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A COMMON FIXED POINT THEOREM FOR WEAKLY RECIPROCALLY CONTINUOUS SYSTEMS OF MAPS SATISFYING A GENERAL CONTRACTIVE CONDITION OF INTEGRAL TYPE

By

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Abstract

In this paper we prove a common fixed point theorem for weakly reciprocally continuous systems of maps satisfying a general contractive inequality of integral type on the finite product of metric spaces. Our result generalizes the results of Branciari [2], Rhoades [25], Matkowski [17] and Gairola [5].

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1 Introduction and Preliminaries

Branciari [2] generalized the celebrated Banach contraction principle for a single-valued self-map using an integral type contraction on a complete metric space. This result has been extended and generalized among others by Rhoades [25] , Vijayaraju et al. [35] , Suzuki [32], Gairola [5] , Gairola-Rawat [12], Vetro [34], Samet-Vetro [26], Stojakovic et al. [31] and others. On the other hand Matkowski [17] -[18] generalized the Banach contraction principal for a system of n maps on the finite product of metric spaces. Several authors have extended and generalized the result of Matkowski [op. cit.] for systems of sigle-valued as well as systems of multi-valued maps on product space (cf. Czerwik [3]-[4], Reddy- Subrahmanyam [23]-[24], Singh-Kulshrestha [30], Singh-Gairola [28]-[29], Baillon-Singh [1], Matkowski-Singh [19], Gairola et al. [13], [14], Gairola-Jangwan [6]-[7], Gairola-Khantwal [8]-[10] and others). Recently Gairola [5] combined the idea of Matkowski [op. cit.] and Branciari [op. cit.] and proved a fixed point theorem for a system of single-valued maps on the finite product of metric spaces. In this paper we extend and generalize the result of Gairola [op. cit.] and prove a common fixed point theorem for two systems of maps on finite product of metric spaces using coordinatewise weak reciprocal continuity. Our result also generalizes the results of Branciari [2], Rhoades [25], Matkowski [17] and others in the literature.

Throughout this paper, we shall follow the following notations and definitions.
Let a_{ik} be non-negative numbers for $i, k = 1, \dots, n$ and $c_{ik}^{(t)}$ square matrix defined in Matkowski [17, 18] (see also [3]).

$$c_{ik}^{(0)} = \begin{cases} a_{ik}, & i \neq k \\ & \\ 1 - a_{ik}, & i = k \end{cases} \qquad i, k = 1, \dots, n \tag{1.1}$$

$$c_{ik}^{(t+1)} = \begin{cases} c_{11}^{(t)} c_{i+1,k+1}^{(t)} + c_{i+1,1}^{(t)} c_{1,k+1}^{(t)}, & i \neq k \\ \\ c_{11}^{(t)} c_{i+1,k+1}^{(t)} - c_{i+1,1}^{(t)} c_{1,k+1}^{(t)}, & i = k \end{cases} \tag{1.2}$$

$t = 0, 1, \ldots, n-2, \qquad i, k = 1, \ldots, n-t-1, \qquad n \geq 2.$

Let (X_i, d_i), $i = 1, \ldots, n$, be metric spaces,

$X = X_1 \times \cdots \times X_n$,
$x = (x_1, \ldots, x_n) \in X$,
$f_i, T_i : X \to X_i, \quad i = 1, \ldots, n$ and
$\{x^m\} = \{(x_1^m, \ldots, x_n^m)\}, \ \ m \in \mathbb{N}$ (natural numbers) be a sequence in X.

Definition 1.1. *[29] Two systems of maps $\{f_1, \ldots, f_n\}$ and $\{T_1, \ldots, T_n\}$ are coordinatewise commuting at a point $x \in X$ if and only if $f_i(T_1x, \ldots, T_nx) = T_i(f_1x, \ldots, f_nx), i = 1, \ldots, n$. Two systems of maps are coordinatewise commuting on X if and only if they are coordinatewise commuting at every point of X.*

Definition 1.2. *[29] Two systems of maps $\{f_1, \ldots, f_n\}$ and $\{T_1, \ldots, T_n\}$ are coordinatewise weakly commuting at a point $x \in X$ if and only if $d_i(f_i(T_1x, \ldots, T_nx), T_i(f_1x, \ldots, f_nx)) \leq d_i(f_ix, T_ix), i = 1, \ldots, n$. Two systems of maps are coordinatewise weakly commuting on X if and only if they are coordinatewise weakly commuting at every point of X.*

Remark 1.1. *Evidently two coordinatewise commuting systems of maps are coordinatewise weakly commuting. However, the weakly commuting systems of maps need not to be commuting (see [28],[29]).*

Definition 1.3. *[14] Two systems of maps $\{f_1, \ldots, f_n\}$ and $\{T_1, \ldots, T_n\}$ are coordinatewise asymptotically commuting or, following the terminology of Jungck [15], coordinatewise compatible, if and only if*

$$\lim_{m\to\infty} d_i(f_i(T_1x^m, \ldots, T_nx^m), T_i(f_1x^m, \ldots, f_nx^m)) = 0,$$

whenever $\lim\limits_{m\to\infty} f_ix^m = \lim\limits_{m\to\infty} T_ix^m = u_i$ for some $u_i \in X_i, i = 1, \ldots, n$.

Definition 1.4. *[8] Two systems of maps $\{f_1, \ldots, f_n\}$ and $\{T_1, \ldots, T_n\}$ are coordinatewise reciprocal continuous if and only if $\lim\limits_{m\to\infty} f_i(T_1x^m, \ldots, T_nx^m) = f_iz$ and $\lim\limits_{m\to\infty} T_i(f_1x^m, \ldots, f_nx^m) = T_iz$, whenever there exist a sequence $\{x^m\}$ in X such that $\lim\limits_{m\to\infty} f_ix^m = \lim\limits_{m\to\infty} T_ix^m = z_i$ for all $i = 1, \ldots, n$.*

If two systems of maps $\{f_1, \ldots, f_n\}$ and $\{T_1, \ldots, T_n\}$ are continuous then they are coordinatewise reciprocal continuous but the converse need not be true (see Example 1.2 [8]).

Definition 1.5. *[9] Two systems of maps $\{f_1, \ldots, f_n\}$ and $\{T_1, \ldots, T_n\}$ are said to be coordinatewise weak reciprocal continuous if and only if $\lim\limits_{m\to\infty} f_i(T_1x^m, \ldots, T_nx^m) = f_iz$ or $\lim\limits_{m\to\infty} T_i(f_1x^m, \ldots, f_nx^m) = T_iz$, whenever there exist a sequence $\{x^m\}$ in X such that $\lim\limits_{m\to\infty} f_ix^m = \lim\limits_{m\to\infty} T_ix^m = z_i$ for all $i = 1, \ldots, n$.*

Remark 1.2. *Notice that the above definitions with $n = 1$ are standard ones for commuting, weakly commuting (see [16] and [27]), asymptotically commuting (see, [33]) (also called compatible [15]), reciprocal continuous maps ([20] and see also [22]) and weak reciprocal continuous maps (see [21]).*

Remark 1.3. *Asymptotically commuting (or compatible) class of maps includes commuting and weakly commuting maps. Commuting maps are necessarily weakly and asymptotically commuting both (see, for instance, [15], [27], [29], [33]).*

Remark 1.4. *The commutativity, weak commutativity and asymptotic commutativity (or compatibility) are equivalent at the point of coincidence of two (or two systems of) maps (see, [1], [13], [15]).*

The following Lemma is due to Matkowski [op. cit.] (see also [3], [30]).

Lemma 1.1. *Let $c_{ik}^{(0)} \geq 0, i, k = 1, \ldots, n, n \geq 2$, then the system of inequalities*

$$\sum_{k=1}^{n} a_{ik} r_k < r_i, \qquad i = 1, \ldots, n \tag{1.3}$$

has a positive solution $r_1, \ldots, r_n$ if and only if the following inequalities hold:

$$c_{ii}^{(t)} > 0, \qquad i = 1, \ldots, n-t; \qquad t = 0, \ldots, n-1, \qquad n \geq 2. \tag{1.4}$$

2 Main Result

Now we state our main result.

Theorem 2.1. *Let $(X_i, d_i), i = 1, \ldots, n$, be complete metric spaces and $T_i, f_i : X \to X_i$, be such that*

$$T_i(X) \subset f_i(X), \qquad i = 1, \ldots, n. \tag{2.1}$$

The systems of maps $(T_1, \ldots, T_n)$ and $(f_1, \ldots, f_n)$ are coordinatewise weakly reciprocally continuous and coordinatewise asymptotically commuting on X.

(2.2)

If there exist non-negative numbers $b < 1$ and $a_{ik}, i, k = 1, \ldots, n$, such that (1.1),(1.2),(1.3) and the following hold:

$$\int_0^{d_i(T_i x, T_i y)} \phi(\xi) d\xi \leq \max_i \left\{ \sum_{k=1}^{n} a_{ik} \int_0^{d_k(f_k x, f_k y)} \phi(\xi) d\xi, b \int_0^{N_i(x,y)} \phi(\xi) d\xi \right\} \tag{2.3}$$

where $N_i(x,y) = \max\left\{ d_i(f_i x, T_i x), d_i(f_i y, T_i y), \frac{d_i(f_i x, T_i y) + d_i(f_i y, T_i x)}{2} \right\}$ for all $x, y \in X$ and $\xi : R^+ \to R^+$ is a Lebesgue-integrable mappings which is summable, non-negative and such that

$$\int_0^{\epsilon} \phi(\xi) d\xi > 0 \quad \text{for each} \quad \epsilon > 0, \tag{2.4}$$

then there exists unique point $z \in X$ such that

$$f_i z = z_i = T_i z, \qquad i = 1, \ldots, n. \tag{2.5}$$

Proof. First we note that the system (1.3) and

$$\sum_{k=1}^{n} a_{ik} r_k < r_i, \qquad i = 1, \dots, n,$$

are equivalent for some positive numbers $r_1, \dots, r_n$. Further if we put

$$h = \max \left\{ r_i^{-1} \sum_{k=1}^{n} a_{ik} r_k \right\}$$

then $h \in (0,1)$ and we may choose positive numbers $r_1, \dots, r_n$ such that

$$\sum_{k=1}^{n} a_{ik} r_k \le h r_i, \qquad i = 1, \dots, n.$$

Pick x_i^0 in $X_i, i = 1, \dots, n$. Since (2.1) holds, we choose a sequence $\{x^m\}$ in X such that

$$f_i x^{m+1} = T_i x^m, \qquad i = 1, \dots, n, \qquad m = 0, 1, \dots.$$

If at any stage $f_i x^{m+1} = f_i x^{m+2}$ then $f_i x^{m+1} = T_i x^{m+1}$ that is, f_i and T_i have a coincidence point at x^{m+1}. Without loss of generality, we may assume that

$$\int_0^{d_i(f_i x^2, f_i x^1)} \phi(\xi) d\xi \le r_i, \qquad i = 1, \dots, n.$$

Then by (2.3), we have

$$\begin{aligned} \int_0^{d_i(f_i x^3, f_i x^2)} \phi(\xi) d\xi &= \int_0^{d_i(T_i x^2, T_i x^1)} \phi(\xi) d\xi \\ &\le \max \left\{ \sum_{k=1}^{n} a_{ik} \int_0^{d_k(f_k x^2, f_k x^1)} \phi(\xi) d\xi, b \int_0^{N_i(x^2, x^1)} \phi(\xi) d\xi \right\} \qquad (2.6) \end{aligned}$$

where

$$\begin{aligned} N_i(x^2, x^1) &= \max \left\{ d_i(f_i x^2, T_i x^2), d_i(f_i x^1, T_i x^1), \frac{d_i(f_i x^2, T_i x^1) + d_i(f_i x^1, T_i x^2)}{2} \right\} \\ &= \max \left\{ d_i(f_i x^2, f_i x^3), d_i(f_i x^1, f_i x^2), \frac{d_i(f_i x^1, f_i x^3)}{2} \right\} \\ &= \max \left\{ d_i(f_i x^2, f_i x^3), d_i(f_i x^1, f_i x^2) \right\}. \end{aligned}$$

Substituting the value of $N_i(x^2, x^1)$ in (2.6), we get

$$\begin{aligned} \int_0^{d_i(f_i x^3, f_i x^2)} \phi(\xi) d\xi &\le \max \left\{ \begin{array}{l} \sum_{k=1}^{n} a_{ik} \int_0^{d_k(f_k x^2, f_k x^1)} \phi(\xi) d\xi, \\ b \int_0^{d_i(f_i x^2, f_i x^3)} \phi(\xi) d\xi, b \int_0^{d_i(f_i x^1, f_i x^2)} \phi(\xi) d\xi \end{array} \right\} \\ &\le \max \left\{ \sum_{k=1}^{n} a_{ik} \int_0^{d_k(f_k x^2, f_k x^1)} \phi(\xi) d\xi, b \int_0^{d_i(f_i x^1, f_i x^2)} \phi(\xi) d\xi \right\} \\ &\le \max \left\{ \sum_{k=1}^{n} a_{ik} r_k, b r_i \right\} \\ &\le \max \{ h r_i, b r_i \} = c r_i, \text{ where } c = \max\{h, b\}. \end{aligned}$$

Similarly

$$\int_0^{d_i(f_i x^4, f_i x^3)} \phi(\xi) d\xi \le c^2 r_i.$$

Inductively

$$\int_0^{d_i(f_i x^{m+1}, f_i x^m)} \phi(\xi)d\xi \le c^{m-1} r_i,$$

which implies that

$$d_i(f_i x^{m+1}, f_i x^m) \to 0 \text{ as } m \to \infty. \tag{2.7}$$

Now we prove that $\{f_i x^m\}$ is a Cauchy sequence in $X_i, i = 1, \ldots, n$. Assume that it is not true then for each $i = 1, \ldots, n$ and positive integer s, there exist a $\epsilon_i > 0$ and positive integers $\{p_i(s)\}$, $\{q_i(s)\}$ with $s < p_i(s) < q_i(s)$ such that

$$d_i(f_i x^{p_i(s)}, f_i x^{q_i(s)}) \ge \epsilon_i, \qquad i = 1, \ldots, n. \tag{2.8}$$

Let $q_i(s)$ be the least integer exceeds $p_i(s)$ and satisfies (2.8), for each positive integer s. Then it is clear that

$$d_i(f_i x^{p_i(s)}, f_i x^{q_i(s)}) \ge \epsilon_i \text{ and } d_i(f_i x^{p_i(s)}, f_i x^{q_i(s)-1}) < \epsilon_i, \forall s \in \mathbb{N}, \qquad i = 1, \ldots, n. \tag{2.9}$$

Now from (2.7) and (2.9), we have

$$\begin{aligned} \epsilon_i &\le d_i(f_i^{p_i(s)} x, f_i^{q_i(s)})x) \\ &\le d_i(f_i^{p_i(s)} x, f_i^{q_i(s)-1} x) + d_i(f_i^{q_i(s)-1} x, f_i^{q_i(s)} x). \end{aligned}$$

Making $s \to \infty$, it follows that

$$\lim_{s\to\infty} d_i(f_i^{p_i(s)} x, f_i^{q_i(s)})x = \epsilon_i.$$

Using triangular inequality,

$$\begin{aligned} d_i(f_i^{p_i(s)} x, f_i^{q_i(s)} x) &\le d_i(f_i^{p_i(s)} x, f_i^{p_i(s)-1} x) + d_i(f_i^{p_i(s)-1} x, f_i^{q_i(s)-1} x) + d_i(f_i^{q_i(s)-1} x, f_i^{q_i(s)} x) \\ &\le 2d_i(f_i^{p_i(s)} x, f_i^{p_i(s)-1} x) + d_i(f_i^{p_i(s)} x, f_i^{q_i(s)-1} x) + d_i(f_i^{q_i(s)-1} x, f_i^{q_i(s)} x). \end{aligned}$$

Making $s \to \infty$ and using (2.7), (2.9), we deduce that

$$\lim_{s\to\infty} d_i(f_i x^{p_i(s)-1}, f_i x^{q_i(s)-1}) = \epsilon_i = \lim_{s\to\infty} d_i(f_i^{p_i(s)} x, f_i^{q_i(s)} x). \tag{2.10}$$

Now we may assume that

$$\int_0^{\epsilon_i} \phi(\xi)d\xi \le r_i, \qquad i = 1, \ldots, n.$$

Then from (2.3),

$$\begin{aligned} \int_0^{d_i(f_i x^{p_i(s)}, f_i x^{q_i(s)})} \phi(\xi)d\xi &= \int_0^{d_i(T_i x^{p_i(s)-1}, T_i x^{q_i(s)-1})} \phi(\xi)d\xi \\ &\le \max \left\{ \begin{array}{l} \sum_{k=1}^{n} a_{ik} \int_0^{d_k(f_k x^{p_i(s)-1}, f_k x^{q_i(s)-1})} \phi(\xi)d\xi, \\ \\ b \int_0^{N_i(x^{p_i(s)-1}, x^{q_i(s)-1})} \phi(\xi)d\xi \end{array} \right\} \end{aligned} \tag{2.11}$$

where

$$\begin{aligned} N_i(x^{p_i(s)-1}, x^{q_i(s)-1}) &= \max \left\{ \begin{array}{l} d_i(f_i x^{p_i(s)-1}, T_i x^{p_i(s)-1}), d_i(f_i x^{q_i(s)-1}, T_i x^{q_i(s)-1}), \\ \frac{d_i(f_i x^{p_i(s)-1}, T_i x^{q_i(s)-1}) + d_i(f_i x^{q_i(s)-1}, T_i x^{p_i(s)-1})}{2} \end{array} \right\} \\ &= \max \left\{ \begin{array}{l} d_i(f_i x^{p_i(s)-1}, f_i x^{p_i(s)}), d_i(f_i x^{q_i(s)-1}, f_i x^{q_i(s)}), \\ \frac{d_i(f_i x^{p_i(s)-1}, f_i x^{q_i(s)}) + d_i(f_i x^{q_i(s)-1}, f_i x^{p_i(s)})}{2} \end{array} \right\} \end{aligned}$$

$$\leq \max\left\{\begin{array}{l} d_i(f_ix^{p_i(s)-1}, f_ix^{p_i(s)}), d_i(f_ix^{q_i(s)-1}, f_ix^{q_i(s)}), \\ \frac{d_i(f_ix^{p_i(s)-1}, f_ix^{q_i(s)-1})+d_i(f_ix^{q_i(s)-1}, f_ix^{q_i(s)})}{2} + \\ \frac{d_i(f_ix^{q_i(s)-1}, f_ix^{p_i(s)-1})+d_i(f_ix^{p_i(s)-1}, f_ix^{p_i(s)})}{2} \end{array}\right\}.$$

Making $s \to \infty$ and using (2.7), (2.10), we deduce that

$$\lim_{s\to\infty} N_i(x^{p_i(s)-1}, x^{q_i(s)-1}) \leq \epsilon_i. \tag{2.12}$$

Taking limit $s \to \infty$ both sides in (2.11) and using (2.10), (2.12), we get

$$\begin{aligned} \int_0^{\epsilon_i} \phi(\xi)d\xi &\leq \max\left\{\sum_{k=1}^{n} a_{ik} \int_0^{\epsilon_k} \phi(\xi)d\xi, b\int_0^{\epsilon_i} \phi(\xi)d\xi\right\} \\ &\leq \max\left\{\sum_{k=1}^{n} a_{ik}r_k, br_i\right\} \leq \max\{hr_i, br_i\} \\ &= cr_i, \text{ where } c = \max\{h, b\} < 1. \end{aligned}$$

Inductively

$$\int_0^{\epsilon_i} \phi(\xi)d\xi \leq c^m r_i.$$

Making $m \to \infty$, we get

$$\int_0^{\epsilon_i} \phi(\xi)d\xi \leq 0,$$

which contradict the condition (2.4). Hence $\{f_ix^m\}$ and $\{T_ix^m\}$ are Cauchy sequences in $X_i, i = 1, \ldots, n$, and since X_i is a complete metric space therefore there exist a point t_i (say) in X_i such that the both sequences $\{f_ix^m\}$ and $\{T_ix^m\}$ converges to t_i .

Since systems of maps $(T_1, \ldots, T_n)$ and $(f_1, \ldots, f_n)$ are coordinatewise weakly reciprocally continuous, so for each $i = 1, \ldots, n$,

$$\text{either } \lim_{m\to\infty} f_i(T_1x^m, \ldots, T_nx^m) = f_it \quad \text{ or } \quad \lim_{m\to\infty} T_i(f_1x^m, \ldots, f_nx^m) = T_it.$$

Case (I): Let us suppose that $\lim_{m\to\infty} f_i(T_1x^m, \ldots, T_nx^m) = f_it, i = 1, \ldots, n$ then coordinatewise asymptotic commutativity of systems of maps $(T_1, \ldots, T_n)$ and $(f_1, \ldots, f_n)$ gives

$$d_i(T_i(f_1x^m, \ldots, f_nx^m), f_i(T_1x^m, \ldots, T_nx^m)) \to 0, \text{ as } m \to \infty,$$

that is

$$\lim_{m\to\infty} T_i(f_1x^m, \ldots, f_nx^m) = \lim_{m\to\infty} f_i(T_1x^m, \ldots, T_nx^m) = f_it. \tag{2.13}$$

Since

$$f_ix^{m+1} = T_ix^m$$

therefore

$$f_i(f_1x^{m+1}, \ldots, f_nx^{m+1}) = f_i(T_1x^m, \ldots, T_nx^m)$$

and

$$\lim_{m\to\infty} f_i(f_1x^{m+1}, \ldots, f_nx^{m+1}) = f_it. \tag{2.14}$$

From (2.3), with $fx^{m+1} := (f_1x^{m+1}, ..., f_nx^{m+1})$,

$$\int_0^{d_i(T_ifx^{m+1}, T_it)} \phi(\xi)d\xi \leq \max\left\{\begin{array}{l} \sum_{k=1}^{n} a_{ik} \int_0^{d_k(f_kfx^{m+1}, f_kt)} \phi(\xi)d\xi, \\ b\int_0^{N_i(fx^{m+1}, t)} \phi(\xi)d\xi \end{array}\right\} \tag{2.15}$$

where

$$N_i(fx^{m+1}, t) = \max\left\{ \begin{array}{l} d_i(f_i f x^{m+1}, T_i f x^{m+1}), d_i(f_i t, T_i t), \\ \frac{d_i(f_i f x^{m+1}, T_i t) + d_i(f_i t, T_i f x^{m+1})}{2} \end{array} \right\}.$$

Making $m \to \infty$ and using (2.13) and (2.14), we get

$$\begin{aligned} \lim_{m\to\infty} N_i(fx^{m+1}, t) &= \max\left\{ d_i(f_i t, f_i t), d_i(f_i t, T_i t), \frac{d_i(f_i t, T_i t) + d_i(f_i t, f_i t)}{2} \right\} \\ &= d_i(f_i t, T_i t). \end{aligned} \tag{2.16}$$

Taking limit $m \to \infty$ both sides in (2.15) and using (2.16), we have

$$\begin{aligned} \int_0^{d_i(f_i t, T_i t)} \phi(\xi) d\xi &\le \max\left\{ \sum_{k=1}^n a_{ik} \int_0^{d_k(f_k t, f_k t)} \phi(\xi) d\xi, b \int_0^{d_i(f_i t, T_i t)} \phi(\xi) d\xi \right\} \\ &= b \int_0^{d_i(f_i t, T_i t)} \phi(\xi) d\xi, \end{aligned}$$

implies that

$$d_i(f_i t, T_i t) = 0.$$

This gives $f_i t = T_i t, i = 1, \ldots, n$. Thus the systems of maps $(f_1, \ldots, f_n)$ and $(T_1, \ldots, T_n)$ has a coincidence point $t = (t_1, \ldots, t_n)$ in X. Since coordinatewise asymptotic commutativity of systems $(f_1, \ldots, f_n)$ and $(T_1, \ldots, T_n)$ is equivalent to thier coordinatewise commutativity at a coincidence point t in X. Therefore

$$f_i(T_1 t, \ldots, T_n t) = T_i(f_1 t, \ldots, f_n t) = f_i(f_1 t, \ldots, f_n t) = T_i(T_1 t, \ldots, T_n t).$$

Now we may assume that

$$\int_0^{d_i(T_i t, T_i(T_1 t, \ldots, T_n t))} \phi(\xi) d\xi \le r_i, \qquad i = 1, \ldots, n.$$

From (2.3), with $Tt := (T_1 t, \ldots, T_n t)$, we obtain

$$\int_0^{d_i(T_i t, T_i T t)} \phi(\xi) d\xi \le \max\left\{ \sum_{k=1}^n a_{ik} \int_0^{d_k(f_k t, f_k T t)} \phi(\xi) d\xi, b \int_0^{N_i(t, Tt)} \phi(\xi) d\xi \right\}, \tag{2.17}$$

where

$$\begin{aligned} N_i(t, Tt) &= \max\left\{ d_i(f_i t, T_i t), d_i(f_i T t, T_i T t), \frac{d_i(f_i t, T_i T t) + d_i(f_i T t, T_i t)}{2} \right\} \\ &= \max\left\{ d_i(T_i t, T_i t), d_i(T_i T t, T_i T t), \frac{d_i(T_i t, T_i T t) + d_i(T_i T t, T_i t)}{2} \right\} \\ &= d_i(T_i t, T_i T t). \end{aligned}$$

Substituting the value of $N_i(t, Tt)$ in (2.17), we get

$$\begin{aligned} \int_0^{d_i(T_i t, T_i T t)} \phi(\xi) d\xi &\le \max\left\{ \sum_{k=1}^n a_{ik} \int_0^{d_k(T_k t, T_k T t)} \phi(\xi) d\xi, b \int_0^{d_i(T_i t, T_i T t)} \phi(\xi) d\xi \right\} \\ &\le \max\left\{ \sum_{k=1}^n a_{ik} r_k, b r_i \right\} \le \max\{h r_i, b r_i\} \\ &= c r_i, \text{ where } c = \max\{h, b\} < 1. \end{aligned}$$

Inductively

$$\int_0^{d_i(T_it,T_iTt)} \phi(\xi)d\xi \le c^m r_i.$$

Making $m \to \infty$, we get

$$d_i(T_it, T_iTt) = 0,$$

that is, $T_it = T_i(T_1t, \ldots, T_nt)$. We also have $T_it = T_i(T_1t, \ldots, T_nt) = f_i(T_1t, \ldots, T_nt)$. Hence T_it is a solution of systems of equation (2.5).

Case (II): Let us assume that $\lim_{m\to\infty} T_i(f_1x^m, \ldots, f_nx^m) = T_it, i = 1, \ldots, n$, then coordinatewise asymptotic commutativity of systems of maps $(T_1, \ldots, T_n)$ and $(f_1, \ldots, f_n)$ yields

$$\lim_{m\to\infty} T_i(f_1x^m, \ldots, f_nx^m) = \lim_{m\to\infty} f_i(T_1x^m, \ldots, T_nx^m) = T_it. \tag{2.18}$$

Since

$$f_ix^{m+1} = T_ix^m$$

therefore

$$f_i(f_1x^{m+1}, \ldots, f_nx^{m+1}) = f_i(T_1x^m, \ldots, T_nx^m)$$

and

$$\lim_{m\to\infty} f_i(f_1x^{m+1}, \ldots, f_nx^{m+1}) = T_it. \tag{2.19}$$

In view of (2.1), there exist a point $v = (v_1, \ldots, v_n) \in X$ such that

$$T_it = f_iv, \qquad i = 1, \ldots, n. \tag{2.20}$$

By (2.3), with $fx^{m+1} := (f_1x^{m+1}, \ldots, f_nx^{m+1})$,

$$\int_0^{d_i(T_ifx^{m+1},T_iv)} \phi(\xi)d\xi \le \max\left\{ \begin{array}{l} \sum\limits_{k=1}^{n} a_{ik} \int_0^{d_k(f_kfx^{m+1},f_kv)} \phi(\xi)d\xi, \\ b\int_0^{N_i(Tx^{m+1},v)} \phi(\xi)d\xi \end{array} \right\}, \tag{2.21}$$

where

$$N_i(fx^{m+1}, v) = \max\left\{ \begin{array}{l} d_i(f_ifx^{m+1}, T_ifx^{m+1}), d_i(f_iv, T_iv), \\ \frac{d_i(f_ifx^{m+1},T_iv)+d_i(f_iv,T_ifx^{m+1})}{2} \end{array} \right\}.$$

Making $m \to \infty$ and using (2.18), (2.19) and (2.20), we have

$$\begin{aligned} \lim_{m\to\infty} N_i(fx^{m+1}, v) &= \max\left\{ d_i(T_it, T_it), d_i(f_iv, T_iv), \tfrac{d_i(T_it,T_iv)+d_i(f_iv,T_it)}{2} \right\} \\ &= d_i(T_it, T_iv). \end{aligned} \tag{2.22}$$

Now taking limit $m \to \infty$ both side in (2.21) and using (2.22), we get

$$\begin{aligned} \int_0^{d_i(T_it,T_iv)} \phi(\xi)d\xi &\le \max\left\{ \sum_{k=1}^{n} a_{ik} \int_0^{d_k(T_kt,f_kv)} \phi(\xi)d\xi, b\int_0^{d_i(T_it,T_iv)} \phi(\xi)d\xi \right\} \\ &= b\int_0^{d_i(T_it,T_iv)} \phi(\xi)d\xi. \end{aligned}$$

This gives

$$d_i(T_it, T_iv) = 0, \tag{2.23}$$

which implies

$$T_i t = T_i v.$$

From (2.20) and (2.23)

$$T_i v = f_i v, \qquad i = 1, \ldots, n.$$

This proves that the systems of maps $(f_1, \ldots, f_n)$ and $(T_1, \ldots, T_n)$ have a coincidence point v in X. Since coordinatewise asymptotic commutativity of systems of maps $(f_1, \ldots, f_n)$ and $(T_1, \ldots, T_n)$ implies their coordinatewise commutativity at a coincidence point v in X. Therefore

$$f_i(T_1 v, \ldots, T_n v) = T_i(f_1 v, \ldots, f_n v) = f_i(f_1 v, \ldots, f_n v) = T_i(T_1 v, \ldots, T_n v).$$

Again we may assume that

$$\int_0^{d_i(T_i v, T_i(T_1 v, \ldots, T_n v))} \phi(\xi) d\xi \leq r_i, \qquad i = 1, \ldots, n.$$

From (2.3), with $Tv := (T_1 v, \ldots, T_n v)$, we obtain

$$\int_0^{d_i(T_i v, T_i T v)} \phi(\xi) d\xi \leq \max \left\{ \sum_{k=1}^{n} a_{ik} \int_0^{d_k(f_k v, f_k T v)} \phi(\xi) d\xi, b \int_0^{N_i(v, Tv)} \phi(\xi) d\xi \right\}, \tag{2.24}$$

where

$$\begin{aligned} N_i(v, Tv) &= \max\left\{ d_i(f_i v, T_i v), d_i(f_i T v, T_i T v), \frac{d_i(f_i v, T_i T v) + d_i(f_i T v, T_i v)}{2} \right\} \\ &= \max\left\{ d_i(T_i v, T_i v), d_i(T_i T v, T_i T v), \frac{d_i(T_i v, T_i T v) + d_i(T_i T v, T_i v)}{2} \right\} \\ &= d_i(T_i v, T_i T v). \end{aligned}$$

Substituting the value of $N_i(v, Tv)$ in (2.24), we get

$$\begin{aligned} \int_0^{d_i(T_i v, T_i T v)} \phi(\xi) d\xi &\leq \max \left\{ \sum_{k=1}^{n} a_{ik} \int_0^{d_k(T_k v, T_k T v)} \phi(\xi) d\xi, b \int_0^{d_i(T_i v, T_i T v)} \phi(\xi) d\xi \right\} \\ &\leq \max \left\{ \sum_{k=1}^{n} a_{ik} r_k, b r_i \right\} \leq \max\{h r_i, b r_i\} \\ &= c r_i, \text{ where } c = \max\{h, b\} < 1. \end{aligned}$$

Inductively

$$\int_0^{d_i(T_i v, T_i T v)} \phi(\xi) d\xi \leq c^m r_i.$$

Making $m \to \infty$,

$$d_i(T_i v, T_i T v) = 0,$$

implies , $T_i v = T_i(T_1 v, \ldots, T_n v)$. We also have $T_i v = T_i(T_1 v, \ldots, T_n v) = f_i(T_1 v, \ldots, T_n v)$. Hence $T_i v$ is a solution of systems of equation (2.5).

Now suppose that the system (2.5) have two distinct solutions z and $\bar{z}$ in X such that

$$f_i z = z_i = T_i z \text{ and } f_i \bar{z} = \bar{z}_i = T_i \bar{z}, \qquad i = 1, \ldots, n.$$

If $z_i \neq \bar{z}_i, i = 1, \ldots, n$, then we may assume that

$$\int_0^{d_i(z_i,\bar{z}_i)} \phi(\xi)d\xi \le r_i, i = 1, \dots, n.$$

From (2.3),

$$\int_0^{d_i(z_i,\bar{z}_i)} \phi(\xi)d\xi == \int_0^{d_i(T_iz,T_i\bar{z})} \phi(\xi)d\xi$$

$$\le \max\left\{ \sum_{k=1}^{n} a_{ik} \int_0^{d_k(f_kz,f_k\bar{z})} \phi(\xi)d\xi, b\int_0^{N_i(z,\bar{z})} \phi(\xi)d\xi \right\} \tag{2.25}$$

where

$$N_i(z,\bar{z}) = \max\left\{ d_i(f_iz,T_iz), d_i(f_i\bar{z},T_i\bar{z}), \frac{d_i(f_iz,T_i\bar{z}) + d_i(f_i\bar{z},T_iz)}{2} \right\}$$
$$= d_i(f_iz, f_i\bar{z}) = d_i(z_i,\bar{z}_i).$$

Substituting this value in (2.25), we get

$$\int_0^{d_i(z,\bar{z}_i)} \phi(\xi)d\xi \le \max\left\{ \sum_{k=1}^{n} a_{ik} \int_0^{d_k(z_k,\bar{z}_k)} \phi(\xi)d\xi, b\int_0^{d_i(z_i,\bar{z}_i)} \phi(\xi)d\xi \right\}$$
$$\le \max\left\{ \sum_{k=1}^{n} a_{ik}r_k, br_i \right\} = \max\{hr_i, br_i\}$$
$$= cr_i, \text{ where } c = \max\{h,b\}.$$

Inductively

$$\int_0^{d_i(z_i,\bar{z}_i)} \phi(\xi)d\xi \le c^m r_i.$$

Making $m \to \infty$, we get

$$d_i(z_i,\bar{z}_i) = 0,$$

which gives $z_i = \bar{z}_i, i = 1, \dots, n$. This completes the proof. □

If we take $\phi(\xi) = 1$ in the Theorem 2.1, we get following result as a special case of Theorem 2.1.

Corollary 2.1. *Let $(X_i, d_i), i = 1, \dots, n$, be complete metric spaces and $T_i, f_i : X \to X_i$, be such that*

$$T_i(X) \subset f_i(X), \qquad i = 1, \dots, n.$$

The systems of maps $(T_1, \dots, T_n)$ and $(f_1, \dots, f_n)$ are coordinatewise weakly reciprocally continuous and coordinatewise asymptotically commuting on X.

If there exist non-negative numbers $b < 1$ and $a_{ik}, i, k = 1, \dots, n$, such that (1.1), (1.2), (1.4) *and the following hold:*

$$d_i(T_ix, T_iy) \le \max_i \left\{ \sum_{k=1}^{n} a_{ik}d_k(f_kx, f_ky), bN_i(x,y) \right\}$$

where $N_i(x,y) = \max\left\{ d_i(f_ix,T_ix), d_i(f_iy,T_iy), \frac{d_i(f_ix,T_iy)+d_i(f_iy,T_ix)}{2} \right\}$ for all $x, y \in X$. Then there exists unique point $z \in X$ such that $f_iz = z_i = T_iz, i = 1, \dots, n$.

Remark 2.1. *If we assume $f_ix = x_i, f_iy = y_i, i = 1, \dots, n$ and $b = 0$ in Corollary* 2.1*, we obtain result of Matkowski [17].*

Corollary 2.2. *Let T and $f : Y \to Y$ are weakly reciprocally continuous and asymptotically commuting (or compatible) self-maps in a complete metric space (Y, d) such that $T(Y) \subset f(Y)$ and satisfying*

$$\int_0^{d(Tx,Ty)} \phi(\xi)d\xi \leq k \int_0^{N(x,y)} \phi(\xi)d\xi$$

where $N(x, y) = \max\left\{d(fx, fy), d(fx, Tx), d(fy, Ty), \frac{d(fx,Ty)+d(fy,Tx)}{2}\right\}$ for all $x, y \in Y$ and $\xi : R^+ \to R^+$ is a Lebesgue-integrable mappings which is summable, non-negative and such that

$$\int_0^{\epsilon} \phi(\xi)d\xi > 0 \quad \textit{for each} \quad \epsilon > 0.$$

Then there exists unique point $z \in Y$ such that $fz = z = Tz$.

Proof. Proof may be completed by putting $(Y, d) = (X_i, d_i), T = T_i, f = f_i, i = 1, \ldots, n$ and $n = 1, k = \max\{a_{11}, b\}$ in the proof of Theorem 2.1. ☐

Remark 2.2. *The result of Rhoades [25] is obtained from Corollary 2.2 by taking f as an identity map. Similarly we can obtain the result of Branciari [2] by assuming $N(x, y) = d(fx, fy)$ and f as an identity map in Corollary 2.2.*

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