bigcap_{r \to 1} \left(\overline{S}_{rx_0} - S_{x_0}\right) = \{x_0\}$$
and
$$\mu\Big(\overline{S}_{rx_0} - S_{x_0}\Big) = \mu(\{x_0\}) = 0$$
whenever $r \to 1$.

Therefore, we can choose a real number r^* such that $S_{r^* x_0} \supset S_{x_0}$ and
$$\int\limits_{\overline{S}_{r^* x_0} - S_{x_0}} h(x)\, d\mu < 1.$$

Let $z^* = r^* x_0$ and Consider the measure p_0 on M_{z^*} which is a continuous extension of the measure q on M_0 defined by
$$p_0(E) = \begin{cases} q(E) & \text{if} \quad E \in M_0, \\ 0 & \text{if} \quad E \notin M_0. \end{cases}$$

Now define a subset $S(\rho)$ of $\mathrm{ca}(S_{z^*}, M_{z^*})$ by

$$S(\rho) = \big\{p \in \mathrm{ca}(S_{z^*}, M_{z^*}) \mid \|p - p_0\| \leq \rho\big\} \tag{4.1}$$

where $\rho = \|q\| + 1$. Clearly, $S(\rho)$ is a closed convex ball in $\mathrm{ca}(S_{z^*}, M_{z^*})$ centered at p_0 of radius ρ and $q \in S(\rho)$.

Define the operator $\mathcal{T} : S(\rho) \to \mathrm{ca}(S_{z^*}, M_{z^*})$ by

$$\mathcal{T}p(E) = \begin{cases} \displaystyle\int_E \Bigg(\int\limits_{\overline{S}_x - S_{x_0}} f\big(\tau, p(\overline{S}_\tau)\big)\, d\mu \Bigg)\, d\mu & \text{if } E \in M_{z^*},\ E \subset \overline{x_0 z^*}, \\ q(E) & \text{if } E \in M_0. \end{cases} \tag{4.2}$$

We shall show that the operator $\mathcal{T}$ satisfies all the conditions of Theorem 3.1 on $S(\rho)$.

Step I : First we show that $\mathcal{T}$ is continuous on $S(\rho)$. Let $\{p_n\}$ be a sequence of vector measures in $S(\rho)$ converging to a vector measure p. Then by dominated convergence theorem,

$$\begin{aligned} \lim_{n\to\infty} \mathcal{T}p_n(E) &= \lim_{n\to\infty} \int_E \Bigg(\int\limits_{\overline{S}_x - S_{x_0}} f\big(\tau, p_n(\overline{S}_\tau)\big)\, d\mu \Bigg)\, d\mu \\ &= \int_E \Bigg(\int\limits_{\overline{S}_x - S_{x_0}} \Big[\lim_{n\to\infty} f\big(\tau, p_n(\overline{S}_\tau)\big)\Big]\, d\mu \Bigg)\, d\mu \\ &= \int_E \Bigg(\int\limits_{\overline{S}_x - S_{x_0}} f\big(\tau, p(\overline{S}_\tau)\big)\, d\mu \Bigg)\, d\mu \\ &= \mathcal{T}p(E) \end{aligned}$$

for all $E \in M_{z^*}$, $E \subset \overline{x_0 z^*}$. Similarly, if $E \in M_0$, then

$$\lim_{n\to\infty} \mathcal{T}p_n(E) = q(E) = \mathcal{T}p(E),$$

and so $\mathcal{T}$ is a pointwise continuous operator on $S(\rho)$.

Next we show that $\{\mathcal{T}p_n : n \in \mathbb{N}\}$ is a equi-continuous sequence in $ca(S_{z^*}, M_{z^*})$. Let $E_1, E_2 \in M_{z^*}$. Then there exist subsets $F_1, F_2 \in M_0$ and $G_1, G_2 \in M_{z^*}$, $G_1 \subset \overline{x_0 z^*}$, $G_2 \subset \overline{x_0 z^*}$ such that

$$E_1 = F_1 \cup G_1 \text{ with } F_1 \cap G_1 = \emptyset$$

and

$$E_2 = F_2 \cup G_2 \text{ with } F_2 \cap G_2 = \emptyset.$$

We know the identities

$$G_1 = (G_1 - G_2) \cup (G_2 \cap G_1), \tag{4.3}$$

and

$$G_2 = (G_2 - G_1) \cup (G_1 \cap G_2). \tag{4.4}$$

Therefore, we have

$$\begin{aligned} \mathcal{T}p_n(E_1) &- \mathcal{T}p_n(E_2) \\ &\leq q(F_1) - q(F_2) + \int\limits_{G_1 - G_2} \Bigg(\int\limits_{\overline{S}_x - S_{x_0}} f\big(\tau, p(\overline{S}_\tau)\big)\, d\mu \Bigg)\, d\mu \end{aligned}$$

$$+\int\limits_{G_2-G_1}\Bigg(\int\limits_{\overline{S}_x-S_{x_0}} f\big(\tau,p(\overline{S}_\tau)\big)\,d\mu\Bigg)\,d\mu.$$

Since $f(x,y)$ is $L^{\mu}_{\mathbb{R}}$- Carathéodory, we have that

$$\begin{aligned}|\mathcal{T}p_n(E_1) \;&-\; \mathcal{T}p_n(E_2)|\\ &\leq\; |q(F_1)-q(F_2)|+\int\limits_{G_1\Delta G_2}\Bigg|\Bigg(\int\limits_{\overline{S}_x-S_{x_0}} f\big(\tau,p(\overline{S}_\tau)\big)\,d\mu\Bigg)\Bigg|\,d\mu\\ &\leq\; |q(F_1)-q(F_2)|+\int\limits_{G_1\Delta G_2} h(x)\,d\mu.\end{aligned}$$

Assume that

$$d(E_1,E_2)=\mu(E_1\Delta E_2)\to 0.$$

Then we have that $E_1\to E_2$. As a result $F_1\to F_2$ and $\mu(G_1\Delta G_2)\to 0$. As q is continuous on compact M_{z^*} , it is uniformly continuous and so

$$\begin{aligned}|\mathcal{T}p_n(E_1)-\mathcal{T}p_n(E_2)| \;&\leq\; |q(F_1)-q(F_2)|+\int\limits_{G_1\Delta G_2} h(x)d\mu\\ &\to\; 0 \text{ as } E_1\to E_2\end{aligned}$$

uniformly for all $n\in\mathbb{N}$. This shows that $\{\mathcal{T}p_n : n\in\mathbb{N}\}$ is a equi-continuous set in $\mathrm{ca}(S_{z^*},M_{z^*})$. As a result, $\{\mathcal{T}p_n\}$ converges to $\mathcal{T}p$ uniformly on M_{z^*} and a fortiori $\mathcal{T}$ is a continuous operator on $S(\rho)$ into $\mathrm{ca}(S_{z^*},M_{z^*})$.

Step II: Next we show that $\mathcal{T}(S(\rho))$ is a totally bounded set in $\mathrm{ca}(S_{z^*},M_{z^*})$. We shall show that the set is uniformly bounded and equi-continuous set in $\mathrm{ca}(S_{z^*},M_{z^*})$. Firstly, we show that $\mathcal{T}(S(\rho))$ is a uniformly bounded set in $\mathrm{ca}(S_{z^*},M_{z^*})$.

Let $\lambda\in\mathcal{T}(S)$ be an arbitrary element. Then, there is a member $p\in S$ such that $\lambda(E)=\mathcal{T}p(E)$ for all $E\in M_{z^*}$. Let $E\in M_{z^*}$. Then there exists two subsets $F\in M_0$ and $G\in M_{z^*}$, $G\subset\overline{x_0 z^*}$ such that

$$E=F\cup G \text{ and } F\cap G=\phi.$$

Hence by definition of $\mathcal{T}$,

$$\begin{aligned}|\lambda(E)| \;&=\; |\mathcal{T}p(E)|\\ &\leq\; |q(F)|+\int_G\Bigg|\Bigg(\int\limits_{\overline{S}_x-S_{x_0}} f\big(\tau,p(\overline{S}_\tau)\big)\,d\mu\Bigg)\Bigg|\,d\mu\\ &\leq\; \|q\|+\int_G h(x)d\mu\\ &\leq\; \|q\|+\int_E h(x)d\mu\\ &<\; \|q\|+1=\rho\end{aligned}$$

for all $E\in M_{z^*}$. From (3.5) it follows that

$$\|\lambda\|=\|\mathcal{T}p\|=|\mathcal{T}p|(E)=\sup_\sigma\sum_{i=1}^{\infty}|Tp(E_i)|\leq\|q\|+1=\rho$$

for all $\lambda \in \mathcal{T}(S(\rho))$. As a result $\mathcal{T}$ defines a mapping $\mathcal{T} : S(\rho) \to S(\rho)$. Moreover, $\mathcal{T}(S(\rho))$ is a uniformly bounded set in $\mathrm{ca}(S_{z^*}, M_{z^*})$.

Next we show that $\mathcal{T}(S(\rho))$ is a equi-continuous set of measures in $ca(S_{z^*}, M_{z^*})$. Let $E_1, E_2 \in M_{z^*}$. Then there exist subsets $F_1, F_2 \in M_0$ and $G_1, G_2 \in M_{z^*}$, $G_1 \subset \overline{x_0 z^*}$, $G_2 \subset \overline{x_0 z^*}$ such that

$$E_1 = F_1 \cup G_1 \text{ with } F_1 \cap G_1 = \emptyset$$

and

$$E_2 = F_2 \cup G_2 \text{ with } F_2 \cap G_2 = \emptyset.$$

We know the identities

$$G_1 = (G_1 - G_2) \cup (G_2 \cap G_1), \tag{4.5}$$

and

$$G_2 = (G_2 - G_1) \cup (G_1 \cap G_2). \tag{4.6}$$

Therefore, we have

$$\begin{aligned} |\lambda(E_1) &- \lambda(E_2)| \\ &= |\mathcal{T}p(E_1) - \mathcal{T}p(E_2)| \\ &\leq |q(F_1) - q(F_2)| + \int\limits_{G_1 - G_2} \left| \left(\int\limits_{\overline{S}_x - S_{x_0}} f\big(\tau, p(\overline{S}_\tau)\big)\, d\mu \right) \right| d\mu \\ &\quad + \int\limits_{G_2 - G_1} \left| \left(\int\limits_{\overline{S}_x - S_{x_0}} f\big(\tau, p(\overline{S}_\tau)\big)\, d\mu \right) \right| d\mu. \end{aligned}$$

Since $g(x, y)$ is $L^{\mu}_{\mathbb{R}}$- Carathéodory, we have that

$$\begin{aligned} |\lambda(E_1) - \lambda(E_2)| &\leq |q(F_1) - q(F_2)| + \int\limits_{G_1 \Delta G_2} \left| \left(\int\limits_{\overline{S}_x - S_{x_0}} f\big(\tau, p(\overline{S}_\tau)\big)\, d\mu \right) \right| d\mu \\ &\leq |q(F_1) - q(F_2)| + \int\limits_{G_1 \Delta G_2} h(x)\, d\mu. \end{aligned}$$

Assume that

$$d(E_1, E_2) = \mu(E_1 \Delta E_2) \to 0.$$

Then we have that $E_1 \to E_2$. As a result $F_1 \to F_2$ and $\mu(G_1 \Delta G_2) \to 0$. As q is continuous on compact M_0, it is uniformly continuous and so

$$\begin{aligned} |\lambda(E_1) - \lambda(E_2)| &\leq |q(F_1) - q(F_2)| + \int\limits_{G_1 \Delta G_2} h(x)\, d\mu \\ &\to 0 \text{ as } E_1 \to E_2 \end{aligned}$$

uniformly for all $\lambda \in \mathcal{T}(S)$. This shows that $\mathcal{T}(S(\rho))$ is a equi-continuous set in $\mathrm{ca}(S_{z^*}, M_{z^*})$. Now an application of the Arzela-Ascolli theorem yields that $\mathcal{T}$ is a totally bounded operator on $S(\rho)$. Now, $\mathcal{T}$ is continuous and totally bounded, it is completely continuous operator on $S(\rho)$ into itself. Now the desired conclusion follows by an application of Theorem 3.1. This completes the proof. □

Next, we prove the uniqueness theorem for the AMIGDE (1.6)-(1.7).

Theorem 4.2. *Assume that the hypotheses (H_0), (H_1), (H_2) and (H_4) hold. Then the AMIGDE* (1.6)-(1.7) *has a unique solution.*

Proof. Now the AMIGDE (1.6)-(1.7) is equivalent to the abstract measure integral equation (1.8) and (1.9). Now, proceeding with the arguments as in the proof of Theorem 4.1, we choose a real number $r^* > 0$ such that $z^* = r^* x_0$ and $\mu(\overline{x_0 z^*}) < 1$. Define the operator $\mathcal{T}$ on $\mathrm{ca}(S_{z^*}, M_{z^*})$ by (4.2). We show that $\mathcal{T}$ is a nonlinear $\mathcal{D}$-contraction on $\mathrm{ca}(S_{z^*}, M_{z^*})$.

Let $p_1, p_2 \in \mathrm{ca}(S_{z^*}, M_{z^*})$ be any two elements. Then, by definition of the operator $\mathcal{T}$, we obtain

$$\mathcal{T}p_1(E) - \mathcal{T}p_2(E) = 0 \quad \text{if } \ E \in M_0,$$

and

$$\mathcal{T}p_1(E) - \mathcal{T}p_2(E) = \int_E \Bigg[\bigg(\int\limits_{\overline{S}_x - S_{x_0}} \big[f\big(\tau, p_1(\overline{S}_\tau)\big) - f\big(\tau, p_2(\overline{S}_\tau)\big)\big]\, d\mu \bigg)\Bigg]\, d\mu$$

for all $E \in M_{z^*}$, $E \subset \overline{x_0 z^*}$.

Therefore, by hypotheses (H_4), we obtain

$$\begin{aligned}
&|\mathcal{T}p_1(E) - \mathcal{T}p_2(E)| \\
&\qquad \le \int_E \Bigg| \int\limits_{\overline{S}_x - S_{x_0}} \big|f\big(\tau, p_1(\overline{S}_\tau)\big) - f\big(\tau, p_2(\overline{S}_\tau)\big)\big|\, d\mu \Bigg|\, d\mu \\
&\qquad \le \int_E \Bigg(\int\limits_{\overline{x_0 x}} \psi_f\left(\big|p_1(\overline{S}_\tau) - p_2(\overline{S}_\tau)\big|\right)\, d\mu \Bigg)\, d\mu \\
&\qquad \le \int_E \Bigg(\int\limits_{\overline{x_0 x}} \psi_f\left(\big|p_1 - p_2\big|(\overline{S}_\tau)\right)\, d\mu \Bigg)\, d\mu \\
&\qquad \le \int_E \Bigg(\int\limits_{\overline{x_0 z^*}} \psi_f\left(\big\|p_1 - p_2\big\|\right)\, d\mu \Bigg)\, d\mu \\
&\qquad \le \int\limits_{\overline{x_0 z^*}} \psi_f\Big(\big\|p_1 - p_2\big\|\Big)\, d\mu \quad (\because \mu(\overline{x_0 z^*}) < 1) \\
&\qquad \le \psi_f\Big(\big\|p_1 - p_2\big\|\Big)
\end{aligned}$$

for all $E \in M_{z^*}$, $E \subset \overline{x_0 z^*}$. This further in view of definition of the norm in $\mathrm{ca}(S_{z^*}, M_{z^*})$ implies that

$$\|\mathcal{T}p_1 - \mathcal{T}p_2\| \le \psi_f\Big(\big\|p_1 - p_2\big\|\Big)$$

for all $E \in M_{z^*}$, $E \subset \overline{x_0 z^*}$. Hence, $\mathcal{T}$ is a nonlinear $\mathcal{D}$-contraction on $\mathrm{ca}(S_{z^*}, M_{z^*})$. Now, an application of Theorem 3.2 yields that the operator $\mathcal{T}$ has a fixed point which corrsponds to the unique solution of the AMIGDE (1.6) and (1.7). This completes the proof. $\square$

Example 4.1. *Given a $p \in \mathrm{ca}(X, M)$ with $p << \mu$, consider the AMIGDE with a linear perturbation of second type of the form*

$$\frac{dp}{d\mu} = \int\limits_{\overline{S}_x - S_{x_0}} \frac{\ln(1 + |p(\overline{S}_\tau)|)}{1 + |p(\overline{S}_\tau)|} \, d\mu \quad a.e.\ [\mu] \ on \ \overline{x_0 z}. \tag{4.7}$$

and

$$p(E)) = 0, \tag{4.8}$$

where $\dfrac{dp}{d\mu}$ is a Radon-Nikodym derivative of p with respect to μ.

Here, $f(x, u) = \dfrac{\ln(1 + |u|)}{1 + |u|}$ for all $x \in \overline{x_0 z}$ and $u \in \mathbb{R}$. Clearly, the function f is continuous and bounded on $S_z \times \mathbb{R}$ with bound $M_f = 1$. Therefore, if the assumptions (H_0)-(H_1) hold, then the AMIGDE (4.7) - (4.8) *has a solution $p(\overline{S}_{x_0}, q)$ defined on $\overline{x_0 z}$.*

Example 4.2. *Given a $p \in \mathrm{ca}(X, M)$ with $p << \mu$, consider the abstract measure integrodifferential equation (AMIGDE) of the form*

$$\frac{dp}{d\mu} = \int\limits_{\overline{S}_x - S_{x_0}} \gamma \sin p(\overline{S}_x) \, d\mu \quad a.e.\ [\mu] \ on \ \overline{x_0 z}. \tag{4.9}$$

and

$$p(E) = 0, \;\; E \in M_0, \tag{4.10}$$

where $\dfrac{dp}{d\mu}$ is a Radon-Nikodym derivative of p with respect to μ and $\gamma \in \mathbb{R}$ with $|\gamma| < 1$.

Here, $f(x, u) = \gamma \sin u$ for all $x \in S_z$ and $u \in \mathbb{R}$. Clearly, the function f is continuous and Lipschitz on $S_z \times \mathbb{R}$ with Lipschitz constant $k = |\gamma| < 1$. Therefore, if the assumptions (H_0)-(H_1) hold, then the AMIGDE (4.9) - (4.10) *has a solution $p(\overline{S}_{x_0}, q)$ defined on $\overline{x_0 z}$.*

Remark 4.1. *Finally, we conjecture that the existence results of this paper may also be extended with appropriate modifications to the perturbed AMIGDE involving the sum of two nonlinearities*

$$\frac{dp}{d\mu} = \int\limits_{\overline{S}_x - S_{x_0}} f\big(\tau, p(\overline{S}_\tau)\big) \, d\mu + \int\limits_{\overline{S}_x - S_{x_0}} g\big(\tau, p(\overline{S}_\tau)\big) \, d\mu \quad a.e.\ [\mu] \ on \ \overline{x_0 z}, \tag{4.11}$$

satisfying the initial vector measure condition

$$p(E) = q(E), \quad E \in M_0. \tag{4.12}$$

The nonlinear AMIGDE (4.11)-(4.12) is a linear perturbation of the nonlinear AMIGDE (1.6)-(1.7) involving the sum of two nonlinearities f and g and so in this case the classical fixed point theorem of Dhage [10, Theorem 2.2] involving the sum of two operators in a Banach space may be used for proving different qualitative behaviour of the solution (see Dhage [7, 8, 9] and references therein).

Acknowledgement

The authors are very much thankful to the referee for his fruitful suggestions.

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Jñānābha, Vol. 49(2) (2019), 94-104

SLIP EFFECT ON MHD STAGNATION POINT FLOW AND HEAT TRANSFER OF CROSS FLUID WITH HEAT GENERATION IN A POROUS MEDIUM

By

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(*Received : November* 03, 2019 ; *Revised: November* 07, 2019)

Abstract

Aim of this paper is to investigate the slip effect on magnetohydrodynamics (MHD) stagnation point flow and heat transfer of cross fluid model with heat generation in presence of porous medium. A cross fluid is a type of generalised Newtonian fluid whose viscosity depends upon shear rate according to equation (2.2). The governing nonlinear partial differential equations are transformed into ordinary differential equations along with boundary conditions by suitable similarity transformations and then reduced to a system of first order ordinary differential equations. Runge-kutta fourth order method with shooting technique is used to solve the system of first order differential equations. The effect of various physical parameters e.g. magnetic parameter, ratio parameter, Prandtl number, Weissenberg number, permeability parameter, heat source/sink parameter on velocity, temperature, skin friction and Nusselt number are presented through graphs, table and discussed numerically.

2010 Mathematics Subject Classification: 76D05, 76D10, 76E25, 76S05, 76W05, 80A20.

Keywords and phrases: Heat transfer, MHD, Stagnation point flow, Cross fluid model, Slip condition, Heat generation, porous medium.

1 Introduction

Firstly the steady of behaviour of boundary layer flow of incompressible fluid due to a stretching surface was analyzed by Crane [5]. Ishak et al. [7] analysed the steady two-dimensional stagnation-point mixed convection flow of an incompressible viscous fluid towards a stretching vertical permeable sheet in its own plane. The stretching velocity and the surface temperature are assumed to vary linearly with the distance from the stagnation-point. Rashad et al. [12] have studied the effect of uniform transpiration velocity on free convection boundary-layer flow of a non-Newtonian fluid over a permeable vertical cone embedded in a porous medium saturated with a nanofluid. Bhattacharyya et al. [2] obtained the solutions of boundary layer flow and heat transfer for two classes of viscoelastic fluid over a stretching sheet with internal heat generation or absorption. A numerical study of the slip effect on the unsteady mixed convection boundary-layer flow near the two-dimensional stagnation point on a vertical permeable surface embedded in a fluid-saturated porous medium with suction has been carried out by Rohni et al. [13]. Effect of thermal radiation on the flow of micropolar fluid and heat transfer past a porous shrinking sheet has been investigated by Bhattacharyya et al. [3]. Bhattacharyya et al. [4] have analyzed the steady boundary layer slip flow and mass transfer over a porous plate embedded in a Darcy porous medium. Makinde [9] has considered the hydromagnetic mixed convection stagnation point flow towards a vertical plate embedded in a highly porous medium with radiation and internal heat generation. Akbar et al. [1] have considered the two-dimensional stagnation-point flow of an incompressible Carreau fluid towards a shrinking surface with dual solutions. Two dimensional boundary layer flow and the heat transfer of a Maxwell fluid past a stretching sheet are studied numerically by Nadeem et al. [10]. Li et al. [8] have analyzed MHD viscoelastic flow due to a vertical stretched surface with Cattaneo-Christov heat flux. Stagnation point flow of Maxwell fluid towards a permeable stretching sheet in the presence of nano particles has been studied by Ramesh et al. [11]. Sharma et al. [14] have discussed MHD slip flow and heat transfer over an exponentially stretching perameable sheet embedded in a porous medium in the presence of heat source. Hayat et al. [6] have analyzed the heat transfer in MHD stagnation point flow of cross fluid model in the presence of magnetic fluid due to stretching sheet. MHD stagnation point flow and heat transfer of a Williamson fluid in the direction of an exponentially stretching sheet embedded in a thermally stratified medium subject to suction is discussed by Vittal et al. [15].

Our main aim in this paper is to investigate of MHD stagnation point flow of cross fluid along stretched surface surface in the presence of porous medium. The slip conditions are imposed on both velocity and temperature. After using similarity transformations, reduced non linear ODE's are solved numerically with shooting technique. The Runge-kutta scheme of fourth order is used for integration. The effects of pertinent parameters on velocity and temperature profiles are discussed in result section.

2 Formulation of the Problem

Consider the steady MHD stagnation point flow of cross fluid with heat generation in the presence of porous medium over an stretched surface. The Cauchy stress tensor τ of a cross fluid model is expressed as

$$\tau = -pI + \mu A_1 \tag{2.1}$$

where p is the pressure, I is the identity matrix, A_1 is the first Rivlin-Erickson tensor and μ is the viscosity of cross fluid model. The viscosity of cross fluid in terms of shear rate is given as

$$\mu = \mu_\infty + (\mu_0 - \mu_\infty)\left[\frac{1}{1 + (\Gamma\dot{\gamma})^{1-n}}\right] \tag{2.2}$$

or

$$\frac{\mu_0 - \mu}{\mu - \mu_\infty} = (\Gamma\dot{\gamma})^{1-n}, \tag{2.3}$$

where μ_0 is the zero shear rate viscosity, μ_∞ is the infinite shear rate viscosity, Γ is the cross time constant and n is the power law index. The shear rate $\dot{\gamma}$ is defined as

$$\dot{\gamma} = \left[4\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial x}\right)^2\right]^{\frac{1}{2}}. \tag{2.4}$$

The first Rivlin-Erickson tensor A_1 is given by

$$A_1 = \dot{L} + \dot{L}^T, \tag{2.5}$$

where $\dot{L}$ is velocity gradient.

Flow equation of momentum and heat transfer for cross fluid model after applying the boundary layer approximation can be written as

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{2.6}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=v\frac{\partial}{\partial y}\left[\frac{\frac{\partial u}{\partial y}}{1+\left\{\Gamma\left(\frac{\partial u}{\partial y}\right)\right\}^{1-n}}\right]+u_e\frac{dU_e}{dx}+\frac{v}{k}(u_e-u)+\frac{\sigma B_0}{\rho}(u_e-u), \tag{2.7}$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\frac{\partial^2 T}{\partial y^2}+\frac{Q_0}{\rho c_p}\left(T-T_\infty\right), \tag{2.8}$$

where u and v are the velocity components along the x and y-axes respectively, t is time, α is the thermal diffusivity, υ is the kinematic viscosity, ρ is the density of fluid, k is the permeability of the porous medium, T is the temperature of the fluid within the boundary layer and T_∞ is the temperature of fluid outside the boundary layer, Q_0 is the heat generation coefficient .

The boundary conditions are

$$\left.\begin{array}{l} u=u_w=c_1x+L\frac{\partial u}{\partial y}, v=0, T=T_w+L'\frac{\partial T}{\partial y}aty=0 \\ u\to u_e=c_2x, T\to T_\infty asy\to\infty \end{array}\right\}. \tag{2.9}$$

3 Method of Solution

Introducing the following transformations, functions and dimensionless quantities

$$\left.\begin{array}{l} \eta=\sqrt{\frac{c_1}{\upsilon}}y, \psi=\sqrt{c_1\upsilon}xf(\eta), \theta(\eta)=\frac{T-T_\infty}{T_w-T_\infty}, \\ u=\frac{\partial\psi}{\partial y}, v=-\frac{\partial\psi}{\partial x}, A=\frac{c_2}{c_1}, K=\frac{\upsilon}{kc_1}, \mathrm{Pr}=\frac{\upsilon}{\alpha}, \\ We=c_1\Gamma Re^{\frac{1}{2}}, M=\frac{\sigma B_0^2}{\rho c_1}, \delta=\frac{Q_0}{\rho c_p c_1} \end{array}\right\}, \tag{3.1}$$

into equation (2.6) to (2.8), we get

$$\left(ff''-f'^2\right)\left[1+(Wef'')^{1-n}\right]^2+\left[A^2+(M+K)(A-f')\right]\left[1+(Wef'')^{1-n}\right]^2 + \left[1+n(Wef'')^{1-n}\right]f'''=0, \tag{3.2}$$

$$\frac{1}{\mathrm{Pr}}\theta''+f\theta'+\delta\theta=0, \tag{3.3}$$

where prime denotes the differentiation with respect to η, A is the velocity ratio parameter, M is magnetic parameter, Pr is Prandtl Number, We is the Weissenberg number, $K$is the permeability parameter,δ is heat source/sink parameter.

It is noted that the equation of continuity (2.6) is identically satisfied.

The corresponding boundary conditions are reduced to

$$\left.\begin{array}{l} f(0)=0, f'(0)=1+Lvf''(0), \theta(0)=1+Lt\theta'(0)at\eta=0 \\ f'(\infty)=A, \theta(\infty)=0at\eta=\infty \end{array}\right\}, \tag{3.4}$$

where $Lv=L\sqrt{\frac{c_1}{\upsilon}}$ and $Lt=L'\sqrt{\frac{c_1}{\upsilon}}$ is the slip parameter.

The physical quantities of interest are the local skin-friction coefficientC_f, heat transfer rate i.e. the local Nusselt number Nu_x, and their mathematical expressions are as follows

$$C_f=\frac{\tau_w}{\frac{1}{2}\rho U_w^2}, Nu_x=\frac{q_w x}{k(T_w-T_\infty)}, \tag{3.5}$$

where the wall shear stress τ_w and the heat flux q_w are given by

$$\tau_w = \left[\mu_0 \frac{\frac{\partial u}{\partial y}}{1 + \left\{ \Gamma \left(\frac{\partial u}{\partial y} \right) \right\}^{1-n}} \right]_{y=0}, q_w = -k \left(\frac{\partial T}{\partial y} \right)_{y=0}. \tag{3.6}$$

Using the similarity transformation given in equation (3.1), we get the following expression

$$\frac{1}{2} C_f Re_x^{1/2} = \frac{f''(0)}{1 + (Wef''(0))^{1-n}}, Nu_x Re_x^{-1/2} = -\theta'(0). \tag{3.7}$$

where $Re_x = \frac{U_w^2}{c_1 v}$ is the local Reynolds number.

The set of equation (3.2) and (3.3) subject to the boundary condition (3.4) are solved numerically using the Runge-Kutta fourth order method with shooting technique. Equations (3.2) and (3.3) are transformed into system of first order differential equations as given below

$$\left. \begin{array}{l} f_1' = f_2, \\ f_2' = f_3, \\ f_3' = -\frac{\left[1+(Wef_3)^{1-n}\right]^2}{\left[1+n(Wef_3)^{1-n}\right]} \left[f_1 f_3 - f_2^2 + A^2 + (M+K)(A - f_2) \right], \\ f_4' = f_5, \\ f_5' = -\Pr \left(f_1 f_5 + \delta f_5 \right), \end{array} \right\} \tag{3.8}$$

where $f = f_1, f' = f_2, f'' = f_3, f''' = f_3', \theta = f_4, \theta' = f_5, \theta'' = f_5'$ subject to the following conditions

$$\left. \begin{array}{l} f_1(0) = 0, f_2(0) = 1 + Lvf_3(0), f_4(0) = 1 + Ltf_5(0) \\ f_2(\infty) = A, f_4(\infty) = 0. \end{array} \right\}. \tag{3.9}$$

In order to get solution, values of $f_3(0)$ and $f_5(0)$ i.e. $f''(0)$ and $\theta'(0)$ are required, but no such values are given in the boundary conditions, therefore according to shooting technique initial guesses for $f_3(0)$ and $f_5(0)$ are taken as $s_1 ands_2$, respectively. Then we compare the calculated values for f_2, f_4 at η_∞ (a suitable gauss ofη) with the given boundary conditions $f_2(\eta_\infty = 4) = 0, f_4(\eta_\infty = 10) = 0$ and adjust the estimated values of$f''(0)$ and $\theta'(0)$ using the Secant method to give a better approximation for the solution. The step-size is taken $h = 0.002$. The above procedure is repeated until we get the converged results within a tolerance limit of 10^{-3}. All the computations are carried out on Matlab software.

4 Results and Discussion

The behaviour of A, which denotes the ratio of free stream velocity to the velocity of the stretching sheet on the velocity field can be observed from Fig. 4.1. The velocity of the fluid and the boundary layer thickness increases when free stream velocity is less than the velocity of the stretching sheet $A < 1$ with an increase in A. However when free stream velocity exceeds the velocity of the stretching sheet $A > 1$, the velocity of the fluid increases where as the boundary layer thickness decreases with an increase in A.

Now, we see the influence of the permeability parameter K on the velocity profiles. Fig. 4.2 shows the variation in velocity field for several values of K. With increase of K the dimensionless velocity $f'(\eta)$ along the plate decreases and consequently the momentum boundary layer thickness decreases. This is due to physical fact that resistance diminishes as permeability of the medium increases. So progressively less drag is experienced by the flow and flow retardation is thereby decreased. Thus the permeability parameter enhances the fluid motion inside the boundary layer.

The effect of magnetic parameter M on velocity profiles are shown by Fig. 4.3. It is observed that the fluid velocity decreases as the values of M increases. This leads to the fact that rate of transport decreases with the increment in M because the Lorentz force which opposes the motion of fluid increases with the increment in M.

Fig. 4.4 illustrates the influence of Weissenberg number We on velocity profiles. It shows that velocity increases as Weissenberg number increases.

The effect of velocity slip parameter on fluid velocity is presented by Fig. 4.5. It is observed that fluid velocity decreases as velocity slip parameter increases.

The influence of heat source ($\delta > 0$) parameter on the dimensionless temperature profiles is illustrated by Fig. 4.6. Due to increase in the strength of the heat source the fluid temperature increases as the thermal boundary layer thickness enlarges.

Fig. 4.7 represented the effect of Prandtl number on the temperature profiles. Beyond the surface the value of the temperature as well as thermal boundary layer thickness rapidly decrease with increasing Pr but near the surface the results are reverse. An increase in Prandtl number means an increase of fluid viscosity which causes a decrease in the flow velocity and the temperature decreases. This is also consistent with the fact that the thermal boundary layer thickness decreases with increasing Prandtl number.

The effect of temperature slip parameter on fluid temperature is shown by Fig. 4.8. It is observed that as the temperature slip parameter increases, fluid temperature decreases.

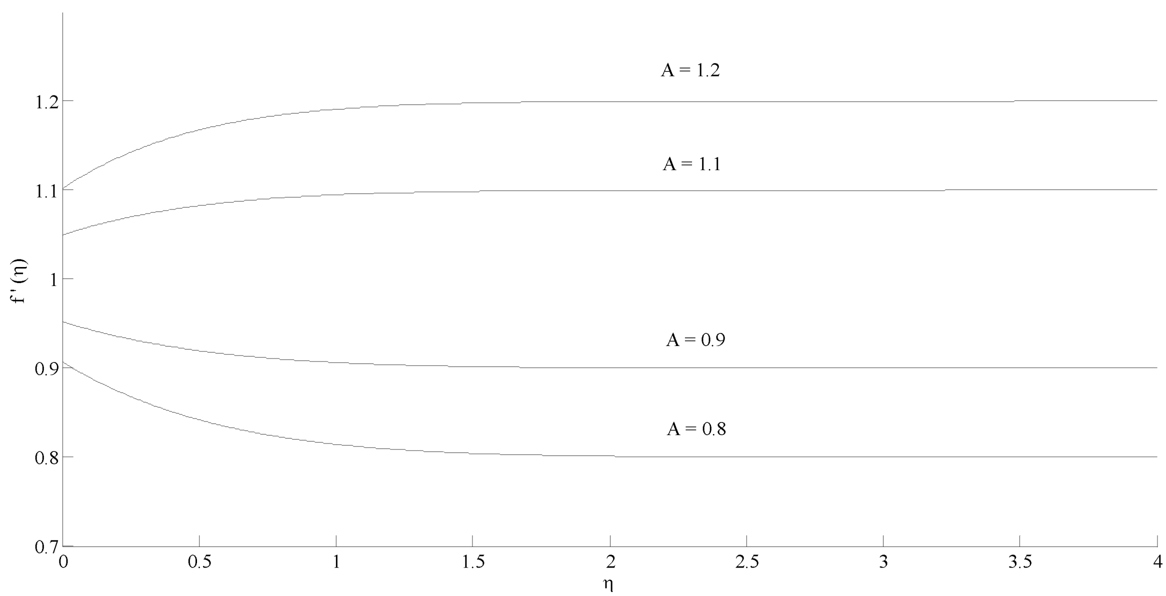

Figure 4.1: Velocity profile against η for various values of ratio parameter A when K=1.0, M = 0.09, Pr = 1.2, We = 0.2, δ = 1.0, Lv = 0.5, Lt = 0.5 and m = 0.1

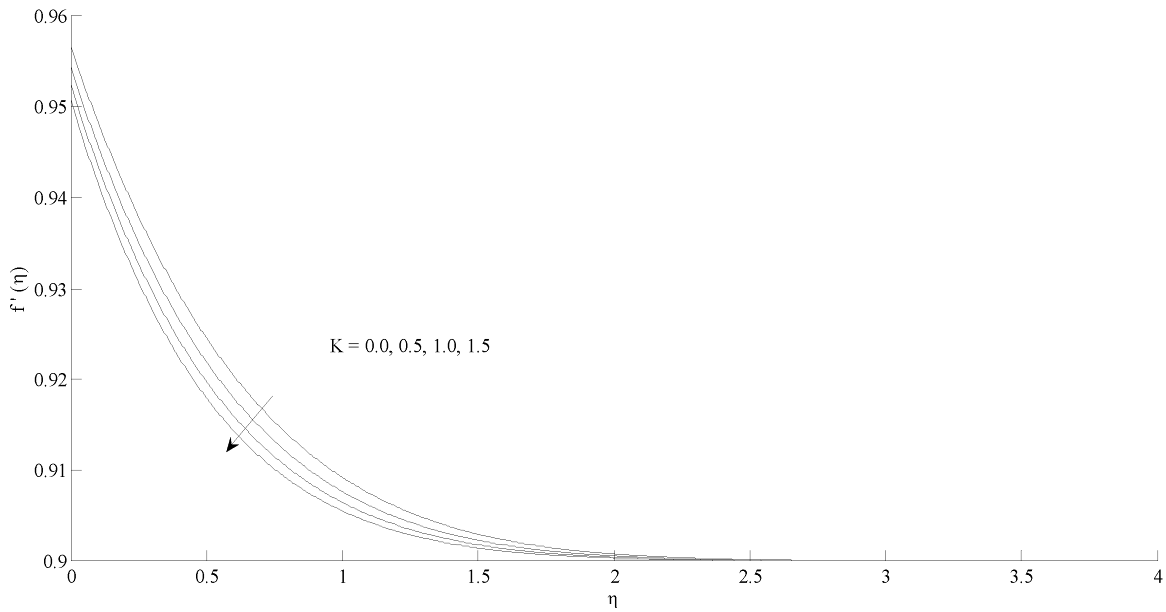

Figure 4.2: Velocity profile against η for various values of K when $M = 0.09$, $A = 0.9$, $\Pr = 1.2$, $We = 0.2$, $\delta = 1.0$, $Lv = 0.5$, $Lt = 0.5$ and $m = 0.1$.

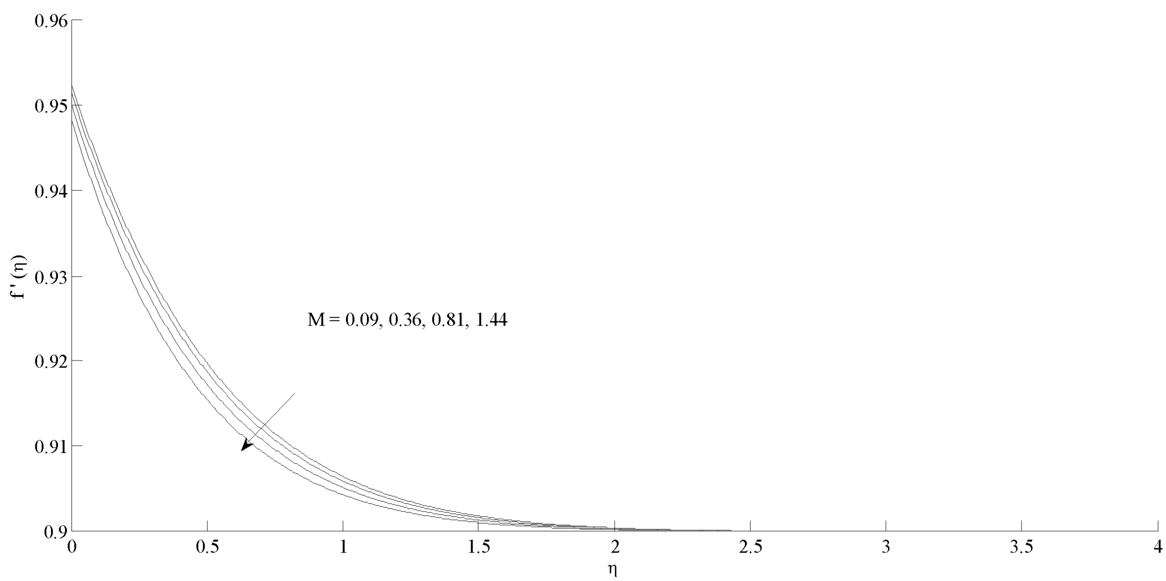

Figure 4.3: Velocity profile against η for various values of Magnetic Parameter M when K=1.0, $A = 0.9$, $\Pr = 1.2$, $We = 0.2$, $\delta = 1.0$, $Lv = 0.5$, $Lt = 0.5$ and $m = 0.1$.

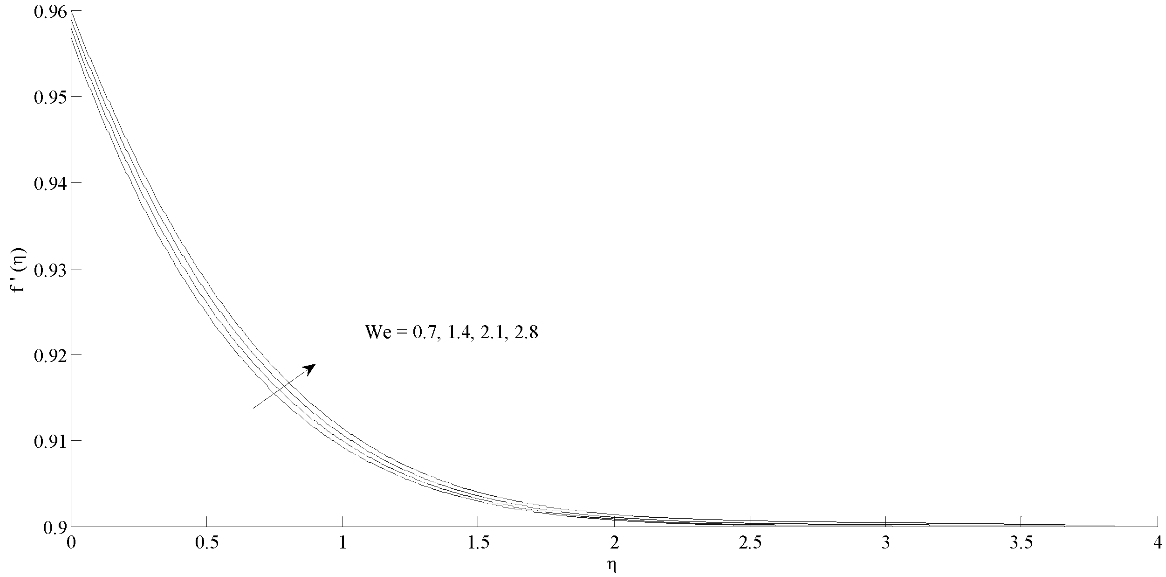

Figure 4.4: Velocity profile against η for various values of Weissenberg Number We when K=0.1, $M = 0.09$,$A = 0.9$, $\Pr = 1.2$, $\delta = 1.0$, $Lv = 0.5$, $Lt = 0.5$ and $m = 0.1$.

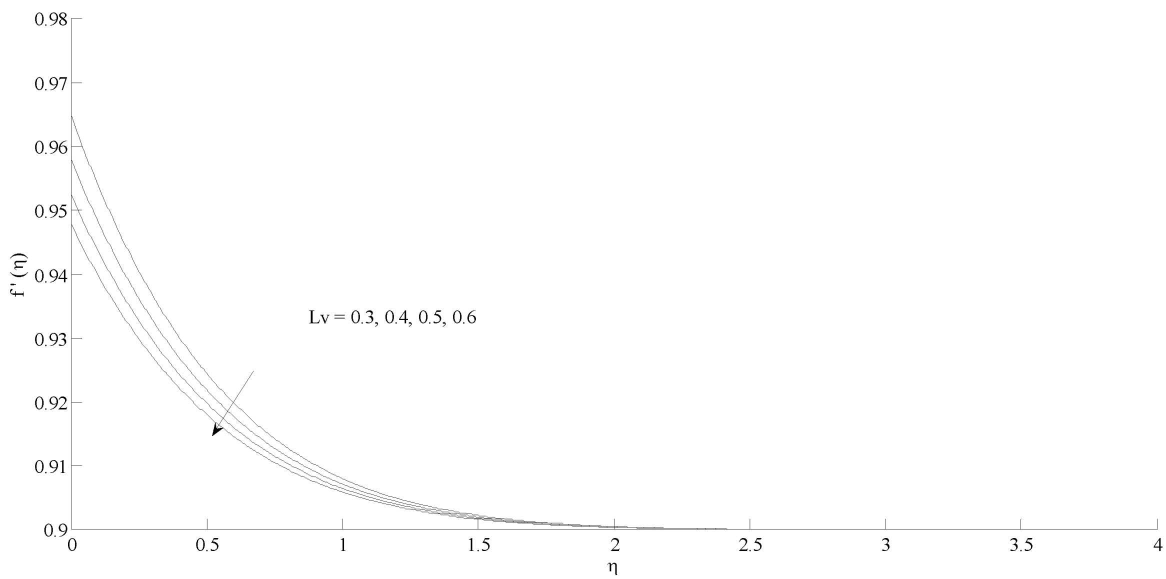

Figure 4.5: Velocity profile against η for various values of Lv when K=1.0, $M = 0.09$, $A = 0.9$, $\Pr = 1.2$, $We = 0.2$, $\delta = 1.0$, $Lt = 0.5$ and $m = 0.1$.

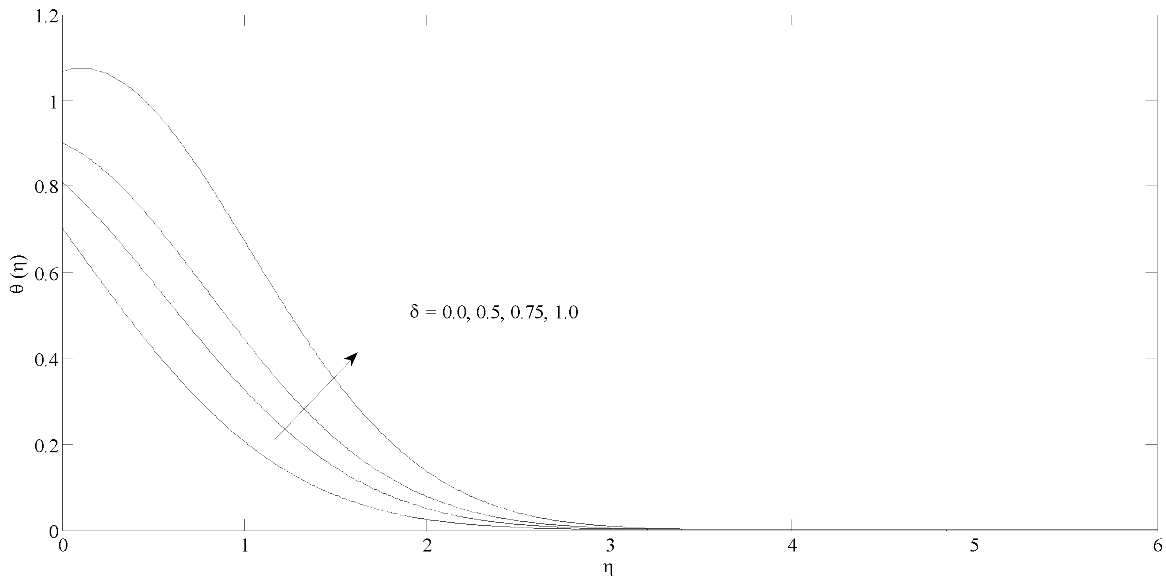

Figure 4.6: Temperature profile against η for various values of δ when K=1.0, $M = 0.09$, $A = 0.9$, $We = 0.2$, $\Pr = 1.2$, $Lv = 0.5$, $Lt = 0.5$ and $m = 0.1$.

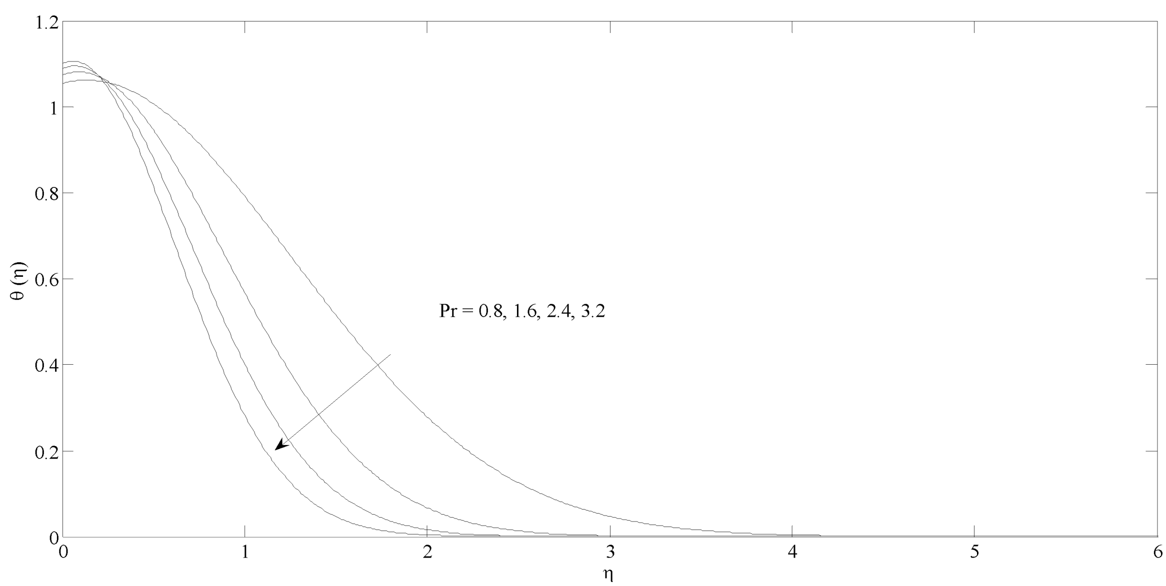

Figure 4.7: Temperature profile against η for various values of Prandtl Number Pr when K=1.0, $M = 0.09$, $A = 0.9$, $We = 0.2$, $\delta = 1.0$, $Lv = 0.5$, $Lt = 0.5$ and $m = 0.1$.

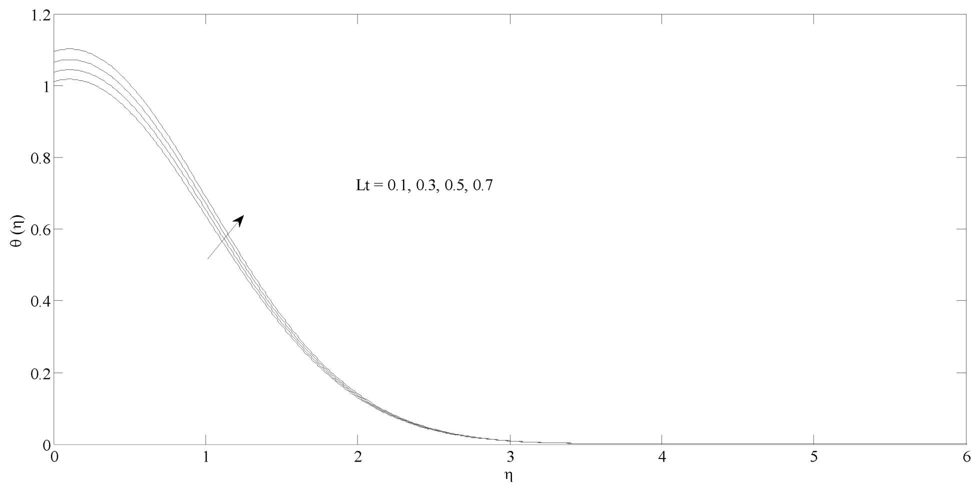

Figure 4.8: Temperature profile against η for various values of Lt when K=1.0, $M = 0.09$, $A = 0.9$, $We = 0.2$, Pr = 1.2, $\delta = 1.0$, $Lv = 0.5$ and $m = 0.1$.

Table 4.1: Numerical values of skin friction coefficient $\{-f''(0)\}$ and local Nusselt number $\{-\theta'(0)\}$ for various values of physical parameters.

A	M	K	We	Lv	Pr	δ	Lt	$-f''(0)$	$-\theta'(0)$
0.8	0.09	1.0	0.2	0.5	1.2	1.0	0.5	0.1859886	-0.13665
0.9								0.0952272	-0.13361
1.1								-0.0998319	0.09788
1.2								-0.2039279	0.17381
0.9	**0.09**							0.0952272	-0.13361
	0.36							0.0970852	-0.13506
	0.81							0.0999093	-0.13719
	1.44							0.1033948	-0.13972
	0.09	**0.0**						0.0868974	-0.12675
		0.5						0.0913896	-0.13052
		1.0						0.0952272	-0.13361
		1.5						0.0985679	-0.13619
		0.1	**0.7**					0.0933752	-0.13232
			1.4					0.0909191	-0.13034
			2.1					0.0884658	-0.12799
			2.8					0.0859435	-0.12508
		1.0	0.2	**0.3**				0.0861875	-0.12629
				0.4				0.0840204	-0.12461
				0.5				0.0819309	-0.12280
				0.6				0.0798756	-0.12095
				0.5	**0.8**			0.0952223	-0.11292
					1.6			0.0952223	-0.15249
					2.4			0.0952223	-0.18125
					3.2			0.0952223	-0.20481
					1.2	**0.0**		0.0952223	0.59195
						0.5		0.0952223	0.37973
						0.75		0.0952223	0.19536
						1.0		0.0952223	-0.13575
						1.0	**0.1**	0.0952223	-0.12786
							0.3	0.0952223	-0.13121
							0.5	0.0952223	-0.13475
							0.7	0.0952223	-0.13848

5 Conclusions

In this paper we have studied slip effect on MHD stagnation point flow and heat transfer of Cross fluid in presence of porous medium. The main findings of this study are as follows:

1. It is observed that velocity decreases as permeability parameter or magnetic parameter increases.
2. Velocity increases as Weisenbreg number increases.
3. Increment in velocity slip parameter and temperature slip parameter enhance both, velocity and temperature profiles.
4. Thermal boundary layer increases as heat source parameter δ increases.
5. The drag force decreases as parameter A increases while it increases as magnetic parameter$M$, permeability parameter K increases. The Nusselt number $(-\theta'(0))$ decreases as temperature slip parameter increases.

Acknowlegement

We are very much thankful to the referee for his valuable suggestions to bring the paper in its present form.

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Jñānābha, Vol. 49(2) (2019), 105-112

THE MULTI-PARAMETER *k*-MITTAG-LEFFLER FUNCTION ASSOCIATED WITH GENERALIZED FRACTIONAL KINETIC EQUATIONS

By

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(*Received : November* 03, 2019 *; Revised: November* 14, 2019)

Abstract

The object of this paper is to establish a new generalized form of the fractional kinetic equation with multiparameter k-Mittag-Leffler function[2]. The solution of fractional kinetic equations are discussed in terms of the Mittag-Leffler function of two parameters[13] by using Sumudu transform[12]. The obtained results are general in nature and can easily get various known and probably new results.

2010 Mathematics Subject Classification: 26A33, 44A20, 33E12, 34A08.

Keywords and phrases: Fractional calculus, Fractional kinetic equation, Multiparameter k-Mittag-Leffler function, Sumudu transform.

1 Introduction and Preliminaries

The Multiparameter k-Mittag-Leffler function is defined by Gehlot [2] as:

$$\begin{aligned} {}_pK_{q,k}^{(\beta,\eta)_m}[z] &= {}_pK_{q,k}^{(\beta,\eta)_m}[a_1,...,a_p;b_1,...,b_q,(\beta_1,\eta_1),...,(\beta_m,\eta_m);z] \\ &= \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p}(a_j)_{n,k}\, z^n}{\prod_{r=1}^{q}(b_r)_{n,k}\prod_{i=1}^{m}\Gamma_k(\eta_i n+\beta_i)}, \end{aligned} \tag{1.1}$$

where $k \in \mathbb{R}^+ = (0,\infty); a_j, b_r, \beta_i \in \mathbb{C};\ \eta_i \in \mathbb{R}\ (j = 1,2,...,p\ ;\ r = 1,2,...,q;\ i = 1,2,...,m)$. $\Gamma_k(x)$ is the k-Gamma function given by

$$\Gamma_k(x) = \int_0^{\infty} e^{-\frac{t^k}{k}}\ t^{x-1}dt, (x \in \mathbb{C}, k \in \mathbb{R}, \Re(x) > 0)\,, \tag{1.2}$$

and $(x)_{n,k}$ is the k-Pochhammer symbol ([1],[4]) given by

$$(x)_{n,k} = x(x+k)(x+2k).....(x+(n-1)k)\,, \qquad (x \in \mathbb{C},\ k \in \mathbb{R}\ and\ n \in \mathbb{N})\,. \tag{1.3}$$

It follows easily that,

$$\Gamma_k(x) = k^{\frac{x}{k}-1}\Gamma\left(\frac{x}{k}\right), \tag{1.4}$$

and

$$(\gamma)_{nq,k} = k^{nq}\left(\frac{\gamma}{k}\right)_{nq}. \tag{1.5}$$

The series (1.1) is valid for none of the parameters $b_r(r=1,2,...,q)$ is negative integer or zero. If any parameter $a_j(j=1,2,...,p)$ in (1.1) is zero or negative, the series terminates into polynomial in z.

Convergent conditions for the series (1.1) are given by

(i) If $p < q+\sum_{i=1}^{m}(\frac{\eta_i}{k})$, then the power series on the right of (1.1) is absolutely convergent for all $z \in \mathbb{C}$.

(ii) If $p = q+\sum_{i=1}^{m}(\frac{\eta_i}{k})$, then the power series on the right of (1.1) is absolutely convergent for all $|k^{p-q-\sum_{i=1}^{m}(\frac{\eta_i}{k})}z| < \prod_{i=1}^{m}(|\frac{\eta_i}{k}|)^{\frac{\eta_i}{k}} and\, |k^{p-q-\sum_{i=1}^{m}(\frac{\eta_i}{k})}z| = \prod_{i=1}^{m}(|\frac{\eta_i}{k}|)^{\frac{\eta_i}{k}}$, $\Re\left(\sum_{r=1}^{q}(\frac{b_r}{k})+\sum_{i=1}^{m}(\frac{\beta_i}{k})-\sum_{j=1}^{p}(\frac{a_j}{k})\right) > \frac{2+q+m-p}{2}$.

Remark 1.1. *If we set $k=1$, equation* (1.1) *reduces to*

$$\begin{aligned} {}_pK_q^{(\beta,\eta)_m}[z] &= {}_pK_{q,1}^{(\beta,\eta)_m}[a_1,...,a_p;b_1,...,b_q,(\beta_1,\eta_1),...,(\beta_m,\eta_m);z] \\ &= \sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_n\, z^n}{\prod_{r=1}^{q}(b_r)_n \prod_{i=1}^{m}\ \Gamma(\eta_i n+\beta_i)}, \end{aligned} \tag{1.6}$$

where ${}_pK_q^{(\beta,\eta)_m}[z]$ is k-series ([3],[5],[8]).

Remark 1.2. *If we take $k=1, m=1$, equation* (1.1) *becomes*

$$\begin{aligned} {}_pM_q^{\eta,\beta}[z] &= {}_pK_{q,1}^{(\beta,\eta)_1}[a_1,...,a_p;b_1,...,b_q,(\beta,\eta);z] \\ &= \sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_n\, z^n}{\prod_{r=1}^{q}(b_r)_n\ \Gamma(\eta n+\beta)}\ , \end{aligned} \tag{1.7}$$

where ${}_pM_q^{\eta,\beta}[z]$ is generalized M-series [10].

Remark 1.3. *If we put $k=1, m=1$ and $\beta=1$, equation* (1.1) *yields*

$$\begin{aligned} {}_pM_q^{\eta}[a_1,...,a_p;\ b_1,...,b_q;\ z] &= {}_pK_{q,1}^{(1,\eta)_1}[a_1,...,a_p;b_1,...,b_q,(1,\eta);z] \\ &= \sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_n\, z^n}{\prod_{r=1}^{q}(b_r)_n\ \Gamma(\eta n+1)}, \end{aligned} \tag{1.8}$$

where ${}_pM_q^{\eta}[z]$ is M-series [11].

Remark 1.4. *If we set $k=1, p=q=m=1, a_1=\delta, b_1=1$* , (1.1) *gives*

$$E_{\eta,\beta}^{\delta}[z] = {}_1K_{1,1}^{(\beta,\eta)_1}[\delta;\ 1,,(\beta,\eta);z] = \sum_{n=0}^{\infty}\frac{(\delta)_n\, z^n}{\Gamma(\eta n+\beta)\ n!}\ , \tag{1.9}$$

where $E_{\eta,\beta}^{\delta}[z]$ is the generalized Mittag-Leffler function [7].

Remark 1.5. *If we take $k=1,\ p=q=m=1,\ a_1=b_1=1$* , (1.1) *takes the form*

$$E_{\eta,\beta}[z] = {}_1K_{1,1}^{(\beta,\eta)_1}[1;\ 1,(\beta,\eta);z] = \sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\eta n+\beta)}, \tag{1.10}$$

where $E_{\eta,\beta}[z]$ is the Mittag-Leffler function [13].

The fractional differential equation between rate of change of reaction was established by Haubold and Mathai [6], and the rate of destruction and rate of production are given as:

$$\frac{dN}{d\theta} = -d\left(N_\theta\right) + p\left(N_\theta\right), \tag{1.11}$$

where $N = N(\theta)$ is the rate of reaction, $d = d(N)$ is the rate of destruction, $p = p(N)$ is the rate of production and N_θ denotes the function defined by

$$N_\theta(\theta^*) = N(\theta - \theta^*);\ \theta^* > 0.$$

The particular case of (1.11), for spatial fluctuations or homogeneities in the quantity $N(\theta)$ are neglected, is given by the equation

$$\frac{dN}{d\theta} = -C_i N_i(\theta), \tag{1.12}$$

with the initial condition $N_i(\theta = 0) = N_0$ is the number of density of the species i at time $\theta = 0$ and $C_i > 0$. If we remove the index i and integrate the standard kinetic equation (1.12), we obtain

$$N(\theta) - N_0 = -C\ {}_0D_\theta^{-1}\ N(\theta), \tag{1.13}$$

where ${}_0D_\theta^{-1}$ is the special case of the Riemann-Liouville integral operator ${}_0D_\theta^{-\nu}$, defined as

$${}_0D_\theta^{-\nu} f(\theta) = \frac{1}{\Gamma(\nu)} \int_0^\theta (\theta - s)^{\nu-1} f(s) ds, \qquad (\theta > 0,\ \Re(\nu) > 0). \tag{1.14}$$

The fractional generalization of the standard kinetic equation (1.13) is given by Haubold and Mathai [6] as

$$N(\theta) - N_0 = -C^\nu\ {}_0D_\theta^{-\nu}\ N(\theta), \tag{1.15}$$

and the solution of (1.15) is given by

$$N(\theta) = N_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(\nu n + 1)} (C\theta)^{\nu n}. \tag{1.16}$$

Further, Saxena and Kalla[9] established the following kinetic equation:

$$N(\theta) - N_0 f(\theta) = -C^\nu\ {}_0D_\theta^{-\nu}\ N(\theta), \qquad (\Re(\nu) > 0), \tag{1.17}$$

where $N(\theta)$ denotes the number density of a given species at time θ, $N_0 = N(0)$ is the number of density at time $\theta = 0$ and C is a constant and $f \in \mathcal{L}(0, \infty)$.

Sumudu transform was studied by Watugala[12] to solve differential and integral equations in the time domain, and to use in various applications of system engineering sciences and applied physics. The sumudu transform is derived from the classical Fourier integral and defined over the set of the functions

$$A = \left\{ f(\theta)\ |\exists\, M, \tau_1, \tau_2 > 0, |f(\theta)| < Me^{|\theta|/\tau_j},\ if\ \ \theta \in (-1)^j \times [0, \infty) \right\}, \tag{1.18}$$

by the following formula

$$G(u) = S\left\{ f(\theta) : u \right\} = \int_0^\infty e^{-\theta} f(u\theta) d\theta, \qquad (-\tau_1 < u < \tau_2), \tag{1.19}$$

where M is a real finite number and τ_1, τ_2 can be finite or infinite (see[12]) and $G(u)$ is known as the Sumudu transform of $f(\theta)$.

2 Solution of generalized fractional kinetic equations by using Sumudu transform

In view of the effectiveness and a great importance of the kinetic equations, we develop a further generalized form of the fractional kinetic equation involving Multiparameter k-Mittag-Leffler function in this section.

Theorem 2.1. *Let $a > 0$, $\nu > 0$, $k \in \mathbb{R}^+ = (0,\infty)$; $\theta, a_j, b_r, \beta_i \in \mathbb{C}; \eta_i \in \mathbb{R}^+$ ($j = 1,2,...,p$; $r = 1,2,...,q$; $i = 1,2,...,m$) with $|u| < a^{-1}$. Then the solution of generalized kinetic equation*

$$N(\theta) - N_0 \ {}_pK_{q,k}^{(\beta,\eta)_m}(\theta) = -a^\nu \ {}_0D_\theta^{-\nu} N(\theta), \tag{2.1}$$

is given by

$$N(\theta) = N_0 \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} (a_j)_{n,k} \ \Gamma(n+1) \ \theta^n}{\prod_{r=1}^{q} (b_r)_{n,k} \ \prod_{i=1}^{m} \Gamma_k(\eta_i n + \beta_i)} \ E_{\nu,n+1}(-a^\nu \ \theta^\nu). \tag{2.2}$$

Proof. By taking Sumudu transform on both the sides of (2.1) and using (1.1) and (1.19), we have

$$S\Big\{N(\theta) : \ u\Big\} - N_0 \ S\Big\{ {}_pK_{q,k}^{(\beta,\eta)_m}(\theta) : u\Big\} = -a^\nu \ S\Big\{ {}_0D_\theta^{-\nu} N(\theta) : u\Big\},$$

$$N(u) - N_0 \ S\Big\{ {}_pK_{q,k}^{(\beta,\eta)_m}(\theta) : u\Big\} = -a^\nu \ u^\nu N(u),$$

where $S\Big\{N(\theta) : \ u\Big\} = N(u)$ and $S\Big\{ {}_0D_\theta^{-\nu} N(\theta) : u\Big\} = u^\nu N(u)$.

$$\begin{aligned} N(u)\left[1 + (au)^\nu\right] &= N_0 \ S\left\{ \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} (a_j)_{n.k} \ \theta^n}{\prod_{r=1}^{q} (b_r)_{n,k} \ \prod_{i=1}^{m} \Gamma_k(\eta_i n + \beta_i)} \right\} \\ &= N_0 \ \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} (a_j)_{n.k}}{\prod_{r=1}^{q} (b_r)_{n,k} \ \prod_{i=1}^{m} \Gamma_k(\eta_i n + \beta_i)} \ S\Big\{\theta^n : u\Big\} \\ &= N_0 \ \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} (a_j)_{n.k}}{\prod_{r=1}^{q} (b_r)_{n,k} \ \prod_{i=1}^{m} \Gamma_k(\eta_i n + \beta_i)} \ \int_0^\infty e^{-\theta} (u\theta)^n \ d\theta, \end{aligned}$$

$$\begin{aligned} N(u)\left[1 + (au)^\nu\right] &= N_0 \ \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} (a_j)_{n.k} \, u^n \Gamma(n+1)}{\prod_{r=1}^{q} (b_r)_{n,k} \ \prod_{i=1}^{m} \Gamma_k(\eta_i n + \beta_i)} \\ N(u) &= \frac{N_0}{[1 + (au)^\nu]} \ \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} (a_j)_{n.k} \, u^n \Gamma(n+1)}{\prod_{r=1}^{q} (b_r)_{n,k} \ \prod_{i=1}^{m} \Gamma_k(\eta_i n + \beta_i)} \\ &= N_0 \ \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} (a_j)_{n.k} \ \Gamma(n+1) \ u^n}{\prod_{r=1}^{q} (b_r)_{n,k} \ \prod_{i=1}^{m} \Gamma_k(\eta_i n + \beta_i)} \ \sum_{\alpha=0}^{\infty} (-1)^\alpha (au)^{\nu\alpha}. \end{aligned}$$

Finally by taking inverse Sumudu transform and make use of the following results

$$S^{-1}\Big\{u^{\nu-1} : \theta\Big\} = \frac{\theta^{\nu-1}}{\Gamma(\nu)}, \qquad (min\{\Re(\nu), \Re(u)\} > 0)$$

and $S^{-1}\Big\{N(u) : \theta\Big\} = N(\theta)$, we get

$$
\begin{aligned}
N(\theta) &= N_0 \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p}(a_j)_{n.k}\, \Gamma(n+1)}{\prod_{r=1}^{q}(b_r)_{n,k}\ \prod_{i=1}^{m}\Gamma_k(\eta_i n+\beta_i)} \sum_{\alpha=0}^{\infty}(-1)^{\alpha} a^{\nu\alpha}\ \frac{\theta^{n+\nu\alpha}}{\Gamma(n+\nu\alpha+1)} \\
&= N_0 \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p}(a_j)_{n.k}\, \Gamma(n+1)\ \theta^n}{\prod_{r=1}^{q}(b_r)_{n,k}\ \prod_{i=1}^{m}\Gamma_k(\eta_i n+\beta_i)}\ \sum_{\alpha=0}^{\infty} \frac{(-1)^{\alpha}(a^{\nu}\theta^{\nu})^{\alpha}}{\Gamma\{(n+1)+\nu\alpha\}} \\
&= N_0 \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p}(a_j)_{n.k}\ \Gamma(n+1)\ \theta^n}{\prod_{r=1}^{q}(b_r)_{n,k}\ \prod_{i=1}^{m}\Gamma_k(\eta_i n+\beta_i)}\ E_{\nu,n+1}(-a^{\nu}\ \theta^{\nu}).
\end{aligned}
$$

This completes the proof of Theorem 2.1. □

If we set $k=1$ in Theorem 1, we arrive at the following result.

Corollary 2.1. *Let $a>0,\ \nu>0;\ \theta, a_j, b_r, \beta_i \in \mathbb{C}; \eta_i \in \mathbb{R}^+$ $(j=1,2,...,p;\ r=1,2,...,q;\ i=1,2,...,m)$ with $|u|<a^{-1}$. Then the solution of the following generalized fractional kinetic equation*

$$
N(\theta) - N_0\ {}_pK_q^{(\beta,\eta)_m}(\theta) = -a^{\nu}\ {}_0D_{\theta}^{-\nu}N(\theta), \tag{2.3}
$$

is given by

$$
N(\theta) = N_0 \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p}(a_j)_n\ \ \Gamma(n+1)\ \theta^n}{\prod_{r=1}^{q}(b_r)_n\ \prod_{i=1}^{m}\Gamma(\eta_i n+\beta_i)}\ E_{\nu,n+1}(-a^{\nu}\ \theta^{\nu}). \tag{2.4}
$$

If we put $k=1, m=1$ in Theorem 2.1, we obtain the following corollary.

Corollary 2.2. *Let $a>0,\ \nu>0;\ \theta, a_j, b_r, \beta \in \mathbb{C}; \eta \in \mathbb{R}^+$ $(j=1,2,...,p;\ r=1,2,...,q\ )$ with $|u|<a^{-1}$. Then the solution of the following generalized fractional kinetic equation*

$$
N(\theta) - N_0\ {}_pM_q^{\eta,\beta}(\theta) = -a^{\nu}\ {}_0D_{\theta}^{-\nu}N(\theta), \tag{2.5}
$$

is given by

$$
N(\theta) = N_0 \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p}(a_j)_n\ \ \Gamma(n+1)\ \theta^n}{\prod_{r=1}^{q}(b_r)_n\ \ \Gamma(\eta n+\beta)}\ E_{\nu,n+1}(-a^{\nu}\ \theta^{\nu}). \tag{2.6}
$$

If we take $k=1, m=1$ and $\beta=1$ in Theorem 2.1, we have

Corollary 2.3. *Let $a>0,\ \nu>0;\ \theta, a_j, b_r, \in \mathbb{C}; \eta \in \mathbb{R}^+$ $(j=1,2,...,p;\ r=1,2,...,q\ )$ with $|u|<a^{-1}$. Then the solution of the following generalized fractional kinetic equation*

$$
N(\theta) - N_0\ {}_pM_q^{\eta}(\theta) = -a^{\nu}\ {}_0D_{\theta}^{-\nu}N(\theta), \tag{2.7}
$$

is given by

$$
N(\theta) = N_0 \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p}(a_j)_n\ \ \Gamma(n+1)\ \theta^n}{\prod_{r=1}^{q}(b_r)_n\ \ \Gamma(\eta n+1)}\ E_{\nu,n+1}(-a^{\nu}\ \theta^{\nu}). \tag{2.8}
$$

If we set $k=1, p=q=m=1, a_1=\delta, b_1=1$ in Theorem 2.1, we obtain

Corollary 2.4. *Let $a>0,\ \nu>0;\ \theta, \delta, \beta \in \mathbb{C}; \eta \in \mathbb{R}^+$ with $|u|<a^{-1}$. Then the solution of the following generalized fractional kinetic equation*

$$
N(\theta) - N_0\ E_{\eta,\beta}^{\delta}(\theta) = -a^{\nu}\ {}_0D_{\theta}^{-\nu}N(\theta), \tag{2.9}
$$

is given by

$$
N(\theta) = N_0 \sum_{n=0}^{\infty} \frac{(\delta)_n\ \theta^n}{\Gamma(\eta n+\beta)}\ E_{\nu,n+1}(-a^{\nu}\ \theta^{\nu}). \tag{2.10}
$$

If we put $k=1, p=q=m=1, a_1=b_1=1$ in Theorem 2.1, we get

Corollary 2.5. *Let $a>0,\ \nu>0;\ \theta,\beta\in\mathbb{C};\eta\in\mathbb{R}^+$ with $|u|<a^{-1}$. Then the solution of the following generalized fractional kinetic equation*

$$N(\theta)-N_0\ E_{\eta,\beta}(\theta)=-a^{\nu}\ {}_0D_{\theta}^{-\nu}N(\theta), \tag{2.11}$$

is given by

$$N\left(\theta\right)=N_0\sum_{n=0}^{\infty}\frac{\Gamma(n+1)\ \theta^n}{\Gamma(\eta n+\beta)}\ E_{\nu,n+1}\left(-a^{\nu}\ \theta^{\nu}\right). \tag{2.12}$$

Theorem 2.2. *Let $a>0,\ \nu>0,\ k\in\mathbb{R}^+=(0,\infty);\ \theta,a_j,b_r,\beta_i\in\mathbb{C};\eta_i\in\mathbb{R}^+$ $(j=1,2,...,p;\ r=1,2,...,q;\ i=1,2,...,m)$ with $|u|<a^{-1}$. Then, the solution of the following generalized fractional kinetic equation*

$$N(\theta)-N_0\ {}_pK_{q,k}^{(\beta,\eta)_m}(a^{\nu}\theta^{\nu})=-a^{\nu}\ {}_0D_{\theta}^{-\nu}N(\theta), \tag{2.13}$$

is given by

$$N\left(\theta\right)=N_0\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}\left(a_j\right)_{n,k}\ \Gamma(\nu n+1)\ (a^{\nu}\theta^{\nu})^n}{\prod_{r=1}^{q}\left(b_r\right)_{n,k}\ \prod_{i=1}^{m}\Gamma_k(\eta_i n+\beta_i)}\ E_{\nu,\nu n+1}\left(-a^{\nu}\ \theta^{\nu}\right). \tag{2.14}$$

Proof. The proof of Theorem 2.2 is similar to that of Theorem 2.1. So, we omit the details of their proof.

If we take $k=1$, then Theorem 2.2 reduces to the following result □

Corollary 2.6. *Let $a>0,\ \nu>0;\ \theta,a_j,b_r,\beta_i\in\mathbb{C};\eta_i\in\mathbb{R}^+$ $(j=1,2,...,p;\ r=1,2,...,q;\ i=1,2,...,m)$ with $|u|<a^{-1}$. Then the solution of following generalized fractional kinetic equation*

$$N(\theta)-N_0\ {}_pK_{q}^{(\beta,\eta)_m}(a^{\nu}\theta^{\nu})=-a^{\nu}\ {}_0D_{\theta}^{-\nu}N(\theta), \tag{2.15}$$

is given by

$$N\left(\theta\right)=N_0\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}\left(a_j\right)_{n}\ \Gamma(\nu n+1)\ (a^{\nu}\theta^{\nu})^n}{\prod_{r=1}^{q}\left(b_r\right)_{n}\ \prod_{i=1}^{m}\Gamma(\eta_i n+\beta_i)}\ E_{\nu,\nu n+1}\left(-a^{\nu}\ \theta^{\nu}\right). \tag{2.16}$$

If we set $k=1, m=1$ then Theorem 2.2 reduces to the following

Corollary 2.7. *Let $a>0,\ \nu>0;\ \theta,a_j,b_r,\beta\in\mathbb{C};\eta\in\mathbb{R}^+$ $(j=1,2,...,p;\ r=1,2,...,q\ )$ with $|u|<a^{-1}$. Then the solution of following generalized fractional kinetic equation*

$$N(\theta)-N_0\ {}_pM_{q}^{\eta,\beta}(a^{\nu}\ \theta^{\nu})=-a^{\nu}\ {}_0D_{\theta}^{-\nu}N(\theta), \tag{2.17}$$

is given by

$$N\left(\theta\right)=N_0\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}\left(a_j\right)_{n}\ \Gamma(\nu n+1)\ (a^{\nu}\theta^{\nu})^n}{\prod_{r=1}^{q}\left(b_r\right)_{n}\ \Gamma(\eta n+\beta)}\ E_{\nu,\nu n+1}\left(-a^{\nu}\ \theta^{\nu}\right). \tag{2.18}$$

If we put $k=1, m=1$ and $\beta=1$, Theorem 2.2 reduces to the following result

Corollary 2.8. *Let $a>0,\ \nu>0;\ \theta,a_j,b_r,\in\mathbb{C};\eta\in\mathbb{R}^+$ $(j=1,2,...,p;\ r=1,2,...,q\ )$ with $|u|<a^{-1}$. Then the solution of following generalized fractional kinetic equation*

$$N(\theta)-N_0\ {}_pM_{q}^{\eta}(a^{\nu}\theta^{\nu})=-a^{\nu}\ {}_0D_{\theta}^{-\nu}N(\theta), \tag{2.19}$$

is given by

$$N\left(\theta\right)=N_0\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}\left(a_j\right)_{n}\ \Gamma(\nu n+1)\ (a^{\nu}\theta^{\nu})^n}{\prod_{r=1}^{q}\left(b_r\right)_{n}\ \Gamma(\eta n+1)}\ E_{\nu,\nu n+1}\left(-a^{\nu}\ \theta^{\nu}\right). \tag{2.20}$$

If we take $k=1, p=q=m=1, a_1=\delta, b_1=1$ then Theorem 2.2 reduce to the following

Corollary 2.9. *Let* $a>0,\ \nu>0;\ \theta,\delta,\beta\in\mathbb{C};\eta\in\mathbb{R}^+$ *with* $|u|<a^{-1}$. *Then the solution of following generalized fractional kinetic equation*

$$N(\theta)-N_0\ E^{\delta}_{\eta,\beta}(a^\nu\theta^\nu)=-a^\nu\ {}_0D_\theta^{-\nu}N(\theta), \tag{2.21}$$

is given by

$$N\left(\theta\right)=N_0\sum_{n=0}^{\infty}\frac{(\delta)_n\Gamma(\nu n+1)\ (a^\nu\theta^\nu)^n}{\Gamma(\eta n+\beta)\ n!}\ E_{\nu,\nu n+1}\left(-a^\nu\ \theta^\nu\right). \tag{2.22}$$

If we set $k=1, p=q=m=1, a_1=b_1=1$ then Theorem 2.2 reduces to the following

Corollary 2.10. *Let* $a>0,\ \nu>0;\ \theta,\beta\in\mathbb{C};\eta\in\mathbb{R}^+$ *with* $|u|<a^{-1}$. *Then the solution of the following generalized fractional kinetic equation*

$$N(\theta)-N_0\ E_{\eta,\beta}(a^\nu\theta^\nu)=-a^\nu\ {}_0D_\theta^{-\nu}N(\theta), \tag{2.23}$$

is given by

$$N\left(\theta\right)=N_0\sum_{n=0}^{\infty}\frac{\Gamma(\nu n+1)\ (a^\nu\theta^\nu)^n}{\Gamma(\eta n+\beta)}\ E_{\nu,\nu n+1}\left(-a^\nu\ \theta^\nu\right). \tag{2.24}$$

3 Conclusion:

The aim of this paper is to give a new generalized form of the fractional kinetic equation associated with multiparameter k-Mittag-Leffler function and obtain solution for the same. From the close relationship of the Multiparameter k-Mittag-Leffler function with many special functions, we can easily construct various known and probably new kinetic equations.

Acknowlegement

The authors are very much thankful to the referee for his valuable suggestions to bring the paper in the present form.

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Jñānābha, Vol. 49(2) (2019), 113-119

CHURCHILL'S DIFFUSION AND EULER TYPE INTEGRAL INVOLVING AN I^*-FUNCTION

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(*Received : August* 02, 2019 *; Revised: November* 18, 2019)

Abstract

In this paper, we obtain an Euler type integral involving an I^*- function with products of a general class of polynomials and exponential function. Then, we use it to express the Churchill's diffusion in terms of a series consisting of the coefficients of that general class of polynomials with a sequence of I^*- functions. Certain particular cases are also discussed.

2010 Mathematics Subject Classification: 33C45, 33C60.

Keywords and phrases: An I^* - function, Churchill's diffusion, a general class of polynomials, Gould and Hopper polynomials.

1 Introduction

Srivastava [1] has applied a general class of polynomials of several variables for evaluation of multilinear generating functions in the Kohnauser sets of bi - orthogonal polynomials. Again, several authors have used this class of polynomials in in representation of the series solution of matter diffusion and wave propagation problems for example Kumar [2] has analyzed the Churchill's diffusion [1] in terms of the series consisting of various parameters along with the coefficients of known various polynomials with the help of a general class of polynomials [10] defined as

$$S_{N_1,\dots,N_m}^{M_1,\dots,M_m}(y_1,\dots,y_m)=\sum_{k_1=0}^{[\frac{N_1}{M_1}]}\cdots\sum_{k_m=0}^{[\frac{N_m}{M_m}]}\frac{(-N_1)_{M_1k_1}}{k_1!}\cdots\frac{(-N_m)_{M_mk_m}}{k_m!}$$
$$\times A[N_1,k_1;...;N_m,k_m]{y_1}^{k_1}...{y_m}^{k_m}, \tag{1.1}$$

where, $N_1,...,N_m; M_1,...,M_m$ are arbitrary positive integers and the coefficients $A[N_1, k_1;...;N_m,k_m]$ are arbitrary parameters real or complex independent of $y_1,\dots,y_m$.

Clearly, on putting $m=1$ in Eqn. (1.1), a generalized class of polynomials of one variable is found (see in [9]) as

$$S_N^M(y)=\sum_{k=0}^{\left[\frac{N}{M}\right]}\frac{(-N)_{Mk}}{k!}A\left[N,k\right]y^k, \tag{1.2}$$

where, N,M are arbitrary positive integers and the coefficients $A[N,k]$ are arbitrary parameters real or complex independent of y.

On application of a generalized class of polynomials (1.2), recently, Kumar and Pathan [5] have derived a generalized Weiertrass Approximation Theorem and obtained various generating functions to make extensions in the approximation theory.

In concerned with the evaluation of some polynomials by Eqn. (1.2), here replace $A[N,k]=2^N, y=(-1)^M h(2x)^{-M}$ to find a relation

$$S_N^M(y)=(x)^{-N} g_N^M(2x,h), \tag{1.3}$$

where, $g_N^M(2x,h)$ are the generalization of Hermite polynomials due to Gould and Hopper [5], defined by

$$g_N^M(2x,h)=\sum_{k=0}^{\left[\frac{N}{M}\right]} \frac{N!}{k!\,(N-Mk)!} h^k (2x)^{N-Mk}. \tag{1.4}$$

Again in Eqn. (1.4) put $h=-1$, it terminates into a relation between Gould - Hopper polynomials and the generalized Hermite polynomials [11]

$$g_N^M(2x,-1)=H_{N,M}(x), \tag{1.5}$$

where, the generalization of Hermite polynomials is given by

$$H_{N,M}(x)=\sum_{k=0}^{\left[\frac{N}{M}\right]} \frac{(-1)^k N! (2x)^{N-Mk}}{k!\,(N-Mk)!}. \tag{1.6}$$

Again to extend our work of (1.1) - (1.6) in applied theory of diffusion problems, we consider the Churchill's diffusion in following form of differential equation as (see [1, p, 131], [4])

$$x\frac{\partial U}{\partial x}+\frac{\partial U}{\partial t}+U=xF(t),\ x>0,\ t>0, U=(x,t), \tag{1.7}$$

with boundary conditions

$$U(x,0)=U(0,t)=0. \tag{1.8}$$

The solution of the problem (1.7) - (1.8) is given by [1, p. 131]

$$U(x,t)=xe^{-2t}\int_0^t e^{2v} F(v)\,dv. \tag{1.9}$$

Here in Eqn. (1.9), $F(t)$ is a known function.

Further to measure our results, by Eqn. (1.9) and due to Kumar [4], a functional by the diffusion distribution for the domain $x>0,\ t>0$, is written as

$$\mathcal{F}(x)=x^{-1}\int_0^\infty U(x,t)dt=\int_0^\infty \left\{\int_0^t e^{-2v} F(t-v)\,dv\right\} dt\ \forall x>0,\ t>0. \tag{1.10}$$

In our present investigation, we make an application of above formulae for straightforward evaluation of the relations between diffusion and the series involving the I^*- functions. In further extensions of above researches, we introduce new parameters $\mu,\sigma,\nu\in\mathbb{R}^+,\ \lambda\geq 0$ and $\rho,\eta,\in\mathbb{C}$ such that $\Re(\rho)>-1,,\Re(\eta)>-1$ and then obtain an integral function in the complex plane for $z\in\mathbb{C},\tau\geq 0,\ X^{\rho,\eta}_{\nu,\lambda,\mu,\sigma}(z,\tau,t)=\int_0^t e^{2\tau\nu}F(\nu)d\nu$ in which $F(s), s>0$, are known function. Again, we analyze the distribution in the complex plane, on considering $F(s), s>0$ involving an I^*- function with products of a general class of polynomials and exponential function and determine a determinant, then use it to express a series representation of the Churchill's diffusion consisting of the coefficients of that general class of polynomials with a sequence of I^*- functions. Certain particular cases are also discussed.

2 Euler Type Integrals involving an I^*- function with products of a general class of polynomials and exponential function.

In this section, we present an I^*- function such that when, $r \geq 2$, an I^*- function is different from Fox's H - function (see [2], [7], [9]) type contour integral in the complex plane, not symmetric in regard of parameters but when $r = 1$, it is identical to that H- function. Then obtain an Euler type integral involving I^*- function with products of a general class of polynomials (1.2) and exponential function to find the diffusion distribution.

The I^*- function is introduced as a contour integral in complex plane given by

$$I^{*m,n}_{\ p_i,q_i;r}\left[z\left|\begin{matrix}(a_j,\alpha_j)_{1,n} & : & (a_{ji},\alpha_{ji})_{1,p_i}\\ (b_j,\beta_j)_{1,m} & : & (b_{ji},\beta_{ji})_{1,q_i}\end{matrix}\right.\right] = \frac{1}{2\pi\omega}\int_L \phi(\xi) z^{\xi} d\xi, \omega = \sqrt{(-1)}, \tag{2.1}$$

in which

$$\phi(\xi) = \frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j\xi)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j\xi)}{\sum_{i=1}^{r}[\prod_{j=1}^{q_i}\Gamma(1-b_{ji}+\beta_{ji}\xi)\prod_{j=1}^{p_i}\Gamma(a_{ji}-\alpha_{ji}\xi)]}. \tag{2.2}$$

Here, for finite value of r; all $p_i, q_i (i = 1, 2, ..., r), m$ and n are the integers, satisfying the inequalities; $0 \leq n \leq p,\ p_i \geq n \geq 1\ (i = 1, 2, \ldots, r), 1 \leq m \leq q, q_i \geq m \geq 1 (i = 1, 2, \ldots, r), a_j (j = 1, \ldots, n), \beta_j (j = 1, \ldots, m), a_{ji} (1 \leq j \leq p_i, (i = 1, 2, \ldots, r), \beta_{ji} (1 \leq j \leq q_i, (i = 1, 2, \ldots, r))$ are real and positive and $a_j (j = 1, \ldots, n) b_j (j = 1, \ldots, m), a_{ji} (1 \leq j \leq p_i, (i = 1, 2, \ldots, r)), b_{ji} (1 \leq j \leq q_i, (i = 1, 2, \ldots, r))$ are complex numbers such that $a_k(b_h + v) \neq \beta_h(a_k - 1 - l)$ for $l, v = 0, 1, 2 \ldots; h = 1, 2, \ldots, m$.

L is contour running form $s - i\infty$ to $s + i\infty$, where s is real in the complex ξ-plane such that the poles. $\xi = \frac{(a_j-1-l)}{\alpha_j}, j = 1, 2, \ldots, n; l = 0, 1, 2, \cdots, \xi = \frac{(b_j+v)}{\beta_j}, j = 1, 2\cdots, m; v = 0, 1, 2, \cdots$ lie to the left and right hand sides of the contour L respectively, the empty product is represented as 1.

The I^*- function converges absolutely in ξ-plane if $|arg(z)| < \frac{\pi}{2}A$, where,

$$A = \sum_{j=1}^{n} a_j + \sum_{j=1}^{m}\beta_j - \max_{1\leq i\leq r}[\sum_{j=1}^{p_i} a_{ji} + \sum_{j=1}^{q_i}\beta_{ji}]\ > 0\ . \tag{2.3}$$

Property 1. The I^*- function is most probably identical to $I-$ function [8].

Property 2. For $r = 1$, the I^*- function defined by (2.1) - (2.3) may become Fox's H - function ([2], [7], [9]) as

$$I^{*m,n}_{\ p_1,q_1;1}\left[z\left|\begin{matrix}(a_j,\alpha_j)_{1,n} & : & (a_{j1},\alpha_{j1})_{1,\ p_1}\\ (b_j,\beta_j)_{1,m} & : & (b_{j1},\beta_{j1})_{1,\ q_1}\end{matrix}\right.\right]$$

$$= H^{m,n}_{p_1+n,q_1+m}\left[z\left|\begin{matrix}(a_j,\alpha_j)_{1,n}, (a_{j1},\alpha_{j1})_{n+1,\ p_1}, (a_{j1},\alpha_{j1})_{1,\ n}\\ (b_j,\beta_j)_{1,m}, (b_{j1},\beta_{j1})_{m+1,\ q_1}, (b_{j1},\beta_{j1})_{1,\ m}\end{matrix}\right.\right],$$

where, $$|\arg(z)| < \frac{\pi}{2}A^*, \text{ and } A^* = \sum_{j=1}^{n}\alpha_j + \sum_{j=1}^{m}\beta_j - \left[\sum_{j=1}^{p_1}\alpha_{j1} + \sum_{j=1}^{q_1}\beta_{j1}\right] > 0\ . \tag{2.4}$$

Now by the diffusion theory, given in Eqns. (1.9) and (1.10), where $F(s)$ is a known function, thus, we consider $F(s)$ for $0 < s < t$, $t > 0$, and the parameters are taken under the conditions $\mu, \sigma, \nu \in \mathbb{R}^+, \lambda \geq 0$, and $\rho, \eta \in \mathbb{C}$, such that $\Re(\rho) > -1, \Re(\eta) > -1$ in the form

$$F(s) = s^{\rho}(t-s)^{\eta} S_N^M(ys^{\mu}(t-s)^{\sigma}) I^{*m,n}_{\ p_i,q_i;r}[zs^{\nu}(t-s)^{\lambda}\left|\begin{matrix}(a_j,a_j)_{1,n} & : & (a_{ji},a_{ji})_{1,\ p_i}\\ (b_j,\beta_j)_{1,m} & : & (b_{ji},\beta_{ji})_{1,\ q_i}\end{matrix}\right.\Bigg], \tag{2.5}$$

and thus we establish following theorems:

Theorem 2.1. *If $\mu, \sigma, \nu \in \mathbb{R}^+, \tau \geq 0, \lambda \geq 0,$ and $\rho, \eta \in \mathbb{C}$, such that $\Re(\rho) > -1, \Re(\eta) > -1$ then by the function $F(s)$ for $0 < s < t, t > 0$, given in Eqn. (2.5), an integral formula $X^{\rho,\eta}_{\nu,\lambda,\mu,\sigma}(z,\tau,t) \forall z \in \mathbb{C}$, given by $X^{\rho,\eta}_{\nu,\lambda,\mu,\sigma}(z,\tau,t) = \int_0^t e^{2\tau\nu} F(\nu) d\nu$ exists, and it may be expressed in the form of the series involving the coefficients of a general class of polynomials with product of a determinant $\mathbb{A}^{\rho,\eta}_{\nu,\lambda,\mu,\sigma}(z,\tau,t)$, as*

$$X^{\rho,\eta}_{\nu,\lambda,\mu,\sigma,l,k}(z,\tau,t) = t^{\eta+\rho+1} \sum_{l=0}^{\infty} \sum_{k=0}^{[\frac{N}{M}]} \frac{(-N)_{Mk}}{k!} A[N,k] \mathbb{A}^{\rho,\eta}_{\nu,\lambda,\mu,\sigma,l,k}(z,\tau,t),$$

where the sequence of functions

$$\mathbb{A}^{\rho,\eta}_{\nu,\lambda,\mu,\sigma,l,k}(z,\tau,t) = \frac{(2\tau t)^l}{l!} (yt^{(\sigma+\mu)})^k \frac{1}{2\pi\omega}$$
$$\times \int_L \frac{\prod_{j=1}^m \Gamma(b_j - \beta_j\xi) \quad \prod_{j=1}^n \Gamma(1 - a_j + a_j\xi) \ \Gamma(1 + l + \rho + \mu k + \nu\xi)\Gamma(1 + \eta + \sigma k + \lambda\xi)}{\Gamma(2 + l + \eta + \rho + (\sigma+\mu)k + (\lambda+\nu)\xi) \sum_{i=1}^r [\prod_{j=1}^{q_i} \Gamma(1 - b_{ji} + \beta_{ji}\xi) \prod_{j=1}^{p_i} \Gamma(a_{ji} - \alpha_{ji}\xi) \ ]}$$
$$(zt^{(\lambda+\nu)})^{\xi} d\xi. \quad (2.6)$$

Proof. By the statement of the Theorem 1, in the integral formula $X^{\rho,\eta}_{\nu,\lambda,\mu,\sigma}(z,\tau,t) = \int_0^t e^{2\tau\nu} F(\nu) d\nu$, use function $F(\nu)$ by Eqn. (2.5) and express the general class of polynomials $S_N^M(.)$ by Eqn. (1.2) and I^*- function by Eqns. (2.1) and (2.2) to find that

$$X^{\rho,\eta}_{\nu,\lambda,\mu,\sigma}(z,\tau,t) = \sum_{k=0}^{[\frac{N}{M}]} \frac{(-N)_{Mk}}{k!} A[N,k] y^k \frac{1}{2\pi\omega} \int_L \phi(\xi) z^{\xi} \int_0^t e^{2\tau\nu} \nu^{\rho+\mu k+\nu\xi} (t-\nu)^{\eta+\sigma k+\lambda\xi} dv d\xi. \tag{2.7}$$

Now in Eqn. (2.7), set $\frac{v}{t} = u, \ dv = tdu$ and use here, the Euler type integral formula for Kummer function ${}_1F_1[.]$, given by [11, p. 37]
${}_1F_1[a;b;z] = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 t^{a-1}(1-t)^{c-a-1} e^{zt} dt, \ \Re(c) > \Re(a) > 0$, to find that $X^{\rho,\eta}_{v,\lambda,\mu,\sigma}(z,\tau,t) = t^{\eta+\rho+1} \sum_{l=0}^{\infty} \sum_{k=0}^{[\frac{N}{M}]} \frac{(-N)_{Mk}}{k!} A[N,k] \mathbb{A}^{\rho,\eta}_{\nu,\lambda,\mu,\sigma,l,k}(z,\tau,t)$, where the sequence of functions

$$\mathbb{A}^{\rho,\eta}_{v,\lambda,\mu,\sigma,l,k}(z,\tau,t) = \frac{(2t\tau)^l}{l!} (yt^{(\sigma+\mu)})^k \frac{1}{2\pi\omega}$$
$$\times \int_L \frac{\prod_{j=1}^m \Gamma(b_j - \beta_j\xi) \prod_{j=1}^n \Gamma(1 - a_j + a_j\xi) \ \Gamma(1 + l + \rho + \mu k + \nu\xi)\Gamma(1 + \eta + \sigma k + \lambda\xi)}{\Gamma(2 + l + \eta + \rho + (\sigma+\mu)k + (\lambda+\nu)\xi) \sum_{i=1}^r [\prod_{j=1}^{q_i} \Gamma(1 - b_{ji} + \beta_{ji}\xi) \prod_{j=1}^{p_i} \Gamma(a_{ji} - \alpha_{ji}\xi)]}$$
$$(zt^{(\lambda+v)})^{\xi} d\xi, \quad (2.8)$$

which proves the Theorem 2.1. □

Theorem 2.2. *If $\mu, \sigma, v \in \mathbb{R}^+, \tau \geq 0, \lambda \geq 0,$ and $\rho, \eta \in \mathbb{C}$, such that $\Re(\rho) > -1, \Re(\eta) > -1$ then by an integral formula $X^{\rho,\eta}_{v,\lambda,\mu,\sigma}(z,\tau,t) \forall z \in \mathbb{C}$, given by (2.6), there holds a functional for $x > 0$, by the diffusion distribution $\mathcal{F}(x,1) = x^{-1} \int_0^{\infty} e^{-2t} X^{\rho,\eta}_{v,\lambda,\mu,\sigma}(z,1,t) dt$ and thus*

$$\mathcal{F}(x,1) = x^{-1} \sum_{k=0}^{[\frac{N}{M}]} \frac{(-N)_{Mk}}{k!} A[N,k] \Psi^{\rho,\eta}_{v,\lambda,\mu,\sigma}(z,1;k) (2^{-(\sigma+\mu)} y)^k \forall x > 0,$$

where, a sequence of functions exists as

$$\Psi^{\rho,\eta}_{\nu,\lambda,\mu,\sigma}(z,1;k)=$$
$$(2)^{-(\eta+\rho+2)}\sum_{l=0}^{\infty}\frac{1}{l!}I^{*m,n+2}_{p_i,q_i;r}\left[2^{-(\lambda+\nu)}z\left|\begin{matrix}(a_j,a_j)_{1,n}(-l-\rho-\mu k,\nu),(-\eta-\sigma k,\lambda); & (a_{ji},\alpha_{ji})_{1,p_i}\\ (b_j,\beta_j)_{1,m}; & (b_{ji},\beta_{ji})_{1,q_i}\end{matrix}\right.\right]$$

provided that $|arg(z)|\ <\frac{\pi}{2}A,\ A=\sum_{j=1}^{n}a_j+\sum_{j=1}^{m}\beta_j-\max_{1\leq i\leq r}[\sum_{j=1}^{p_i}a_{ji}+\sum_{j=1}^{q_i}\beta_{ji}]>0\ ,r\geq 1.$

$$\tag{2.9}$$

Proof. To prove Theorem 2.2, in reference of the Theorem 2.1, we define a function by integral transforms of the function (2.6), as $\mathcal{F}(x,\tau)=x^{-1}\int_0^{\infty}e^{-2\tau t}X^{\rho,\eta}_{v,\lambda,\mu,\sigma}(z,\tau,t)dt,\forall x>0$, to get

$$\mathcal{F}(x,\tau)=x^{-1}\int_0^{\infty}e^{-2\tau t}t^{\eta+\rho+1}\sum_{l=0}^{\infty}\frac{(2\tau t)^l}{l!}\sum_{k=0}^{[\frac{N}{M}]}\frac{(-N)_{Mk}}{k!}A[N,k](yt^{(\sigma+\mu)})^k$$
$$\times\frac{1}{2\pi\omega}\int_L\frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j\xi)\quad\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j\xi)\ \Gamma(1+l+\rho+\mu k+\nu\xi)\Gamma(1+\eta+\sigma k+\lambda\xi)}{\Gamma(2+l+\eta+\rho+(\sigma+\mu)k+(\lambda+v)\xi)\sum_{i=1}^{r}[\prod_{j=m+1}^{q_i}\Gamma(1-b_{ji}+\beta_{ji}\xi)\prod_{j=n+1}^{p_i}\Gamma(a_{ji}-a_{ji}\xi)]}$$
$$(zt^{(\lambda+v)})^{\xi}d\xi dt.\quad(2.10)$$

Then, in the Eqn. (2.10), change the order of integration with summation and integration, then apply the formula of the Gamma function [6] and by the Eqns. (2.1) - (2.3), we get

$$\mathcal{F}(x,\tau)=x^{-1}\sum_{k=0}^{[\frac{N}{M}]}\frac{(-N)_{Mk}}{k!}A[N,k]\Psi^{\rho,\eta}_{v,\lambda,\mu,\sigma}(z,\tau,k)(2^{-(\sigma+\mu)}y)^k,$$

where,

$$\Psi^{\rho,\eta}_{\nu,\lambda,\mu,\sigma}(z,\tau;k)=(2)^{-(\eta+\rho+2)}$$
$$\times\sum_{l=0}^{\infty}\frac{(\tau)^l}{l!}I^{*m,n+2}_{p_i,q_i;r}\left[2^{-(\lambda+\nu)}z\left|\begin{matrix}(a_j,a_j)_{1,n},(-l-\rho-\mu k,\nu),(-\eta-\sigma k,\lambda); & (a_{ji},\alpha_{ji})_{1,p_i}\\ (b_j,\beta_j)_{1,m}; & (b_{ji},\beta_{ji})_{1,q_i}\end{matrix}\right.\right].\tag{2.11}$$

Again, the Eqns. (1.9) and (1.10) show that the $\mathcal{F}(x,1)=x^{-1}\int_0^{\infty}e^{-2t}X^{\rho,\eta}_{\nu,\lambda,\mu,\sigma}(z,1,t)dt,x>0$, represent the diffusion distribution of Churchill's problem by Eqns. (1.7) - (1.8). So that by the Eqn. (2.11) for $\tau=1$, and the prescribed conditions $|arg(z)|<\frac{\pi}{2}A$, and $A=\sum_{j=1}^{n}\alpha_j+\sum_{j=1}^{m}\beta_j-\max_{1\leq i\leq r}[\sum_{j=1}^{p_i}\alpha_{ji}+\sum_{j=1}^{q_i}\beta_{ji}]>0$, and $\mu,\sigma,v\in\mathbb{R}^+,\tau\geq 0,\lambda\geq 0$, and $\rho,\eta\in\mathbb{C}$, such that $\Re(\rho)>-1,\Re(\eta)>-1$, we obtain the result (2.9). □

Corollary 2.1. *If* $r=1,\mu,\sigma,v\in\mathbb{R}^+,t\geq 0,\lambda\geq 0$, *and* $\rho,\eta\in\mathbb{C}$, *such that* $\Re(\rho)>-1,\Re(\eta)>-1$ *then by an integral formula* $X^{\rho,\eta}_{v,\lambda,\mu,\sigma}(z,1,t)\forall z\in\mathbb{C}$ *given by* (2.6), *there holds a functional for* $x>0$, *by the diffusion distribution* $\mathcal{F}(x,1)=x^{-1}\int_0^{\infty}e^{-2t}X^{\rho,\eta}_{v,\lambda,\mu,\sigma}(z,1,t)dt$ *and thus by Eqns.* (2.4) *and* (2.9)

$$\mathcal{F}(x,1)=x^{-1}\sum_{k=0}^{[\frac{N}{M}]}\frac{(-N)_{Mk}}{k!}A[N,k]\Psi^{*\rho,\eta}_{v,\lambda,\mu,\sigma}(z,1;k)(2^{-(\sigma+\mu)}y)^k\forall x>0,$$

where, a sequence of functions exists as

$$\Psi^{*\rho,\eta}_{\nu,\lambda,\mu,\sigma}(z,1;k) = (2)^{-(\eta+\rho+2)} \sum_{l=0}^{\infty} \frac{1}{l!}$$

$$\times H^{m,n+2}_{p_1+n+2,q_1+m}\left[2^{-(\lambda+\nu)}z \middle| \begin{array}{c} (a_j,\alpha_j)_{1,n},(a_{j1},\alpha_{j1})_{n+1,p_1},(a_{j1},\alpha_{j1})_{1,\ n},(-l-\rho-\mu k,\nu),(-\eta-\sigma k,\lambda) \\ (b_j,\beta_j)_{1,m},(b_{j1},\beta_{j1})_{m+1,\ q_1},(b_{j1},\beta_{j1})_{1,m} \end{array}\right]$$

$$\textit{provided that } |arg(z)| < \frac{\pi}{2}A^{*} \text{and } A^{*} = \sum_{j=1}^{n} a_j + \sum_{j=1}^{m} \beta_j - [\sum_{j=1}^{p_1} \alpha_{j1} + \sum_{j=1}^{q_1} \beta_{j1}] > 0. \quad (2.12)$$

3 Some particular results.

In this section, by Theorem 2.2, we derive some particular cases as:

In Eqn. (2.11), set $A[N,k] = 2^{2N}, y = (-1)^{M} h(2x)^{-M}, \sigma = 0, \mu = 1, \rho = N, v = 1, \lambda = 0, \Re(\eta) > -1$ and $\tau = 0$, and use Eqns. (1.3) and (1.4), for $x > 0$, we get

$$\mathcal{F}_1(x,0,z) = (2)^{-(\eta+1)} \sum_{k=0}^{[\frac{N}{M}]} \frac{N!\Gamma(\eta+1)}{k!(N-Mk)!}(2x)^{N-Mk-1}(2^{-1}h)^{k}$$

$$\times I^{*m,n+1}_{p_i,q_i;r}\left[2^{-1}z \middle| \begin{array}{cc} (a_j,\alpha_j)_{1,n},(-N-k,1); & (a_{ji},\alpha_{ji})_{1,p_i} \\ (b_j,\beta_j)_{1,m}; & (b_{ji},\beta_{ji})_{1,q_i} \end{array}\right], \quad (3.1)$$

provided that

$$|\arg(z)| < \frac{\pi}{2} A, A = \sum_{j=1}^{n} \alpha_j + \sum_{j=1}^{m} \beta_j - \max_{1\le i\le r}[\sum_{j=1}^{p_i} \alpha_{ji} + \sum_{j=1}^{q_i} \beta_{ji}] > 0 \ , r \ge 1.$$

Now, in Eqn. (2.11), set $A[N,k] = 2^{2N}, y = (-1)^{M+1}h(2x)^{-M}, \sigma = 0, \mu = -M, \rho = N, v = 1, \lambda = 0, \tau = 0$, and use Eqns. (1.3) and (1.4), for $x > 0$, we get

$$\mathcal{F}_2(x,0,z) = (2)^{-(\eta+1)} \sum_{k=0}^{[\frac{N}{M}]} \frac{N!\Gamma(\eta+1)}{k!(N-Mk)!}(2x)^{N-Mk-1}(-2^{M}h)^{k}$$

$$\times I^{*m,n+1}_{p_i,q_i;r}\left[2^{-1}z| \begin{array}{cc} (a_j,\alpha_j)_{1,n},(Mk-N,1); & (a_{ji},\alpha_{ji})_{1,p_i} \\ (b_j,\beta_j)_{1,m}; & (b_{ji},\beta_{ji})_{1,q_i} \end{array}\right], \quad (3.2)$$

provided that

$$|\arg(z)| < \frac{\pi}{2} A, A = \sum_{j=1}^{n} \alpha_j + \sum_{j=1}^{m} \beta_j - \max_{1\le i\le r}[\sum_{j=1}^{p_i} \alpha_{ji} + \sum_{j=1}^{q_i} \beta_{ji}] > 0 \ , r \ge 1.$$

Again, put $x = 1$, $h = 1$, in Eqn. (3.2), for $x > 0$, it terminates into the formula

$$\mathcal{F}_3(1,0,z) = (2)^{-(\eta+1)} \sum_{k=0}^{[\frac{N}{M}]} \frac{N!\Gamma(\eta+1)}{k!(N-Mk)!}(2)^{N-Mk\ -1}\ (-2^{M})^{k}$$

$$\times I^{*m,n+1}_{p_i,q_i;r}\left[2^{-1}z \middle| \begin{array}{cc} (a_j,\alpha_j)_{1,n},(Mk-N,1); & (a_{ji},\alpha_{ji})_{1,\ p_i} \\ (b_j,\beta_j)_{1,m}; & (b_{ji},\beta_{ji})_{1,\ q_i} \end{array}\right], \quad (3.3)$$

provided that

$$|arg(z)| < \frac{\pi}{2} A, A = \sum_{j=1}^{n} a_j + \sum_{j=1}^{m} \beta_j - \max_{1\le i\le r}[\sum_{j=1}^{p_i} \alpha_{ji} + \sum_{j=1}^{q_i} \beta_{ji}] > 0 \ , r \ge 1.$$

Then, putting $r = 1$ in the Eqn. (3.3), for $x = 1$, it gives us the diffusion

$$\mathcal{F}_4(1,0,z) = (2)^{-(\eta+1)} \sum_{k=0}^{[\frac{N}{M}]} \frac{N!\Gamma(\eta+1)}{k!(N-Mk)!} (2)^{N-Mk\ -1}\ (-2^M)^k$$
$$\times H^{m,n+1}_{p_1+n+1,q_1+m}\left[2^{-1}z\left|\begin{array}{c}(a_j,\alpha_j)_{1,n},(a_{j1},\alpha_{j1})_{n+1,\ p_1},(a_{j1},\alpha_{j1})_{1,\ n},(Mk-N,1)\\(b_j,\beta_j)_{1,m},(b_{j1},\beta_{j1})_{m+1,\ q_1},(b_{j1},\beta_{j1})_{1,\ m}\end{array}\right.\right] \tag{3.4}$$

provided that

$$|\arg(z)| < \frac{\pi}{2}A^* \text{and } A^* = \sum_{j=1}^{n}\alpha_j + \sum_{j=1}^{m}\beta_j - \left[\sum_{j=1}^{p_1}\alpha_{j1} + \sum_{j=1}^{q_1}\beta_{j1}\right] > 0\ .$$

Concluding Remarks

The H- function has applied in computational work of various problems in the work of many authors for example ([6], [7]), thus the Eqns. (2.12) and (3.1) - (3.4) may do great job in computation and estimations of the physical problems. The relations between theory of approximation and generating functions [5] may help in finding out other results.

Acknowlegement

We are very much thankful to the referee for his valuable comments to bring the paper in its present form.

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ISSN 0304-9892 (Print) ISSN 2455-7463 (Online)

Jñānābha

Statement of ownership and particulars about the journal

1.	Place of Publication	D.V. Postgraduate College Orai-285001, U.P., India
2.	Periodicity of Publication	Bi-annual
3.	Printer's Name Nationality Address	Creative Laser Graphics (Iqbal Ahmad) Indian IIT Gate, Kanpur
4.	Publisher's Name	Dr. R.C. Singh Chandel For Vijñāna Parishad of India
	Nationality Address	Indian D.V. Postgraduate College Orai-285001, U.P. India
5.	Editor's Name Nationality Address	Dr. R.C. Singh Chandel Indian D.V. Postgraduate College Orai-285001, U.P. India
6.	Name and Address of the individuals who own the journal and partners of share holders holding more than one percent of the total capital	**Vijñāna Parishad of India** Address: D.V. Postgraduate College Orai-285001, U.P. India

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