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SOME COUPLED FIXED POINT THEOREMS IN CONVEX METRIC SPACES

By
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Abstract

We establish some coupled fixed point theorems in convex cone metric spaces for $(k, \mu)-$Lipschitzian and $(k, \mu, L)-$Lipschitzian mappings. We assume that the cone has nonempty interior. Our results generalize and extend several known results in the existing literature.
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1 Introduction

The term *'coupled fixed point'* was coined from the seminal paper of Guo and Lakshmikantham [13] in which the concept was introduced in 1987. Several coupled fixed point theorems established in cone metric space and partially ordered metric space are available in the literature. In addition to [13], some of the various authors whose contributions are also of colossal value in the study of the notion of coupled fixed point are Abbas and Beg [1], Bhaskar and Lakshmikantham [4], Beg *et al.* [7], Chang and Ma [8], Chang *et al.* [9], Dua and Li [11], Guu [14] as well as a host of others in the literature.

Bhaskar and Lakshmikantham [4] established a coupled fixed point theorem in a metric space endowed with partial order by employing a weak contractive type condition. More recently, the result of [4] was further generalized and extended by Lakshmikantham and Ciric [16]. See also Ciric and Lakshmikantham [10].

We shall employ the following definitions in the sequel:

Definition 1.1. *Let E be a real Banach space. A nonempty convex closed subset $P \subset E$ is called a cone in E if it satisfies the following:*
(i) P is closed, nonempty and $P \neq \{0\}$;
(ii) $a,\ b \in I\!R,\ a,\ b \geq 0$ and $x,\ y \in P \implies ax + by \in P$;
(iii) $x \in P$ and $-x \in P \implies x = 0$.

For a given cone $P \subset E$, the partial ordering $\preceq$ with respect to P is defined by $x \preceq y$ if and only if $y - x \in P$. If $y - x \in \text{int } P$, we write $x \prec\prec y$ (where int P denotes the interior of P). Also, we use $x \prec y$ if $x \preceq y$ and $x \neq y$.

Definition 1.2. *Let X be a nonempty set and let E be a real Banach space equipped with the partial ordering $\preceq$ with respect to the cone $P \subset E$. Suppose that the mapping $d\colon X \times X \to E$ satisfies the following conditions:*
(i) $0 \preceq d(x, y),\ \forall\ x,\ y \in X$ and $d(x, y) = 0 \iff x = y$;

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(ii) $d(x,y) = d(y,x),\ \forall\ x,\ y \in X$;
(iii) $d(x,y) \preceq d(x,z) + d(z,y),\ \forall\ x,\ y,\ z \in X$.
Then, d is called a cone metric on X, and the pair (X,d) is called a cone metric space.

The notion of convex metric spaces was introduced by Takahashi [23] and he established that all normed spaces and their convex subsets are convex metric spaces. In addition, he also gave several examples of convex metric spaces which are not imbedded in any normed space. Several papers have been devoted to the study of fixed point theory in convex metric spaces in the literature. See Agarwal et al [2], Beg [5, 6] and Guay et al [12]. The aim of this paper is to prove coupled fixed point theorems in convex metric spaces for $(k,\mu)-$Lipschitzian and $(k,\mu,L)-$Lipschitzian mappings.

The following definitions are also pertinent to the study:

Definition 1.3. *[6, 23] Let (X,d) be a metric space. A mapping $W\colon X \times X \times [0,1] \to X$ is said to be a convex structure on X if for each $(x,\ y,\ \lambda) \in X \times X \times [0,1]$ and $u \in X$,*

$$d(u, W(x,\ y,\ \lambda)) \preceq \lambda d(u,x) + (1-\lambda) d(u,y). \tag{1.1}$$

A metric space X having the convex structure W is called a convex metric space.

Let (X,d,W) be a convex metric space. A nonempty subset C of X is said to be convex if $W(x,\ y,\ \lambda) \in C$ whenever $(x,\ y,\ \lambda) \in C \times C \times [0,1]$.

Definition 1.4. *[4, 16]Let (X,d) be a metric space. An element $(x,y) \in X \times X$ is said to be a coupled fixed point of the mapping $T\colon X \times X \to X$ if $T(x,y) = x$ and $T(y,x) = y$.*

Definition 1.5. *[15, 21] Let (X,d) be a cone metric space. Let $\{x_n\}_{n=1}^{\infty} \subseteq X$ and $x \in X$. Then,*
(i) $\{x_n\}_{n=1}^{\infty}$ converges to x, that is, $\lim\limits_{n\to\infty} x_n = x$, if for every $c \in E$ with $0 \prec\prec c$ there exists a natural number N such that $d(x_n,x) \prec\prec c$ for all $n \succeq N$;
(ii) $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence if for every $c \in E$ with $0 \prec\prec c$, there exists a natural number N such that $d(x_n,x_m) \prec\prec c$ for all $n,\ m \succeq N$.

A cone metric space (X,d) is said to be complete if every Cauchy sequence in X converges to a point $x \in X$.

Definition 1.6. *Let C be a nonempty subset of a convex metric space (X,d,W).*
(i) A mapping $T\colon C \times C \to C$ is said to be a $(k,\mu)-$Lipschitzian if and only if there exist two constants $k,\ \mu \in [0,\infty)$, such that

$$d(T(x,y), T(u,v)) \preceq kd(x,u) + \mu d(y,v),\ \forall\ x,\ y,\ u,\ v \in C. \tag{1.2}$$

(ii) A mapping $T\colon C \times C \to C$ is said to be $(k,\mu,L)-$Lipschitzian if and only if there exist constants $k,\ \mu \in [0,\infty)$ and $L \succeq 0$, such that

$$d(T(x,y), T(u,v)) \preceq kd(x,u) + \mu d(y,v) + Ld(T(x,y),x),\ \forall\ x,\ y,\ u,\ v \in C. \tag{1.3}$$

Definition 1.7. *[3] : (a) A function $\phi : I\!R^+ \to I\!R^+$ is called a comparison function if: (i) ϕ is monotone increasing; (ii) $\lim\limits_{n\to\infty} \phi^n(t) = 0,\ \forall\ t \geq 0$.*
(b) A comparison function satisfying $\sum_{n=0}^{\infty} \phi^n(t)$ converges for all $t \succ 0$ is called a $(c)-$comparison function.

In addition, we shall require the following iterative process in the sequel:

For $(x_0, y_0) \in C \times C$, define the sequence $\{(x_n, y_n)\} \subset C \times C$ in terms of a convex structure by

$$x_{n+1} = W(x_n, T(x_n, y_n); 1 - \alpha_n), \quad y_{n+1} = W(y_n, T(y_n, x_n); 1 - \alpha_n), \tag{1.4}$$

$n = 0, 1, 2, \cdots,\ \alpha_n \in [0, 1]$. *The iterative process defined in (1.4) above is a Mann-type iterative process.*

Remark 1.1. *If $\alpha_n = 1$ in (1.4), then we obtain the iterative process in Sabetghadam et al. [21] and some others in the reference section which is again stated below:*

For $(x_0, y_0) \in C \times C$, define the sequence $\{(x_n, y_n)\} \subset C \times C$ by

$$x_{n+1} = T(x_n, y_n), \quad y_{n+1} = T(y_n, x_n). \tag{1.4*}$$

2 Main results

Theorem 2.1. *Let (X, d, W) be a complete convex cone metric space, C a nonempty closed convex subset of X and $T\colon C \times C \to C$ is a $(k, \mu, L)-$Lipschitzian mapping. Suppose that $\phi\colon I\!R^+ \to I\!R^+$ is a $(c)-$comparison function such that, for arbitrary $x,\ y \in C$, there exist $u,\ v \in C$ with*

(i) $d(T(u, v), u) \preceq \phi(d(T(x, y), x))$;

(ii) $d(u, x) \preceq bd(T(x, y), x),\ b \succ 0$.

For $(x_0, y_0) \in C \times C$, let $\{(x_n, y_n)\} \subset C \times C$ be defined by $(4^\star)$.

Then, T has a coupled fixed point in C.

Proof. Let $(x_0, y_0) \in C \times C$ be an arbitrary point. We consider sequences $\{x_n\}_{n=0}^{\infty},\ \{y_n\}_{n=0}^{\infty} \subset C$ such that by conditions (i) and (ii) of the theorem above, we have

$$d(T(x_{n+1}, y_{n+1}), x_{n+1}) \preceq \phi(d(T(x_n, y_n), x_n)),\ n = 0, 1, 2, \cdots, \tag{2.1}$$

and

$$d(x_{n+1}, x_n) \preceq bd(T(x_n, y_n), x_n),\ b \succ 0,\ n = 0, 1, 2, \cdots. \tag{2.2}$$

We obtain by induction in (2.1) that

$$\begin{aligned} lld(T(x_{n+1}, y_{n+1}), x_{n+1}) &\preceq \phi(d(T(x_n, y_n), x_n)) \\ &\preceq \phi^2(d(T(x_{n-1}, y_{n-1}), x_{n-1})) \preceq \cdots \preceq \phi^{n+1}(d(T(x_0, y_0), x_0)). \end{aligned} \tag{2.3}$$

Using (2.3) in (2.2) gives

$$d(x_{n+1}, x_n) \preceq b\phi^n(d(T(x_0, y_0), x_0)). \tag{2.4}$$

In a similar manner, we obtain by conditions (i) and (ii) of the theorem that

$$d(T(y_{n+1}, x_{n+1}), y_{n+1}) \preceq \phi(d(T(y_n, x_n), y_n)),\ n = 0, 1, 2, \cdots, \tag{2.5}$$

and

$$d(y_{n+1}, y_n) \preceq bd(T(y_n, x_n), y_n),\ b \succ 0,\ n = 0, 1, 2, \cdots. \tag{2.6}$$

By induction in (2.5), we have that

$$\begin{aligned} d(T(y_{n+1}, x_{n+1}), y_{n+1}) &\preceq \phi(d(T(y_n, x_n), y_n)) \\ &\preceq \phi^2(d(T(y_{n-1}, x_{n-1}), y_{n-1})) \preceq \cdots \preceq \phi^{n+1}(d(T(y_0, x_0), y_0)). \end{aligned} \tag{2.7}$$

Using (2.7) in (2.6) yields

$$d(y_{n+1}, y_n) \preceq b\phi^n(d(T(y_0, x_0), y_0)). \tag{2.8}$$

For each $r \in \mathbb{N}$, we obtain by using (2.4) and (2.8) in the repeated application of the triangle inequality that

$$\begin{aligned} d(x_n, x_{n+r}) + d(y_n, y_{n+r}) &\preceq [d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{n+r-1}, x_{n+r})] \\ &\quad + [d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + \cdots + d(y_{n+r-1}, y_{n+r})] \\ &\preceq b[\textstyle\sum_{k=n}^{n+r-1} \phi^k(d(T(x_0, y_0), x_0)) + \sum_{k=n}^{n+r-1} \phi^k(d(T(y_0, x_0), y_0))] \\ &= b[\textstyle\sum_{k=0}^{n+r-1} \phi^k(d(T(x_0, y_0), x_0)) - \sum_{k=0}^{n-1} \phi^k(d(T(x_0, y_0), x_0))] \\ &\quad + b[\textstyle\sum_{k=0}^{n+r-1} \phi^k(d(T(y_0, x_0), y_0)) - \sum_{k=0}^{n-1} \phi^k(d(T(y_0, x_0), y_0))] \\ &\to\ 0 \text{ as } n \to \infty, \end{aligned} \tag{2.9}$$

and noting that ϕ is a $(c)-$comparison function. It follows from (**??**) that for $c \in E,\ 0 \prec\prec c$ and for large n, we have $b[\sum_{k=n}^{n+r-1} \phi^k(d(T(x_0, y_0), x_0)) + \sum_{k=n}^{n+r-1} \phi^k(d(T(y_0, x_0), y_0))] \prec\prec c$, thus, leading to the fact that $d(x_n, x_{n+r}) + d(y_n, y_{n+r}) \prec\prec c$.

Therefore, it follows that the sequence $\{(x_n,\ y_n)\}$ is a Cauchy sequence in $C \times C$. Since X is complete, we have that C is also a complete subspace of X. Suppose that there exist $x^*,\ y^* \in C$ such that $\lim\limits_{n\to\infty} x_n = x^*$ and $\lim\limits_{n\to\infty} y_n = y^*$. That is, for $c \in E,\ 0 \prec\prec c$, there exists $q \in \mathbb{N}$ such that

$$d(x_q, x^*) \prec\prec \frac{c}{3(1+k)} \text{ and } d(y_q, y^*) \prec\prec \frac{c}{3(1+\mu)},\ \text{for all } n \geq q, \tag{2.10}$$

with $k \prec 1 + \mu,\ \mu \prec 1 + k$.

Using (1.3), (2.3), (2.10) as well as the triangle inequality give

$$\begin{aligned} d(T(x^*, y^*), x^*) &\preceq d(T(x^*, y^*), T(x_q, y_q)) + d(T(x_q, y_q), x^*) \\ &= d(T(x_q, y_q), T(x^*, y^*)) + d(x_{q+1}, x^*) \\ &\preceq kd(x_q, x^*) + \mu d(y_q, y^*) + Ld(T(x_q, y_q), x_q) + d(x_{q+1}, x^*) \\ &\preceq L\phi^q(d(T(x_0, y_0), x_0)) + kd(x_q, x^*) + \mu d(y_q, y^*) + d(x_{q+1}, x^*), \\ &\preceq kd(x_q, x^*) + \mu d(y_q, y^*) + d(x_{q+1}, x^*), \end{aligned} \tag{2.11}$$

since $\phi^q(d(T(x_0, y_0), x_0)) \to\ 0$ as $q \to \infty$, noting that ϕ is a comparison function. Therefore, we have from (2.11) that

$$\begin{aligned} d(T(x^*, y^*), x^*) &\preceq kd(x_q, x^*) + \mu d(y_q, y^*) + d(x_{q+1}, x^*) \\ &\prec\prec \tfrac{kc}{3(1+k)} + \tfrac{\mu c}{3(1+\mu)} + \tfrac{c}{3(1+k)} \prec \tfrac{c}{3} + \tfrac{c}{3} + \tfrac{c}{3} = c, \end{aligned}$$

from which it follows that $d(T(x^*, y^*), x^*) \prec\prec c$.

Thus, $d(T(x^*, y^*), x^*) = 0$, that is, $T(x^*, y^*) = x^*$.

Similarly, from (1.3), (2.7), (2.10) and the triangle inequality, we get

$$\begin{aligned} d(T(y^*, x^*), y^*) &\preceq d(T(y^*, x^*), T(y_q, x_q)) + d(T(y_q, x_q), y^*) \\ &= d(T(y_q, x_q), T(y^*, x^*)) + d(y_{q+1}, y^*) \\ &\preceq kd(y_q, y^*) + \mu d(x_q, x^*) + Ld(T(y_q, x_q), y_q) + d(y_{q+1}, y^*) \\ &\preceq L\phi^q(d(T(y_0, x_0), y_0)) + kd(y_q, y^*) + \mu d(x_q, x^*) + d(y_{q+1}, y^*), \\ &\preceq kd(y_q, y^*) + \mu d(x_q, x^*) + d(y_{q+1}, y^*), \end{aligned} \tag{2.12}$$

since $\phi^q(d(T(y_0, x_0), y_0)) \to\ 0$ as $q \to \infty$, and observing that ϕ is a comparison function. Now, since $k < 1 + \mu,\ \mu < 1 + k$, then, we obtain from (2.12) that

$$\begin{aligned} d(T(y^*, x^*), y^*) &\preceq kd(y_q, y^*) + \mu d(x_q, x^*) + d(y_{q+1}, y^*) \\ &\prec\prec \tfrac{kc}{3(1+\mu)} + \tfrac{\mu c}{3(1+k)} + \tfrac{c}{3(1+\mu)} \prec \tfrac{(1+\mu)c}{3(1+\mu)} + \tfrac{(1+k)c}{3(1+k)} + \tfrac{c}{3(1+\mu)} \prec \tfrac{c}{3} + \tfrac{c}{3} + \tfrac{c}{3} = c, \end{aligned}$$

from which it follows again that $d(T(y^*,x^*),y^*) \prec\prec c$.
Therefore, $d(T(y^*,x^*),y^*) = 0$, that is, $T(y^*,x^*) = y^*$.
Hence, (x^*,y^*) is a coupled fixed point of T.

Theorem 2.1 can be extended also to the next result under the assumption of two metrics ρ and d such that (X,ρ) is complete and $T\colon C\times C \to C$ is a $(k,\mu,L)-$Lipschitzian mapping with respect to d. □

Theorem 2.2. *Let X be a nonempty set, $C \subset X$, ρ and d are two metrics on X and $T\colon C\times C\to C$ is a mapping. Suppose that:*
(H_1) there exists a $(c)-$comparison function $\phi\colon I\!R^+ \to I\!R^+$ such that, for arbitrary $x,\ y\in C$, there exist $u,\ v\in C$ with
(i) $d(T(u,v),u) \preceq \phi(d(T(x,y),x))$;
(ii) $d(u,x) \preceq bd(T(x,y),x),\ b\succ 0$;
(H_2) there exist real numbers $A\succeq 0,\ 0\prec R\prec 1$ such that, for arbitrary $x,\ y\in C$, there exists $u\in C$ with
(i) $\rho(u,x)\preceq Ad(u,x)$;
(ii) $\rho(T(x,y),x)\preceq Rd(T(x,y),x)$;
(H_3) (X,ρ,W) is a complete convex cone metric space and C is a closed convex subset of (X,ρ,W);
(H_4) $T\colon (C,d)\times(C,d)\to(C,d)$ is a $(k,\mu,L)-$Lipschitzian.
For $(x_0,y_0)\in C\times C$, let $\{(x_n,y_n)\}\subset C\times C$ be defined by $(1.4^\star)$.
Then, T has a coupled fixed point in C.

Proof. Let $x_0\in C$ be an arbitrary point. By using conditions $(H_1)(i)$ and $(H_1)(ii)$, we obtain as in Theorem 2.1 that for $r\in I\!N$,

$$\begin{aligned} d(x_n,x_{n+r})+d(y_n,y_{n+r}) &\preceq b[\textstyle\sum_{k=n}^{n+r-1}\phi^k(d(T(x_0,y_0),x_0))+\sum_{k=n}^{n+r-1}\phi^k(d(T(y_0,x_0),y_0))]\\ &= b[\textstyle\sum_{k=0}^{n+r-1}\phi^k(d(T(x_0,y_0),x_0))-\sum_{k=0}^{n-1}\phi^k(d(T(x_0,y_0),x_0))]\\ &\quad + b[\textstyle\sum_{k=0}^{n+r-1}\phi^k(d(T(y_0,x_0),y_0))-\sum_{k=0}^{n-1}\phi^k(d(T(y_0,x_0),y_0))]\\ &\quad \to\ 0 \text{ as } n\to\infty, \end{aligned} \tag{2.13}$$

from which it follows by the right-hand side expression in (2.13) that for $c\in E,\ 0\prec\prec c$ and for large n, we have

$$b\left[\sum_{k=n}^{n+r-1}\phi^k(d(T(x_0,y_0),x_0))+\sum_{k=n}^{n+r-1}\phi^k(d(T(y_0,x_0),y_0))\right]\prec\prec c, \tag{2.14}$$

thus, showing that

$$d(x_n,x_{n+r})+d(y_n,y_{n+r})\prec\prec c. \tag{2.15}$$

It follows again that the sequence $\{(x_n,\ y_n)\}$ is a Cauchy sequence in $C\times C$ with respect to d.

We now show that $\{(x_n,\ y_n)\}$ is a Cauchy sequence in $C\times C$ with respect to ρ too:
By Condition $(H_2)(i)$, we have that for $r\in I\!N$,

$$\rho(x_n,x_{n+r})\preceq Ad(x_n,x_{n+r}) \ \text{ and }\ \rho(y_n,y_{n+r})\preceq Ad(y_n,y_{n+r}). \tag{2.16}$$

We obtain from (2.13) and (2.16) that

$$\begin{aligned} \rho(x_n,x_{n+r})+\rho(y_n,y_{n+r}) &\preceq A[d(x_n,x_{n+r})+d(y_n,y_{n+r})]\\ &\preceq bA[\textstyle\sum_{k=n}^{n+r-1}\phi^k(d(T(x_0,y_0),x_0))+\sum_{k=n}^{n+r-1}\phi^k(d(T(y_0,x_0),y_0))]\\ &\quad \to\ 0 \text{ as n}\to\infty. \end{aligned} \tag{2.17}$$

Using (2.17), we obtain as in (2.14) and (2.15) that for $c \in E,\ 0 \prec\prec c$ and for large n,

$$\rho(x_n, x_{n+r}) + \rho(y_n, y_{n+r}) \prec\prec c. \tag{2.18}$$

Thus, we obtain from (22) that $\{(x_n,\ y_n)\}$ is a Cauchy sequence in $C \times C$ with respect to ρ too.

By (H_3), (X, ρ, W) is a complete convex cone metric space. Therefore, (C, ρ, W) is a complete subspace of complete convex cone metric space (X, ρ, W). By this reason, there exist $x^*,\ y^* \in C$ such that $\lim_{n\to\infty} x_n = x^*$ and $\lim_{n\to\infty} x_n = y^*$. That is, for $c \in E,\ 0 \prec\prec c$, there exists $q \in \mathbb{N}$ such that the inequality Conditions (14) hold.

Using (1.3), (2.3), Conditions $(H_2)(ii)$ and (H_4) as well as the triangle inequality, we have that

$$\begin{aligned}
\rho(T(x^*, y^*), x^*) &\preceq Rd(T(x^*, y^*), x^*)\\
&\preceq R[d(T(x^*, y^*), T(x_q, y_q)) + d(T(x_q, y_q), x^*)]\\
&= R[d(T(x_q, y_q), T(x^*, y^*)) + d(x_{q+1}, x^*)]\\
&\preceq R[kd(x_q, x^*) + \mu d(y_q, y^*) + Ld(T(x_q, y_q), x_q) + d(x_{q+1}, x^*)]\\
&\preceq R[L\phi^q(d(T(x_0, y_0), x_0)) + kd(x_q, x^*) + \mu d(y_q, y^*) + d(x_{q+1}, x^*)],\\
&\preceq R[L\phi^q(d(T(x_0, y_0), x_0)) + kd(x_q, x^*) + \mu d(y_q, y^*) + d(x_{q+1}, x^*)]\\
&\prec kd(x_q, x^*) + \mu d(y_q, y^*) + d(x_{q+1}, x^*),\ 0 \prec R \prec 1,
\end{aligned} \tag{2.19}$$

where $\phi^q(d(T(x_0, y_0), x_0)) \to\ 0$ as $q \to \infty$ (since ϕ is a comparison function again). Therefore, we have from (2.10) and (2.19) that

$$\begin{aligned}
\rho(T(x^*, y^*), x^*) &\prec kd(x_q, x^*) + \mu d(y_q, y^*) + d(x_{q+1}, x^*)\\
&\prec\prec \tfrac{kc}{3(1+k)} + \tfrac{\mu c}{3(1+\mu)} + \tfrac{c}{3(1+k)} \prec \tfrac{c}{3} + \tfrac{c}{3} + \tfrac{c}{3} = c,
\end{aligned}$$

from which it follows that $\rho(T(x^*, y^*), x^*) \prec\prec c$. Thus, $\rho(T(x^*, y^*), x^*) = 0$, that is, $T(x^*, y^*) = x^*$.

Similarly, using (1.3), (11), Conditions $(H_2)(ii)$ and (H_4) as well as the triangle inequality, we obtain

$$\begin{aligned}
\rho(T(y^*, x^*), y^*) &\preceq Rd(T(y^*, x^*), y^*)\\
&\preceq R[d(T(y^*, x^*), T(y_q, x_q)) + d(T(y_q, x_q), y^*)]\\
&= R[d(T(y_q, x_q), T(y^*, x^*)) + d(y_{q+1}, y^*)]\\
&\preceq R[kd(y_q, y^*) + \mu d(x_q, x^*) + Ld(T(y_q, x_q), y_q) + d(y_{q+1}, y^*)]\\
&\preceq R[L\phi^q(d(T(y_0, x_0), y_0)) + kd(y_q, y^*) + \mu d(x_q, x^*) + d(y_{q+1}, y^*)],\\
&\prec kd(y_q, y^*) + \mu d(x_q, x^*) + d(y_{q+1}, y^*),\ \text{since } 0 \prec R \prec 1,
\end{aligned} \tag{2.20}$$

where $\phi^q(d(T(y_0, x_0), y_0)) \to\ 0$ as $q \to \infty$ (since ϕ is a comparison function again). Since $k \prec 1 + \mu,\ \mu \prec 1 + k$, then, we obtain from (2.10) and (2.20) that

$$\begin{aligned}
d(T(y^*, x^*), y^*) &\prec kd(y_q, y^*) + \mu d(x_q, x^*) + d(y_{q+1}, y^*)\\
&\prec\prec \tfrac{kc}{3(1+\mu)} + \tfrac{\mu c}{3(1+k)} + \tfrac{c}{3(1+\mu)} \prec \tfrac{(1+\mu)c}{3(1+\mu)} + \tfrac{(1+k)c}{3(1+k)} + \tfrac{c}{3(1+\mu)} \prec \tfrac{c}{3} + \tfrac{c}{3} + \tfrac{c}{3} = c,
\end{aligned}$$

from which it follows again that $d(T(y^*, x^*), y^*) \prec\prec c$. Therefore, $d(T(y^*, x^*), y^*) = 0$, that is, $T(y^*, x^*) = y^*$.

Hence, (x^*, y^*) is a coupled fixed point of T. □

Theorem 2.3. *Let (X, d, W) be a complete convex cone metric space, C a nonempty closed convex subset of X and $T\colon C \times C \to C$ is a $(k, \mu)-$Lipschitzian mapping. Suppose that $\phi\colon \mathbb{R}^+ \to \mathbb{R}^+$ is a $(c)-$comparison function such that, for arbitrary $x,\ y \in C$, there exist*

$u,\ v \in C$ with
(i) $d(T(u,v),u) \preceq \phi(d(T(x,y),x))$;
(ii) $d(u,x) \preceq bd(T(x,y),x),\ b \succ 0$.
For $(x_0,y_0) \in C \times C$, let $\{(x_n,y_n)\} \subset C \times C$ be defined by $(4^\star)$.
Then, T has a coupled fixed point in C.

Proof. The proof is similar to that of Theorem 2.1, but with $L = 0$. □

Theorem 2.4. *Let (X,d,W) be a complete convex metric space, C a nonempty closed convex subset of X and $T\colon C \times C \to C$ is a $(k,\mu,L)-$Lipschitzian mapping. Suppose that for arbitrary $x,\ y \in C$, there exist $u,\ v \in C$ such that*
(i) $d(T(u,v),x) \preceq rd(T(x,y),x)),\ 0 \preceq r \prec 1$;
(ii) $d(T(x,y),x) \succeq \frac{kd(x,u)+\mu d(y,v)}{k+\mu+w},\ w \succ 0$.
For $(x_0,y_0) \in C \times C$, let $\{(x_n,y_n)\} \subset C \times C$ be defined by $(4^\star)$, with $\alpha_n \preceq \alpha,\ \forall\ n$, $\alpha \in [0,1]$. Then, T has a coupled fixed point in C, if $\alpha \prec \frac{1-r}{k+\mu+w+L-r}$.

Proof. For any $x,\ y \in C$, let $u = W(x,T(x,y);1-\alpha_n)$. Then,

$$\begin{aligned} d(u,x) &= d(W(x,T(x,y);1-\alpha_n),x) \\ &\preceq (1-\alpha_n)d(x,x) + \alpha_n d(T(x,y),x) = \alpha_n d(T(x,y),x). \end{aligned} \tag{2.21}$$

Also,

$$\begin{aligned} d(u,T(u,v)) &= d(W(x,T(x,y);1-\alpha_n),T(u,v)) \\ &\preceq (1-\alpha_n)d(x,T(u,v)) + \alpha_n d(T(x,y),T(u,v)) \\ &\preceq r(1-\alpha_n)d(T(x,y),x) \\ &\quad + \alpha_n[\ kd(x,u) + \mu d(y,v) + Ld(T(x,y),x)\ ] \\ &\preceq r(1-\alpha_n)d(T(x,y),x) + \alpha_n(k+\mu+w+L)d(T(x,y),x) \\ &= [r + (k+\mu+w+L-r)\alpha_n]d(T(x,y),x) \\ &= \beta d(T(x,y),x), \end{aligned} \tag{2.22}$$

where $\beta = r + (k+\mu+w+L-r)\alpha_n$ and $0 \leq \beta < 1$ since $\alpha \prec \frac{1-r}{k+\mu+w+L-r}$.

From (**??**), we have by a similar process as in Theorem 2.1 that

$$\begin{aligned} d(T(x_{n+1},y_{n+1}),x_{n+1}) &\preceq \beta d(T(x_n,y_n),x_n) \\ &\leq \beta^2 d(T(x_{n-1},y_{n-1}),x_{n-1}) \preceq \cdots \leq \beta^{n+1}d(T(x_0,y_0),x_0). \end{aligned} \tag{2.23}$$

Using (2.23) in (2.21) gives

$$d(x_{n+1},x_n) \preceq \alpha_n d(T(x_n,y_n),x_n) \preceq \alpha\beta^n d(T(x_0,y_0),x_0). \tag{2.24}$$

Similarly, we have from (2.22) again that

$$\begin{aligned} d(T(y_{n+1},x_{n+1}),y_{n+1}) &\preceq \beta d(T(y_n,x_n),y_n) \\ &\preceq \beta^2 d(T(y_{n-1},x_{n-1}),y_{n-1}) \preceq \cdots \preceq \beta^{n+1}d(T(y_0,x_0),y_0), \end{aligned} \tag{2.25}$$

from which we get by using (2.25) in (2.21) that

$$d(y_{n+1},y_n) \preceq \alpha_n d(T(y_n,x_n),y_n) \leq \alpha\beta^n d(T(y_0,x_0),y_0). \tag{2.26}$$

For each $r \in \mathbb{N}$, we obtain by using (2.24) and (2.26) in the repeated application of the triangle inequality that

$$\begin{aligned} d(x_n,x_{n+r}) + d(y_n,y_{n+r}) &\preceq [d(x_n,x_{n+1}) + d(x_{n+1},x_{n+2}) + \cdots + d(x_{n+r-1},x_{n+r})] \\ &\quad + [d(y_n,y_{n+1}) + d(y_{n+1},y_{n+2}) + \cdots + d(y_{n+r-1},y_{n+r})] \\ &\preceq \alpha\beta^n[1 + \beta + \beta^2 + \cdots + \beta^{r-1}][d(T(x_0,y_0),x_0) + d(T(y_0,x_0),y_0)] \\ &= \frac{\alpha\beta^n(1-\beta^r)}{1-\beta}\left[d(T(x_0,y_0),x_0) + d(T(y_0,x_0),y_0)\right], \\ &\quad \text{(since } \alpha_n \leq \alpha,\ \forall\ n\text{)},\ \to\ 0 \text{ as } n \to \infty \text{ since } \beta \in [0,1). \end{aligned} \tag{2.27}$$

It follows from (2.27) that for $c \in E,\ 0 \prec\prec c$ and for large n, we have

$$\frac{\alpha\beta^n(1-\beta^r)}{1-\beta}[d(T(x_0,y_0),x_0)+d(T(y_0,x_0),y_0)] \prec\prec c,$$

and thus, leads to the fact that $d(x_n,x_{n+r})+d(y_n,y_{n+r}) \prec\prec c$.

Hence, it follows that the sequence $\{(x_n,\ y_n)\}$ is a Cauchy sequence in $C\times C$. Since X is complete, then C is also complete as a subspace of X. Suppose that there exist $x^*,\ y^* \in C$ such that $\lim_{n\to\infty} x_n = x^*$ and $\lim_{n\to\infty} y_n = y^*$. That is, for $c \in E,\ 0 \prec\prec c$, there exists $q \in \mathbb{N}$ such that

$$d(x_q,x^*) \prec\prec \frac{c}{2(1+k)} \text{ and } d(y_q,y^*) \prec\prec \frac{c}{2(1+\mu)}, \text{ for all } n \succeq q, \tag{2.28}$$

with $k \prec \mu$ and $\mu \prec 1+k$.

Using (1.3), (2.23), (2.28) as well as the triangle inequality give

$$\begin{aligned} d(T(x^*,y^*),x^*) &\preceq d(T(x^*,y^*),T(x_q,y_q))+d(T(x_q,y_q),x_q)+d(x_q,x^*)\\ &= d(T(x_q,y_q),T(x^*,y^*))+d(T(x_q,y_q),x_q)+d(x_q,x^*)\\ &\preceq kd(x_q,x^*)+\mu d(y_q,y^*)+Ld(T(x_q,y_q),x_q)\\ &\qquad +d(T(x_q,y_q),x_q)+d(x_q,x^*)\\ &= (1+k)d(x_q,x^*)+\mu d(y_q,y^*)+(1+L)d(T(x_q,y_q),x_q)\\ &= \beta^q(1+L)d(T(x_0,y_0),x_0)+(1+k)d(x_q,x^*)+\mu d(y_q,y^*)\\ &\preceq (1+k)d(x_q,x^*)+\mu d(y_q,y^*), \end{aligned} \tag{2.29}$$

since $\beta^q d(T(x_0,y_0),x_0)) \to\ 0$ as $q\to\infty$.

It follows from (2.28) and (2.29) that

$$\begin{aligned} d(T(x^*,y^*),x^*) &\preceq (1+k)d(x_q,x^*)+\mu d(y_q,y^*)\\ &\prec\prec (1+k)\tfrac{c}{2(1+k)}+\tfrac{\mu c}{2(1+\mu)} \prec \tfrac{c}{2}+\tfrac{c}{2}=c, \end{aligned}$$

from which it follows that $d(T(x^*,y^*),y^*) \prec\prec c$. Thus, $d(T(x^*,y^*),x^*)=0$, that is, $T(x^*,y^*)=x^*$.

In a similar manner, from (1.3), (2.25), (2.28) and the triangle inequality, we get

$$\begin{aligned} d(T(y^*,x^*),y^*) &\preceq d(T(y^*,x^*),T(y_q,x_q))+d(T(y_q,x_q),y_q)+d(y_q,y^*)\\ &= d(T(y_q,x_q),T(y^*,x^*))+d(T(y_q,x_q),y_q)+d(y_q,y^*)\\ &\preceq kd(y_q,y^*)+\mu d(x_q,x^*)+Ld(T(y_q,x_q),y_q)+d(y_q,y^*)\\ &\qquad +d(T(y_q,x_q),y_q)+d(y_q,y^*)\\ &= (1+L)d(T(y_q,x_q),y_q)+(1+k)d(y_q,y^*)+\mu d(x_q,x^*)\\ &\preceq \beta^q(1+L)d(T(y_0,x_0),y_0))+(1+k)d(y_q,y^*)+\mu d(x_q,x^*)\\ &\preceq (1+k)d(y_q,y^*)+\mu d(x_q,x^*), \end{aligned} \tag{2.30}$$

since $\beta^q(d(T(y_0,x_0),y_0)) \to\ 0$ as $q\to\infty$.

Since $k \prec \mu,\ \mu \prec 1+k$, then, we obtain from (2.30) that

$$\begin{aligned} d(T(y^*,x^*),y^*) &\preceq kd(y_q,y^*)+\mu d(x_q,x^*)\\ &\prec\prec (1+k)\tfrac{c}{2(1+\mu)}+\tfrac{\mu c}{2(1+k)} \prec (1+\mu)\tfrac{c}{2(1+\mu)}+\tfrac{(1+k)c}{2(1+k)}=\tfrac{c}{2}+\tfrac{c}{2}=c, \end{aligned}$$

from which it follows again that $d(T(y^*,x^*),y^*) \prec\prec c$. Therefore, $d(T(y^*,x^*),y^*)=0$, that is, $T(y^*,x^*)=y^*$.

Hence, (x^*,y^*) is a coupled fixed point of T. □

Theorem 2.5. *Let (X, d, W) be a complete convex metric space, C a nonempty closed convex subset of X and $T: C\times C \to C$ is a $(k,\mu)-$Lipschitzian mapping. Suppose that for arbitrary $x,\ y \in C$, there exist $u,\ v \in C$ such that*
(i) $d(T(u,v),x) \preceq rd(T(x,y),x)),\ 0 \preceq r \prec 1;$
(ii) $d(T(x,y),x) \succeq \frac{kd(x,u)+\mu d(y,v)}{k+\mu+w},\ w \succ 0.$
For $(x_0, y_0) \in C\times C$, let $\{(x_n, y_n)\} \subset C\times C$ be defined by (1.4), with $\alpha_n \preceq \alpha,\ \forall\ n$, $\alpha \in [0,1]$. Then, T has a coupled fixed point in C, if $\alpha \prec \frac{1-r}{k+\mu+w-r}$.

Proof. The Proof of Theorem 2.5 is complete by setting $L = 0$ in the Proof of Theorem 2.4. □

Remark 2.1. *Our results generalize and extend analogous ones in the literature. For instance, see [18, 21] and some others.*

Example 2.1. *Suppose that $E = I\!R^2$, $P = \{(x,y) \in I\!R^2 \mid x,\ y \geq 0\} \subseteq I\!R^2$, and let $X = [0,\ 1]$. We define $d: X\times X \to X$ by $d(x,y) = (|x-y|, |x-y|)$.*

Therefore, (X, d) is a complete cone metric space. Define the mapping $T: X\times X \to X$ by $T(x,y) = \frac{3x+2y}{10}$. Then, T satisfies the $(k,\mu)-$Lipschitzian contractive condition (1.2) with $k = \frac{3}{10}$ and $\mu = \frac{1}{5}$ in the following sense:

$$\begin{aligned} d(T(x,y),T(u,v)) &= |T(x,y)-T(u,v)| \\ &= |\tfrac{3x+2y}{10} - \tfrac{3u+2v}{10}| = |\tfrac{3(x-u)}{10} + \tfrac{2(y-v)}{10}| \\ &\preceq \tfrac{3}{10}|x-u| + \tfrac{1}{5}|y-v| = \tfrac{3}{10}d(x,u) + \tfrac{1}{5}d(y,v), \end{aligned}$$

where $k = \frac{3}{10}$ and $\mu = \frac{1}{5}$.

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