Jñānābha, Vol. 49(2) (2019), 1-5

FIXED POINT THEOREMS FOR GENERALIZED NON-EXPANSIVE MAPPINGS

By
N. Chandra, Mahesh C. Joshi and Narendra K. Singh
Department of Mathematics
D. S. B. Campus, Kumaun University, Nainital, India-263002
Email: cnaveen329@gmail.com, mcjoshi69@gmail.com & nsijwali@gmail.com
(*Received : June* 06, 2018 *; Revised: October* 20, 2019)

Abstract

In this paper, we obtain a fixed point theorem for the mappings satisfying non-expansive type conditions.
2010 Mathematics Subject Classifications: 47H10, 54H25.
Keywords and phrases: Generalized non-expansive mappings and Fixed points.

1 Introduction and Preliminaries

A mapping T be on a metric space (X, d) is said to be non-expansive if $d(Tx, Ty) \leq d(x, y)$ for all $x, y \in X$, while mapping T is called contraction if there exists a non-negative real number $k < 1$ such that $d(Tx, Ty) \leq k\, d(x, y)$ for all $x, y \in X$. It is well known that every contraction on a complete metric space has a unique fixed point (Banach Contraction Principle). But, this is not true for non-expansive mappings on complete metric space, e.g., $Tx = x + 1$ in usual metric space $X = [0, \infty)$. It is well known that for the existence of fixed points for non-expansive mappings one needs the convex structure of the space X. Many researchers tried to explore the existence of fixed points for non-expansive type mappings and have done a good work in this direction for Banach spaces as well as metric spaces (see, [1, 4, 6, 2, 3, 5, 7, 8, 9] and references therein).

Bogin [1] proved the following result.

Theorem 1.1. *Let (X, d) be a complete metric space and $T : X \to X$ a mapping satisfying*

$$d(Tx, Ty) \leq a\, d(x, y) + b\, [d(x, Tx) + d(y, Ty)] + c\, [d(x, Ty) + d(y, Tx)], \tag{1.1}$$

where $a \geq 0$, $b > 0$, $c > 0$ and $a + 2b + 2c = 1$. Then T has a unique fixed point.

Greguš [10] considered a class of self mappings on X which satisfy (1.1) with $c = 0$. In fact, he proved the following theorem.

Theorem 1.2. *Let C be a non-empty closed convex subset of a Banach space X and T : $C \to C$ a mapping satisfying*

$$||Tx - Ty|| \leq a\, ||x - y|| + b\, [||x - Tx|| + ||y - Ty||], \tag{1.2}$$

for all $x, y \in C$, where $a > 0$, $b > 0$ and $a + 2b = 1$. Then T has a unique fixed point.

Greguš's result has inspired many authors for further investigations in this direction, (see Abdelijawad et al.[11], Ćirić [3, 5, 7, 8, 6], Jungck[13] and references therein). In 1993, Ćirić [4] proved the the following result which is a proper generalization of the above theorems.

Theorem 1.3. *Let* (X,d) *be a complete metric space and* $T : X \to X$ *a mapping satisfying*

$$d(Tx,Ty) \leq a \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{1}{2}[d(x,Ty)+d(y,Tx)]\right\} + b \max\{d(x,Tx), d(y,Ty)\} + c\,[d(x,Ty)+d(y,Tx)], \quad (1.3)$$

where $a \geq 0$, $b > 0$, $c > 0$ *and* $a+b+2c=1$. *Then* T *has a unique fixed point and* T *is continuous at that fixed point.*

In 2008, Suzuki [14] introduced a weaker notion of contractions and proved the following theorem.

Theorem 1.4 ([14])**.** *Let* (X,d) *be a complete metric space,* T *be a mapping on* X. *Define a non-increasing function* $\theta : [0,1) \to (\frac{1}{2}, 1]$ *by*

$$\theta(r) = \begin{cases} 1 & \text{if } 0 \leq r \leq \frac{\sqrt{5}-1}{2}, \\ \frac{1-r}{r^2} & \text{if } \frac{\sqrt{5}-1}{2} \leq r \leq \frac{1}{\sqrt{2}}, \\ \frac{1}{1+r} & \text{if } \frac{1}{\sqrt{2}} \leq r < 1. \end{cases}$$

Assume that there exists $r \in [0,1)$ *such that*

$$\theta(r)d(x,Tx) \leq d(x,y) \text{ implies } d(Tx,Ty) \leq r\,d(x,y), \quad (1.4)$$

for each $x,y \in X$. *Then there exists a unique fixed point* z *of* T. *Moreover,* $\lim_{n\to\infty} T^n x = z$ *for all* $x \in X$.

Since $\lim_{r\to 1-0} \theta(r) = \frac{1}{2}$, it is very natural to consider the following condition.

Definition 1.1 ([15])**.** *Let* T *be a mapping on a subset* A *of a Banach Space* X. *Then* T *is said to satisfy condition (C) if*

$$\frac{1}{2}||x - Tx|| \leq ||x-y|| \quad \text{implies} \quad ||Tx - Ty|| \leq ||x-y||, \quad (1.5)$$

for all $x,y \in A$.

The condition (C) is weaker than non-expansive (see, Proposition 1 of [15]).

Using above condition (C), many results have been obtained by researchers for the existence of fixed points, see [12, 16] and references therein. Recently, Popescu [16] generalized the result due to Bogin [1] for non-expansive mappings in the setting of condition (C). In fact, he proved the following.

Theorem 1.5. *Let* (X,d) *be a nonempty complete metric space and* $T : X \to X$ *a mapping satisfying*

$$\frac{1}{2}d(x,Tx) \leq d(x,y) \text{ implies}$$
$$d(Tx,Ty) \leq a\,d(x,y) + b\,[d(x,Tx)+d(y,Ty)] + c\,[d(x,Ty)+d(y,Tx)],$$

where $a \geq 0, b > 0, c > 0$ *and* $a+2b+2c=1$. *Then* T *has a unique fixed point.*

Let (X,d) be a metric space and $T : X \to X$ a self-mapping of X. For $x,y \in X$, we use the following notation:

$$M(Tx,Ty) = a \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{1}{2}[d(x,Ty)+d(y,Tx)]\right\} + b \max\{d(x,Tx), d(y,Ty)\} + c\,[d(x,Ty)+d(y,Tx)].$$

Now, we investigate a new generalized class of self-mappings T on metric space X which satisfy the following generalized non-expansive type condition:

$$\frac{1}{2}d(x,Tx) \leq d(x,y) \text{ implies } d(Tx,Ty) \leq M(Tx,Ty), \quad (1.6)$$

for all $x,y \in X$, where a,b and c are non-negative real numbers such that

$$a+b+2c=1. \quad (1.7)$$

2 Main Results

Theorem 2.1. *Let (X, d) be a complete metric space and $T : X \to X$ satisfying (1.6), where $a \geq 0, b > 0, c > 0$ and such that (1.7) holds. Then T has a unique fixed point, i.e., there is a unique $z \in X$ such that $Tz = z$.*

Proof. Let $x \in X$ be an arbitrary point. Since $\frac{1}{2}d(x, Tx) \leq d(x, Tx)$, by condition (1.6) and (1.7), we obtain

$$\begin{aligned} d(Tx, T^2x) &\leq M(Tx, T^2x) \\ &\leq (a+b)\max\{d(x, Tx), d(Tx, T^2x)\} + c\,d(x, T^2x) \\ &\leq (a+b+2c)\max\{d(x, Tx), d(Tx, T^2x)\} \\ &= \max\{d(x, Tx), d(Tx, T^2x)\}. \end{aligned}$$

Hence, $d(Tx, T^2x) \leq d(x, Tx) \quad \forall x \in X$.

Thus, if we define a sequence $\{x_n\}$ in X such that $x_n = T^n x$, where $x \in X$ is arbitrary. Then

$$d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n) \quad \forall n \geq 1. \tag{2.1}$$

That is, the sequence $\{d(x_n, x_{n+1})\}_{n=0}^{\infty}$ is a non-increasing, where $x_0 = T^0 x = x$.

Now, we show that there is a non-negative real number $m < 2$ such that $d(Tx, T^3x) \leq m\,d(x, Tx)$. If $d(x, Tx) \leq d(x, T^2x)$, then $\frac{1}{2}d(x, Tx) \leq d(x, T^2x)$ and using (2.1), we have

$$\begin{aligned} d(Tx, T^3x) &\leq M(Tx, T^3x) \\ &\leq a\,[d(x, Tx) + d(Tx, T^2x)] + b\,d(x, Tx) \\ &\quad +c\,[d(x, Tx) + d(Tx, T^2x) + d(Tx, T^3x)] \\ &\leq (2a+b+2c)d(x, Tx) + c\,d(Tx, T^3x) \\ \Rightarrow \qquad d(Tx, T^3x) &\leq \frac{2a+b+2c}{1-c}\,d(x, Tx) = \frac{1+a}{1-c}\,d(x, Tx). \end{aligned}$$

Setting $m_1 = \frac{1+a}{1-c} < 2$, we get $d(Tx, T^3x) \leq m_1\,d(x, Tx)$.

Now suppose that $d(x, Tx) > d(x, T^2x)$. Since $\frac{1}{2}d(x, Tx) \leq d(x, Tx)$,

$$d(Tx, T^2x) \leq M(Tx, T^2x) < (a+b+c)\,d(x, Tx).$$

Then,

$$\begin{aligned} d(Tx, T^3x) &\leq d(Tx, T^2x) + d(T^2x, T^3x) \leq 2\,d(Tx, T^2x) \\ &< (1+a+b)\,d(x, Tx). \end{aligned}$$

Setting $m_2 = (1+a+b) < 2$, we get $d(Tx, T^3x) < m_2\,d(x, Tx)$.

Thus taking $m = \max\{m_1, m_2\}$, we obtain $0 < m < 2$ and

$$d(Tx, T^3x) \leq m\,d(x, Tx) \quad \forall\, x \in X. \tag{2.2}$$

Since $\frac{1}{2}d(Tx, T^2x) \leq d(Tx, T^2x)$, using the condition (2.2) we have

$$\begin{aligned} d(T^2x, T^3x) &\leq M(T^2x, T^3x) \\ &\leq (a+b)\,d(Tx, T^2x) + c\,d(Tx, T^3x) \\ &\leq (a+b+mc)\,d(x, Tx). \end{aligned}$$

Taking $k = (a+b+mc) < 1$, we get

$$d(T^2x, T^3x) \leq k\,d(x, Tx) \quad \forall x \in X.$$

Hence, by induction we have

$$d(x_n, x_{n+1}) \leq k^{[n/2]} d(x, Tx) \quad \forall\, n \geq 0, \tag{2.3}$$

where $[n/2]$ means the greatest integer not exceeding $n/2$. Since $0 < k < 1$, condition (2.3) implies that $\{x_n\}$ is a Cauchy sequence and by completeness of X, there exist $z \in X$ such that $x_n \to z$ as $n \to \infty$.

Next, we will prove that $z \in X$ is a fixed point of T. Assuming that there is some n such that

$$\frac{1}{2} d(x_n, x_{n+1}) > d(z, x_n) \text{ and } \frac{1}{2} d(x_{n+1}, x_{n+2}) > d(z, x_{n+1}).$$

Then,

$$d(x_n, x_{n+1}) \leq d(z, x_n) + d(z, x_{n+1}) < \frac{1}{2}[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})] \leq d(x_n, x_{n+1}),$$

which is a contradiction, so for all $n \geq 0$, we get

$$\text{either } \frac{1}{2} d(x_n, x_{n+1}) \leq d(z, x_n) \text{ or } \frac{1}{2} d(x_{n+1}, x_{n+2}) \leq d(z, x_{n+1}).$$

Thus, there exist a subsequence $\{n_j\}$ of $\{n\}$ such that $\frac{1}{2} d(x_{n_j}, x_{n_j+1}) \leq d(z, x_{n_j})$ for all $j \geq 0$. Then, we have

$$d(Tz, x_{n_j+1}) \leq M(Tz, Tx_{n_j}).$$

Now, taking $n \to \infty$, we get

$$d(z, Tz) \leq (a + b + c)\, d(z, Tz) \quad \Rightarrow \quad d(z, Tz) = 0 \quad \Rightarrow \quad Tz = z.$$

For uniqueness of fixed point, let z' be another fixed point of T. Then, $\frac{1}{2} d(z, Tz) = 0 \leq d(z, z')$ and hence

$$d(Tz, Tz') \leq M(Tz, Tz') \quad \Rightarrow \quad d(z, z') \leq (a + 2c)\, d(z, z') \quad \Rightarrow \quad z = z'.$$

□

Remark 2.1. *It is to be noted that, Theorem* 2.1 *is a weaker version of the result due to Ćirić [4] (Theorem* 2.1*, page 148) for non-expansive mappings.*

From our main result, following corollaries which are generalizations of results due to Popescu [16] (Theorem 2.1, page 3913) and Suzuki [15] (Theorem 4, page 1094) for non-expansive mappings in metric sense respectively.

Corollary 2.1. *Let (X, d) be a complete metric space and $T : X \to X$ a mapping satisfying $\frac{1}{2} d(x, Tx) \leq d(x, y)$ implies*

$$d(Tx, Ty) \leq a\, d(x, y) + b\,[d(x, Tx) + d(y, Ty)] + c\,[d(x, Ty) + d(y, Tx)],$$

where $a \geq 0$ and $b > 0, c > 0$ such that $a + b + 2c = 1$. Then T has a unique fixed point.

Corollary 2.2. *Let (X, d) be a complete metric space and $T : X \to X$ a mapping satisfying $\frac{1}{2} d(x, Tx) \leq d(x, y)$ implies*

$$d(Tx, Ty) \leq a\, d(x, y) + b \max\{d(x, Tx), d(y, Ty)\} + c\,[d(x, Ty) + d(y, Tx)],$$

where $a \geq 0$ and $b > 0, c > 0$ such that $a + b + 2c = 1$. Then T has a unique fixed point.

Taking $a = 0$ in Theorem 2.1, we get the following corollary.

Corollary 2.3. *Let (X, d) be a complete metric space and $T : X \to X$ a mapping satisfying $\frac{1}{2} d(x, Tx) \leq d(x, y)$ implies*

$$d(Tx, Ty) \leq b \max\{d(x, Tx), d(y, Ty)\} + c\,[d(x, Ty) + d(y, Tx)],$$

where $b > 0, c > 0$ such that $b + 2c = 1$. Then T has a unique fixed point.

Acknowledgment
First author thanks to UGC, New Delhi for support under RGNF. We are also thankful to the referee for his valuable suggestions to improve the paper in its present form.

References

[1] J. Bogin, A generalization of a fixed point theorem of Goebel, Kirk and Shimi, *Canad. Math. Bull.*, **19** (1976), 7-12.

[2] S. K. Chatterjea, Applications of an extension of a theorem of Disher on common fixed point, *Pure Math. Manuscript*, **6** (1987), 35-38.

[3] Lj. B. Ćirić, On some discontinuous fixed point mappings in convex metric spaces, *Czechoslovak Math. J.*, **43(118)** (1993), 319-326.

[4] Lj. B. Ćirić, On some non-expansive type mappings and fixed points, *Indian J. Pure Appl. Math*, **24** (1993), 145-149.

[5] Lj. B. Ćirić, Non-expansive type mappings and fixed point theorems in convex metric spaces, *Rend. Accad. Naz. Sci. XI Mem. Appl.*, **5(XIX)** (1995), 263-271.

[6] Lj. B. Ćirić, On a common fixed point theorem of a Greguš type, *Publ. Inst. Math.*, **49** (1991), 117-178.

[7] Lj. B. Ćirić, A new class of non-expansive type mappings and fixed points, *Czechoslovak Math. J.*, **49(124)** (1999), 891-899.

[8] Lj. B. Ćirić, On generalization of Greguš fixed point theorem, *Czechoslovak Math. J.*, **50** (2000), 449-458.

[9] M. Chandra, S. N. Mishra, S. L. Singh and B. E. Rhoades, Coincidences and fixed points of nonexpansive type multivalued and single valued maps, *Indian J. Pure appl. Math.*, **26(5)** (1995), 393-401.

[10] M. Greguš, A fixed point theorem in Banach spaces, *Boll. Unione Mat. Ital. Sez. A*, **5(17)** (1980), 193-198.

[11] T. Abdeljawad, E. Karapinar, A common fixed point theorem of a Greguš type on convex cone metric spaces, *J. Comput. Anal. Appl.*, **13(4)** (2011), 609-621.

[12] E. Karapinar, K. Tas, Generalized (C)-condition and related fixed point theorems, *Comput. Anal. Appl.*, **61(11)** (2011), 3370-3380.

[13] G. Jungck, On a fixed point theorem of Fisher and Sessa, *Int. J. Math. Math. Sci.*, **13** (1990), 497-500.

[14] T. Suzuki, A generalized Banach contraction principle that characterizes metric completeness, *Proc. Amer. Math. Soc.*, **136** (2008), 1861-1869.

[15] T. Suzuki, Fixed point theorem and convergence theorem for some generalized non-expansive mappings, *J. Math. Anal. Appl.*, **340** (2008), 1088-1095.

[16] O. Popescu, Two generalizations of some fixed point theorems, *Comput. Math. Appl.*, **62** (2011), 3912-3919.