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EXISTENCE THEOREMS FOR A PBVP OF FIRST ORDER FUNCTIONAL RANDOM INTEGRODIFFERENTIAL INCLUSIONS

By

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Abstract

In this paper, some existence theorems for a periodic boundary value problem of first order ordinary functional random integrodifferential inclusions are proved for convex and nonconvex cases of random multi-valued functions involved in the inclusion. The existence theorems for extremal random solutions are also proved under certain monotonic conditions of the multi-valued function. The multi-valued random fixed point theoretic approach of Dhage (2011) is used while establishing the results concerning the existence of extremal random solutions.

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1 Statement of the Problem

Let $(\Omega, \mathcal{A}, \mu)$ be a complete σ-finite measure space. For a given Banach space X, let $\mathcal{P}(X)$ denote the class of all subsets of X, called the power set of X. Denote

$$\mathcal{P}_p(X) = \{A \subset X \mid A \text{ is non-empty and has the property } p\}.$$

Here, p may be p =closed (in short cl) or p =convex (in short cv) or p =bounded (in short bd) or p =compact (in short cp). Thus $\mathcal{P}_{cl}(X)$, $\mathcal{P}_{cv}(X)$, $\mathcal{P}_{bd}(X)$ and $\mathcal{P}_{cp}(X)$ denote respectively, the classes of all closed, convex, bounded and compact subsets of X. Similarly, $\mathcal{P}_{cl,bd}(X)$ and $\mathcal{P}_{cv,cp}(X)$ denote respectively, the classes of closed-bounded and compact-convex non-empty subsets of the Banach space X.

Let $\mathbb{R}$ be the real line and let $J = [0, T]$ be a closed and bounded interval in $\mathbb{R}$ for some $T > 0$. Now, consider the first order random periodic boundary value problem (in short RPBVP) of integro-differential inclusion,

$$\left.\begin{aligned} x'(t,\omega) + \lambda x(t,\omega) &\in F\left(t, x(\theta(t),\omega), \int_0^{\sigma(t)} k(t,\tau,x(\eta(\tau),\omega),\omega)\,d\tau, \omega\right) \quad \text{a.e.} \quad \omega \in \Omega, \\ x(0,\omega) &= x(T,\omega) \end{aligned}\right\} \tag{1.1}$$

for all $t \in J$, where $\theta, \sigma, \eta : J \to J$ are continuous, $k : J \times J \times \mathbb{R} \times \Omega \to \mathbb{R}$ and $F : J \times \mathbb{R} \times \mathbb{R} \times \Omega \to \mathcal{P}_p(\mathbb{R})$.

By a *random solution* of the RPBVP (1.1) on $J \times \Omega$ we mean a measurable function $x : \Omega \to C(J, \mathbb{R})$ satisfying for each $\omega \in \Omega$, $x'(t,\omega) + \lambda x(t,\omega) = v(t)$ for some function $v \in L^1(J, \mathbb{R})$ such that

$$v(t) \in F\left(t, x(\theta(t),\omega), \int_0^{\sigma(t)} k(t,\tau,x(\eta(\tau),\omega),\omega)\,d\tau, \omega\right) \quad \text{a.e. } \omega \in \Omega,$$

for all $t \in J$, where $C(J,\mathbb{R})$ and $L^1(J,\mathbb{R})$ are respectively the Banach spaces of continuous and Lebesgue integrable real-valued functions defined on J.

The RPBVP (1.1) includes several known random differential inclusions already studied in the literature as special cases. The special case in the form of differential inclusion

$$\left.\begin{aligned} x'(t,\omega)+\lambda x(t,\omega) &\in F(t,x(t,\omega),\omega) \text{ a.e. } \omega\in\Omega,\\ x(0,\omega) &= x(T,\omega), \end{aligned}\right\} \tag{1.2}$$

for all $t \in J$, where $F : J\times\mathbb{R}\times\Omega \to \mathcal{P}_p(\mathbb{R})$ is a special case of the RPBVP (1.1) and can be discussed for various aspects of the solutions via multi-valued fixed point techniques. See Deimling [1], Hu and Papapgeorgiou [13], Papapgeorgiou [18] and the reference therein. In this chapter we prove the existence theorems for RPBVP (1.1) under convex and nonconvex case of the multi-valued function F involved in RPBVP (1.1).

2 Auxiliary Results

Let $\mathcal{M}(J,\mathbb{R})$ denote the class of real-valued measurable functions on J and let $L^1(J,\mathbb{R})$ denote the Banach space of Lebesgue integrable functions on J with norm $\|\cdot\|_{L^1}$ defined by

$$\|x\|_{L^1} = \int_0^T |x(t)|\,dt.$$

Definition 2.1. *Given a measurable space $(\Omega,\mathcal{A})$ and given a Banach space E, let β_E denote the class of all Borel subsets of E. A single-valued mapping $f : \Omega\to E$ is called measurable if for any $B\in\beta_E$, we have*

$$f^{-1}(B) = \big\{\omega\in\Omega \mid f(\omega)\in B\big\}\in\mathcal{A}.$$

It is known that scalar multiplication, addition and product of single-valued measurable functions are measurable in a separable Banach space.

Let $F : J\times\mathbb{R}\times\mathbb{R}\times\Omega\to\mathcal{P}_p(\mathbb{R})$ be a multi-valued mapping. Then for any function $x:\Omega\to\in C(J,\mathbb{R})$, let $S^1_F:\Omega\times C(J,\mathbb{R})\to\mathcal{P}_p(L^1(J,\mathbb{R}))$ be a multi-valued operator defined by

$$\begin{aligned} S^1_F(\omega)(x) = \Bigg\{ & v\in L^1(J,\mathbb{R}) \;\Bigg|\; \\ & v(t)\in F\Big(t, x(\theta(t),\omega), \int_0^{\sigma(t)} k(t,s,x(\eta(t),\omega),\omega)\,ds,\omega\Big) \text{ a.e. } \omega\in\Omega\Bigg\}. \end{aligned} \tag{2.1}$$

This is our set of *selection functions* for the multi-valued function F on $J\times\mathbb{R}\times\mathbb{R}\times\Omega$. When there is no confusion, we denote $S^1_F(\omega)(x) = S^1_F(\omega)(y)$, where $y(t,\omega)=x(\theta(t),\omega)$ for some continuous function $\theta : J\to J$. The integral of the random multi-valued function F is defined as

$$\begin{aligned} \int_0^T F\Big(s, & x(\theta(s),\omega), \int_0^{\sigma(s)} k(s,\tau,x(\eta(\tau),\omega),\omega)d\tau,\omega\Big)\,ds \\ &= \left\{\int_0^T v(s)\,ds : v\in S^1_F(\omega)(x)\right\}. \end{aligned}$$

Furthermore, if the integral on the left hand side of above expression exists for every measurable function $x:\Omega\to C(J,\mathbb{R})$, then we say that multi-valued mapping F is Lebesgue integrable on J.

We need the following definitions in the sequel.

Definition 2.2. *A multi-valued mapping $\beta : \Omega \to \mathcal{P}_{cp}(\mathbb{R})$ is said to be measurable if for any $y \in \mathbb{R}$, the function $\omega \mapsto d(y, \beta(\omega)) = \inf\{|y - x| : x \in \beta(\omega)\}$ is measurable.*

Definition 2.3. *A multi-valued mapping $\beta : J \times \mathbb{R} \times \mathbb{R} \times \Omega \to \mathcal{P}_{cp}(\mathbb{R})$ is called strong random Carathéodory if*

(i) $\omega \mapsto \beta(t, x, y, \omega)$ *is measurable for each* $t \in J$ *and* $x, y \in \mathbb{R}$, *and*
(ii) $(t, x, y) \mapsto \beta(t, x, y, \omega)$ *is jointly Hausdorff continuous for almost everywhere* $\omega \in \Omega$,

Again, a strong random Carathéodory multi-valued function β is called strong random L^1-Carathéodory if

(iii) for each real number $r > 0$ there exists a measurable function $h_r : \Omega \to L^1(J, \mathbb{R})$ such that for almost everywhere $\omega \in \Omega$,

$$\|\beta(t, x, y, \omega)\|_{\mathcal{P}} = \sup\{|u| : u \in \beta(t, x, y, \omega)\} \leq h_r(t, \omega),$$

for all $t \in J$ and $x, y \in \mathbb{R}$ with $|x| \leq r$ and $|y| \leq r$.

Then we have the following lemmas which are well-known in the literature.

Lemma 2.1 ([15, 16])**.** *Let E be a Banach space. If dim$(E) < \infty$ and $\beta : J \times E \times \Omega \to \mathcal{P}_{cp}(E)$ is strong L^1-Carathéodory, then $S^1_\beta(x) \neq \emptyset$ for each $x \in E$.*

Lemma 2.2 ([15, 16])**.** *Let E be a Banach space. If $\beta : J \times E \to \mathcal{P}_{cp}(E)$ is strong Carathéodory, then the multi-valued mapping $t \mapsto \beta(t, x(t))$ is measurable for any measurable function $x : J \to E$.*

Let X be a metric space and let $Q : X \to \mathcal{P}_p(X)$ be a multi-valued mapping. Q is called bounded if $\bigcup Q(S)$ is bounded subset of X for all bounded subsets S of X. Q is called compact if $\overline{Q(X)} = \overline{\bigcup_{x \in X} Qx}$ is a compact subset of X. Again, Q is called totally bounded if $Q(S) = \bigcup_{x \in S} Qx$ is totally bounded subset of X for all bounded sets S in X. It is clear that every compact mapping is totally bounded, but the converse may not be true. However, these two notions are equivalent on bounded subsets of X. Q is called an upper semi-continuous at $x \in X$ if for each open set V in X containing $Q(x)$, there exists a neighborhood $N(x)$ in X such that $\bigcup Q(N(x)) \subset V$. Q is called upper semi-continuous on X if it is upper semi-continuous at each point of X. Finally, Q is called completely continuous on X if it is upper semi-continuous and totally bounded on X. It is known that if Q is a closed multi-valued mapping with compact values on X, then for any sequences $\{x_n\}$ and $\{y_n\}$ in X such that $x_n \to x_*$, $y_n \to y_*$ and $y_n \in Qx_n$, $n \in \mathbb{N}$, we have that $y_* \in Qx_*$. The converse of this statement holds if Q is a compact multi-valued mapping on X. The details of all these definitions appear in Deimling [1] and Hu and Papageorgiou [13].

A multi-valued mapping $Q : \Omega \to \mathcal{P}_p(X)$ is called measurable (respectively weakly measurable) if

$$Q^-(B) = \{\omega \in \Omega \mid Q(\omega) \cap B \neq \emptyset\} \in \mathcal{A} \tag{2.2}$$

for all closed (respectively open) subsets B in X. A multi-valued mapping $Q : \Omega \times X \to \mathcal{P}_p(X)$ is called a multi-valued random operator if $Q(\cdot, x)$ is measurable for each $x \in X$, and we write $Q(\omega, x) = Q(\omega)x$. A measurable function $\xi : \Omega \to X$ is called a random fixed point of the multi-valued random operator $Q(\omega)$ if $\xi(\omega) \in Q(\omega)\xi(\omega)$ for a.e. $\omega \in \Omega$. The set of all random fixed points of the multi-valued random operator $Q(\omega)$ is denoted by $\mathcal{F}_Q(\omega)$. A multi-valued random operator $Q : \Omega \times X \to \mathcal{P}_p(X)$ is called bounded resp. totally bounded, compact, closed, completely continuous) if the multi-valued mapping $Q(\omega, \cdot)$ is bounded (resp. totally bounded, compact, closed, completely continuous) almost everywhere for $\omega \in \Omega$.

We employ the following well-known random fixed point theorem for compact and continuous multi-valued random mappings in a separable Banach spaces or Polish space. See Dhage [7, 8, 9] and references therein.

Theorem 2.1 (Dhage [8])**.** *Let $(\Omega, \mathcal{A})$ be a measurable space and let $\mathcal{B}_r(0)$ and $\overline{\mathcal{B}_r(0)}$ be respectively the open and closed balls in a separable Banach space X centered at origin of radius r. Let $Q : \Omega \times \overline{\mathcal{B}_r(0)} \to \mathcal{P}_{cp,cv}(X)$ be continuous and condensing multi-valued random operator. If there does not exist a function $u : \Omega \to \partial\mathcal{B}_r$ with $\|u(\omega)\| = r$ such that $\lambda(\omega)u \in Q(\omega)u$ for all $\omega \in \Omega$ and for all measurable functions $\lambda : \Omega \to \mathbb{R}$ satisfying $\lambda(\omega) > 1$ on Ω, then $Q(\omega)$ has a random fixed point in $\overline{\mathcal{B}_r(0)}$.*

Remark 2.1. *It is known that the compact and totally bounded multi-valued random operators are condensing, but the converse may not be true.*

Corollary 2.1 (Dhage [8])**.** *Let $(\Omega, \mathcal{A})$ be a measurable space and let $\mathcal{B}_r(0)$ and $\overline{\mathcal{B}_r(0)}$ be respectively the open and closed balls in a separable Banach space X centered at origin of radius r. Let $Q : \Omega \times \overline{\mathcal{B}_r(0)} \to \mathcal{P}_{cp,cv}(X)$ be continuous and compact multi-valued random operator. If there does not exist a function $u : \Omega \to \partial\mathcal{B}_r$ with $\|u(\omega)\| = r$ such that $\lambda(\omega)u \in Q(\omega)u$ for all $\omega \in \Omega$ and for all measurable functions $\lambda : \Omega \to \mathbb{R}$ satisfying $\lambda(\omega) > 1$ on Ω, then $Q(\omega)$ has a random fixed point in $\overline{\mathcal{B}_r(0)}$.*

3 Existence Results

We seek the random solutions of RPBVP (1.1) in the Banach space $C(J, \mathbb{R})$ with usual supremum norm given by

$$\|x\| = \sup_{t \in J} |x(t)|. \tag{3.1}$$

Clearly, $C(J, \mathbb{R})$ becomes a separable Banach space with respect to the above norm with some nice algebraic and topological properties in $C(J, \mathbb{R})$.

We need the following definition a growth function in what follows.

Definition 3.1. *A scalar function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is called a* **submultiplicative $\mathcal{D}$-function** *if*

- *(i)* *ψ is continuous,*
- *(ii)* *ψ is nondecreasing , and*
- *(iii)* *ψ is scalarly submultiplicative, that is, $\psi(\lambda r) \le \lambda\psi(r)$ for all $\lambda \ge 0$ and $r \in \mathbb{R}^+$.*

The class of all $\mathcal{D}$-functions on $\mathbb{R}^+$ is denoted by Ψ. There do exist $\mathcal{D}$-functions on $\mathbb{R}$. Indeed, the function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ defined by $\psi(r) = \ell\, r$, $\ell > 0$ satisfies the conditions $(i) - (iii)$ mentioned above and hence a $\mathcal{D}$-function on $\mathbb{R}^+$. Note also that if $\psi \in \Psi$, then $\psi(0) = 0$.

We consider the following set of hypotheses in what follows.

(H_0) The functions $\theta, \sigma, \eta : J \to J$ are continuous and the single-valued mapping $q : \Omega \to \mathbb{R}$ is measurable.

(H_1) The single-valued mapping $k : \Omega \to C(J \times J \times \mathbb{R}, \mathbb{R})$ is measurable and there exists a measurable function $\alpha : \Omega \to L^1(J, \mathbb{R}^+)$ such that

$$\left| \int_0^{\sigma(t)} k(t, s, y, \omega)\, ds \right| \le \alpha(t, \omega)|y| \quad \text{a.e.} \quad \omega \in \Omega,$$

for all $t, s \in J$ and $y \in \mathbb{R}$, where the measurability of the function α is understood in the sense of Definition 2.1.

(H_2) For each $(t, x, y) \in J \times \mathbb{R} \times \mathbb{R}$, $F(t, x, y, \omega)$ is a compact-convex subset of $\mathbb{R}$ almost everywhere for $\omega \in \Omega$.

(H_3) F is strong random Carathéodory.

(H_4) There exists a measurable function $\gamma : \Omega \to L^1(J, \mathbb{R})$ with $\gamma(t, \omega) > 0$ a.e. $\omega \in \Omega$ and a function $\psi \in \Psi$ such that

$$\|F(t, x, y, \omega)\|_{\mathcal{P}} \le \gamma(t, \omega)\, \psi(|x| + |y|) \quad \text{a.e.} \quad \omega \in \Omega,$$

for all $t \in J$ and $x, y \in \mathbb{R}$.

We frequently make use of the following estimate concerning the multi-valued function $F(t, x, y, \omega)$ in the sequel. If the hypotheses (H_1) and (H_4) hold, then for any measurable functions $x, y : \Omega \to C(J, \mathbb{R})$ with $\|x(\omega)\| \le r$ and $\|y(\omega)\| \le r$, we obtain a bound of the multi-valued function F independent of x and y and in terms of the growth functions ψ, γ and α.

We need the following well-known result in what follows. See Dhage [7], Nieto and Lopez [17] and references therein.

Lemma 3.1. *For any function $\sigma \in L^1(J, \mathbb{R})$, x is a solution to the differential equation*

$$\left.\begin{aligned} x'(t) + \lambda x(t) &= \sigma(t), \quad t \in J, \\ x(0) &= x(T), \end{aligned}\right\} \tag{3.2}$$

if and only if it is a solution of the integral equation

$$x(t) = \int_0^T G_\lambda(t, s)\, \sigma(s)\, ds, \quad t \in J, \tag{3.3}$$

where, the Green's function $G(t, s)$ is given by

$$G_\lambda(t, s) = \begin{cases} \dfrac{e^{\lambda s - \lambda t + \lambda T}}{e^{\lambda T} - 1}, & \text{if} \quad 0 \le s \le t \le T, \\[2ex] \dfrac{e^{\lambda s - \lambda t}}{e^{\lambda T} - 1}, & \text{if} \quad 0 \le t < s \le T. \end{cases} \tag{3.4}$$

Notice that the Green's function G_λ is continuous and nonnegative on $J \times J$ and therefore, the number

$$M_{G_\lambda} := \max\{\, |G_\lambda(t, s)| \,:\, t, s \in [0, T] \,\}$$

exists for all $\lambda \in \mathbb{R}^+$. For the sake of convenience, we write $G_\lambda(t, s) = G(t, s)$ and $M_{G_\lambda} = M_G$.

Theorem 3.1. *Assume that the hypotheses $(H_0)-(H_4)$ hold. Furthermore, if there exists a real number $r > 0$ such that*

$$r \geq \int_0^T \gamma(t,\omega)[1+\alpha(t,\omega)]\psi(r)\,dt \tag{3.5}$$

for all $\omega \in \Omega$, then the RPBVP (1.1) has a random solution in $C(J,\mathbb{R})$ defined on $J \times \Omega$.

Proof. Let,

$$\Delta_1 = \big\{\omega \in \Omega \mid \text{The condition in hypothesis } (\mathrm{H}_1) \text{ is true}\big\},$$
$$\Delta_2 = \big\{\omega \in \Omega \mid \text{The condition in hypothesis } (\mathrm{H}_2) \text{ is true}\big\},$$
$$\Delta_3 = \big\{\omega \in \Omega \mid \text{The condition in hypothesis } (\mathrm{H}_3) \text{ is true}\big\}$$

and

$$\Delta_4 = \big\{\omega \in \Omega \mid \text{The condition in hypothesis } (\mathrm{H}_4) \text{ is true}\big\}.$$

Set

$$\Sigma = \Delta_1 \cap \Delta_2 \cap \Delta_3 \cap \Delta_4,$$

so that,

$$\Sigma^c = \Delta_1^c \cup \Delta_2^c \cup \Delta_3^c \cup \Delta_4^c.$$

Therefore,

$$\mu\left(\Sigma^c\right) = \mu(\Delta_1^c) + \mu(\Delta_2^c) + \mu(\Delta_3^c) + \mu(\Delta_4^c) = 0.$$

Set $X = C(J,\mathbb{R})$. Define a multi-valued operator $Q : \Omega \times X \to \mathcal{P}_p(X)$ by

$$\begin{aligned} Q(\omega)x &= \Bigg\{u \in X \,\Big|\, u(t,\omega) = \int_0^T G(t,s)v(s)\,ds,\ t \in J \\ &\qquad\qquad\qquad \text{and}\ \ v \in S_F^1(\omega)(x)\Bigg\} \\ &= \big(\mathcal{L} \circ S_F^1(\omega)\big)(x) \end{aligned} \tag{3.6}$$

where $\mathcal{L} : L^1(J,\mathbb{R}) \to C(J,\mathbb{R})$ is a continuous operator defined by

$$\mathcal{L}v(t) = \int_0^T G(t,s)\,v(s)\,ds. \tag{3.7}$$

Clearly, the operator $Q(\omega)$ is well defined in view of hypothesis (H_3). We shall show that $Q(\omega)$ satisfies all the conditions of Theorem 2.1.

Step I : First, we show that $Q(\omega)$ is a multi-valued random operator on X. First, we show that the multi-valued mapping $(\omega, x) \mapsto S_F^1(\omega)(x)$ is jointly measurable. Let $f \in L^1(J,\mathbb{R})$ be arbitrary. Then we have

$$\begin{aligned} d(f, S_F^1(\omega)(x)) &= \inf\{\|f-h\|_{L^1} : h \in S_F^1(\omega)(x)\} \\ &= \inf\left\{\int_0^T |f(t)-h(t)|\,dt : h \in S_F(\omega)(x)\right\} \\ &= \int_0^T \inf\left\{|f(t)-z| : z \in F\Big(t, x(\theta(t),\omega), \int_0^{\sigma(t)} k(t,s,x(\eta(s),\omega),\omega)\,ds, \omega\Big)\right\} dt \\ &= \int_0^T d\Big(f(t), F\Big(t, x(\theta(t),\omega), \int_0^{\sigma(t)} k(t,s,x(\eta(s),\omega),\omega)\,ds, \omega\Big)\Big)\,dt. \end{aligned}$$

Let

$$y(t,\omega)=\int_0^{\sigma(t)} k(t,s,x(\eta(s),\omega),\omega)\,ds,\ \ (t,\omega)\in J\times\Omega.$$

Since the function $x(t,\omega)$ is measurable in ω and continuous in t, by Carathéodory theorem, the mapping $(t,\omega)\mapsto x(\eta(t),\omega)$ is jointly measurable. Again since the mapping $k(t,s,x,\omega)$ is measurable in ω and continuous in t,s and x, the mapping

$$(t,s,x,\omega)\mapsto k(t,s,x(\eta(s),\omega),\omega)$$

is jointly measurable from $J\times J\times X\times\Omega$ into $\mathbb{R}$. Now the integral

$$\int_0^{\sigma(t)} k(t,s,x(\eta(s),\omega),\omega)\,ds$$

is the limit of the finite sum of measurable functions, so it is measurable, and consequently the mapping $(t,\omega)\mapsto y(t,\omega)$ is jointly measurable.

Next, the multi-valued mapping $F(t,x,y,\omega)$ is measurable in ω and d_H-continuous in t, x and y, so the multi-valued mapping $\omega\mapsto F(t,x(t,\omega),y(t,\omega),\omega)$ is measurable in view of Carathéodory theorem. This further implies that the mapping

$$\omega\mapsto F\left(t,x(\theta(t),\omega),\int_0^{\sigma(t)} k(t,s,x(\eta(s),\omega),\omega)\,ds,\omega\right)$$

is measurable for all t and $x,y\in X$. Now the function $z\mapsto d(z,F(t,x,y,\omega))$ is continuous, and, so, the the mapping

$$(t,x,\omega,f)\mapsto d\left(f(t),F\Big(t,x(\theta(t),\omega),\int_0^{\sigma(t)} k(t,s,x(\eta(s),\omega),\omega)\,ds,\omega\Big)\right)$$

is jointly measurable from $J\times X\times\Omega\times L^1(J,\mathbb{R})$ into $\mathbb{R}^+$. Now the integral is the limit of the finite sum of measurable functions, and so, $d(f,S^1_F(\omega)(x))$ is measurable. As a result, the multi-valued mapping $(\cdot,\cdot)\to S^1_{F(\cdot)}(\cdot)$ is jointly measurable.

Define the multi-valued map ϕ on $J\times X\times\Omega$ by

$$\phi(t,x,\omega)=\int_0^T G(t,s)F\Big(s,x(\theta(s),\omega),\int_0^{\sigma(s)} k(s,\tau,x(\eta(\tau),\omega),\omega)\,d\tau,\omega\Big)\,ds.$$

We shall show that $\phi(t,x,\omega)$ is continuous in t in the Hausdorff metric on $\mathbb{R}$. Let $\{t_n\}$ be a sequence in J converging to $t\in J$. Then, we have

$$\begin{aligned}
&d_H(\phi(t_n,x,\omega),\phi(t,x,\omega))\\
&=d_H\left(\int_0^T G(t_n,s)F\Big(s,x(\theta(s),\omega),\int_0^{\sigma(s)} k(s,\tau,x(\eta(\tau),\omega),\omega)\,d\tau,\omega\Big)\,ds,\right.\\
&\qquad\left.\int_0^T G(t,s)F\Big(s,x(\theta(s),\omega),\int_0^{\sigma(s)} k(s,\tau,x(\eta(\tau),\omega),\omega)\,d\tau,\omega\Big)\,ds\right)\\
&=\int_0^T |G(t_n,s)-G(t,s)|\Big\|F\Big(s,x(\theta(s),\omega),\int_0^{\sigma(s)} k(s,\tau,x(\eta(\tau),\omega),\omega)\,d\tau,\omega\Big)\Big\|_{\mathcal{P}}\,ds\\
&=\int_0^T |G(t_n,s)-G(t,s)|\,\gamma(s,\omega)\psi\Big(|x(\eta(s),\omega)|
\end{aligned}$$

$$+\Big|\int_0^{\sigma(s)} k(s,\tau,x(\eta(\tau),\omega),\omega),\ d\tau)\Big|\Big)\,ds$$

$$=\int_0^T |G(t_n,s)-G(t,s)|\,\gamma(s,\omega)\psi\Big(|x(\theta(s),\omega)|+\alpha(s,\omega)|x(\eta(s),\omega)|\Big)\,ds$$

$$\to 0 \quad \text{as} \quad n\to\infty.$$

Thus, the multi-valued map $t\mapsto \phi(t,x,\omega)$ is continuous and hence, by Lemma 2.2, the map

$$(t,x,\omega)\mapsto \int_0^t G(t,s)F\Big(s,x(\eta(s),\omega),\int_0^{\sigma(s)} k(s,\tau,x(\eta(\tau),\omega),\omega)\,d\tau,\omega\Big)\,ds$$

is measurable. Consequently, $Q(\omega)$ is a random multi-valued random operator on X into itself.

Step II : Next, we show that $Q(\omega)$ is compact and continuous for each fixed $\omega\in\Sigma$. First, we show that $Q(\omega)$ is a continuous multi-valued random operator on X. Let $\{x_n\}$ be a given sequence of points in X converging to a point x. Then by d_H-continuity of the multi-valued mapping $F(t,x,y,\omega)$ in x and y and by the dominated convergence theorem, we obtain

$$\begin{aligned}
&\lim_{n\to\infty} Q(\omega)x_n(t)\\
&=\lim_{n\to\infty}\int_0^T G(t,s)F\Big(s,x_n(\theta(s),\omega),\int_0^{\sigma(s)} k(s,\tau,x_n(\eta(\tau),\omega),\omega)\,d\tau,\omega\Big)ds\\
&=\int_0^T G(t,s)\lim_{n\to\infty} F\Big(s,x_n(\theta(s),\omega),\int_0^{\sigma(s)} k(s,\tau,x_n(\eta(\tau),\omega),\omega)\,d\tau,\omega\Big)ds\\
&=Q(\omega)x(t)
\end{aligned}$$

for all $t\in J$ and $\omega\in\Sigma$. This shows that $Q(\omega)$ is a Hausdorff continuous multi-valued random operator on X.

Next we how that $Q(\omega)$ is compact operator on X for each fixed $\omega\in\Sigma$. If S be a bounded set in X, then there is a constant $r>0$ such that $\|x\|\le r$ for all $x\in S$. Let $\{y_n(\omega)\}$ be a sequence sequence in $\bigcup Q(\omega)(S)$ for above fixed $\omega\in\Sigma$. We will show that $\{y_n(\omega)\}$ has a cluster point. This is achieved by showing that $\{y_n(\omega)\}$ is uniformly bounded and equi-continuous sequence in X.

Case I : First, we show that $\{y_n(\omega)\}$ is uniformly bounded sequence. By the definition of $\{y_n(\omega)\}$, we have a $v_n\in S^1_F(\omega)(x_n)$ for some $x_n\in S$ such that $y_n(t,\omega)=\int_0^T G(t,s)v_n(s)\,ds,\ t\in J.$

Therefore,

$$\begin{aligned}
|y_n(t,\omega)| &\le \int_0^T G(t,s)|v_n(s)|\,ds\\
&\le \int_0^T G(t,s)\Big\|F\Big(s,x(\theta(s),\omega),\int_0^{\sigma(s)} k(s,\tau,x(\eta(\tau),\omega),\omega)\,d\tau,\omega\Big)\Big\|_{\mathcal{P}}\,ds\\
&\le \int_0^T G(t,s)\gamma(s,\omega)\psi(|x(\theta(s),\omega)|+\alpha(s,\omega)|x(\eta(s),\omega)|)\,ds\\
&\le M_G\,\|\delta(\omega)\|_{L^1}\psi(r)
\end{aligned}$$

for all $t \in J$, where $\delta(t,\omega) = \gamma(t,\omega)(1+\alpha(t,\omega))$ for all $(t,\omega) \in J \times \Sigma$. Taking the supremum over t in the above inequality yields,

$$\|y_n(\omega)\| \leq M_G \, \|\delta(\omega)\|_{L^1} \psi(r)$$

which shows that $\{y_n(\omega)\}$ is a uniformly bounded sequence in $Q(\omega)(X)$.

Case II : Next we show that $\{y_n(\omega)\}$ is an equi-continuous sequence in $Q(\omega)(S)$. Let $t, \tau \in J$. Then, for each fixed $\omega \in \Sigma$, we have

$$\begin{aligned}
&|y_n(t,\omega) - y_n(\tau,\omega)| \\
&= \Big| \int_0^T G(t,s) v_n(s)\, ds - \int_0^T G(\tau,s) v_n(s)\, ds \Big| \\
&\leq \Big| \int_0^T \big| G(t,s) - G(\tau,s) \big| \, |v_n(s)|\, ds \Big| \\
&\leq \Big| \int_0^T \big| G(t,s) - G(\tau,s) \big| \, \Big\| F\Big(s, x(\theta(s),\omega), \int_0^{\sigma(s)} k(s,\tau,x(\eta(\tau),\omega),\omega)\, d\tau, \omega\Big) \Big\|_{\mathcal{P}} ds \Big| \\
&\leq \Big| \int_0^T \big| G(t,s) - G(\tau,s) \big| \gamma(s,\omega) \psi(\|x(\omega)\| + \alpha(s,\omega)\|x(\omega)\|)\, ds \Big| \\
&\leq \Big| \int_0^T \big| G(t,s) - G(\tau,s) \big| \, \gamma(s,\omega)[1+\alpha(s,\omega)]\psi(r)\, ds \Big|,
\end{aligned}$$

for all $n \in \mathbb{N}$. From the above inequality, it follows that

$$|y_n(t,\omega) - y_n(\tau,\omega)| \to 0 \quad \text{as} \quad t \to \tau.$$

uniformly for all $n \in \mathbb{N}$. This shows that $\{y_n(\omega)\}$ is an equi-continuous sequence in $Q(\omega)(S)$. Now $\{y_n(\omega)\}$ is a uniformly bounded and equi-continuous sequence for each $\omega \in \Sigma$, so it has a cluster point in view of Arzelà-Ascoli theorem. Thus Q is a continuous and compact and hence completely continuous multi-valued random operator on $\Omega \times X$ into X.

Step III : Next, we show that $Q(\omega)$ has convex values on X for each $\omega \in \Sigma$. Again, let $u_1, u_2 \in Q(\omega)x$. Then there are $v_1, v_2 \in S^1_G(\omega)(x)$ such that

$$u_1(t) = \int_0^T G(t,s) v_1(s)\, ds, \;\; t \in J,$$

and

$$u_2(t) = \int_0^T G(t,s) v_2(s)\, ds, \;\; t \in J.$$

Now for any $\lambda \in [0,1]$, one has

$$\begin{aligned}
&\lambda u_1(t) + (1-\lambda) u_2(t) \\
&\qquad = \lambda \left(\int_0^T G(t,s) v_1(s))\, ds \right) + (1-\lambda) \left(\int_0^T G(t,s) v_2(s)\, ds \right) \\
&\qquad = \int_0^T G(t,s) \big[\lambda v_1(s) + (1-\lambda) v_2(s) \big]\, ds.
\end{aligned}$$

Since $S^1_F(\omega)$ has convex values on X (because F has convex values), we have that $v(t) = \lambda v_1(t) + (1-\lambda) v_2(t) \in S^1_F(\omega)(x)(t)$ for all $t \in J$. Hence, $\lambda u_1 + (1-\lambda) u_2 \in Q(\omega)x$ and consequently $Q(\omega)x$ is a convex subset of X for each $x \in X$. As a result, $Q(\omega)$ defines a multi-valued random operator $Q : \Omega \times X \to \mathcal{P}_{cp,cv}(X)$.

Step IV : Finally, let r be a fixed positive real number and consider the closed ball $B[0,r]$ in the separable Banach space $C(J,\mathbb{R})$. Let there exists an $u:\Omega\to\partial B_r$ be such that $\|u\|=r$ and $\lambda u(t,\omega)\in Q(\omega)u(t,\omega)$ on $J\times\Omega$ for all $\lambda>1$. Then there is a $v\in S^1_F(\omega)(u)$ such that

$$\lambda\, u(t,\omega)=\int_0^T G(t,s)v(s)\,ds$$

for all $t\in J$ and $\omega\in\Omega$. Therefore,

$$\begin{aligned}
\lambda\,|u(t,\omega)| &\le \int_0^T |G(t,s)|\,|v(s)|\,ds\\
&\le \int_0^T G(t,s)\Big\|F\Big(s,u(\theta(s),\omega),\int_0^{\sigma(s)}k(s,\tau,u(\eta(\tau),\omega),\omega)\,d\tau,\omega\Big)\Big\|_{\mathcal{P}}\,ds\\
&\le \int_0^T \gamma(s,\omega)\psi(|u(\theta(t),\omega)|+\alpha(s,\omega)|u(\eta(s),\omega)|)\,ds\\
&\le \int_0^T \gamma(s,\omega)\psi(\|u(\omega)\|+\alpha(s,\omega)\|u(\omega)\|)\,ds\\
&\le \int_0^T \gamma(s,\omega)\big[1+\alpha(s,\omega)\big]\psi(\|u(\omega)\|)\,ds\\
&\le \int_0^T \gamma(s,\omega)\big[1+\alpha(s,\omega)\big]\psi(r)\,ds \qquad (3.8)
\end{aligned}$$

for all $t\in J$ and $\omega\in\Omega$. Taking the supremum over t in the above inequality (3.8), we obtain

$$\lambda\|u(\omega)\|\le\int_0^T\gamma(s,\omega)\big[1+\alpha(s,\omega)\big]\psi(r)\,ds$$

or

$$\lambda r\le\int_0^T\gamma(s,\omega)\big[1+\alpha(s,\omega)\big]\psi(r)\,ds.$$

This is a contraction since $\lambda>1$ and the inequality

$$\int_0^T\gamma(s,\omega)\big[1+\alpha(s,\omega)\big]\psi(r)\,ds\le r$$

for all $s\in J$ and $\omega\in\sum$. Now a direct application of Corollary 2.1 yields the desired result.
□

4 Non-Convex Case

In this section, we obtain the existence of the random solutions for nonconvex case of RPBVP (1.1) defined on $J\times\Omega$. This is achieved under certain monotonicity conditions of the multi-valued function F with respect to certain order relation in $C(J,\mathbb{R})$. Define an order relation $\preceq$ in $C(J,\mathbb{R})$ by

$$x\preceq y\iff y-x\in K \qquad (4.1)$$

where, the order cone K in $C(J,\mathbb{R})$ defined by

$$K=\{x\in C(J,\mathbb{R})\mid x(t)\ge 0\ \text{ for all }\ t\in J\}. \qquad (4.2)$$

Clearly, $C(J,\mathbb{R})$ becomes an ordered Banach space with the order cone K which is normal in it. Let $a,b\in X=C(J,\mathbb{R})$ be such that $a\preceq b$. Then by an order interval $[a,b]$ we mean a set in X defined by

$$[a,b]=\{x\in X\mid a\preceq x\preceq b\}. \qquad (4.3)$$

Let $a, b : \Omega \to X$ be two measurable functions. By $a \preceq b$ on Ω, we mean $a(\omega) \preceq b(\omega)$ for all $\omega \in \Omega$. Then the sector $[a, b]$ defined by

$$\begin{aligned} [a,b] &= \{x \in X \mid a(\omega) \preceq x \preceq b(\omega) \text{ for all } \omega \in \Omega\} \\ &= \bigcap_{\omega \in \Omega} [a(\omega), b(\omega)] \end{aligned}$$

is called the **random order interval** in X.

Now we define the different notions of order relations in $\mathcal{P}_p(X)$ as follows. These order theoretic notions are useful to define different monotonic concepts of the multi-valued random mappings on the ordered Banach space X.

Let $A, B \in \mathcal{P}_p(X)$. Then by $A \overset{i}{\preceq} B$ we mean "for every element $a \in A$ there exists an element $b \in B$ such that $a \preceq b$." Again $A \overset{d}{\preceq} B$ means "for each $b \in B$ there exists $a \in A$ such that $a \preceq b$". Further, we have $A \overset{id}{\preceq} B \iff A \overset{i}{\preceq} B$ and $A \overset{d}{\preceq} B$. Finally, $A \preceq B$ implies that $a \preceq b$ for all $a \in A$ and $b \in B$. Note that if $A \preceq A$, then it follows that A is a singleton set.

Definition 4.1. *A multi-valued random operator $Q : \Omega \times X \to \mathcal{P}_{cl}(X)$ is called right monotone increasing if $x, y \in X$, $x \preceq y$, then $S_Q(\omega)(x) \overset{i}{\preceq} S_Q(\omega)(y)$ a.e. $\omega \in \Omega$.*

We employ the following random fixed point theorem of Dhage [2, 3, 4, 5, 6] for right monotone increasing multi-valued random operators in ordered Banach spaces in what follows.

Theorem 4.1. *Let $(\Omega, \mathcal{A})$ be a measurable space and let $[a, b]$ be a random interval in a separable ordered Banach space X. If $Q : \Omega \times [a, b] \to \mathcal{P}_{cp}([a, b])$ is a compact, upper semi-continuous, right monotone increasing multi-valued random operator and the cone $\mathcal{K}$ in X is normal, then $Q(\omega)$ has a random fixed point in $[a, b]$.*

Definition 4.2. *A multi-valued $F : J \times \mathbb{R} \times \mathbb{R} \times \Omega \to \mathcal{P}_{cp}(\mathbb{R})$ is called random Carathéodory if*

(i) $\omega \mapsto F(t, x, y, \omega)$ is measurable for each $t \in J$ and $x, y \in \mathbb{R}$, and
(ii) $(t, x, y) \mapsto F(t, x, y, \omega)$ is an upper semi-continuous almost everywhere for $\omega \in \Omega$.

Again, a random Carathéodory multi-valued function F is called random L^1-Carathéodory if

(iii) for each real number $r > 0$ there exists a measurable function $h_r : \Omega \to L^1(J, \mathbb{R})$ such that for a.e. $\omega \in \Omega$,

$$\|F(t, x, y, \omega)\|_{\mathcal{P}} = \sup\{|u| : u \in F(t, x, y, \omega)\} \leq h_r(t, \omega)$$

for all $t \in J$ and $x, y \in \mathbb{R}$ with $|x| \leq r$ and $|y| \leq r$.

Then we have the following lemmas which are well-known in the literature.

Lemma 4.1. *Let E be a Banach space. If $dim(E) < \infty$ and $F : J \times E \times \Omega \to \mathcal{P}_{cp}(E)$ is random L^1-Carathéodory, then $S^1_F(x) \neq \emptyset$ for each $x \in E$.*

Lemma 4.2. *Let E be a Banach space, F a Carathéodory classical multi-valued operator with $S^1_F \neq \emptyset$ and let $\mathcal{L} : L^1(J,E) \to C(J,E)$ be a continuous linear mapping. Then the composite operator*

$$\mathcal{L} \circ S^1_F : C(J,E) \to \mathcal{P}_{bd,cl}(C(J,E))$$

is a closed graph operator on $C(J,E) \times C(J,E)$.

We need the following definition in what follows.

Definition 4.3. *A measurable function $a : \Omega \to C(J,\mathbb{R})$ is a strict lower random solution for the RPBVP (1.1) if for all $v \in S^1_F(\omega)(a)$, we have*

$$a'(t,\omega) + \lambda a(t,\omega) \leq v(t), \;\; a(0,\omega) \leq a(T,\omega) \quad a.e. \quad \omega \in \Omega,$$

for all $t \in J$. Similarly, a strict upper random solution for the RPBVP (1.1) on $J \times \Omega$ is defined.

We frequently use the following fundamental results while establishing the existence and approximation theorems for random solution of the RPBVP (1.1) in what follows.

Lemma 4.3 (Dhage [10, 11])**.** *If there exists a differentiable function $u \in C(J,\mathbb{R})$ such that*

$$\left.\begin{aligned} u'(t) + \lambda u(t) \leq \sigma(t), \;\; t \in J,\\ u(0) \leq u(T), \end{aligned}\right\} \tag{4.4}$$

for all $t \in J$, where $\lambda \in \mathbb{R}$, $\lambda > 0$ and $\sigma \in L^1(J,\mathbb{R})$, then

$$u(t) \leq \int_0^T G(t,s)\,\sigma(s)\,ds, \tag{4.5}$$

for all $t \in J$, where $G(t,s)$ is a Green's function given by the expression (3.4) on $J \times J$.

Lemma 4.4 (Dhage [10, 11])**.** *If there exists a differentiable function $v \in C(J,\mathbb{R})$ such that*

$$\left.\begin{aligned} v'(t) + \lambda v(t) \geq \sigma(t), \;\; t \in J,\\ v(0) \geq v(T), \end{aligned}\right\} \tag{4.6}$$

for all $t \in J$, where $\lambda \in \mathbb{R}$, $\lambda > 0$ and $\sigma \in L^1(J,\mathbb{R})$, then

$$v(t) \geq \int_0^T G(t,s)\,\sigma(s)\,ds, \tag{4.7}$$

for all $t \in J$, where $G(t,s)$ is a Green's function given by expression (3.4) on $J \times J$.

We consider the following set of hypotheses in the sequel.

(H_5) F is random L^1-Carathéodory.
(H_6) The multi-valued mapping $x \mapsto S^1_F(\omega)(x)$ is right monotone increasing in $C(J,\mathbb{R})$.
(H_7) RPBVP (1.1) has a strict lower random solution a and a strict upper random solution b with $a \preceq b$ defined on $J \times \Omega$.

Hypotheses (H_5) is common in the literature. Some nice sufficient conditions for guarantying $S^1_F(\omega) \neq \emptyset$ appear in Deimling [32,33] and references therein. A mild hypothesis of (H_5) has been used in Halidias and Papageorgiou [60]. Hypotheses (H_6) and (H_7) are relatively new to the literature, but the special forms have already been appeared in the works of several authors.

Theorem 4.2. *Assume that the assumptions (H_0)-(H_1) and (H_5)-(H_7) hold. Then the RPBVP (1.1) has a random solution in $[a, b]$ defined on $J \times \Omega$.*

Proof. Let,

$$\Gamma_1 = \big\{\omega \in \Omega \mid \text{The condition in hypothesis } (\mathrm{H}_1) \text{ is true}\big\},$$

$$\Gamma_5 = \big\{\omega \in \Omega \mid \text{The condition in hypothesis } (\mathrm{H}_5) \text{ is true}\big\},$$

$$\Gamma_6 = \big\{\omega \in \Omega \mid \text{The condition in hypothesis } (\mathrm{H}_6) \text{ is true}\big\}$$

and

$$\Gamma_7 = \big\{\omega \in \Omega \mid \text{The condition in hypothesis } (\mathrm{H}_7) \text{ is true}\big\}.$$

Set

$$\Gamma = \Gamma_1 \cap \Gamma_5 \cap \Gamma_6 \cap \Gamma_7,$$

so that,

$$\Gamma^c == \Gamma_1^c \cup \Gamma_5^c \cup \Gamma_6^c \cup \Gamma_7^c.$$

Therefore,

$$\mu\left(\Gamma^c\right) = \mu(\Gamma_1^c) + \mu(\Gamma_5^c) + \mu(\Gamma_6^c) + \mu(\Gamma_7^c) = 0.$$

Let $X = C(J, \mathbb{R})$. Define a random order interval $[a, b]$ in X which is well defined in view of hypothesis (H_7). Now the RPBVP (1.1) is equivalent to the random integral inclusion

$$x(t,\omega) \in \int_0^T G(t,s) F\Big(s, x(\eta(s),\omega), \int_0^{\sigma(s)} k(s,\tau,x(\eta(\tau),\omega),\omega)\, d\tau, \omega\Big) ds \tag{4.8}$$

for all $t \in J$.

Define a multi-valued operator $Q : \Omega \times [a, b] \to \mathcal{P}_p(X)$ by (3.6). Clearly, the operator $Q(\omega)$ is well defined in view of hypothesis (H_5). We shall show that $Q(\omega)$ satisfies all the conditions of Theorem 4.1.

Step I : First, we show that Q is closed valued multi-valued random operator on $\Omega \times [a, b]$. Let $\omega \in \Gamma$ be fixed. Observe that the operator $Q(\omega) = F^1(\omega)$. To show $Q(\omega)$ has closed values, it then suffices to prove that the composition operator $\mathcal{L} \circ S_F^1(\omega)$ has closed values on $[a, b]$. Let $x \in [a, b]$ be arbitrary and let $\{v_n\}$ be a sequence in $S_F^1(\omega)(x)$ converging to v in measure. Then, by the definition of $S_F^1(\omega)$,

$$v_n(t) \in F\bigg(t, x(\theta(t),\omega), \int_0^{\sigma(t)} k(t,\tau,x(\eta(\tau),\omega),\omega)\, d\tau, \omega\bigg) \quad \text{a.e. } \omega \in \Omega.$$

for all $t \in J$. Since $F\Big(t, x(\theta(t),\omega), \int_0^{\sigma(t)} k(t,\tau,x(\eta(\tau),\omega),\omega)\, d\tau, \omega\Big)$ is closed, we have that

$$v(t) \in F\bigg(t, x(\theta(t),\omega), \int_0^{\sigma(t)} k(t,\tau,x(\eta(\tau),\omega),\omega)\, d\tau, \omega\bigg) \quad \text{a.e. } \omega \in \Omega$$

for all $t \in J$. Hence, $v \in S_F^1(\omega)(x)$. As a result, $S_F^1(\omega)(x)$ is a closed set in $L^1(J, \mathbb{R})$ for each $\omega \in \Sigma$. From the continuity of $\mathcal{L}$, it follows that $(\mathcal{L} \circ S_F^1(\omega)(x)$ is a closed set in X. Therefore, $Q(\omega)$ is a closed-valued multi-valued operator on $[a, b]$ for each $\omega \in \Gamma$.

Next, proceeding with the arguments as in Theorem 4.2, it can be shown that $Q(\omega)$ is a multi-valued random operator on $[a, b]$ into X.

Step II: Secondly, we show that $Q(\omega)$ is right monotone increasing and multi-valued random operator on $[a,b]$ into itself. Let Let $\omega \in \Gamma$ be fixed and let $x, y \in [a,b]$ be such that $x \preceq y$. Since (H_6) holds, we have that $S^1_F(\omega)(x) \overset{i}{\preceq} S^1_F(\omega)(y)$. Hence $Q(\omega)(x) \overset{i}{\preceq} Q(\omega)(y)$ for all $\omega \in \Sigma$. From Lemmas 4.3 and 4.4 it follows that $a \preceq Q(\omega)a$ and $Q(\omega)b \preceq b$ for all $\omega \in \Gamma$. Therefore, $Q(\omega)$ is a right monotone increasing, so we have for almost everywhere $\omega \in \Omega$,

$$a \preceq Q(\omega)a \overset{i}{\preceq} Q(\omega)x \overset{i}{\preceq} Q(\omega)b \preceq b$$

for all $x \in [a,b]$. Hence Q defines a right monotone increasing multi-valued random operator $Q : \Omega \times [a,b] \to \mathcal{P}_{cp}([a,b])$.

Step III : Next, we show that $Q(\omega)$ is completely continuous random multi-valued map on $\Omega \times [a,b]$. First, we show that $Q(\omega)([a,b])$ is compact for each $\omega \in \Gamma$. Let $\{y_n(\omega)\}$ be a sequence in $Q(\omega)([a,b])$ for above fixed $\omega \in \Gamma$. We will show that $\{y_n(\omega)\}$ has a cluster point. This is achieved by showing that $\{y_n(\omega)\}$ is uniformly bounded and equi-continuous sequence in X.

Case I : First, we show that $\{y_n(\omega)\}$ is uniformly bounded sequence. Since the cone K in X is normal, the random order interval $[a,b]$ is norm-bounded. Hence there is a real number $r > 0$ such that $\|y_n(\omega)\| \le r$ for all $n \in \mathbb{N}$. By the definition of $\{y_n(\omega)\}$, we have a $v_n(\omega) \in S^1_F(\omega)(x)$ for some $x \in [a,b]$ such that

$$y_n(t,\omega) = \int_0^T G(t,s)v_n(s)\,ds,\ \ t \in J.$$

Therefore,

$$\begin{aligned}
|y_n(t,\omega)| &\le \int_0^T G(t,s)|v_n(s)|\,ds \\
&\le \int_0^T G(t,s)\Big\|F\Big(s, x(\theta(s),\omega), \int_0^{\sigma(s)} k(s,\tau,x(\eta(\tau),\omega),\omega)\,d\tau,\omega\Big)\Big\|_{\mathcal{P}}\,ds \\
&\le M_G \int_0^T h_r(s,\omega)\,ds \\
&\le M_G\|h_r(\omega)\|_{L^1}
\end{aligned}$$

for all $t \in J$, where $r = \|a(\omega)\| + \|b(\omega)\|$. Taking the supremum over t in the above inequality yields,

$$\|y_n(\omega)\| \le M_G\|h_r(\omega)\|_{L^1}$$

which shows that $\{y_n(\omega)\}$ is a uniformly bounded sequence in $Q(\omega)([a,b])$.

Next we show that $\{y_n(\omega)\}$ is an equi-continuous sequence in $Q(\omega)([a,b])$. Let $t, \tau \in J$. Then we have

$$\begin{aligned}
|y_n(t,\omega) - y_n(\tau,\omega)| &\le \Big|\int_0^T G(t,s)v_n(s)\,ds - \int_0^T G(\tau,s)v_n(s)\,ds\Big| \\
&\le \Big|\int_0^T |G(t,s) - G(\tau,s)|\,|v_n(s)|\,ds\Big| \\
&\le \Big|\int_0^T |G(t,s) - G(\tau,s)|\,h_r(s,\omega)\,ds\Big|
\end{aligned}$$

$$\to 0 \quad \text{as} \quad t \to \tau$$

for each $n \in \mathbb{N}$ and $\omega \in \Omega$.

This shows that $\{y_n(\omega)\}$ is an equi-continuous sequence in $Q(\omega)([a,b])$. So $\{y_n(\omega)\}$ has a cluster point in view of Arzelà-Ascoli theorem. As a result, $Q(\omega)$ is a compact multi-valued random operator on $\Omega \times [a,b]$ into $\mathcal{P}_{cp}([a,b])$.

Case II : Next, let $\{x_n(\omega)\}$ be a sequence in $[a,b]$ such that $x_n(\omega) \to x_*(\omega)$. Let $\{y_n(\omega)\}$ be a sequence such that $y_n(\omega) \in Q(\omega)x_n$ and $y_n(\omega) \to y_*(\omega)$. We show that $y_*(\omega) \in Q(\omega)x_*$. Since $y_n(\omega) \in Q(\omega)x_n$, there exists a $v_n \in S^1_F(\omega)(x_n)$ such that

$$y_n(t,\omega) = \int_0^T G(t,s)v_n(s)\,ds, \ \ t \in J.$$

From lemma 4.2, it follows that $\mathcal{L} \circ S^1_F(\omega)$ is a closed graph operator. Also from the definition of $\mathcal{L}$, we have

$$y_n(t,\omega) \in \big(\mathcal{L} \circ S^1_F(\omega)\big)(x_n).$$

Since $y_n(\omega) \to y_*(\omega)$, there is a point $v_* \in S^1_F(x_*)$ such that

$$y_*(t,\omega) = \int_0^T G(t,s)v_*(s)\,ds, \ \ t \in J.$$

This shows that $Q(\omega)$ is a upper semi-continuous multi-valued random operator on $[a,b]$. Therefore, $Q(\omega)$ defines an upper semi-continuous and compact and hence completely continuous multi-valued random operator on $[a,b]$. Now an application of Theorem 4.1 yields that $Q(\omega)$ has a random fixed point which further implies that the RPBVP (1.1) has a random solution on $J \times \Omega$. This completes the proof. □

Next, we prove the existence of the extremal random solutions for the RPBVP (1.1) on $J \times \Omega$.

Definition 4.4. *A multi-valued random operator $Q : \Omega \times X \to \mathcal{P}_p(X)$ is called strict monotone increasing if $x, y \in X$ $x \preceq y$, $x \neq y$, then $Q(\omega)x \preceq Q(\omega)y$ a.e. $\omega \in \Omega$. Similarly, the multi-valued random operator $Q(\omega)$ is called strict monotone decreasing if $x, y \in X$ $x \preceq y$, $x \neq y$, then $Q(\omega)x \succeq Q(\omega)y$ a.e. $\omega \in \Omega$. Finally, $Q(\omega)$ is called strict monotone if it is either a strict monotone increasing or strict monotone decreasing multi-valued random operator on X.*

Remark 4.1. *We remark that every strict monotone increasing multi-valued random operator in an ordered Banach space with compact values is right monotone increasing, but the converse may not be true.*

Below we prove a random fixed point theorem for strict monotone increasing multi-valued random operators on a separable ordered Banach space into itself. We use the the following multi-valued random fixed point theorem of Dhage [6] in what follows.

Theorem 4.3 (Dhage [6])**.** *Let $(\Omega, \mathcal{A})$ be a measurable space and let $[a,b]$ be a random interval in a separable ordered Banach space X. If $Q : \Omega \times [a,b] \to \mathcal{P}_{cp}([a,b])$ is completely continuous and strict monotone increasing multi-valued random operator and the cone K in*

X is normal, then $Q(\omega)$ has a least random fixed point $x_*(\omega)$ and a greatest random fixed point $y^*(\omega)$ in $[a,b]$ and the sequences $\{x_n(\omega)\}$ and $\{y_n(\omega)\}$ defined by

$$x_0(\omega) = a(\omega),\ x_{n+1}(\omega) \in Q(\omega)x_n,\ n = 0, 1, 2, ...,$$

and

$$y_0(\omega) = b(\omega),\ y_{n+1}(\omega) \in Q(\omega)y_n,\ n = 0, 1, 2, ...,$$

converge to $x_*(\omega)$ and $y^*(\omega)$ respectively.

We consider the following hypothesis in the sequel.

(H_8) For each $t, s \in J$, the single-valued mapping $k(t,s,x,\omega)$ is nondecreasing in x almost everywhere for $\omega \in \Omega$.

(H_9) For each $t \in J$, the multi-valued map $(F(t,x,y,\omega)$ is strict monotone increasing in x and y almost everywhere for $\omega \in \Omega$.

Theorem 4.4. *Assume that (H_5), (H_7) and (H_8)-(H_9) hold. Then the RPBVP (1.1) has a minimal random solution and a maximal random solution in $[a,b]$ defined on $J \times \Omega$.*

Proof. The proof is quite similar to that of Theorem 4.2. We briefly sketch the outline of the proof. Let,

$$\Upsilon_5 = \big\{\omega \in \Omega \mid \text{The condition in hypothesis } (\mathrm{H}_5) \text{ is true}\big\},$$
$$\Upsilon_7 = \big\{\omega \in \Omega \mid \text{The condition in hypothesis } (\mathrm{H}_7) \text{ is true}\big\},$$
$$\Upsilon_8 = \big\{\omega \in \Omega \mid \text{The condition in hypothesis } (\mathrm{H}_8) \text{ is true}\big\}$$

and

$$\Upsilon_9 = \big\{\omega \in \Omega \mid \text{The condition in hypothesis } (\mathrm{H}_9) \text{ is true}\big\}.$$

Set

$$\Upsilon = \Upsilon_5 \cap \Upsilon_7 \cap \Upsilon_8 \cap \Upsilon_9,$$

so that,

$$\Upsilon^c = \Upsilon_5^c \cup \Upsilon_7^c \cup \Upsilon_8^c \cup \Upsilon_9^c.$$

Therefore,

$$\mu\left(\Upsilon^c\right) = \mu(\Upsilon_5^c) + \mu(\Upsilon_7^c) + \mu(\Upsilon_8^c) + \mu(\Upsilon_9^c) = 0.$$

Set $X = C(J,\mathbb{R})$ and consider the random order interval $[a,b]$ in X which does exist in view of hypothesis (H_7). Here $S_F^1(\omega)(x) \neq \emptyset$ for each $x \in [a,b]$ and $\omega \in \Upsilon$ in view of hypothesis (H_5). Also by hypotheses (H_8) and (H_9), the multi-valued mapping $x \mapsto S_F^1(\omega)(x)$ is strict monotone increasing on $[a,b]$. Consequently, the multi-valued random operator $Q(\omega)$ defined by (3.6) is strict monotone increasing on $[a,b]$. The rest of the proof is similar to Theorem 4.2 and now the desired result follows by an application of Theorem 4.3. □

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