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GENERALIZED HERMITE POLYNOMIAL FAMILIES

By

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Abstract

In recent papers, new sets of Sheffer and Brenke polynomials based on higher order Bell numbers, and several integer sequences related to them have been studied. In this article we find families of Sheffer polynomials, including as a particular case generalized Hermite polynomial families, deriving their recurrence relations and differential equations, which are sometimes of fractional type.

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1 Introduction

In recent articles [8, 18], new sets of Sheffer [21] and Brenke [7] polynomials based on higher order Bell numbers [15, 16, 17, 18] have been studied. Furthermore, several integer sequences associated with the considered polynomials sets both of exponential and logarithmic type have been introduced [8].

In this article we find families of Sheffer polynomials, including as a particular case generalized Hermite polynomial families.

The classical Hermite polynomials have been previously considered by Laplace and subsequently studied by Chebyshev and Hermite. V.A. Steklov proved their density in the weighted space $L^2_{w(x)}(-\infty, +\infty)$, with $w(x) = e^{-x^2}$.

It is worth to recall that these polynomials have been introduced by the beginning in the multi-dimensional case, and were deeply analyzed by Appell and Kampé de Fériet in classical book [1] and widely studied and applied by G. Dattoli and his collaborators (see e.g. [9, 11]).

Many extensions of the Hermite polynomials have been proposed in literature by several authors, so that it seems quite impossible to list all of them. We just remember the most important contributions, limiting ourselves to the one-dimensional case [12, 14, 6, 20, 23].

2 Sheffer and Appell polynomial families

We start recalling the particular meaning of the term *set* in the framework of polynomial theory.

Definition 2.1. *A polynomial family* $\{P_n(x)\}_{n\geq 0}$ *is called a polynomial set*

$$\textit{iff}\ \forall n,\ \deg P_n = n.$$

In what follows, we are dealing with polynomial families that, in several cases, do not satisfy the above condition.

The Appell polynomials $\{A_n(x)\}$ *are introduced [21] by means of the exponential generating function [24] of the type:*

$$\mathcal{A}(t)\exp(xt) = \sum_{n=0}^{\infty} A_n(x)\,\frac{t^n}{n!}\ , \tag{2.1}$$

where

$$\mathcal{A}(t) = \sum_{n=0}^{\infty} a_n\,\frac{t^n}{n!}\ , \quad (a_0 \neq 0)\,. \tag{2.2}$$

The Sheffer polynomials [21] $\{s_n(x)\}$ *generalize the Appell family, by considering the generating function [24]:*

$$A(t)\exp(xH(t)) = \sum_{n=0}^{\infty} s_n(x)\,\frac{t^n}{n!}\ , \tag{2.3}$$

where

$$A(t) = \sum_{n=0}^{\infty} a_n\,\frac{t^n}{n!}\ , \quad (a_0 \neq 0)\,, \qquad H(t) = \sum_{n=0}^{\infty} h_n\,\frac{t^n}{n!}\ , \quad (h_0 = 0)\,. \tag{2.4}$$

Obviously, when $H(t) \equiv t$*, the Sheffer polynomials give back the Appell sets.*

According to a different characterization (see [19, p. 18]), the same Sheffer sequence $\{s_n(x)\}$ *can be defined by means of the pair* $(g(t), f(t))$*, where* $g(t)$ *is an invertible series and* $f(t)$ *is a delta series:*

$$g(t) = \sum_{n=0}^{\infty} g_n\,\frac{t^n}{n!}\ , \quad (g_0 \neq 0)\,, \qquad f(t) = \sum_{n=0}^{\infty} f_n\,\frac{t^n}{n!}\ , \quad (f_0 = 0,\ f_1 \neq 0)\,. \tag{2.5}$$

Denoting by $f^{-1}(t)$ *the compositional inverse of* $f(t)$ *(i.e. such that* $f\left(f^{-1}(t)\right) = f^{-1}\left(f(t)\right) = t$*), the exponential generating function of the sequence* $\{s_n(x)\}$ *is given by*

$$\frac{1}{g[f^{-1}(t)]}\ \exp\left(xf^{-1}(t)\right) = \sum_{n=0}^{\infty} s_n(x)\,\frac{t^n}{n!}\ , \tag{2.6}$$

so that

$$A(t) = \frac{1}{g[f^{-1}(t)]}\,, \qquad H(t) = f^{-1}(t)\,. \tag{2.7}$$

When $g(t) \equiv 1$*, the Sheffer sequence corresponding to the pair* $(1, f(t))$ *is called the associated Sheffer sequence* $\{\sigma_n(x)\}$ *for* $f(t)$*, and its exponential generating function is given by*

$$\exp\left(xf^{-1}(t)\right) = \sum_{n=0}^{\infty} \sigma_n(x)\,\frac{t^n}{n!}\ . \tag{2.8}$$

A list of known Appel and Sheffer polynomial sequences and their associated ones can be found in [5].

Remark 2.1. *It is well known [7, 13] that there is a natural link between the function* $H(t)$ *and the degree of polynomials* $s_n(x)$ *in expansion (2.3). Namely,*

$$\deg s_n = n \ \text{ iff, } \ \textit{in equation (2.4),} \ \ h_1 \neq 0.$$

Actually, in what follows, if $H(t)$ is a polynomial of degree m*, we have found that* $\deg s_n \leq [\frac{n}{m}]$*, where* $[\cdot]$ *denotes the integral part.*

In general, we are dealing with a Sheffer polynomial set iff *the condition* $h_1 \neq 0$ *is satisfied.*

3 An extensions of the Hermite polynomials

We put, in this Section:

$$A(t) = \exp[\alpha\,(t+\gamma)^m]\,, \qquad H(t) = \beta\, t^n\,, \tag{3.1}$$

where α, β are real numbers and m, n positive integer numbers.

We consider the generalized Hermite polynomial families $\mathcal{H}_k(\alpha, \beta, \gamma, m, n; x)$, defined by the generating function

$$G(t,x) = \exp[\alpha\,(t+\gamma)^m + x\,\beta\, t^n] = \sum_{k=0}^{\infty} \mathcal{H}_k(\alpha, \beta, \gamma, m, n; x)\,\frac{t^k}{k!}\,. \tag{3.2}$$

Remark 3.1. *Note that the fourth parameter β is unessential, since it only produces a change of scale on the x-axis. Actually it has been introduced for recovering exactly the Hermite polynomial set by putting $\alpha = -1$, $\beta = 2$, $\gamma = 0$, $m = 2$, $n = 1$.*

The above generating function (3.2) includes, as particular cases, several other families of classical polynomials generalizing the Hermite set.

We find:

- *the Gould Hopper polynomials ($G(t,x) = \exp[xt + ht^r]$), assuming $\alpha = h$, $\beta = 1$, $\gamma = 0$, $m = r$, $n = 1$;*
- *the L.R. Bragg polynomials ($G(t,x) = \exp[pxt - t^p]$), assuming $\alpha = -1$, $\beta = p$, $\gamma = 0$, $m = p$, $n = 1$;*
- *the Lahiri polynomials ($G(t,x) = \exp[\nu xt - t^m]$), assuming $\alpha = -1$, $\beta = \nu$, $\gamma = 0$, $m = m$, $n = 1$;*
- *the two variables higher order Hermite-Kampé de Fériet polynomials, ($G(t,x) = \exp[xt + yt^m]$), assuming $\alpha = y$, $\beta = 1$, $\gamma = 0$, $m = m$, $n = 1$.*

4 Properties of the polynomials $\mathcal{H}_k(\alpha, \beta, \gamma, m, n; x)$

4.1 A differential identity

Theorem 4.1. *For any $k \geq 0$, the polynomials $\mathcal{H}_k(\alpha, \beta, \gamma, m, n; x)$ satisfy the differential identity:*

$$\mathcal{H}'_k(\alpha, \beta, \gamma, m, n; x) = \beta\,(k)_n\,\mathcal{H}_{k-n}(\alpha, \beta, \gamma, m, n; x)\,, \tag{4.1}$$

where we have used the falling factorial symbol $(k)_n := k(k-1)\cdots(k-n+1)$.

Proof. Differentiating $G(t,x)$ with respect to x, we have:

$$\frac{\partial G}{\partial x} = \beta\, t^n\, G(t,x) = \beta \sum_{k=0}^{\infty} \mathcal{H}_k(\alpha, \beta, \gamma, m, n; x)\,\frac{t^{k+n}}{k!}\,, \tag{4.2}$$

and therefore:

$$\frac{\partial G}{\partial x} = \sum_{k=0}^{\infty} \mathcal{H}'_k(\alpha, \beta, \gamma, m, n; x)\,\frac{t^k}{k!} = \beta \sum_{k=0}^{\infty} [k(k-1)\cdots(k-n+1)]\,\mathcal{H}_{k-n}(\alpha, \beta, \gamma, m, n; x)\,\frac{t^k}{k!}\,,$$

so that the differential identity (4.1) is proved. □

4.2 Recurrence relation

Theorem 4.2. *For any $k \geq 0$, the polynomials $\mathcal{H}_k(\alpha,\beta,\gamma,m,n;x)$ satisfy the recurrence relation:*

$$\begin{aligned}\mathcal{H}_{k+1}(\alpha,\beta,\gamma,m,n;x) = \alpha m \sum_{h=0}^{m-1} \binom{m-1}{h} \gamma^{m-h-1} (k)_h \, \mathcal{H}_{k-h}(\alpha,\beta,\gamma,m,n;x) + \\ + \beta n \, (k)_{n-1} \, x \, \mathcal{H}_{k-n+1}(\alpha,\beta,\gamma,m,n;x)\,,\end{aligned} \tag{4.3}$$

where $(k)_h := k(k-1)\cdots(k-h+1)$ and $(k)_{n-1} := k(k-1)\cdots(k-n+2)$.

Proof. Differentiating $G(t,x)$ with respect to t, and putting, for shortness: $\mathcal{H}_k(x) := \mathcal{H}_k(\alpha,\beta,\gamma,m,n;x)$, we find:

$$\frac{\partial G}{\partial t} = \left[\alpha m \,(t+\gamma)^{m-1} + \beta n \, x \, t^{n-1}\right] G(t,x) = \sum_{k=0}^{\infty} \mathcal{H}_{k+1}(x) \, \frac{t^k}{k!}\,, \tag{4.4}$$

and therefore

$$\sum_{k=0}^{\infty} \mathcal{H}_{k+1}(x) \, \frac{t^k}{k!} = \alpha m \sum_{k=0}^{\infty} \mathcal{H}_k(x) \, \frac{t^k}{k!} \sum_{h=0}^{m-1} \binom{m-1}{h} \gamma^{m-h-1} t^h + \beta n \sum_{k=0}^{\infty} x \, \mathcal{H}_k(x) \, \frac{t^{k+n-1}}{k!}\,,$$

i.e.

$$\sum_{k=0}^{\infty} \mathcal{H}_{k+1}(x) \, \frac{t^k}{k!} = \alpha m \sum_{k=0}^{\infty} \mathcal{H}_k(x) \sum_{h=0}^{m-1} \binom{m-1}{h} \gamma^{m-h-1} \, \frac{t^{k+h}}{k!} + \beta n \sum_{k=0}^{\infty} x \, \mathcal{H}_k(x) \, \frac{t^{k+n-1}}{k!}\,,$$

$$\begin{aligned}\sum_{k=0}^{\infty} \mathcal{H}_{k+1}(x) \, \frac{t^k}{k!} = \alpha m \sum_{k=0}^{\infty} \left[\sum_{h=0}^{m-1} \binom{m-1}{h} \gamma^{m-h-1} (k)_h\right] \mathcal{H}_{k-h}(x) \frac{t^k}{k!} + \\ + \beta n \sum_{k=0}^{\infty} x \, (k)_{n-1} \mathcal{H}_{k-n+1}(x) \, \frac{t^k}{k!}\,,\end{aligned}$$

so that the recurrence (4.3) follows. □

4.3 Shift operators

We recall that a polynomial set $\{p_n(x)\}$ is called quasi-monomial if and only if there exist two operators $\hat{P}$ and $\hat{M}$ such that

$$\hat{P}\,(p_k(x)) = k p_{k-1}(x)\,, \qquad \hat{M}\,(p_k(x)) = p_{k+1}(x)\,, \quad (k = 1, 2, \dots). \tag{4.5}$$

$\hat{P}$ is called the *derivative* operator and $\hat{M}$ the *multiplication* operator, as they act in the same way of classical operators on monomials.

This definition, tracing back to a paper by J.F. Steffensen [25], has been recently improved by G. Dattoli [9] and widely used in several applications (see e.g. [10, 11]).

Y. Ben Cheikh [3] proved that every polynomial set is quasi-monomial under the action of suitable derivative and multiplication operators. In particular, in the same article (Corollary 3.2), the following result is proved

Theorem 4.3. *Let $(p_k(x))$ denote a Boas-Buck polynomial set, i.e. a set defined by the generating function*

$$A(t)\psi(xH(t)) = \sum_{k=0}^{\infty} p_k(x) \, \frac{t^k}{k!}\,, \tag{4.6}$$

where

$$A(t) = \sum_{k=0}^{\infty} \tilde{a}_k t^k \ , \quad (\tilde{a}_0 \neq 0)\,, \qquad \psi(t) = \sum_{k=0}^{\infty} \tilde{\gamma}_k t^k \ , \quad (\tilde{\gamma}_k \neq 0 \quad \forall k)\,, \tag{4.7}$$

with $\psi(t)$ *not a polynomial, and lastly*

$$H(t) = \sum_{k=0}^{\infty} \tilde{h}_k\, t^{k+1} \ , \quad (\tilde{h}_0 \neq 0)\,. \tag{4.8}$$

Let $\sigma \in \Lambda^{(-)}$ *the lowering operator defined by*

$$\sigma(1) = 0\,, \qquad \sigma(x^k) = \frac{\tilde{\gamma}_{k-1}}{\tilde{\gamma}_k}\ x^{k-1}\,, \quad (k = 1, 2, \dots). \tag{4.9}$$

Put

$$\sigma^{-1}(x^k) = \frac{\tilde{\gamma}_{k+1}}{\tilde{\gamma}_k}\ x^{k+1}, \quad (k = 0, 1, 2, \dots). \tag{4.10}$$

Denoting, as before, by $f(t)$ *the compositional inverse of* $H(t)$*, the Boas-Buck polynomial set* $\{p_k(x)\}$ *is quasi-monomial under the action of the operators*

$$\hat{P} = f(\sigma)\,, \qquad \hat{M} = \frac{A'[f(\sigma)]}{A[f(\sigma)]} + x D_x H'[f(\sigma)]\sigma^{-1}\,, \tag{4.11}$$

where prime *denotes the ordinary derivatives with respect to* t*.*

Remark 4.1. *It is worth to note that the above mentioned result (Corollary 3.2 in [3]), given for polynomial sets, never uses in proof the condition* $h_1 \neq 0$*. Therefore, it can be applied even to polynomials defined by Sheffer generating functions (2.3), i.e. to Sheffer polynomial families.*

Note that in our case we are dealing with a Sheffer polynomial family, so that, since we have $\psi(t) = e^t$*, the operator* σ *defined by equation (4.9) simply reduces to the derivative operator* D_x*. Furthermore, we have:*

$$A'(t) = A(t)\,\alpha\, m\,(t+\gamma)^{m-1}\,, \qquad \frac{A'(t)}{A(t)} = \alpha\, m\,(t+\gamma)^{m-1}\,,$$

$$H'(t) = \beta\, n\, t^{n-1}\,, \qquad H^{-1}(t) = f(t) = \beta^{-1/n}\, t^{1/n}\,,$$

so that we have the theorem

Theorem 4.4. *The generalized Hermite polynomial family* $\{\mathcal{H}_k(\alpha,\beta,\gamma,m,n;x)\}$ *is quasi-monomial under the action of the operators*

$$\hat{P} = \beta^{-1/n}\, D_x^{1/n}\,, \qquad \hat{M} = \alpha\, m\, \left(\beta^{-1/n} D_x^{1/n} + \gamma\right)^{m-1} + n\, x\, \beta^{1/n} D_x^{(n-1)/n}\,. \tag{4.12}$$

so that

$$\hat{M}\hat{P} = \alpha\, m\, \beta^{-1/n}\left(\beta^{-1/n} D_x^{1/n} + \gamma\right)^{m-1} D_x^{1/n} + n\, x\, D_x\,.$$

4.4 Differential equation

According to the results of monomiality principle [9, 11], the quasi-monomial polynomials $\{p_n(x)\}$ satisfy the differential equation

$$\hat{M}\hat{P}\, p_k(x) = k\, p_k(x)\,. \tag{4.13}$$

In the present case, we have

Theorem 4.5. *Putting, for shortness:* $\mathcal{H}_k(x) := \mathcal{H}_k(\alpha,\beta,\gamma,m,n;x)$*, the generalized Hermite polynomials* $\{\mathcal{H}_k(x)\}$ *satisfy the differential equation*

$$\left[\alpha\, m\, \beta^{-1/n}\left(\beta^{-1/n} D_x^{1/n} + \gamma\right)^{m-1} D_x^{1/n} + n\, x\, D_x\right] \mathcal{H}_k(x) = k\, \mathcal{H}_k(x)\,, \tag{4.14}$$

i.e.

$$\left[\alpha\, m \sum_{h=0}^{m-1} \binom{m-1}{h} \beta^{-(h+1)/n} \gamma^{m-h-1} D_x^{(h+1)/n} + n\, x\, D_x\right] \mathcal{H}_k(x) = k\, \mathcal{H}_k(x)\,. \tag{4.15}$$

5 The particular case when $\gamma = 0$

Since all the classical extension of the Hermite polynomials recalled in Remark 2 are obtained by letting $\gamma = 0$, it is convenient to rewrite the preceding formulas in this particular case.

We put, in this Section:

$$A(t) = \exp(\alpha\, t^m)\,, \qquad H(t) = \beta t^n\,, \tag{5.1}$$

where α, β are real numbers and m, n positive integer numbers.
Then, putting $\mathcal{H}_k(\alpha,\beta,m,n;x) := \mathcal{H}_k(\alpha,\beta,0,m,n;x)$, we consider the generalized Hermite polynomial families $\mathcal{H}_k(\alpha,\beta,m,n;x)$, defined by the generating function

$$G(t,x) = \exp[\alpha\, t^m + x\,\beta t^n] = \sum_{k=0}^{\infty} \mathcal{H}_k(\alpha,\beta,m,n;x)\,\frac{t^k}{k!}\,. \tag{5.2}$$

By using the same techniques of the preceding Section, we find the following results.

5.1 A differential identity

Theorem 5.1. *For any $k \geq 0$, the polynomials $\mathcal{H}_k(\alpha,\beta,m,n;x)$ satisfy the differential identity:*

$$\mathcal{H}'_k(\alpha,\beta,m,n;x) = \beta\,(k)_n\,\mathcal{H}_{k-n}(\alpha,\beta,m,n;x)\,, \tag{5.3}$$

where we have used the falling factorial symbol $(k)_n := k(k-1)\cdots(k-n+1)$.

5.2 Recurrence relation

Theorem 5.2. *For any $k \geq 0$, the polynomials $\mathcal{H}_k(\alpha,\beta,m,n;x)$ satisfy the recurrence relation:*

$$\mathcal{H}_{k+1}(\alpha,\beta,\gamma,m,n;x) = \alpha m\,(k)_{m-1}\,\mathcal{H}_{k-m+1}(\alpha,\beta,m,n;x) + \beta n\,(k)_{n-1}\,x\,\mathcal{H}_{k-n+1}(\alpha,\beta,m,n;x)\,, \tag{5.4}$$

where $(k)_{m-1} := k(k-1)\cdots(k-m+2)$ and $(k)_{n-1} := k(k-1)\cdots(k-n+2)$.

5.3 Shift operators

Theorem 5.3. *The generalized Hermite polynomial family $\{\mathcal{H}_k(\alpha,\beta,m,n;x)\}$ is quasi-monomial under the action of the operators*

$$\hat{P} = \beta^{-1/n}\,D_x^{1/n}\,, \qquad \hat{M} = \alpha\,m\,\beta^{(1-m)/n}D_x^{(m-1)/n} + n\,x\,\beta^{1/n}D_x^{(n-1)/n}\,. \tag{5.5}$$

so that

$$\hat{M}\hat{P} = \alpha\,m\,\beta^{-m/n}D_x^{m/n} + n\,x\,D_x\,.$$

5.4 Differential equation

Theorem 5.4. *The generalized Hermite polynomials $\{\mathcal{H}_k(\alpha,\beta,m,n;x)\}$ satisfy the differential equation*

$$\left[\alpha\,m\,\beta^{-m/n}D_x^{m/n} + n\,x\,D_x\right]\,\mathcal{H}_k(\alpha,\beta,m,n;x) = k\,\mathcal{H}_k(\alpha,\beta,m,n;x)\,. \tag{5.6}$$

Remark 5.1. *Note that the equation (5.6), when $n = 1$, is an ordinary differential equation of order m. The corresponding polynomials constitute a standard polynomial set (for any index k we find a polynomial of degree $n = k$).*

When $n = 1$ and $m = 2$ *we have a three term recurrence relation so that the polynomials are orthogonal with respect to a suitable measure.*

When $n = 1$ and $m > 2$ *we find again a the three term recurrence relation but the indexes are delayed.*

When $n > 1$ the differential equation is of fractional order, so that the polynomial degree n does not correspond to the index k.

For example, if $n = 3$ and $m = 1$ the polynomials are arranged in groups containing each three polynomials, and the polynomial degree n increases every three steps, according to the rule $k = 3n + j$, $(j = 0, 1, 2)$, i.e. $k - j \equiv n$, (mod 3).

If $(n, m) = 1$ the degree of polynomials appear in different order, depending on the values of n and m.

This can be easily checked by using the computer algebra program Alpha©.

5.5 First few values, in particular cases, when $\alpha = \beta = 1$ and $\gamma = 0$

As an example, assuming $\alpha = \beta = 1$, and by using a simplified notation, we consider here the polynomials $\mathcal{H}_0(1, 1, 3, 1; x) =: \mathcal{H}_0(3, 1; x)$ (a Gould-Hopper case):

$$\begin{aligned}
\mathcal{H}_0(3, 1; x) &= 1 \\
\mathcal{H}_1(3, 1; x) &= x \\
\mathcal{H}_2(3, 1; x) &= x^2 \\
\mathcal{H}_3(3, 1; x) &= x^3 + 6 \\
\mathcal{H}_4(3, 1; x) &= x^4 + 24x \\
\mathcal{H}_5(3, 1; x) &= x^5 + 60x^2 \\
\mathcal{H}_6(3, 1; x) &= x^6 + 120x^3 + 360 \\
\mathcal{H}_7(3, 1; x) &= x^7 + 210x^4 + 2520x \\
\mathcal{H}_8(3, 1; x) &= x^8 + 336x^5 + 10080x^2
\end{aligned}$$

and the polynomials $\mathcal{H}_0(1, 1, 1, 3; x) =: \mathcal{H}_0(1, 3; x)$:

$$\begin{aligned}
\mathcal{H}_0(1, 3; x) &= 1 \\
\mathcal{H}_1(1, 3; x) &= 1 \\
\mathcal{H}_2(1, 3; x) &= 1 \\
\mathcal{H}_3(1, 3; x) &= 6x + 1 \\
\mathcal{H}_4(1, 3; x) &= 24x + 1 \\
\mathcal{H}_5(1, 3; x) &= 60x + 1 \\
\mathcal{H}_6(1, 3; x) &= 360x^2 + 120x + 1 \\
\mathcal{H}_7(1, 3; x) &= 2520x^2 + 210x + 1 \\
\mathcal{H}_8(1, 3; x) &= 10080x^2 + 336x + 1.
\end{aligned}$$

Remark 5.2. *It is worth to note that interchanging $(m, 1)$ (the Gould-Hopper case) with $(1, m)$ (the fractional derivative case), the corresponding polynomials can be easily reconstructed, since the coefficients appear in reverse order and the degree obey the increasing order rule observed in the preceding Remark 4. This is a general phenomenon, not only appearing when $m = 3$.*

5.6 First few values in a particular case, when $\gamma = 1$

Putting, for shortness, $\tilde{\mathcal{H}}_k(x) := \mathcal{H}_k(1, 1, 1, 2, 1; x)$, we consider here the polynomials defined by the generating function

$$G(t, x) = \exp[(t + 1)^2 + x\,t] = \sum_{k=0}^{\infty} \tilde{\mathcal{H}}_k(x)\ \frac{t^k}{k!}\ .$$

We find:

$$\tilde{\mathcal{H}}_0(x) = e$$
$$\tilde{\mathcal{H}}_1(x) = e\ (x+2)$$
$$\tilde{\mathcal{H}}_2(x) = e\ (x^2+4x+6)$$
$$\tilde{\mathcal{H}}_3(x) = e\ (x^3+6x^2+18x+20)$$
$$\tilde{\mathcal{H}}_4(x) = e\ (x^4+8x^3+36x^2+80x+76)$$
$$\tilde{\mathcal{H}}_5(x) = e\ (x^5+10x^4+60x^3+200x^2+380x+312)\,.$$

Remark 5.3. *Note that the sequence* $\{1,\ 2,\ 6,\ 20,\ 76,\ 312,\ \dots\}$ *appears in the Encyclopedia of Integer Sequences [22] under # A000898, namely, the sequence defined by the recurrence relation:* $a(n) = 2[a(n-1)+(n-1)a(n-2)]$, $a(0)=1$. *(Formerly M1648 N0645). That is the value of the* n*-th derivative of* $\exp(t^2)$ *evaluated at* $t=1$. *– N. Calkin, Apr. 22, 2010.*

References

[1] P. Appell and J. Kampé de Fériet, *Fonctions Hypergéométriques et Hypersphériques. Polynômes d'Hermite*, Gauthier-Villars, Paris, 1926.

[2] E.T. Bell, Exponential polynomials, *Annals of Mathematics*, **35** (1934), 258-277.

[3] Y. Ben Cheikh, Some results on quasi-monomiality, *Appl. Math. Comput.*, **141** (2003), 63-76.

[4] R.P. Boas and R.C. Buck, Polynomials defined by generating relations, *Amer. Math. Monthly*, **63** (1958), 626-632.

[5] R.P. Boas and R.C. Buck, *Polynomial Expansions of Analytic Functions*, Springer-Verlag, Berlin, Gottingen, Heidelberg, New York, 1958.

[6] L.R. Bragg, Products of certain generalized Hermite polynomials; associated relations, *Boll. Un. Mat. Ital.* (4) **1**, (1968), 347-355.

[7] W.C. Brenke, On generating functions of polynomial systems, *Amer. Math. Monthly*, **52** (1945), 297–301.

[8] G. Bretti, P. Natalini and P.E. Ricci, A new set of Sheffer-Bell polynomials and logarithmic numbers, *Georgian Math. J.*, 2018, (to appear).

[9] G. Dattoli, Hermite-Bessel and Laguerre-Bessel functions: A by-product of the monomiality principle, in *Advanced Special Functions and Applications* (Proceedings of the Melfi School on Advanced Topics in Mathematics and Physics; Melfi, 9-12 May, 1999) (D. Cocolicchio, G. Dattoli and H.M. Srivastava, Editors), Aracne Editrice, Rome, 2000, pp. 147–164.

[10] G. Dattoli, P.E. Ricci and H.M. Srivastava, Editors), Advanced Special Functions and Related Topics in Probability and in Differential Equations, (Proceedings of the Melfi School on Advanced Topics in Mathematics and Physics; Melfi, June 24-29, 2001), in: *Applied Mathematics and Computation*, **141**, (No. 1) (2003), 1-230.

[11] G. Dattoli, B. Germano, M.R. Martinelli and P.E. Ricci, Monomiality and partial differential equations, *Math. Comput. Modelling*, **50** (2009), 1332-1337.

[12] H.W. Gould and A.T. Hopper, Operational formulas connected with two generalizations of Hermite Polynomials, *Duke Math. J.*, **29** (1962), 51-62.

[13] W.N. Huff, The type of the polynomials generated by $f(x\,t)\phi(t)$, *Duke Math. J.*, **14(4)** (1947), 1091-1104.

[14] M. Lahiri, On a generalisation of Hermite polynomials, *Proc. Amer. Math. Soc.*, **27** (1971), 117-121.
[15] P. Natalini and P.E. Ricci, An extension of the Bell polynomials, *Comput. Math. Appl.*, **47** (2004), 719-725.
[16] P. Natalini and P.E. Ricci, Higher order Bell polynomials and the relevant integer sequences, *Appl. Anal. Discrete Math.*, **11** (2017), 327-339.
[17] P. Natalini and P.E. Ricci, Remarks on Bell and higher order Bell polynomials and numbers, *Cogent Mathematics*, **3** (2016), 1-15.
[18] P.E. Ricci, P. Natalini and G. Bretti, Sheffer and Brenke polynomials associated with generalized Bell numbers, *Jñānābha, Vijnana Parishad of India*, **47** (No2-2017), 337-352.
[19] S.M. Roman, *The Umbral Calculus*, Academic Press, New York, 1984.
[20] B.B. Saha, On a generating function of generalized Hermite polynomials, *C.R. Acad. Bulgare Sci.* **27**, (1974), 889-892.
[21] I.M. Sheffer, Some properties of polynomials sets of zero type, *Duke Math. J.*, **5** (1939), 590-622.
[22] N.J.A. Sloane, *The On-Line Encyclopedia of Integer Sequences*. Published electronically at http://oeis.org, 2016.
[23] H.M. Srivastava, A Note on a Generating Function for Generalized Hermite Polynomials, *Indag. Math.* (Proceed.), **79**, (1976), 457-461.
[24] H.M. Srivastava and H.L. Manocha, *A Treatise on Generating Functions*, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane and Toronto, 1984.
[25] J.F. Steffensen, The poweroid, an extension of the mathematical notion of power, *Acta Math.*, **73** (1941), 333-366.