Jñānābha, Vol. 49(1) (2019), 26-39

NUMERICAL SOLUTION OF THE CONVECTION DIFFUSION EQUATION BY THE LEGENDRE WAVELET METHOD

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(*Received : January* 08, 2019 ; *Revised: June* 03, 2019)

Abstract

In this paper, a numerical method is proposed for solving Convection Diffusion equations. The method is based upon the Legendre wavelet expansion. The Legendre wavelets operational matrix of integration is derived and utilized to transform the equation to a system of algebraic equations by combining collocation method. The proposed method is very convenient for solving such problems since conditions are taken into account automatically. Illustrative examples are included to demonstrate the validity and applicability of the proposed Legendre wavelets method.

2010 Mathematics Subject Classifications: 42C40, 35A08, 65L60.

Keywords and phrases: Two - dimensional Legendre wavelets, Operational matrix of integration, Partial differential equations, Convection diffusion equation, Collocation method.

1 Introduction

Convection diffusion equations are widely used for modeling and simulations of various complex phenomena in science and engineering, such as dispersion of chemicals in reactors, smoke plume in atmosphere, tracer dispersion in a porous medium, migration of contaminants in a stream etc. Since it is impossible to solve Convection diffusion equations analytically for most application problems, efficient numerical algorithms are becoming increasingly important to numerical simulations involving Convection diffusion equations. For this model, some authors have studied the numerical techniques such as the Crank - Nicholson method [19], ADI method [12], the Bessel collocation method [20], the Wavelet - Galerkin method [3, 8], the finite difference method [1, 5, 18], the finite element method [6, 7] and the Piecewise - analytical method [16].

Among these methods, the Wavelets method is more attractive. Wavelet theory is relatively new and an emerging area in mathematical research. It has been applied in a wide range of engineering disciplines. Wavelets are used in system analysis, signal analysis for wave - form representation and segmentations, optimal - control, numerical analysis, time - frequency analysis and fast algorithms for easy implementation [17]. However wavelets are just another basis set which offers considerable advantages over alternative basis sets and allows us to attack problems not accessible with conventional numerical methods. Their main advantages are given in [9].

In the last two decades wavelets methods have been applied for solving partial differential equations [2, 4, 10, 14, 15]. It is worth mentioning that Legendre wavelets have both of spectral accuracy, orthogonality and other properties of wavelets.

In this work, a numerical method based on the two - dimensional Legendre wavelets is proposed for solving Convection diffusion equation with Dirichlet initial boundary conditions given as follows

$$\frac{\partial u}{\partial t}+a(x)\frac{\partial u}{\partial x}+b(x)\frac{\partial^2 u}{\partial x^2}=g(x,t), 0\leq x\leq 1, 0\leq t\leq 1 \tag{1.1}$$

with the conditions

$$u(x,0)=f_0(x), u(x,1)=f_1(x), 0\leq x\leq 1 \tag{1.2}$$

and

$$u(0,t)=g_0(t), u(1,t)=g_1(t), 0\leq t\leq 1 \tag{1.3}$$

where $a(x)$ and $b(x)(\neq 0)$ are continuous functions.

In the proposed method, both of the operational matrices of integration and derivative are mutually employed to obtain numerical solutions of the mentioned problem. The proposed method is very convenient for solving such problems since the given conditions are taken into account automatically. Numerical results demonstrate the efficiency of this Legendre wavelets method in solving convection diffusion equation.

2 Basic Definitions, Mathematical Preliminaries and Notations

In this section some necessary definitions and mathematical preliminaries of Wavelet theory and Legendre wavelets are given which will be used further in this paper.

2.1 Wavelets and Legendre Wavelets

Wavelets is a family of functions constructed from dilation parameter 'a' and translation parameter 'b' of a single function called the 'mother wavelet' $\psi(t)$. They are defined by

$$\psi_{a,b}(t)=\frac{1}{\sqrt{|a|}}\psi(\frac{t-b}{a}), a,b\in R, a\neq 0.$$

Now for the discrete values of a and b, $a=a_0^{-k}$, $b=nb_0a_0^{-k}, a_0>1, b_0>0$, where n and k are positive integers. We have the following family of discrete wavelets:

$$\psi_{k,n}(t)=|a|^{-\frac{1}{2}}\psi(a_0^k t-nb_0)$$

where $\psi_{k,n}(t)$ forms a basis of $L^2(R)$.

Legendre Wavelets: Legendre wavelet $\psi_{nm}(t)=\psi(k,n,m,t)$ have four arguments $n=2n-1$, $n=1,2,3,...,2^{k-1}, k$ can be any positive integer, m is order for Legendre polynomials and t is the normalized time [11]. They are defined on the interval [0, 1) by

$$\psi_{nm}(t)=\begin{cases}2^{\frac{k}{2}}\sqrt{m+\frac{1}{2}}P_m(2^kt-n) for \frac{n-1}{2^k}\leq t\leq\frac{n+1}{2^k}\\ 0 \qquad\qquad\qquad\qquad \text{otherwise.}\end{cases}$$

The coefficient $\sqrt{m+\frac{1}{2}}$ is for orthonormality, the dilation parameter is 2^{-k} and the translation parameter is $n2^{-k}$. Here $P_m(t)$ are the well known Legendre polynomials of order m which are orthogonal with respect to the weight function $w(t) = 1$ on the interval $[-1, 1]$ and satisfy the following formulae,

$$P_0(t) = 1, P_1(t) = t$$

and

$$P_{m+1}(t) = (\frac{2m+1}{m+1})tP_m(t) - (\frac{m}{m+1})P_{m-1}(t), m = 1, 2, 3, ...$$

2.2 Function Approximation:

A function $f(t)$ defined over $[0, 1]$ may be expanded by Legendre wavelets as

$$f(t) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{nm}\psi_{nm}(t), \tag{2.1}$$

where $c_{nm} = (f(t), \psi_{nm}(t))$ in which, denotes the inner product.

If the infinite series in equation (2.1) is truncated, then it can be rewritten as

$$f(t) \approx \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{nm}\psi_{nm}(t) = C^T\Psi(t), \tag{2.2}$$

where T indicates transposition and C and $\Psi(t)$ are $m = 2^{k-1}M$ column vectors.

For simplicity eq. (2.2) can be written as

$$f(t) \approx \sum_{i=1}^{m} c_i\psi_i(t) = C^T\Psi(t), \tag{2.3}$$

where $c_i = c_{nm}, \psi_i(t) = \psi_{nm}(t)$.

The index i, is determined by the relation $i = M(n-1) + m + 1$, thus we have

$$C \triangleq [c_1, c_2, ..., c_m]^T,$$
$$\Psi(t) = [\psi_1(t), \psi_2(t), ..., \psi_m(t)]^T. \tag{2.4}$$

In the same way, an arbitrary function of two variables $u(x, y)$ defined over $[0\ ,\ 1)\times[0\ ,\ 1)$ may be expanded into Legendre wavelets basis as

$$u(x, y) \approx \sum_{i=1}^{m} \sum_{j=1}^{m} u_{ij}\psi_i(x)\psi_j(y) = \Psi^T(x)U\Psi(y),$$

where $U = [u_{ij}]$ and $u_{ij} = (\psi_i(x), (u(x, y), \psi_j(y)))$.

By taking the collocation points $t_i = \frac{2i-1}{2m}, (i = 1, 2, ..., m)$ in the $\Psi(t)$.

Now define Legendre wavelets matrix $\varphi_{m\times m}$ as

$$\varphi_{m\times m} \triangleq [\Psi(\frac{1}{2m}), \Psi(\frac{3}{2m}), ..., \Psi(\frac{2m-1}{2m})].$$

Here $\varphi_{m\times m}$ has a diagonal form.

3 Operational Matrices

The integration of integer order α of the vector $\Psi(t)$, defined in (2.4) can be expressed as

$$(I^{\alpha}\Psi)(t) \approx P^{\alpha}\Psi(t),$$

where P^{α} is the $m \times m$ Legendre wavelet operational matrix of integration of integer order α. This matrix P^{α} can be approximated as

$$P^{\alpha} \approx \varphi_{m\times m} P_B^{\alpha} \varphi_{m\times m}^{-1},$$

where P_B^{α} is the operational matrix of integration of integer order α of the Block - Pulse functions (BPFs), which is given by [13].

$$P_B^{\alpha} = \frac{1}{m^{\alpha}} \frac{1}{(\alpha+1)!} \begin{bmatrix} 1 & \xi_1 & \xi_2 & \cdots & \xi_{m-1} \\ 0 & 1 & \xi_1 & \cdots & \xi_{m-2} \\ 0 & 0 & 1 & \cdots & \xi_{m-3} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix},$$

where $\xi_i = (i+1)^{\alpha+1} - 2i^{\alpha+1} + (i-1)^{\alpha+1}$.

We define a m - set of Block pulse functions (BPF) as:

$$b_i(t) = \begin{cases} 1, \frac{i}{m} \le t < \frac{i+1}{m} \\ 0, \qquad \text{otherwise} \end{cases}$$

where $i = 0, 1, 2, ..., m-1$

The functions $b_i(t)$ are disjoint and orthogonal, that is

$$b_i(t)b_j(t) = \begin{cases} 0, i \neq j \\ b_i(t), i = j \end{cases},$$

$$\int_0^1 b_i(t)b_j(t)dt = \begin{cases} 0, i \neq j \\ \frac{1}{m}, i = j \end{cases}.$$

On taking the derivative of integer order α of the vector $\Psi(t)$, defined in (2.4), we have

$$(D^{\alpha}\Psi)(t) \approx Q^{\alpha}\Psi(t),$$

where Q^{α} is Legendre wavelet operational matrix of derivative of integer order α. Here $Q^{\alpha}$is inverse of matrix P^{α}, so it can be expressed as

$$Q^{\alpha} = \varphi_{m\times m} P_B^{-\alpha} \varphi_{m\times m}^{-1},$$

where $P_B^{-\alpha}$ is the operational matrix of derivative of integer order α of the BPFs, which is given by

$$P_B^{-\alpha} = m^{\alpha}(\alpha+1)! \begin{bmatrix} 1 & d_1 & d_2 & \cdots & d_{m-1} \\ 0 & 1 & d_1 & \cdots & d_{m-2} \\ 0 & 0 & 1 & \cdots & d_{m-3} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix},$$

where $d_i = -\sum_{j=1}^{i} \xi_j d_{i-j}$ for all $i = 1, 2, ..., m-1$ and $d_0 = 1$.

4 The Proposed Method

In this section we apply the Legendre wavelets operational matrices of integer order for solving convection diffusion equation with variable coefficients given by equation (1.1) along with the conditions given in (1.2) and (1.3).

For solving the given problem we approximate

$$\frac{\partial^3 u}{\partial t \partial x^2} = \Psi(t)^T U \Psi(x), \tag{4.1}$$

where $U = [u_{i,j}]_{m\times m}$ is an unknown matrix which should be found and $\Psi(.)$ is the vector which is defined in (2.4).

By integration of order 2 of (4.1) with respect to x we have

$$\frac{\partial u}{\partial t} \approx \Psi(t)^T U P^2 \Psi(x) + [\frac{\partial u}{\partial t} + x\frac{\partial}{\partial x}(\frac{\partial u}{\partial t})]_{x=0}. \tag{4.2}$$

On putting $x = 1$ and using (1.3), we have by (4.2)

$$[\frac{\partial}{\partial x}(\frac{\partial u}{\partial t})]_{x=0} \approx \frac{\partial g_1}{\partial t} - \frac{\partial g_0}{\partial t} - \Psi(t)^T U P^2 \Psi(1). \tag{4.3}$$

Now on substituting (4.3) into (4.2), we have

$$\frac{\partial u}{\partial t} \approx \Psi(t)^T U P^2 \Psi(x) + (1-x)\frac{\partial g_0}{\partial t} - x\Psi(t)^T U P^2 \Psi(1) + x\frac{\partial g_1}{\partial t}. \tag{4.4}$$

Moreover by integrating (4.1) with respect to t, we have

$$\frac{\partial^2 u}{\partial x^2} \approx (P\Psi(t))^T U \Psi(x) + [\frac{\partial^2 u}{\partial x^2} + t\frac{\partial}{\partial t}(\frac{\partial^2 u}{\partial x^2})]_{t=0}. \tag{4.5}$$

By putting $t = 1$ in (4.5) and using (1.2), we have by (4.5)

$$[\frac{\partial}{\partial t}(\frac{\partial^2 u}{\partial x^2})]_{t=0} \approx \frac{\partial^2 f_1}{\partial x^2} - \frac{\partial^2 f_0}{\partial x^2} - (P\Psi(t))^T U \Psi(x). \tag{4.6}$$

On substituting (4.6) into (4.5), we have

$$\frac{\partial^2 u}{\partial x^2} \approx (P\Psi(t))^T U \Psi(x) + (1-t)\frac{\partial^2 f_0}{\partial x^2} + t\frac{\partial^2 f_1}{\partial x^2} - t(P\Psi(1))^T U \Psi(x). \tag{4.7}$$

Now by integration of (4.4) with respect to t and using conditions (1.2) and (1.3), we get

$$\begin{aligned} u(x,t) \approx\ & (P\Psi(t))^T U P^2 \Psi(x) - x(P\Psi(t))^T U P^2 \Psi(1) - t(P\Psi(1))^T U P^2 \Psi(x) \\ & + xt(P\Psi(1))^T U P^2 \Psi(1) + H(x,t), \end{aligned} \tag{4.8}$$

where

$$\begin{aligned} H(x,t) &= f_0(x) + (1-x)(g_0(t) - g_0(0) - tg_0'(0)) + x(g_1(t) - g_1(0) - tg_1'(0)) \\ &+t(f_1(x) - f_0(x)) - t(1-x)g_0(1) - g_0(0) - g_0'(0) - xt(g_1(1) - g_1(0) - g_1'(0)). \end{aligned}$$

On differentiating (4.8) with respect to x, we have

$$\begin{aligned} \frac{\partial u}{\partial x} \approx\ & (P\Psi(t))^T U P\Psi(x) - (P\Psi(t))^T U P^2 \Psi(1) - t(P\Psi(1))^T U P \Psi(x) \\ & + t(P\Psi(1))^T U P^2 \Psi(1) + \frac{\partial}{\partial x} H(x,t). \end{aligned} \tag{4.9}$$

Now by substituting (4.4), (4.7), (4.9) into (1.1) and taking collocation points

$$x_i, t_i = \frac{2i-1}{2m}, i = 1, 2, ..., m,$$

into the obtained equation, we have a nonlinear system of algebraic equations. This nonlinear system can be solved by using an iterative method such as Newton iteration method. By solving this system and finding U, we obtain the numerical solution of the problem by substituting U into (4.8).

5 Numerical Examples

In this section, we will use the proposed method to solve the convection diffusion equation with variable or constant coefficients. The following numerical examples are given to show the efficiency and reliability of the proposed method and the results have been compared with the exact solution.

Example 5.1. *Consider convection diffusion eq.* (1.1) *with* $a(x) = -0.1, b(x) = -0.01$ *and* $g(x,t) = 0$.

The given conditions are

$$u(x,0) = e^{-x}, u(x,1) = e^{-x-0.09}$$

and

$$u(0,t) = e^{-0.09t}, u(1,t) = e^{-1-0.09t}.$$

The exact solution of this problem is $u(x,t) = e^{-x-0.09t}$.

The space - time diagram of the numerical solution for $M = 6, k = 2$ *is shown in figure* 5.1.

Absolute errors between the numerical and analytical solution are shown in figure 5.2.

The graph of analytical and approximate solutions for some nodes on $[0,1) \times [0,1)$ *is presented in figure* 5.3.

Absolute errors between the numerical and analytical solutions at different times are shown in figure 5.4.

Example 5.2. *Consider convection diffusion eq.* (1.1) *with* $a(x) = -\frac{x}{6}, b(x) = -\frac{x^2}{12}$ *and* $g(x,t) = 0$.

The given conditions are

$$u(x,0) = x^3, u(x,1) = x^3 e$$

and

$$u(0,t) = 0, u(1,t) = e^t.$$

The exact solution of this problem is $u(x,t) = x^3 e^t$.

The space - time diagram of the numerical solution for $M = 6, k = 2$ *is shown in figure* 5.5.

Absolute errors between the numerical and analytical solution are shown in figure 5.6.

The graph of analytical and approximate solutions for some nodes on $[0,1) \times [0,1)$ *is presented in figure* 5.7.

Absolute errors between the numerical and analytical solutions at different times are shown in figure 5.8.

Example 5.3. *Consider convection diffusion eq.* (1.1) *with* $a(x) = 2, b(x) = -1$ *and* $g(x,t) = -2e^{t-x}$.

The given conditions are

$$u(x,0) = e^{-x}, u(x,1) = e^{1-x}$$

and

$$u(0,t) = e^t, u(1,t) = e^{t-1}.$$

The exact solution of this problem is $u(x,t) = e^{t-x}$.

The space - time diagram of the numerical solution for $M = 6, k = 2$ *is shown in figure* 5.9.

Absolute errors between the numerical and analytical solutions are shown in figure 5.10.

The graph of analytical and approximate solutions for some nodes on $[0, 1) \times [0, 1)$ *is presented in figure* 5.11.

Absolute errors between the numerical and analytical solutions at different times are shown in figure 5.12.

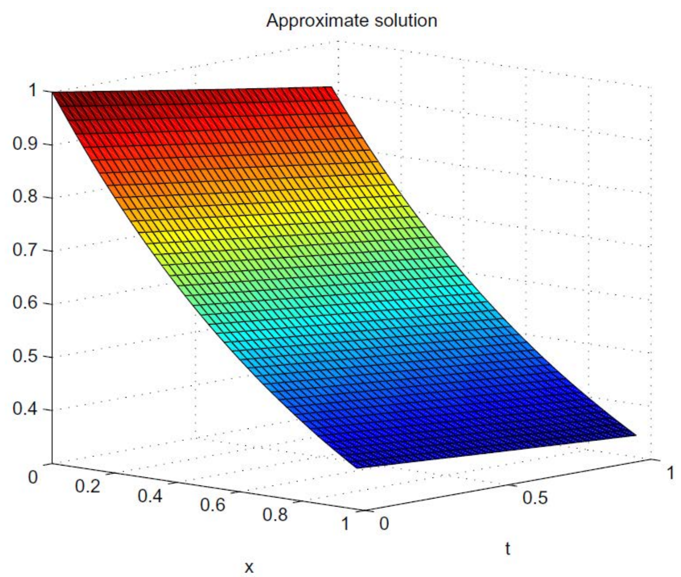

Figure 5.1: Approximate Solution of Example 5.1.

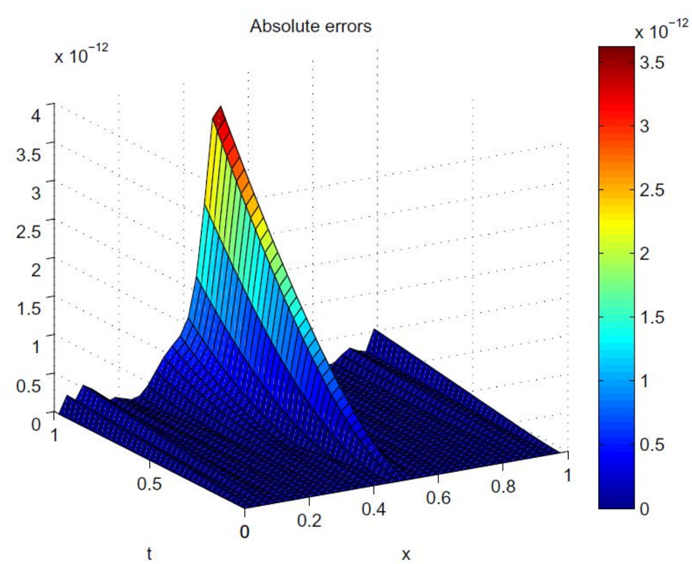

Figure 5.2: Absolute Errors of Example 5.1.

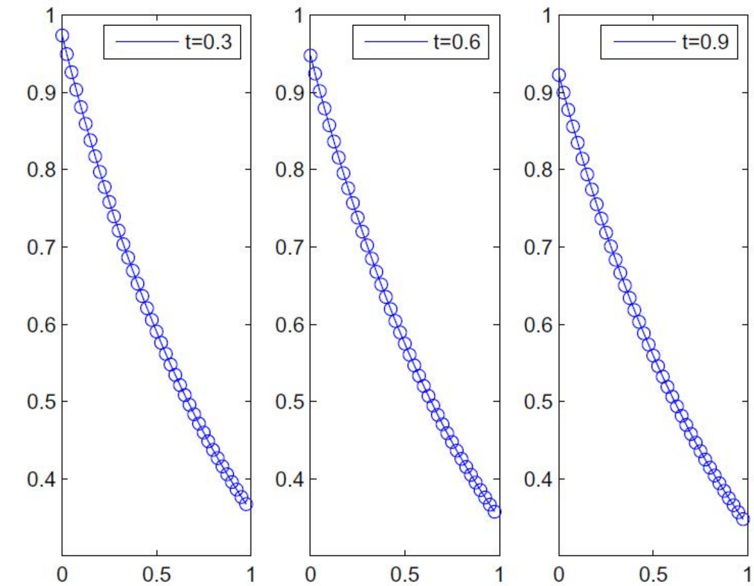

Figure 5.3: Numerical and Exact Solutions in different values of t for Example 5.1.

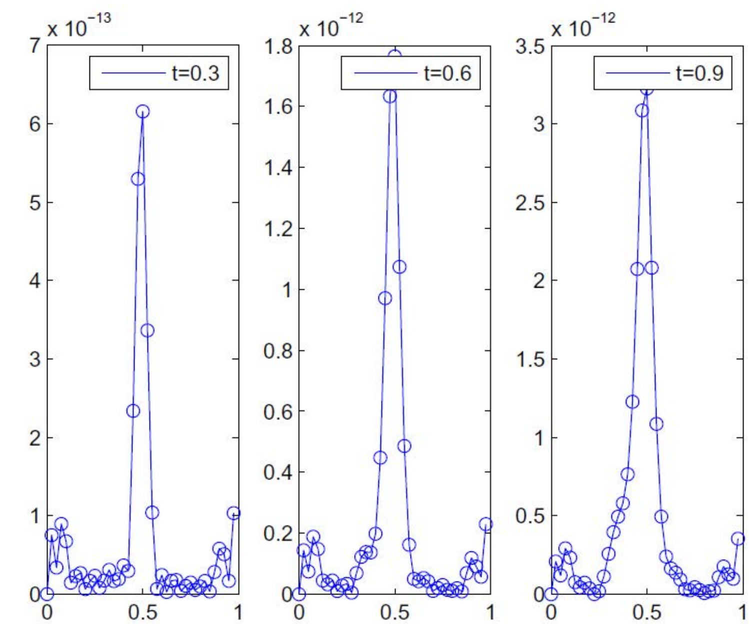

Figure 5.4: Absolute Errors in different values of t for Example 5.1.

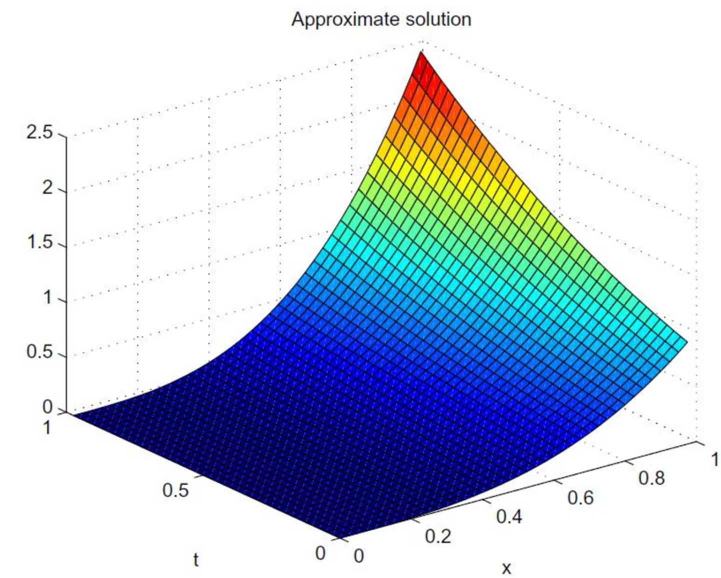

Figure 5.5: Approximate Solution of Example 5.2.

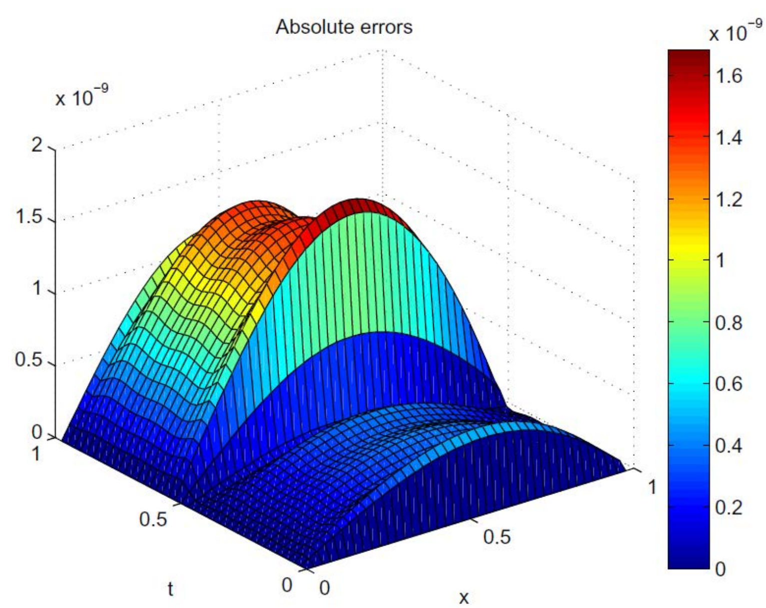

Figure 5.6: Absolute Errors of Example 5.2.

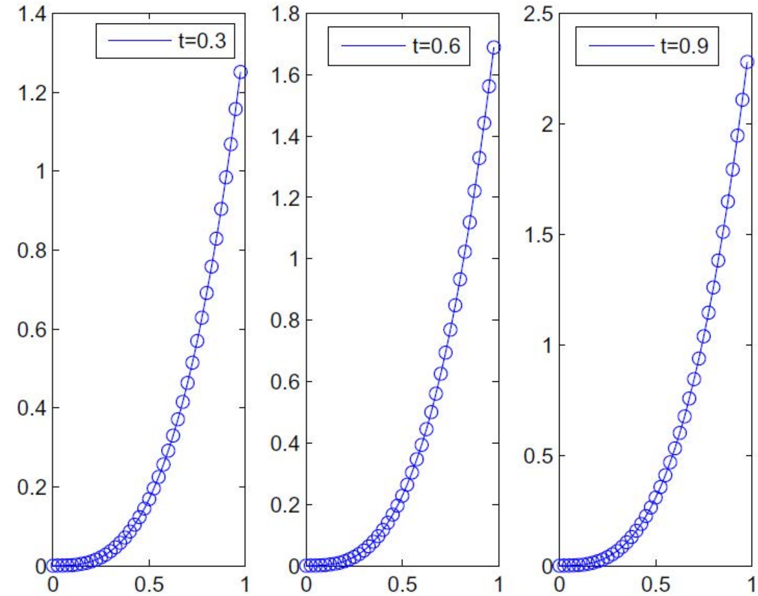

Figure 5.7: Numerical and Exact Solutions in different values of t for Example 5.2.

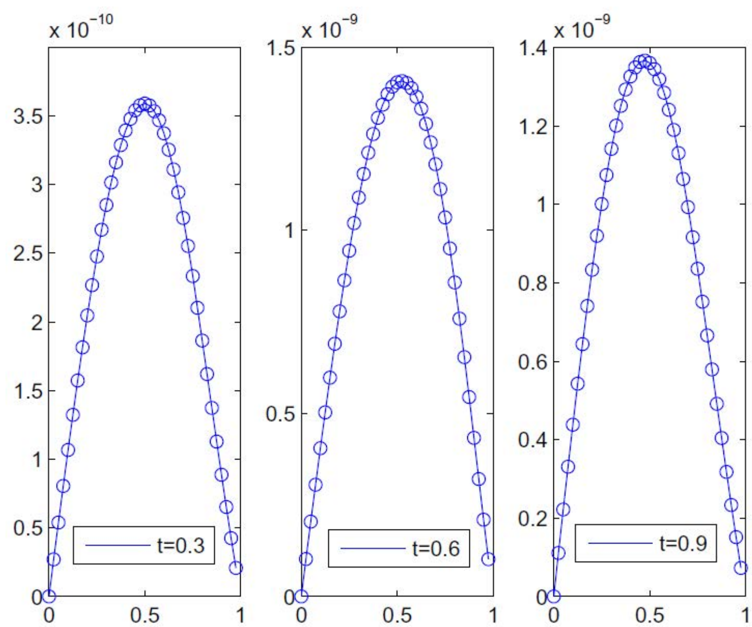

Figure 5.8: Absolute Errors in different values of t for Example 5.2.

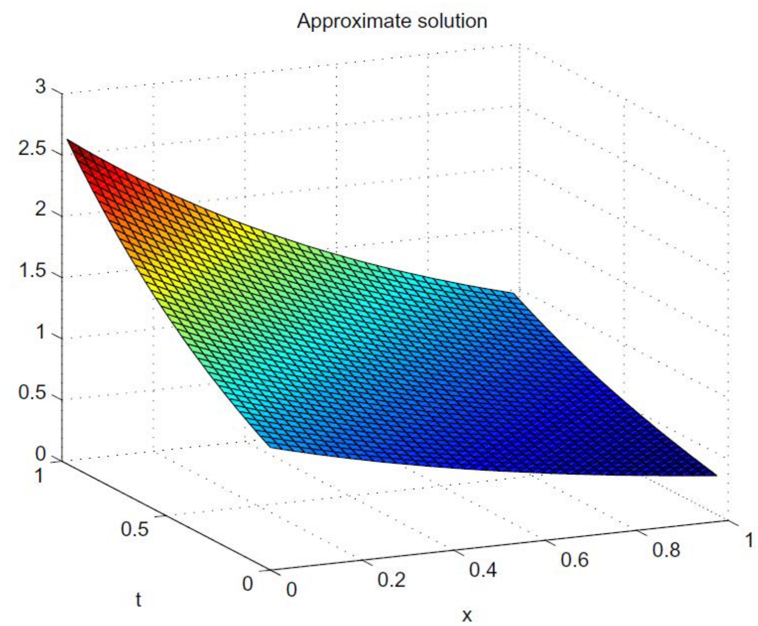

Figure 5.9: Approximate Solution of Example 5.3.

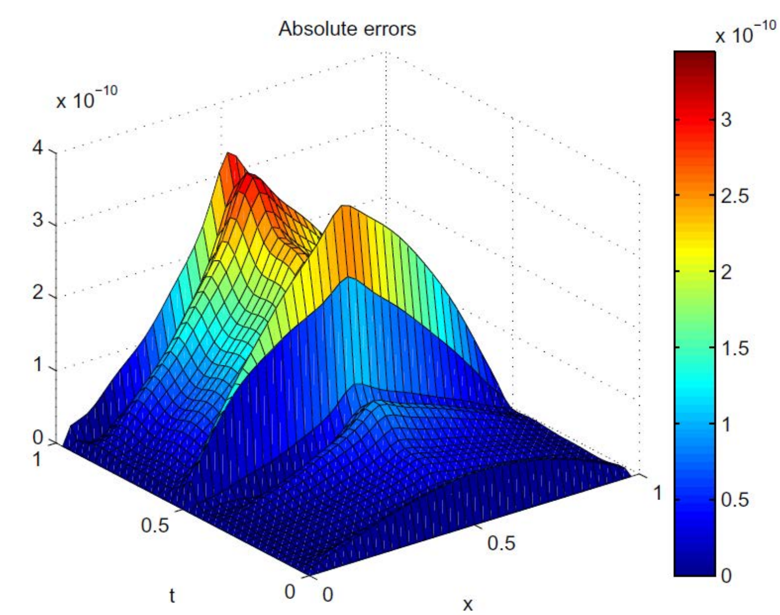

Figure 5.10: Absolute Errors of Example 5.3.

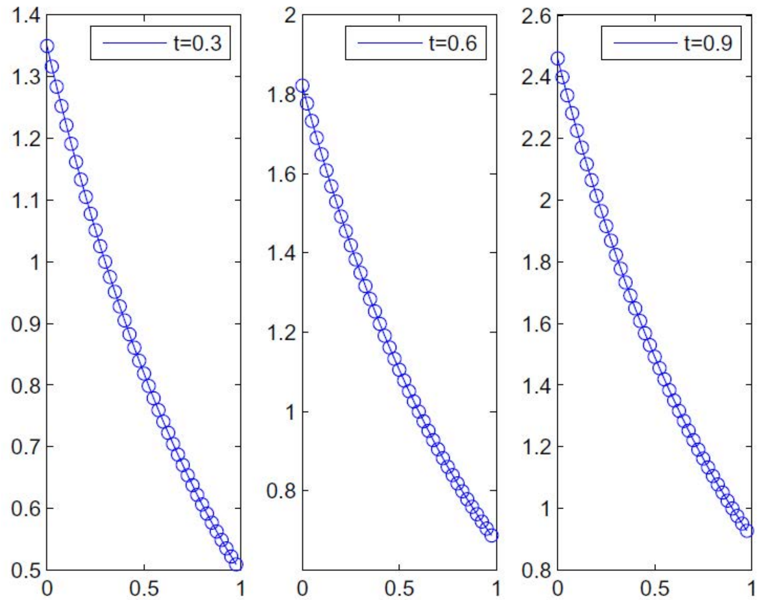

Figure 5.11: Numerical and Exact Solutions in different values of t for Example 5.3.

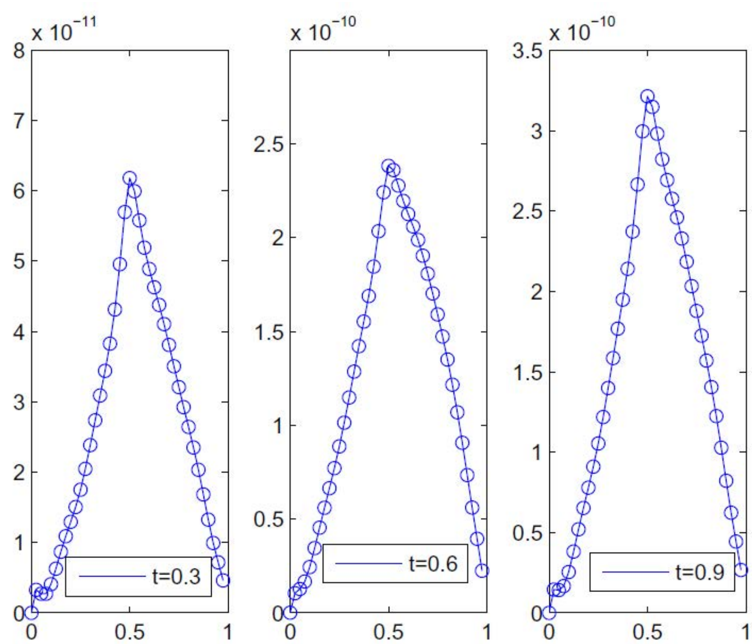

Figure 5.12: Absolute Errors in different values of t for Example 5.3.

6 Conclusion

In this paper, we have derived the two dimensional Legendre wavelets operational matrices of integration and differentiation and proposed a numerical method to approximate the solution of convection diffusion equation with variable or constant coefficients. The method is computationally efficient and the algorithm can be implemented easily on a computer. The advantage of the method is that only small size operational matrix is required to provide the solution at high accuracy. Numerical examples are given to show that the proposed method is applicable, efficient and accurate.

Acknowledgement. We are very much thankful to the referee for his valuable suggestions to bring the paper in its present form.

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