Jñānābha, Vol. 49(2) (2019), 113-119

CHURCHILL'S DIFFUSION AND EULER TYPE INTEGRAL INVOLVING AN I^*-FUNCTION

By

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(*Received : August* 02, 2019 *; Revised: November* 18, 2019)

Abstract

In this paper, we obtain an Euler type integral involving an I^*- function with products of a general class of polynomials and exponential function. Then, we use it to express the Churchill's diffusion in terms of a series consisting of the coefficients of that general class of polynomials with a sequence of I^*- functions. Certain particular cases are also discussed.

2010 Mathematics Subject Classification: 33C45, 33C60.

Keywords and phrases: An I^* - function, Churchill's diffusion, a general class of polynomials, Gould and Hopper polynomials.

1 Introduction

Srivastava [1] has applied a general class of polynomials of several variables for evaluation of multilinear generating functions in the Kohnauser sets of bi - orthogonal polynomials. Again, several authors have used this class of polynomials in in representation of the series solution of matter diffusion and wave propagation problems for example Kumar [2] has analyzed the Churchill's diffusion [1] in terms of the series consisting of various parameters along with the coefficients of known various polynomials with the help of a general class of polynomials [10] defined as

$$S_{N_1,\ldots,N_m}^{M_1,\ldots,M_m}(y_1,\ldots,y_m)=\sum_{k_1=0}^{[\frac{N_1}{M_1}]}\cdots\sum_{k_m=0}^{[\frac{N_m}{M_m}]}\frac{(-N_1)_{M_1k_1}}{k_1!}\cdots\frac{(-N_m)_{M_mk_m}}{k_m!}$$
$$\times A[N_1,k_1;...;N_m,k_m]{y_1}^{k_1}...{y_m}^{k_m}, \tag{1.1}$$

where, $N_1,...,N_m;M_1,...,M_m$ are arbitrary positive integers and the coefficients $A[N_1, k_1;...;N_m,k_m]$ are arbitrary parameters real or complex independent of $y_1,\ldots,y_m$.

Clearly, on putting $m=1$ in Eqn. (1.1), a generalized class of polynomials of one variable is found (see in [9]) as

$$S_N^M(y)=\sum_{k=0}^{\left[\frac{N}{M}\right]}\frac{(-N)_{Mk}}{k!}A\left[N,k\right]y^k, \tag{1.2}$$

where, N,M are arbitrary positive integers and the coefficients $A\left[N,k\right]$ are arbitrary parameters real or complex independent of y.

On application of a generalized class of polynomials (1.2), recently, Kumar and Pathan [5] have derived a generalized Weiertrass Approximation Theorem and obtained various generating functions to make extensions in the approximation theory.

In concerned with the evaluation of some polynomials by Eqn. (1.2), here replace $A[N,k]=2^N, y=(-1)^M h(2x)^{-M}$ to find a relation

$$S_N^M(y)=(x)^{-N}g_N^M(2x,h), \tag{1.3}$$

where, $g_N^M(2x,h)$ are the generalization of Hermite polynomials due to Gould and Hopper [5], defined by

$$g_N^M(2x,h)=\sum_{k=0}^{\left[\frac{N}{M}\right]}\frac{N!}{k!\,(N-Mk)!}h^k(2x)^{N-Mk}. \tag{1.4}$$

Again in Eqn. (1.4) put $h=-1$, it terminates into a relation between Gould - Hopper polynomials and the generalized Hermite polynomials [11]

$$g_N^M(2x,-1)=H_{N,M}(x), \tag{1.5}$$

where, the generalization of Hermite polynomials is given by

$$H_{N,M}(x)=\sum_{k=0}^{\left[\frac{N}{M}\right]}\frac{(-1)^k N!(2x)^{N-Mk}}{k!\,(N-Mk)!}. \tag{1.6}$$

Again to extend our work of (1.1) - (1.6) in applied theory of diffusion problems, we consider the Churchill's diffusion in following form of differential equation as (see [1, p, 131], [4])

$$x\frac{\partial U}{\partial x}+\frac{\partial U}{\partial t}+U=xF(t),\ x>0,\ t>0, U=(x,t), \tag{1.7}$$

with boundary conditions

$$U(x,0)=U(0,t)=0. \tag{1.8}$$

The solution of the problem (1.7) - (1.8) is given by [1, p. 131]

$$U(x,t)=xe^{-2t}\int_0^t e^{2v}F(v)\,dv. \tag{1.9}$$

Here in Eqn. (1.9), $F(t)$ is a known function.

Further to measure our results, by Eqn. (1.9) and due to Kumar [4], a functional by the diffusion distribution for the domain $x>0,\ t>0$, is written as

$$\mathcal{F}(x)=x^{-1}\int_0^\infty U(x,t)dt=\int_0^\infty\left\{\int_0^t e^{-2v}F(t-v)\,dv\right\}dt\ \forall x>0,\ t>0. \tag{1.10}$$

In our present investigation, we make an application of above formulae for straightforward evaluation of the relations between diffusion and the series involving the I^*- functions. In further extensions of above researches, we introduce new parameters $\mu,\sigma,\nu\in\mathbb{R}^+,\ \lambda\geq 0$ and $\rho,\eta,\in\mathbb{C}$ such that $\Re(\rho)>-1,,\Re(\eta)>-1$ and then obtain an integral function in the complex plane for $z\in\mathbb{C},\tau\geq 0,\ X_{\nu,\lambda,\mu,\sigma}^{\rho,\eta}(z,\tau,t)=\int_0^t e^{2\tau\nu}F(\nu)d\nu$ in which $F(s), s>0$, are known function. Again, we analyze the distribution in the complex plane, on considering $F(s), s>0$ involving an I^*- function with products of a general class of polynomials and exponential function and determine a determinant, then use it to express a series representation of the Churchill's diffusion consisting of the coefficients of that general class of polynomials with a sequence of I^*- functions. Certain particular cases are also discussed.

2 Euler Type Integrals involving an I^*- function with products of a general class of polynomials and exponential function.

In this section, we present an I^*- function such that when, $r \geq 2$, an I^*- function is different from Fox's H - function (see [2], [7], [9]) type contour integral in the complex plane, not symmetric in regard of parameters but when $r = 1$, it is identical to that H- function. Then obtain an Euler type integral involving I^*- function with products of a general class of polynomials (1.2) and exponential function to find the diffusion distribution.

The I^*- function is introduced as a contour integral in complex plane given by

$$I^{*m,n}_{\ p_i,q_i;r}\left[z\left|\begin{array}{l}(a_j,\alpha_j)_{1,n}\ :\ (a_{ji},\alpha_{ji})_{1,p_i}\\(b_j,\beta_j)_{1,m}:\ (b_{ji},\beta_{ji})_{1,q_i}\end{array}\right.\right]=\frac{1}{2\pi\omega}\int_L\phi(\xi)z^{\xi}d\xi,\omega=\sqrt{(-1)},\qquad(2.1)$$

in which

$$\phi(\xi)=\frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j\xi)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j\xi)}{\sum_{i=1}^{r}[\prod_{j=1}^{q_i}\Gamma(1-b_{ji}+\beta_{ji}\xi)\prod_{j=1}^{p_i}\Gamma(a_{ji}-\alpha_{ji}\xi)]}.\qquad(2.2)$$

Here, for finite value of r; all $p_i, q_i (i = 1, 2, ..., r), m$ and n are the integers, satisfying the inequalities; $0 \leq n \leq p$, $p_i \geq n \geq 1$ $(i = 1, 2, \ldots, r), 1 \leq m \leq q, q_i \geq m \geq 1 (i = 1, 2, \ldots, r), a_j (j = 1, \ldots, n), \beta_j (j = 1, \ldots, m), a_{ji} (1 \leq j \leq p_i, (i = 1, 2, \ldots, r), \beta_{ji} (1 \leq j \leq q_i, (i = 1, 2, \ldots, r))$ are real and positive and $a_j (j = 1, \ldots, n) b_j (j = 1, \ldots, m), a_{ji} (1 \leq j \leq p_i, (i = 1, 2, \ldots, r)), b_{ji} (1 \leq j \leq q_i, (i = 1, 2, \ldots, r))$ are complex numbers such that $a_k(b_h + v) \neq \beta_h(a_k - 1 - l)$ for $l, v = 0, 1, 2 \ldots; h = 1, 2, \ldots, m$.

L is contour running form $s - i\infty$ to $s + i\infty$, where s is real in the complex ξ-plane such that the poles. $\xi = \frac{(a_j-1-l)}{\alpha_j}, j = 1, 2, \ldots, n; l = 0, 1, 2, \cdots, \xi = \frac{(b_j+v)}{\beta_j}, j = 1, 2 \cdots, m; v = 0, 1, 2, \cdots$ lie to the left and right hand sides of the contour L respectively, the empty product is represented as 1.

The I^*- function converges absolutely in ξ-plane if $|arg(z)| < \frac{\pi}{2}A$, where,

$$A=\sum_{j=1}^{n}a_j+\sum_{j=1}^{m}\beta_j-\max_{1\leq i\leq r}[\sum_{j=1}^{p_i}a_{ji}+\sum_{j=1}^{q_i}\beta_{ji}]\ >0\ .\qquad(2.3)$$

Property 1. The I^*- function is most probably identical to $I-$ function [8].

Property 2. For $r = 1$, the I^*- function defined by (2.1) - (2.3) may become Fox's H - function ([2], [7], [9]) as

$$I^{*m,n}_{\ p_1,q_1;1}\left[z\left|\begin{array}{l}(a_j,\alpha_j)_{1,n}\ :\ (a_{j1},\alpha_{j1})_{1,\ p_1}\\(b_j,\beta_j)_{1,m}:\ (b_{j1},\beta_{j1})_{1,\ q_1}\end{array}\right.\right]$$

$$=H^{m,n}_{p_1+n,q_1+m}\left[z\left|\begin{array}{l}(a_j,\alpha_j)_{1,n},(a_{j1},\alpha_{j1})_{n+1,\ p_1},(a_{j1},\alpha_{j1})_{1,\ n}\\(b_j,\beta_j)_{1,m},(b_{j1},\beta_{j1})_{m+1,\ q_1},(b_{j1},\beta_{j1})_{1,\ m}\end{array}\right.\right],$$

where, $|\arg(z)| < \frac{\pi}{2}A^*$, and $A^* = \sum_{j=1}^{n}\alpha_j + \sum_{j=1}^{m}\beta_j - \left[\sum_{j=1}^{p_1}\alpha_{j1} + \sum_{j=1}^{q_1}\beta_{j1}\right] > 0$.

(2.4)

Now by the diffusion theory, given in Eqns. (1.9) and (1.10), where $F(s)$ is a known function, thus, we consider $F(s)$ for $0 < s < t$, $t > 0$, and the parameters are taken under the conditions $\mu, \sigma, \nu \in \mathbb{R}^+, \lambda \geq 0$, and $\rho, \eta \in \mathbb{C}$, such that $\Re(\rho) > -1, \Re(\eta) > -1$ in the form

$$F(s)=s^{\rho}(t-s)^{\eta}S_N^M(ys^{\mu}(t-s)^{\sigma})I^{*m,n}_{\ p_i,q_i;r}[zs^{\nu}(t-s)^{\lambda}\left|\begin{array}{l}(a_j,a_j)_{1,n}\ :\ (a_{ji},a_{ji})_{1,\ p_i}\\(b_j,\beta_j)_{1,m}:\ (b_{ji},\beta_{ji})_{1,\ q_i}\end{array}\right.\Bigg],\qquad(2.5)$$

and thus we establish following theorems:

Theorem 2.1. *If $\mu, \sigma, \nu \in \mathbb{R}^+, \tau \geq 0, \lambda \geq 0,$ and $\rho, \eta \in \mathbb{C},$ such that $\Re(\rho) > -1, \Re(\eta) > -1$ then by the function $F(s)$ for $0 < s < t, t > 0,$ given in Eqn. (2.5), an integral formula $X^{\rho,\eta}_{\nu,\lambda,\mu,\sigma}(z,\tau,t) \forall z \in \mathbb{C},$ given by $X^{\rho,\eta}_{\nu,\lambda,\mu,\sigma}(z,\tau,t) = \int_0^t e^{2\tau\nu} F(\nu) d\nu$ exists, and it may be expressed in the form of the series involving the coefficients of a general class of polynomials with product of a determinant $\mathbb{A}^{\rho,\eta}_{\nu,\lambda,\mu,\sigma}(z,\tau,t),$ as*

$$X^{\rho,\eta}_{\nu,\lambda,\mu,\sigma,l,k}(z,\tau,t) = t^{\eta+\rho+1} \sum_{l=0}^{\infty} \sum_{k=0}^{[\frac{N}{M}]} \frac{(-N)_{Mk}}{k!} A[N,k] \mathbb{A}^{\rho,\eta}_{\nu,\lambda,\mu,\sigma,l,k}(z,\tau,t),$$

where the sequence of functions

$$\mathbb{A}^{\rho,\eta}_{\nu,\lambda,\mu,\sigma,l,k}(z,\tau,t) = \frac{(2\tau t)^l}{l!} (y t^{(\sigma+\mu)})^k \frac{1}{2\pi\omega}$$
$$\times \int_L \frac{\prod_{j=1}^m \Gamma(b_j - \beta_j\xi) \quad \prod_{j=1}^n \Gamma(1 - a_j + a_j\xi) \ \Gamma(1 + l + \rho + \mu k + \nu\xi) \Gamma(1 + \eta + \sigma k + \lambda\xi)}{\Gamma(2 + l + \eta + \rho + (\sigma+\mu)k + (\lambda+\nu)\xi) \sum_{i=1}^r [\prod_{j=1}^{q_i} \Gamma(1 - b_{ji} + \beta_{ji}\xi) \prod_{j=1}^{p_i} \Gamma(a_{ji} - \alpha_{ji}\xi) \ ]}$$
$$(z t^{(\lambda+\nu)})^{\xi} d\xi. \quad (2.6)$$

Proof. By the statement of the Theorem 1, in the integral formula $X^{\rho,\eta}_{\nu,\lambda,\mu,\sigma}(z,\tau,t) = \int_0^t e^{2\tau\nu} F(\nu) d\nu$, use function $F(\nu)$ by Eqn. (2.5) and express the general class of polynomials $S_N^M(.)$ by Eqn. (1.2) and I^*- function by Eqns. (2.1) and (2.2) to find that

$$X^{\rho,\eta}_{\nu,\lambda,\mu,\sigma}(z,\tau,t) = \sum_{k=0}^{[\frac{N}{M}]} \frac{(-N)_{Mk}}{k!} A[N,k] y^k \frac{1}{2\pi\omega} \int_L \phi(\xi) z^{\xi} \int_0^t e^{2\tau\nu} \nu^{\rho+\mu k+\nu\xi} (t-\nu)^{\eta+\sigma k+\lambda\xi} dv d\xi. \tag{2.7}$$

Now in Eqn. (2.7), set $\frac{v}{t} = u, \ dv = t du$ and use here, the Euler type integral formula for Kummer function ${}_1F_1[.]$, given by [11, p. 37]
${}_1F_1[a;b;z] = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 t^{a-1} (1-t)^{c-a-1} e^{zt} dt, \ \Re(c) > \Re(a) > 0,$ to find that $X^{\rho,\eta}_{v,\lambda,\mu,\sigma}(z,\tau,t) = t^{\eta+\rho+1} \sum_{l=0}^{\infty} \sum_{k=0}^{[\frac{N}{M}]} \frac{(-N)_{Mk}}{k!} A[N,k] \mathbb{A}^{\rho,\eta}_{\nu,\lambda,\mu,\sigma,l,k}(z,\tau,t),$ where the sequence of functions

$$\mathbb{A}^{\rho,\eta}_{v,\lambda,\mu,\sigma,l,k}(z,\tau,t) = \frac{(2t\tau)^l}{l!} (y t^{(\sigma+\mu)})^k \frac{1}{2\pi\omega}$$
$$\times \int_L \frac{\prod_{j=1}^m \Gamma(b_j - \beta_j\xi) \prod_{j=1}^n \Gamma(1 - a_j + a_j\xi) \ \Gamma(1 + l + \rho + \mu k + \nu\xi) \Gamma(1 + \eta + \sigma k + \lambda\xi)}{\Gamma(2 + l + \eta + \rho + (\sigma+\mu)k + (\lambda+\nu)\xi) \sum_{i=1}^r [\prod_{j=1}^{q_i} \Gamma(1 - b_{ji} + \beta_{ji}\xi) \prod_{j=1}^{p_i} \Gamma(a_{ji} - \alpha_{ji}\xi)]}$$
$$(z t^{(\lambda+v)})^{\xi} d\xi, \quad (2.8)$$

which proves the Theorem 2.1. ∎

Theorem 2.2. *If $\mu, \sigma, v \in \mathbb{R}^+, \tau \geq 0, \lambda \geq 0,$ and $\rho, \eta \in \mathbb{C},$ such that $\Re(\rho) > -1, \Re(\eta) > -1$ then by an integral formula $X^{\rho,\eta}_{v,\lambda,\mu,\sigma}(z,\tau,t) \forall z \in \mathbb{C},$ given by (2.6), there holds a functional for $x > 0,$ by the diffusion distribution $\mathcal{F}(x,1) = x^{-1} \int_0^{\infty} e^{-2t} X^{\rho,\eta}_{v,\lambda,\mu,\sigma}(z,1,t) dt$ and thus*

$$\mathcal{F}(x,1) = x^{-1} \sum_{k=0}^{[\frac{N}{M}]} \frac{(-N)_{Mk}}{k!} A[N,k] \Psi^{\rho,\eta}_{v,\lambda,\mu,\sigma}(z,1;k) (2^{-(\sigma+\mu)} y)^k \forall x > 0,$$

where, a sequence of functions exists as

$$\Psi^{\rho,\eta}_{\nu,\lambda,\mu,\sigma}(z,1;k) =$$
$$(2)^{-(\eta+\rho+2)}\sum_{l=0}^{\infty}\frac{1}{l!}I^{*m,n+2}_{p_i,q_i;r}\left[2^{-(\lambda+\nu)}z\left|\begin{array}{ccc}(a_j,a_j)_{1,n}(-l-\rho-\mu k,\nu),(-\eta-\sigma k,\lambda); & (a_{ji},\alpha_{ji})_{1,p_i}\\ (b_j,\beta_j)_{1,m}; & (b_{ji},\beta_{ji})_{1,q_i}\end{array}\right.\right]$$
$$\textit{provided that } |arg(z)| \ < \frac{\pi}{2}A,\ A=\sum_{j=1}^{n}a_j+\sum_{j=1}^{m}\beta_j-\max_{1\le i\le r}[\sum_{j=1}^{p_i}a_{ji}+\sum_{j=1}^{q_i}\beta_{ji}]>0\ ,r\ge 1. \tag{2.9}$$

Proof. To prove Theorem 2.2, in reference of the Theorem 2.1, we define a function by integral transforms of the function (2.6), as $\mathcal{F}(x,\tau)=x^{-1}\int_0^\infty e^{-2\tau t}X^{\rho,\eta}_{v,\lambda,\mu,\sigma}(z,\tau,t)dt,\forall x>0$, to get

$$\mathcal{F}(x,\tau)=x^{-1}\int_0^\infty e^{-2\tau t}t^{\eta+\rho+1}\sum_{l=0}^{\infty}\frac{(2\tau t)^l}{l!}\sum_{k=0}^{[\frac{N}{M}]}\frac{(-N)_{Mk}}{k!}A[N,k](yt^{(\sigma+\mu)})^k$$
$$\times\frac{1}{2\pi\omega}\int_L\frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j\xi)\ \ \prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j\xi)\ \Gamma(1+l+\rho+\mu k+\nu\xi)\Gamma(1+\eta+\sigma k+\lambda\xi)}{\Gamma(2+l+\eta+\rho+(\sigma+\mu)k+(\lambda+v)\xi)\sum_{i=1}^{r}[\prod_{j=m+1}^{q_i}\Gamma(1-b_{ji}+\beta_{ji}\xi)\prod_{j=n+1}^{p_i}\Gamma(a_{ji}-a_{ji}\xi)]}$$
$$(zt^{(\lambda+v)})^{\xi}d\xi dt. \tag{2.10}$$

Then, in the Eqn. (2.10), change the order of integration with summation and integration, then apply the formula of the Gamma function [6] and by the Eqns. (2.1) - (2.3), we get

$$\mathcal{F}(x,\tau)=x^{-1}\sum_{k=0}^{[\frac{N}{M}]}\frac{(-N)_{Mk}}{k!}A[N,k]\Psi^{\rho,\eta}_{v,\lambda,\mu,\sigma}(z,\tau,k)(2^{-(\sigma+\mu)}y)^k,$$

where,

$$\Psi^{\rho,\eta}_{\nu,\lambda,\mu,\sigma}(z,\tau;k)=(2)^{-(\eta+\rho+2)}$$
$$\times\sum_{l=0}^{\infty}\frac{(\tau)^l}{l!}I^{*m,n+2}_{p_i,q_i;r}\left[2^{-(\lambda+\nu)}z\left|\begin{array}{ccc}(a_j,a_j)_{1,n},(-l-\rho-\mu k,\nu),(-\eta-\sigma k,\lambda); & (a_{ji},\alpha_{ji})_{1,p_i}\\ (b_j,\beta_j)_{1,m}; & (b_{ji},\beta_{ji})_{1,q_i}\end{array}\right.\right]. \tag{2.11}$$

Again, the Eqns. (1.9) and (1.10) show that the $\mathcal{F}(x,1)=x^{-1}\int_0^\infty e^{-2t}X^{\rho,\eta}_{\nu,\lambda,\mu,\sigma}(z,1,t)dt, x>0$, represent the diffusion distribution of Churchill's problem by Eqns. (1.7) - (1.8). So that by the Eqn. (2.11) for $\tau=1$, and the prescribed conditions $|arg(z)|<\frac{\pi}{2}A$, and $A=\sum_{j=1}^{n}\alpha_j+\sum_{j=1}^{m}\beta_j-\max_{1\le i\le r}[\sum_{j=1}^{p_i}\alpha_{ji}+\sum_{j=1}^{q_i}\beta_{ji}]>0$, and $\mu,\sigma,v\in\mathbb{R}^+,\tau\ge 0,\lambda\ge 0$, and $\rho,\eta\in\mathbb{C}$, such that $\Re(\rho)>-1,\Re(\eta)>-1$, we obtain the result (2.9). □

Corollary 2.1. *If* $r=1,\mu,\sigma,v\in\mathbb{R}^+,t\ge 0,\lambda\ge 0$, *and* $\rho,\eta\in\mathbb{C}$, *such that* $\Re(\rho)>-1,\Re(\eta)>-1$ *then by an integral formula* $X^{\rho,\eta}_{v,\lambda,\mu,\sigma}(z,1,t)\forall z\in\mathbb{C}$ *given by* (2.6), *there holds a functional for* $x>0$, *by the diffusion distribution* $\mathcal{F}(x,1)=x^{-1}\int_0^\infty e^{-2t}X^{\rho,\eta}_{v,\lambda,\mu,\sigma}(z,1,t)dt$ *and thus by Eqns.* (2.4) *and* (2.9)

$$\mathcal{F}(x,1)=x^{-1}\sum_{k=0}^{[\frac{N}{M}]}\frac{(-N)_{Mk}}{k!}A[N,k]\Psi^{*\rho,\eta}_{v,\lambda,\mu,\sigma}(z,1;k)(2^{-(\sigma+\mu)}y)^k\forall x>0,$$

where, a sequence of functions exists as

$$\Psi^{*\rho,\eta}_{\nu,\lambda,\mu,\sigma}(z,1;k) = (2)^{-(\eta+\rho+2)}\sum_{l=0}^{\infty}\frac{1}{l!}$$

$$\times H^{m,n+2}_{p_1+n+2,q_1+m}\left[2^{-(\lambda+\nu)}z\left|\begin{array}{c}(a_j,\alpha_j)_{1,n},(a_{j1},\alpha_{j1})_{n+1,p_1},(a_{j1},\alpha_{j1})_{1,\ n},(-l-\rho-\mu k,\nu),(-\eta-\sigma k,\lambda)\\(b_j,\beta_j)_{1,m},(b_{j1},\beta_{j1})_{m+1,\ q_1},(b_{j1},\beta_{j1})_{1,m}\end{array}\right.\right]$$

provided that $|arg(z)| < \frac{\pi}{2}A^*$ and $A^* = \sum_{j=1}^{n}a_j+\sum_{j=1}^{m}\beta_j - [\sum_{j=1}^{p_1}\alpha_{j1}+\sum_{j=1}^{q_1}\beta_{j1}] > 0.$ (2.12)

3 Some particular results.

In this section, by Theorem 2.2, we derive some particular cases as:

In Eqn. (2.11), set $A[N,k]=2^{2N}, y=(-1)^{M}h(2x)^{-M}, \sigma=0, \mu=1, \rho=N, v=1, \lambda=0, \Re(\eta)>-1$ and $\tau=0$, and use Eqns. (1.3) and (1.4), for $x>0$, we get

$$\mathcal{F}_1(x,0,z) = (2)^{-(\eta+1)}\sum_{k=0}^{[\frac{N}{M}]}\frac{N!\Gamma(\eta+1)}{k!(N-Mk)!}(2x)^{N-Mk-1}(2^{-1}h)^k$$

$$\times I^{*m,n+1}_{p_i,q_i;r}\left[2^{-1}z\left|\begin{array}{cc}(a_j,\alpha_j)_{1,n},(-N-k,1); & (a_{ji},\alpha_{ji})_{1,p_i}\\(b_j,\beta_j)_{1,m}; & (b_{ji},\beta_{ji})_{1,q_i}\end{array}\right.\right], \tag{3.1}$$

provided that

$$|\arg(z)| < \frac{\pi}{2}A, A = \sum_{j=1}^{n}\alpha_j+\sum_{j=1}^{m}\beta_j - \max_{1\le i\le r}[\sum_{j=1}^{p_i}\alpha_{ji}+\sum_{j=1}^{q_i}\beta_{ji}] > 0\ , r\ge 1.$$

Now, in Eqn. (2.11), set $A[N,k]=2^{2N}, y=(-1)^{M+1}h(2x)^{-M}, \sigma=0, \mu=-M, \rho=N, v=1, \lambda=0, \tau=0$, and use Eqns. (1.3) and (1.4), for $x>0$, we get

$$\mathcal{F}_2(x,0,z) = (2)^{-(\eta+1)}\sum_{k=0}^{[\frac{N}{M}]}\frac{N!\Gamma(\eta+1)}{k!(N-Mk)!}(2x)^{N-Mk-1}(-2^{M}h)^k$$

$$\times I^{*m,n+1}_{p_i,q_i;r}\left[2^{-1}z|\begin{array}{cc}(a_j,\alpha_j)_{1,n},(Mk-N,1); & (a_{ji},\alpha_{ji})_{1,p_i}\\(b_j,\beta_j)_{1,m}; & (b_{ji},\beta_{ji})_{1,q_i}\end{array}\right], \tag{3.2}$$

provided that

$$|\arg(z)| < \frac{\pi}{2}A, A = \sum_{j=1}^{n}\alpha_j+\sum_{j=1}^{m}\beta_j - \max_{1\le i\le r}[\sum_{j=1}^{p_i}\alpha_{ji}+\sum_{j=1}^{q_i}\beta_{ji}] > 0\ , r\ge 1.$$

Again, put $x=1,\ h=1$, in Eqn. (3.2), for $x>0$, it terminates into the formula

$$\mathcal{F}_3(1,0,z) = (2)^{-(\eta+1)}\sum_{k=0}^{[\frac{N}{M}]}\frac{N!\Gamma(\eta+1)}{k!(N-Mk)!}(2)^{N-Mk\ -1}\ (-2^{M})^k$$

$$\times I^{*m,n+1}_{p_i,q_i;r}\left[2^{-1}z\left|\begin{array}{cc}(a_j,\alpha_j)_{1,n},(Mk-N,1); & (a_{ji},\alpha_{ji})_{1,\ p_i}\\(b_j,\beta_j)_{1,m}; & (b_{ji},\beta_{ji})_{1,\ q_i}\end{array}\right.\right], \tag{3.3}$$

provided that

$$|arg(z)| < \frac{\pi}{2}A, A = \sum_{j=1}^{n}a_j+\sum_{j=1}^{m}\beta_j - \max_{1\le i\le r}[\sum_{j=1}^{p_i}\alpha_{ji}+\sum_{j=1}^{q_i}\beta_{ji}] > 0\ , r\ge 1.$$

Then, putting $r = 1$ in the Eqn. (3.3), for $x = 1$, it gives us the diffusion

$$\mathcal{F}_4(1,0,z) = (2)^{-(\eta+1)} \sum_{k=0}^{[\frac{N}{M}]} \frac{N!\Gamma(\eta+1)}{k!(N-Mk)!} (2)^{N-Mk\ -1} \left(-2^M\right)^k$$
$$\times H^{m,n+1}_{p_1+n+1,q_1+m}\left[2^{-1}z \middle| \begin{matrix} (a_j,\alpha_j)_{1,n}, (a_{j1},\alpha_{j1})_{n+1,\ p_1}, (a_{j1},\alpha_{j1})_{1,\ n}, (Mk-N,1) \\ (b_j,\beta_j)_{1,m}, (b_{j1},\beta_{j1})_{m+1,\ q_1}, (b_{j1},\beta_{j1})_{1,\ m} \end{matrix}\right] \tag{3.4}$$

provided that

$$|\arg(z)| < \frac{\pi}{2}A^* \text{and } A^* = \sum_{j=1}^{n}\alpha_j + \sum_{j=1}^{m}\beta_j - \left[\sum_{j=1}^{p_1}\alpha_{j1} + \sum_{j=1}^{q_1}\beta_{j1}\right] > 0\ .$$

Concluding Remarks

The H- function has applied in computational work of various problems in the work of many authors for example ([6], [7]), thus the Eqns. (2.12) and (3.1) - (3.4) may do great job in computation and estimations of the physical problems. The relations between theory of approximation and generating functions [5] may help in finding out other results.

Acknowlegement

We are very much thankful to the referee for his valuable comments to bring the paper in its present form.

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