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STARLIKENESS OF THE $p-$VALENT GAUSS HYPERGEOMETRIC FUNCTIONS

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ABSTRACT

In this paper, we find conditions on the parameters a,b,c in the Gauss hypergeometric function $F(a,b;c;z)$ so that $z^p(a,b;c;z)$ to be in the class $S_p^*(\alpha)$ of p–valent starlike functions of order $\alpha(0 \le \alpha \le p)$. Some consequent results, including the result showing that the function $z^pF\left(a,b;c;z^2\right) \in S_p^*$ and by applying the Alexander property, the conditions for which $z^pF(a,b;c;z) \in K_p(\alpha)$ are also obtained. Further, result on the parametric conditions under which $z^pF(a,b;c;z)$ to be in a subclass of S_p^* is derived. In particular, results for incomplete beta functions are also given.

2010 Mathematics Subject Classification : 30C45, 30C55
Key Words and Phrases : Multivalent starlike (convex) functions, Analytic functions, Gauss hypergeometric functions, Incomplete beta functions.

1. Introduction. Let A_p denotes the class of functions of the form

(1.1) $f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \left(p \in \mathbb{N} = \{1,2,3.....\}\right)$

which are analytic and p–valent in the open unit disk

$= \{z : z \in \mathbb{C} \text{ and } |z| < 1\}.$

A function $f \in \ _p$ is called p–valent starlike of order α if

$$(1.2)\qquad \Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha,\ \ 0 \le \alpha < p,\ \ z \in \quad .$$

By $S_p^*(\alpha)$ we denote the class of all p–valent starlike functions of order α, By $S_p(\alpha)$ we denote the subclass of $S_p^*(\alpha)$ consisting of function $f \in \ _p$ for which

$$(1.3)\qquad \left|\frac{zf'(z)}{f(z)} - p\right| < p - \alpha, 0 \le \alpha < p, z \in \quad .$$

Also, a function $f \in \ _p$ is called p–valent convex of order a if $f(z)$ satisfies

$$(1.4)\qquad \Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha, 0 \le \alpha < p, z \in \quad .$$

By $K_p(\alpha)$ we denote the class of all p-valent convex functions of order α. It follows from (1.2) and (1.4) that

$$(1.5)\qquad f(z) \in K_p(\alpha) \Leftrightarrow \frac{zf'(z)}{p} \in S_p^*(\alpha).$$

Also, we denote $S_p^*(0) = S_p^*$, $S_p(0) = S_p$ and $K_p(0) = K_p$.

The Gauss hypergeometric function $F(a,b;c;z)$, for complex numbers a,b,c with $c \ne 0,-1,-2,\ldots.$ is defined by

$$(1.6)\qquad F(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n} z^n, z \in$$

where the symbole $(\lambda)_n$ is the Pochhammer symbol and is defined in terms of Gamma function by

$$(1.7)\qquad (\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{cases} 1 & (n = 0, \lambda \in \mathbb{C}\setminus\{0\}) \\ \lambda(\lambda+1)(\lambda+2)\ldots(\lambda+n-1) & (n \in \mathbb{N}, \lambda \in \mathbb{C}). \end{cases}$$

The hypergeometric functions are involvd in the literature in many situations including in the proof of well known Bieberbach conjecture [2] and they contributed to various fields including conformal mappings quasiconformal theory, and continued fraction [1], [9]. Various identities involving $F(a\ ,b;\ c;\ z)$ and summation formula for $F(a,b;c;1)$ if $Re(c-a-b)>0$ have been derived in [18] ([13]).

Starlikeness and convexity of $zF(a,b;c;z)\in \ _1 =$ was studied by Hasto, Ponnusamy and Vuorinen in [4] (see also [6], [16]). Also, $zF(a,b;c;z)\in \ =$ was studied by Swaminathan in [17]. Some other properties of $zF(a,b;c;z)$ have also been discussed in [5],[8],[10]-[12],[14]. Some necessary and sufficient coefficient inequalities for $zF(a,b;c;z)$ to be in certain subclasses of are derived in [15]. Recently El-Ashwah in [3] have derived certain sufficient coefficient conditions for p-valent Gauss hypergeometric function $z^pF(a,b;c;z)$ to be starlike and convex, respectively, in the unit disk .

In this paper, we find the conditions on the real constants *a,b,c* so that the p-valent Gauss hypergeometric function $z^pF(a,b;c;z)$ is in $S_p^*(\alpha)$ class. Some of its consequent results, including the result showing by the square transformation that the p-valent function $z^pF(a,b;c;z^2)\in S_p^*$ and by applying the Alexander property, the conditions for which $z^pF(a,b;c;z)\in K_p(\alpha)$ are also obtained. Further, a result on the parametric conditions under which $z^pF(a,b;c;z)\in S_p$ is derived. Results for incomplete beta functions are also given.

2. Main Results. We need following lemma due to Miller and Mocanu [7] to prove our main results.

Lemma 1. *Let* $\Omega\subset\mathbb{C}$. *Suppose that* $\psi:\mathbb{C}^2\times \ \to\mathbb{C}$ *satisfies the condition* $\psi\,(ir,s;z)\notin\Omega$ *when r is real and* $s\le-(1+r^2)/2$. *If q is analytic in* , *with q(0)=1 and* $\psi(q(z),zq'(z);z)\in\Omega$ *for* $z\in$, *then* $\Re(q(z))>0$ in .

Theorem 1. *Let a, b and c be non-zero real numbers such that* $F(a,b;c;z)\neq 0$ in *and* $0\le\alpha<p$. Let $\tilde{\alpha}=p-\alpha, A=\tilde{\alpha}^2-\tilde{\alpha}(a+b)+ab$, $B=\tilde{\alpha}(a+b)-2\tilde{\alpha}^2$, $C=\tilde{\alpha}\tilde{c}+ab$, $D=\tilde{\alpha}\tilde{c}$ *and* $\tilde{c}=c-1-(a+b)$. *If inequalities* :

(i) $c\ge 1+a+b-ab/\tilde{\alpha}$ (or, *equivalently* $C\ge 0$)

(ii) $C+\tilde{\alpha}\ge 2A$

(iii) $(\tilde{\alpha}+2\tilde{\alpha}^2)C+2BD+D^2\ge 0,$

hold, then $z^pF(a,b;c;z)\in S_p^*(a)$.

Proof. Let

(2.1) $\varphi(z) = z^p F(a,b;c;z)$

and $q(z)$ be defined by

(2.2) $\dfrac{z\varphi'(z)}{\varphi(z)} = \alpha + (p-\alpha)q(z),\ 0 \le \alpha < p.$

Then $q(z)$ is analytic in and $q(0)=1$.

It is well known that hypergeometric function $F(a,b;c;z) =: F(z)$ satisfies the second order (hypergeometric) differential equation

(2.3) $z(1-z)F''(z) + [c-(1+a+b)z]F'(z) - abF(z) = 0.$

By some simple computations on using (2.1) and (2.2), the differential equation (2.3) yields that

(2.4) $(1-z)(p-\alpha)zq'(z) + (1-z)(p-\alpha)^2 q^2(z) + [(p-\alpha)\{c-1-(a+b)z\} -$

$$2(p-\alpha)^2(1-z)]q(z) + (p-\alpha)^2(1-z) - (p-\alpha)\{c-1-(a+b)z\} - abz = 0.$$

Let $\psi(r,s;z) = (1-z)\tilde{\alpha}s + (1-z)\tilde{\alpha}^2 r^2 + [\tilde{\alpha}\{c-1-(a+b)z\} - 2\tilde{\alpha}^2(1-z)]r$

$$+\tilde{\alpha}^2(1-z) - \tilde{\alpha}\{c-1-(a+b)z\} - abz.$$

Then (2.4) is equivalent to

$$\psi(q(z), zq'(z); z) = 0.$$

Now, we need to prove $\Re(q(z)) > 0, z \in$ by applying Lemma 1. Thus, we must show by the assumptions of the theorem that $\psi(ir,s;z) \ne 0$ for $z \in$ and for $s \le -(a+r^2)/2\,(r \in \mathbb{R})$. Consider

$\psi(ir,s;z) = [\tilde{\alpha}s - \tilde{\alpha}^2 r^2 + \tilde{\alpha}^2 - \tilde{\alpha}(a+b) + ab + i\{\tilde{\alpha}(a+b) - 2\tilde{\alpha}^2\}r](1-z) - \tilde{\alpha}\tilde{c} - ab + i\tilde{\alpha}\tilde{c}r$

$$= [\tilde{\alpha}s - \tilde{\alpha}^2 r^2 + A + iBr](1-z) - C + iDr,$$

where $A = \tilde{\alpha}^2 - \tilde{\alpha}(a+b) + ab, B = \tilde{\alpha}(a+b) - 2\tilde{\alpha}^2, C = \tilde{\alpha}\tilde{c} + ab, D = \tilde{\alpha}\tilde{c}.$ We find that $\psi(ir,s;z) = 0$ at $z = z_0$ where

$$z_0 = 1 + \frac{-C + iDr}{\tilde{\alpha}s - \tilde{\alpha}^2 r^2 + A + iBr}.$$

Further, we see that

$$|z_0|^2 = 1 + \frac{-2\left(\tilde{\alpha}s - \tilde{\alpha}^2 r^2 + A\right)C + C^2 + \left(2BD + D^2\right)r^2}{\left(\tilde{\alpha}s - \tilde{\alpha}^2 r^2 + A\right)^2 + B^2 r^2}.$$

In order to satisfy the conditions of lemma 1, we have to show $|z_0|^2 \geq 1$ for all the relevant parametric values or equivalently. For this, let

(2.5) $\quad P = -2\left(\tilde{\alpha}s - \tilde{\alpha}^2 r^2 + A\right)C + C^2 + \left(2BD + D^2\right)r^2,$

for all $s \leq -\left(1 + r^2\right)/2$ and for all real *r*. Since $\tilde{\alpha} > 0$ and $C \geq 0$ by (i), we see that the inequality needs only to be checked for the largest value of s, i.e. for $s = -\left(1 + r^2\right)/2$. In this case, we get that

$$P \geq \left(\tilde{\alpha}\left(1 + r^2\right) + 2\tilde{\alpha}^2 r^2 - 2A\right)C + C^2 + \left(2BD + D^2\right)r^2$$

$$= \left[\left(\tilde{\alpha} + 2\tilde{\alpha}^2\right)C + 2BD + D^2\right]r^2 - 2AC + C^2 + \tilde{\alpha}C \geq 0,$$

by (ii) and (iii). Hence, by Lemma 1, we conclude that $\Re\left(q(z)\right) > 0, z \in U$ which shows that $\varphi \in S_p^*(\alpha)$. This proves Theorem 1.

Taking $\alpha = p/2$ in Theorem 1, we get following result :

Corollary 1. *Let a, b and c be non-zero real numbers such that* $F(a,b;c;z) \neq 0$ *in* *. If inequality*

(2.6) $c \geq \max\left\{1 + a + b - 2ab/p, p + 2ab/p - (a+b)\right\}$

with

(2.7) $\left\{c - 1 - (a+b)\right\}\left\{c - p + a + b\right\} \geq -2(p+1)ab/p$

holds, then $z^p F(a,b;c;z) \in S_p^*(p/2)$.

Remark 1. We remark that for $p=1$, condition (2.7) reduces to the condition:

$$c \geq 1 + |a - b|$$

and then Corollary 1 coincides with the result obtained by Hasto et al. ([4], Corollary 1.7, p.3).

Further, by using the square transformation, Corollary 1 proves following result.

Corollary 2. S*uppose that* $F(a,b;c;z) \neq 0$ *in* *, and in addition a,b,c satisfy the* *conditions* (2.6) *and* (2.7). *Then* $z^pF\left(a,b;c;z^2\right)\in S_p^*$.

Proof. Let $\varphi(z)=z^pF(a,b;c;z)$ and $h(z)=\dfrac{\varphi\left(z^2\right)}{z^p}$. Then, we have

$$\frac{zh'(z)}{h(z)}=z\frac{z^2\varphi'\left(z^2\right)}{\varphi\left(z^2\right)}-p,$$

and by Corollary 1, we have φ is starlike of order $p/2$ and therefore, it concludes that $h\in S_p^*$.

Also, Theorem 1 provides following result for $\alpha=p-\dfrac{a+b}{2}(0<a+b\le 2p)$.

Corollary 3. *Let a,b and c be non-zero real numbers such that* $F(z,b;c;z)\neq 0$ *in* .
If $0<a+b\le 2p$ *and inequality:*

(2.8) $$c\ge 1+\frac{a^2+b^2}{a+b},$$

holds, then $z^pF(a,b;c;z)\in S_p^*\left(p-\dfrac{a+b}{2}\right)$.

Proof. Since, on taking $\alpha=p-\dfrac{a+b}{2}$, we get from the assumptions of Theorem 1 that B=0, and by (2.8), we get $C\ge 0$. Also, condition (ii) which is equivalent to

$$c\ge-\frac{(a+b)}{2}-\frac{(a-b)^2}{2}$$

holds if (2.8) is true. Further, as $C\ge 0$ and B=0, the condition (iii) obviously holds. Thus the conclusion follows.

Further, for α=0, Theorem 1 yields follwing result.

Corollary 4. *Let a,b and c be non zero real numbers such that* $F(a,b;c;z)\neq 0$ *in* .
if inequality

$$c\ge\max\left\{1+a+b-ab/p,2p+ab/p-(a+b)\right\}$$

with

$$(c-1)(c-2p)\ge a^2+b^2-ab/p+(1-2p)(a+b)$$

holds, then $z^pF(a,b;c;z)\in S_p^*$.

Since, hypergeometric function $F(a,b;c;z)$ satisfies the identity

(2.9) $$\frac{z\left(z^pF(p,b;c;z)\right)^1}{p}=z^pF(p+1,b;c;z),$$

we can directly prove our next result by using (2.9) and the Alexander property (1.5), on taking $a=p+1$ in Theorem 1.

Theorem 2. *Let a,b and c be non zero real numbers such that* $F(p+1,b;c;z)\neq 0$ *in* .

Let $\tilde{\alpha}=p-\alpha, A=\tilde{\alpha}^2-\tilde{\alpha}(1+p+b)+(p+1)b$, $B=\tilde{\alpha}(p+1+b)-2\tilde{\alpha}^2$, $C=\tilde{\alpha}\tilde{c}+(p+1)b$, $D=\tilde{\alpha}\tilde{c}$ *and* $\tilde{c}=c-2-b-p$. *If inequalities :*

(i) $c\geq 2+p+b-b(p+1)/\tilde{\alpha}$(*equivalently* $C\geq 0$)

(ii) $C+\tilde{\alpha}\geq 2A$; *and*

(iii) $\left(\tilde{\alpha}+2\tilde{\alpha}^2\right)C+2BD+D^2\geq 0$,

hold, then $z^pF(p,b;c;z)\in K_p(\alpha)$.

For $\alpha=0$, *Theorem 2 yields following result.*

Corollary 5. *Let a,b and c be non zero real numbers such that* $F(p+1,b;c;z)\neq 0$ *in* . *Then* $z^pF(p,b;c;z)\in K_p$ *if*

$$c\geq \max\{2+p-b/p, p+b/p-1\}.$$

For the subclass S_p of S_p^* we prove our next result.

Theorem 3. *Let for real nubers a,b,c if* $F(p,b;c;z)\neq 0$ *in* *and*

let $u:=\frac{(a-p)(b-p)}{p}, v:=\frac{ab-p^2}{p}, w:=\frac{(a+b)(b+p)}{p}$ *satisfy :*

(i) $c+p\geq|1+w|$,

(ii) $c-p\geq|1-u|$, *and*

(iii) $2c^2+2p^2-4v^2-2(1-u)(1+w)>-\left[\left((c-p)^2-(1-u)^2\right)\left((c+p)^2-(1+w)^2\right)\right]^{1/2}$

then $z^pF(a,b;c;z)\in S_p$.

Proof. Let $\varphi(z)=z^pF(a,b;c;z)=:z^pF(z)$ and $\omega(z)=\dfrac{z\varphi'(z)}{p\varphi(z)}-1$. To Prove the result we need to show from (1.3) that for $z\in$,

$$|\omega(z)|<1\Leftrightarrow \Re\left(\frac{1-\omega(z)}{1+\omega(z)}\right)>0,$$

or, equivalently

$$2\Re\left(\frac{p\varphi(z)}{z\varphi'(z)}-\frac{1}{2}\right)>0.$$

Define

(2.10) $$q(z)=2\left\{\frac{p\varphi(z)}{z\varphi'(z)}-\frac{1}{2}\right\},$$

then $q(z)$ is analytic in with $q(0)=1$, From (2.10), we get that

(2.11) $$\frac{zF'(z)}{pF(z)}=\frac{1-q(z)}{1+q(z)}$$

and

(2.12) $$\frac{zF''(z)}{F(z)}=\frac{(p^2+p)q^2(z)-2pzq'(z)-2p^2q(z)+p^2-p}{z(1+q(z))^2}.$$

Applying (2.11) and (2.12) in (2.3), we get

$$\psi(q(z),zq'(z);z)=0,$$

where

$$\psi(r,s;z)=2s(1-z)+r^2\left\{c-(p+1)-\frac{(a-p)(b-p)z}{p}\right\}$$

$$+2r\left\{p+\frac{(ab-p^2)z}{p}\right\}-\left\{c+(p-1)-\frac{(a+p)(b+p)z}{p}\right\}.$$

Again, we apply Lemma 1 to conclude the proof of the theorem. Therefore, to prove $\Re(q(z))>0$ in U, we must show from the assumption of the theorem that

$\psi(ir,s;z)\neq 0$ for $z\in$ and $s\leq\dfrac{-(1+r^2)}{2}$ with all $r\in\mathbb{R}$.

By the defined notations u, v, w, we get

$$\psi(ir, s; z) = 2s(1-z) - r^2\{c-(p+1)-uz\} + 2ir\{p+vz\} - c-(p-1)+wz.$$

For z satisfying $\psi(ir, s; z) = 0$, we get

$$|z|^2 = \frac{\left[2s - r^2\{c-(p+1)\} - c-(p-1)\right]^2 + 4r^2p^2}{\left(2s+ur^2-w\right)^2 + 4v^2r^2}.$$

Thus $\psi(ir, s; z) \neq 0$ for $z \in$ if and only if

$$\frac{\left[2s - r^2\{c-(p+1)\} - c-(p-1)\right]^2 + 4r^2p^2}{\left(2s+ur^2-w\right)^2 + 4v^2r^2} \geq 1,$$

which after simplification gives

(2.13) $\left[2s - r^2\{c-(p+1)\} - c-(p-1)\right]^2 + 4r^2\left(p^2-v^2\right) - \left(2s+ur^2-w\right) \geq 0.$

In the expression (2.13), we see that the coefficient of s^2 is 0 and coefficient of s is

$$4\left[\left(-u-c+(p+1)\right)r^2 + w-c-(p-1)\right].$$

By assumptions (i) and (ii), we have

$$-u-c+(p+1) < 0 \text{ and } w-c-p+1 < 0,$$

so the coefficient of s is negative, and it suffices to check the inequality (2.13) for largest value of s i.e. $s = \frac{-\left(1+r^2\right)}{2}$. Setting $t = r^2$ and $2s = -(1+t)$, the inequality (2.13) can be written as

(2.14) $\{t(c-p)+c+p\}^2 + 4\left(p^2-v^2\right)t - \{(1-u)t+w+1\}^2 \geq 0,$

which is a second degree polynomial inequality in t.
Let

$$A = (c-p)^2 - (1-u)^2,$$

$$B = c^2 - p^2 + 2\left(p^2-v^2\right) - (1-u)(1+w)$$

$$C = (c+p)^2 - (1+w)^2,$$

then inequality (2.14), is of the form $At^2 + 2Bt + C \geq 0$ for all $t \geq 0$, which holds true if and only if $A \geq 0$, $C \geq 0$ and $B \geq -\sqrt{AC}$, which are exactly the conditions of the theorem. This proves the Theorem 3.

3. Special Cases. On taking $b=1$ in (1.6), we get an incomplete beta function denoted by $\phi(a,c;z)$ and is defined by

$$\phi(a,c;z)=\sum_{n=0}^{\infty}\frac{(a)_n}{(c)_n}z^n, z\in \quad .$$

In this section, we mention the results for p-valent incomplete beta function $z^p\phi(a,c;z)$ which can directly be obtained by taking $b=1$ in the Main Results.

Corollary 6. *Let for non-zero real numbers a and* c, $\phi(a,c;z)\neq 0$ in . *Let for* $0\leq\alpha<p, \tilde{\alpha}=p-\alpha, A_1=\tilde{\alpha}^2-\tilde{\alpha}(a+1)+a, B_1=\tilde{\alpha}(a+1)-2\tilde{\alpha}^2, C_1=\tilde{\alpha}\tilde{c}+a, D_1=\tilde{\alpha}\tilde{c}$ *and* $\tilde{c}=c-2-a$. *If inequalities:*

(i) $c\geq 2+a-a/\tilde{\alpha}$ (or, *equivalently* $C_1\geq 0$),

(ii) $C_1+\tilde{\alpha}\geq 2A_1$,

(iii) $(\tilde{\alpha}+2\tilde{\alpha}^2)C_1+2B_1D_1+D_1^2\geq 0$,

hold, then $z^p\phi(a,c;z)\in S_p^*(\alpha)$.

Corollary 7. *Let a and c be non-zero real numbers such that* $\phi(a,c;z)\neq 0$ *in* . *If inequality*

(3.1) $c\geq\max\{2+a-2a/p, p+2a/p-(a+1)\}$

with

(3.2) $(c-2-a)(c-p+a+1)\geq 2(p+1)a/p$

holds, then $z^p\phi(a,c;z)\in S_p^*(p/2)$.

Corollary 8. *Suppose that* $\phi(a,c;z)\neq 0$ *in* , *and in addition a,c satisfy the conditions* (3.1) *and* (3.2). *Then* $z^p\phi(a,c;z^2)\in S_p^*$.

Corollary 9. *Let a and c be non-zero real number such that* $\phi(a,c;z)\neq 0$ *in* . If $0<a+1\leq 2p$ *and inequality*:

(3.3) $c\geq 1+\dfrac{a^2+1}{a+1}$,

holds, then $z^p\phi(a,c;z)\in S_p^*\left(p-\dfrac{a+1}{2}\right)$.

Corollary 10. *Let a and c be non zero number such that* $\phi(a,c;z)\neq 0$ *in* . *If inequality*

$$c \geq \max\{2+a-a/p, 2p+a/p-(a+1)\}$$

with

$$(c-1)(c-2p)\geq a^2+1-a/p+(1-2p)(a+1)$$

holds, then $z^p\phi(a,c;z^2)\in S_p^*$.

Corollary 11. Let a and c be non zero real numbers such that $\phi(p+1,c;z)\neq 0$ in . Let for $0\leq a<p$, $\tilde{\alpha}=p-\alpha$, $A_1=\tilde{\alpha}^2-\tilde{\alpha}(2+p)+(p+1)$, $B_1=\tilde{\alpha}(p+2)-2\tilde{\alpha}^2$, $C_1=\tilde{\alpha}\tilde{c}+(p+1)$, $D_1=\tilde{\alpha}\tilde{c}$ *and* $\tilde{c}=c-3-p$. *If inequalities:*

(i) $C\geq 3+p-(p+1)/\tilde{\alpha}$ (or, *equivalently* $C_1\geq 0$)

(ii) $C_1+\tilde{\alpha}\geq 2A_1$; *and*

(iii) $(\tilde{\alpha}+2\tilde{\alpha}^2)C_1+2B_1D_1+D_1^2\geq 0$,

hold, then $z^p\phi(a,c;z)\in K_p(\alpha)$.

Corollary 12. *Let a and c be non zero real nubers such that* $\phi(p+1,c;z)\neq 0$ *in* . *Then* $z^p\phi(a,c;z)\in K_p$ *if*

$$c\geq \max\{2+p-1/p, p+1/p-1\}.$$

Corollary 13. *Let for real numbers* $a,c,\phi(a,c;z)\neq 0$ *in and let* $u_1:=\dfrac{(a-p)(1-p)}{p}$, $v_1:=\dfrac{a-p^2}{p}$, $w_1:=\dfrac{(a+p)(p+1)}{p}$ *satisfy :*

(i) $C+p\geq |1+w_1|$,

(ii) $C-p\geq |1-u_1|$, *and*

(iii) $2c^2+2p^2-4v_1^2-c(1-u_1)(1+w_1)>-\sqrt{\left((c-p)^2-(1-u_1)^2\right)\left((c+p)^2-(1+w_1)^2\right)}$

then $z^p\phi(a,c;z)\in S_p$.

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SETS OF ORTHOGONAL POLYNOMIALS OF DISCRETE VARIABLES

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ABSTRACT

In the present paper, sets of polynomials $A_n(x,w)$ have been defined by means of the n^{th} difference formulae through the finite differences:

$$\text{(i)} \qquad A_n^{a+\overline{\lambda-1}\,n,\ \beta+\overline{\mu-1}\,n}(x,\gamma,\delta,w) = \frac{\underset{w}{\overset{n}{\Delta}} x\left[(x-1+\gamma w)^{[\alpha+\lambda n\ w]}(x+1-\delta w)^{[\beta+\mu n\ w]}\right]}{n!2^n(x-1+\gamma w)^{[aw]}(x+1-\delta w)^{[\beta w]}},$$

λ,μ being integers not both being zero simultaneously, whereas for $\lambda = 0 = \mu,$ we define it as

$$\text{(ii)} \qquad A_n^{\alpha-n,\beta-n}(x,\gamma,\delta,w) = \frac{\underset{w}{\overset{n}{\Delta}} x\left[(x-1+\gamma w)^{[\alpha w]}(x+1-\delta w)^{[\beta w]}\right]}{n!2^n(x-1+\gamma w)^{[\alpha-nw]}(x+1-\delta w)^{[\beta-nw]}}.$$

These polynomials are valid for discrete values of the varieble at equasl intervsls viz. x, $x-w$, $x-2w$,...,$x-mw$. Obviously, as $w \to 0,$ the variable becomes continuous and hence, as will be seen, these polynomials reduce to classical Jacobi and Modified Jacobi polynomials in the limiting case. Other conditions are mentioned in the paper that follows.

After obtaining its explicit forms, orthogonality and Pseudo orthogonality properties for these polynomials have been established, whenever they happen to be precisely of degree n in x. Finally, expansions of certain factorial functions have been obtained in series of our polynomiels. As $w \to 0,$ the orthogonality, Pseudo orthogonality properties as well as the expansions reduce to the corresponding known properties.

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Keywords and Phrases : Discrete variable, Orthogonal polynomial, Pseudo orthogonality, Jump function, Explicit, form, Jacobi polynomials, Modified Jacobi polynomials.

1. Introduction. We define our polynomial set $A_n(x,\gamma,\delta,w)$ by means of the n^{th} difference formulae through the finite differences :

$$(1.1)\quad A_n^{\alpha+\overline{\lambda-1}\,n\,,\,\beta+\overline{\mu-1}\,n}(x,\gamma,\delta,w)=\frac{\overset{n}{\underset{w}{\Delta}}{}_x\left[(x-1+\gamma w)^{\left[\overline{\alpha+\eta}\,w\right]}(x+1-\delta w)^{\left[\overline{\beta+\mu}\,w\right]}\right]}{n!2^n(x-1+\gamma w)^{[aw]}(x+1-\delta w)^{[\beta w]}},$$

λ,μ being integers not both being zero simultaneously, whereas for $\lambda=0=\mu$, we define it as :

$$(1.2)\quad A_n^{\alpha-n,\beta-n}(x,\gamma,\delta,w)=\frac{\overset{n}{\underset{w}{\Delta}}{}_x\left[(x-1+\gamma w)^{[\alpha w]}(x+1-\delta w)^{[\beta w]}\right]}{n!2^n(x-1+\gamma w)^{\left[\overline{\alpha-n}\,w\right]}(x+1-\delta w)^{\left[\overline{\beta-n}\,w\right]}}.$$

These polynomials are valid for discrete values or the variable at equal intervals viz. x, $x-w$, $x-2w$,...,$x-mw$. Obviously, as $w\to 0$, the variable becomes continuous and hence, as will be seen from their explicit forms, these polynomials, in the limiting case, reduce to the classical Jacobi and modified Jacobi polynomials. Here α and β may take any integral value (see, for details, [2,p.71 et seq.] and [3, Chapter 4]).

Some main results in the theory of finite differences [1] of ten used, are:

$$(1.3)\quad \underset{w}{\Delta}{}_x f(x)=\frac{f(x+w)-f(x)}{w}\Rightarrow \lim_{w\to o}\left[\underset{w}{\Delta}{}_x f(x)\right]=\frac{d}{dx}f(x),$$

$$(1.4)\quad x^{[\alpha w]}=x(x-w)(x-2w)...\left(x-\overline{\alpha-1}\,w\right)$$

$$(1.5)\quad x^{[-\beta w]}=\frac{1}{(x+w)(x+2w)...(x+\beta w)},$$

$$(1.6)\quad \overset{r}{\underset{w}{\Delta}}{}_x\left(x^{[\alpha w]}\right)=\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-r)}x^{\left[\overline{\alpha-r}\,W\right]}=(-1)^{\Gamma}(-\alpha)_r\,x^{\left[\overline{\alpha-r}\,W\right]}\ ;\ 0\le r\le\alpha,$$

$$(1.7)\quad \overset{n}{\underset{w}{\Delta}}{}_x(u_x.v_x)=\sum_{r=o}^{n}\binom{n}{r}\overset{n-r}{\underset{w}{\Delta}}{}_x(u_{x+rW})\overset{r}{\underset{w}{\Delta}}{}_x(v_x),$$

$$(1.8)\quad \underset{w}{\Delta}{}_x\left(\frac{x}{w}\right)_R=\frac{R}{W}\left(\frac{x}{w}+1\right)_{R-1},$$

$$(1.9)\quad \underset{w}{\Delta}{}_x\left(-\frac{x}{w}\right)_H=\frac{R}{W}\left(-\frac{x}{w}\right)_{R-1},$$

$$(1.10)\quad \underset{w}{\Delta}{}_x(-x)^{[\alpha w]}=-\alpha(-x-w)^{\left[\overline{\alpha-1}\,w\right]},$$

$$(1.11)\quad \underset{w}{\Delta}{}_x f(x)=\phi(x)\Rightarrow \overset{-1}{\underset{w}{\Delta}}{}_x\phi(x)=f(x).$$

2. Explicit Forms. For brevity, writing y for $x-1+\gamma w$ and z for $x+1-\delta w$ after making appeals to (1.1) and (1.7), we get

(2.1) $$A_n^{\gamma+\overline{\lambda-1}\,n\ ,\ \beta+\overline{\mu-1}\,n}(x,\gamma,\delta,w)=\frac{1}{n!2^n y^{[\alpha w]} z^{[\beta w]}}$$

$$\sum_{r=0}^{n}\frac{(-1)^r(-n)_r}{r!}\frac{\Gamma(1+\gamma+\lambda n)}{\Gamma\left(1+r+\overline{\lambda-1}\,n+r\right)}(y+rw)^{\left[\alpha+\overline{\lambda-1}\,n+r\ w\right]}\frac{\Gamma(1+\beta+\mu n)}{\Gamma(1+\beta+\mu n-r)}z^{\left[\beta+\mu n-r\ w\right]}$$

$$=\frac{\left(1+\gamma+\overline{\lambda-1}\,n\right)_n}{n!2^n y^{[\alpha w]} z^{[\beta w]}}\sum_{r=0}^{n}\frac{(-n)_r(-\beta-\mu r)_r}{r!\left(1+\alpha+\overline{\lambda-1}\,n\right)_r}(-w)^{\overline{\lambda+\mu-1}\,n}$$

$$\left(\frac{y}{w}+1\right)_r y^{[\alpha w]}\left(\alpha-\frac{y}{w}\right)_{\overline{\lambda-1}\,n}\frac{z^{[\beta w]}(\beta-z/w)_{\mu n}}{(z/w+1-\beta-\mu n)_r}$$

$$=\frac{\left(1+\alpha+\overline{\lambda-1}\,n\right)_n}{n!2^n}(-w)^{\overline{\lambda+\mu-1}\,n}\left(\gamma-\frac{y}{w}\right)_{\overline{\lambda-1}\,n}\left(\beta-\frac{z}{w}\right)_{\mu n}$$

$$\sum_{r=0}^{n}\frac{(-n)_r(-\beta-\mu n)_r(y/w+1)_r}{r!\left(1+\alpha+\overline{\lambda-1}\,n\right)_r(1-\beta-\mu n+z/w)_r}$$

$$=\frac{\left(1+\alpha+\overline{\lambda-1}\,n\right)_n}{n!2^n}(-w)^{\overline{\lambda+\mu-1}\,n}\left(\alpha-\gamma-\frac{x-1}{w}\right)_{\overline{\lambda-1}\,n}\left(\beta+\delta-\frac{x-1}{w}\right)_{\mu n}$$

$$ {}_3F_2\left[\begin{matrix}-n,-\beta-\mu n,1+\gamma+\dfrac{x-1}{w}\\ 1+\alpha+\overline{\lambda-1}\,n,1-\beta-\mu n-\delta+\dfrac{x+1}{w}\end{matrix};1\right]$$

Special Cases.

(2.2) $$\lim_{w\to 0}A_n^{\alpha+\overline{\lambda-1}\,n,\ \beta+\overline{\mu-1}\,n}(x,\gamma,\delta,w)$$

$$=\frac{\left(1+\alpha+\overline{\lambda-1}\,n\right)_n}{n!2^n}(x-1)^{\overline{\lambda-1}\,n}(x+1)^{\mu n}\ {}_2F_1\left[\begin{matrix}-\beta-\mu n,-n\\ 1+\alpha+\overline{\lambda-1}\,n\end{matrix};\frac{x-1}{x+1}\right]$$

(2.3) $$=(x-1)^{\overline{\lambda-1}\,n}(x+1)^{\overline{\mu-1}\,n}P_n^{\alpha+\overline{\lambda-1}\,n,\beta+\overline{\mu-1}\,n}(x)$$

$\equiv$ Modified Jacobi polynomials, unless $\lambda=1=\mu$, when it is converted into the classical Jacobi polynomials $P_n^{\alpha,\beta}(x)$. Other two intresting special cases, amongst many others, are

$$\left(\frac{x+1}{x-1}\right)^n P_n^{\alpha+n,\beta-n}(x) \text{ for } \lambda=2,\mu=0 \text{ and } \left(\frac{x+1}{x-1}\right)^n P_n^{\alpha-n,\beta+n}(x) \text{ for } \lambda=0,\mu=2.$$

In a similar way, making appeals to (1.2) and (1.7) we have

(2.4) $$A_n^{\alpha-n,\beta-n}(x,\gamma,\delta,w)=\frac{1}{n;2^n y^{\left[\overline{\alpha-n}\, w\right]} z^{\left[\overline{\beta-n}\, w\right]}}$$

$$\sum_{r=0}^{n}\frac{(-1)^r(-n)_r}{r!}\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-n+r)}(y+\gamma w)^{\left[\overline{\alpha-n+r}\, w\right]}\frac{\Gamma(1+\beta)}{\Gamma(1+\beta-r)}z^{\left[\overline{\beta-r}\, w\right]}$$

$$=\frac{(-w)^n\left(\frac{z}{w}+1-\beta\right)_n(-\alpha)_n}{n!2^n y^{\left[\overline{\alpha-n}\, w\right]} z^{[\beta w]}}\sum_{r=0}^{n}\frac{(-n)_r(-\beta)_r w^r}{r;(1-\alpha-n)_r}\left(1+\frac{y}{w}\right)_r y^{\left[\overline{\alpha-n}\, w\right]}\frac{z^{[\beta w]}}{w^r\left(1+\frac{z}{w}-\beta\right)_r}$$

$$=\frac{(-\alpha)_n(-w)^n\left(1-\beta-\delta+\frac{x+1}{w}\right)_n}{n!2^n}\; {}_3F_2\left[\begin{matrix}-n,-\beta,1+\gamma+\frac{x-1}{w} \\ 1+\alpha-n,1-\beta-\delta+\frac{x+1}{w}\end{matrix};1\right].$$

Special Case.

(2.5) $$\lim_{w\to 0} A_n^{\alpha+n,\beta-n}(x,\gamma,\delta,w)=\frac{(-1)^n(-\alpha)_n}{n!2^n}(x+1)^n\; {}_2F_1\left[\begin{matrix}-n,-\beta \\ 1+\alpha-n\end{matrix};\frac{x-1}{x+1}\right]$$

(2.6) $$=p_n^{(\alpha-n,\beta-n)}(x).$$

$\equiv$ Modified Jacobi polynomials.

3. Orthogonality.

Case I. For the orthogonality property of (2.1) with $\lambda=1=\mu$, we have

(3.1) $$A_n^{\alpha,\beta}(x,\gamma,\delta,w)=\frac{(1+\alpha)_n}{n!2^n}(-w)^n\left(\beta+\delta-\frac{x+1}{w}\right)_n\; {}_3F_2\left[\begin{matrix}-n,-\beta-n,1+\gamma+\frac{x-1}{w} \\ 1+\alpha,1-\beta-\delta-n+\frac{x+1}{w}\end{matrix};1\right],$$

which, obviously, is a polynomial precisely of degree n in x and that its coefficient is

$$(3.2)\qquad \frac{(1+\alpha)_n w^n}{n!2^n}\sum_{r=0}^{n}\frac{(-n)_r(-\beta-n)_r}{r!(1+\alpha)_r}=\frac{(1+\alpha)_n w^n}{n!2^n}\,{}_2F_1\left[\begin{matrix}-n,-\beta-n\\1+\alpha\end{matrix};1\right]=\frac{w^n(1+\alpha+\beta)_{2n}}{n!2^n(1+\alpha+\beta)_n},$$

which implies

$$(3.3)\qquad A_n^{\alpha,\beta}(x,\gamma,\delta,w)=\frac{(1+\alpha+\beta)_{2n}(-w)^n}{n!2^n(1+\alpha+\beta)_n}\left(\beta+\delta-\frac{x-1}{w}\right)_n+\prod\nolimits_{n-1}(x),$$

where $\prod_{n-1}(x)$ is a polynomial of degree n–1 in x.

Let us take the Jump function as

$$(3.4)\qquad \psi(x,\alpha,\beta,\gamma,\delta,w)=(x-1+\gamma w)^{[\alpha w]}(x+1-\delta w)^{[\beta w]}.$$

Therefore,

$$\boldsymbol{A}_n^{\alpha,\beta}(x,\gamma,\delta,w)\psi(x,\alpha,\beta,\gamma,\delta,w)=\frac{1}{n!2^n}\overset{n}{\underset{w}{\Delta}}{}_x\left[(x-1+\gamma w)^{[\overline{\alpha+n}\,w]}(x+1-\delta w)^{[\overline{\beta+n}\,w]}\right]$$

$$=\underset{w}{\Delta}{}_x\left[\frac{1}{n!2^n}\overset{n-1}{\underset{w}{\Delta}}x\left\{(x-1+\gamma w)^{[\overline{\alpha+n}\,w]}(x+1-\delta w)^{[\overline{\beta+n}\,w]}\right\}\right].$$

Hence

$$(3.5)\qquad \overset{-1}{\underset{w}{\Delta}}{}_x\left[\boldsymbol{A}_n^{\alpha,\beta}(x,\gamma,\delta,w)\psi(x,\alpha,\beta,\gamma,\delta,w)\right]$$

$$=\frac{1}{n!2^n}\overset{n-1}{\underset{w}{\Delta}}{}_x\left\{(x-1+\gamma w)^{[\overline{\alpha+n}\,w]}(x+1-\delta w)^{[\overline{\beta+n}\,w]}\right\}.$$

Now

$$(3.6)\qquad \sum_{x=-1+\delta w}^{1-\gamma w}\boldsymbol{A}_n^{\alpha,\beta}(x,\gamma,\delta,w)\psi(x,\alpha,\beta,\gamma,\delta,w)$$

$$=\left[\overset{-1}{\underset{w}{\Delta}}{}_x\left\{\boldsymbol{A}_n^{\alpha,\beta}(x,\gamma,\delta,w)\Psi(x,\alpha,\beta,\gamma,\delta,w)\right\}\right]_{x=-1+\delta w}^{1-\gamma w}$$

$$=\frac{1}{n!2^n}\left[\overset{n-1}{\underset{w}{\Delta}}x\left\{(x-1+\gamma w)^{[\overline{\alpha+n}\,w]}(x+1-\delta w)^{[\overline{\beta+n}\,w]}\right\}\right]_{x=-1+\delta w}^{1-\gamma w}$$

$$=\frac{1}{n!2^n}\left[\sum_{r=0}^{n-1}\binom{n-1}{r}\overset{n-r-1}{\underset{w}{\Delta}}x\left(x-1+\overline{\gamma+r}\,w\right)^{[\overline{\alpha+n}\,w]}\overset{r}{\underset{w}{\Delta}}x(x+1-\delta w)^{[\overline{\beta+n}\,w]}\right]_{x=-1+\delta w}^{1-\gamma w}.$$

(3.7) $$= \frac{(-1)^{n-1}}{n!2^n}\left[\sum_{r=0}^{n-1}\binom{n-1}{r}(-\alpha-n)_{n-r-1}(-\beta-n)_r w^r\left(\frac{x-1}{w}+r+1\right)_r\right.$$

$$\left.(x-1+\gamma w)^{\left[\overline{\alpha+n}\; w\right]}(x+1-\delta w)^{\left[\overline{\beta+n}\; w\right]}\right]_{x=-1+\delta w}^{1-\gamma w}$$

(3.8) $=0.$

Now, in the indefinite sum obtained by summation by parts [1; p.34], i.e.

(3.9) $$\underset{w}{\overset{-1}{\Delta}}_x(u_x.v_x)=u_x\underset{w}{\overset{-1}{\Delta}}_x(v_x)-\underset{w}{\overset{-1}{\Delta}}_x\left[\underset{w}{\Delta}_x(u_x)\underset{w}{\overset{-1}{\Delta}}_x(v_{x+w})\right].$$

Taking $u_x=\left(\beta+\delta-\frac{x+1}{w}\right)_m$ and $v_x=A_n^{\alpha,\beta}(x,\gamma,\delta,w)\Psi(x,\alpha,\beta,\gamma,\delta,w)$, we get

$$\left[\underset{w}{\overset{-1}{\Delta}}_x\left[\left(\beta+\delta-\frac{x+1}{w}\right)_m\left\{A_n^{\alpha,\beta}(x,\gamma,\delta,w)\psi(x,\alpha,\beta,\gamma,\delta,w)\right\}\right]\right]_{x=-1+\delta w}^{1-\gamma w}$$

$$=\left[\left(\beta+\delta-\frac{x+1}{w}\right)_m\underset{w}{\overset{-1}{\Delta}}_x\left\{A_n^{\alpha,\beta}(x,\gamma,\delta,w)\Psi(x,\alpha,\beta,\gamma,\delta,w)\right\}\right]_{x=-1+\delta w}^{1-\gamma w}$$

$$-\left[\underset{w}{\overset{-1}{\Delta}}_x\left[\underset{w}{\Delta}_x\left\{\left(\beta+\delta-\frac{x+1}{w}\right)_m\right\}\underset{w}{\overset{-1}{\Delta}}_x\left\{A_n^{\alpha,\beta}(x+w,\alpha,\gamma,\delta,w)\psi(x+w,\alpha,\beta,\gamma,\delta,w)\right\}\right]\right]_{x=-1+\delta w}^{1-\gamma w}$$

$$=\left[\frac{\left(\beta+\delta-\frac{x+1}{w}\right)_m}{n!2^n}\underset{w}{\overset{n-1}{\Delta}}_x\left\{(x-1+\gamma w)^{\left[\overline{\alpha+n}\; w\right]}(x+1-\delta w)^{\left[\overline{\beta+n}\; w\right]}\right\}\right]_{x=-1+\delta w}^{1-\gamma w} \quad \text{using (3.5)}$$

$$-\left[\underset{w}{\overset{-1}{\Delta}}_x\left[-\frac{m}{w}\left(\beta+\delta-\frac{x+1}{w}\right)_{m-1}\frac{1}{x;2^n}\underset{w}{\overset{n-1}{\Delta}}_x\left(x-1+\overline{\gamma+1}\;\; w\right)^{\left[\overline{\alpha+1+\overline{n-1}}\; w\right]}\left(x+1-\overline{\delta-1}w\right)^{\left[\overline{\beta+1+\overline{n-1}}\; w\right]}\right]\right]_{x=-1+\delta w}^{1-\gamma w}$$

$$=\left[\frac{m}{2nw}\underset{w}{\overset{-1}{\Delta}}_x\left\{\left(\beta+\delta-\frac{x+1}{w}\right)_{m-1}A_{n-1}^{\alpha+1,\beta+1}(x,w,\gamma,\delta,w)\Psi(x+w,\alpha+1,\beta+1,\gamma,\delta,w)\right\}\right]_{x=-1+\delta w}^{1-\gamma w}$$

as the first part vaniwhes because of (3.6) and (3.8).

Therefore,

(3.10) $$\sum_{x=1+\delta w}^{1-\gamma w}\left[\left(\beta+\delta-\frac{x+1}{w}\right)_m A_n^{\alpha,\beta}(x,\gamma,\delta,w)\psi(x,\alpha,\beta,\gamma,\delta,w)\right]$$

$$=\frac{m}{2nw}\sum_{x=-1+\delta w}^{1-\gamma w}\left(\beta+\delta-\frac{x+1}{w}\right)_{m-1} A_{n-1}^{\alpha+1,\beta+1}(x+w,\alpha,\delta,w)\psi(x+w,\alpha+1,\beta+1,\gamma,\delta,w),$$

(3.11) $$=\frac{m!(n-m)!}{2^m w^m n!}\sum_{x=-1+\delta w}^{1-\gamma w} A_{n-m}^{\alpha+m,\beta+m}(x+mw,\gamma,\delta,w)\psi(x+mw,\alpha+m,\beta+m,\gamma,\delta,w),$$

by continuing the process of (3.10) m-times in all.

But the right hand side of (3.11) vanishes due to (3.8), if $n>m$. Therefore,

(3.12) $$\sum_{x=-1+\delta w}^{1-\gamma w}\left[\left(\beta+\delta-\frac{x+1}{w}\right)_m A_n^{\alpha,\beta}(x,\gamma,\delta,w)\Psi(x,\alpha,\beta,\gamma,\delta,w)\right]=0\ ;\ n>m.$$

Hence, if P_m by any polynomial whatsoever, of degree $m<n$, we get

(3.13) $$\sum_{x=1+\delta w}^{1-\gamma w} P_m A_n^{\alpha,\beta}(x,\gamma,\delta,w)\Psi(x,\alpha,\beta,\gamma,\delta,w)=0;n>m.$$

Also, in particular, as $A_m^{\alpha,\beta}(x,\gamma,\delta,w)$ is a polynomial precisely of degree m in x we may equally write

(3.14) $$\sum_{x=1+\delta w}^{1-\gamma w}\left[A_m^{\alpha,\beta}(x,\gamma,\delta,w)A_n^{\alpha,\beta}(x,\gamma,\delta,w)\Psi(x,\alpha,\beta,\gamma,\delta,w)\right]=0\ ;\ n>m.$$

Similarly, reversing the process, we have

(3.15) $$\sum_{x=-1+\delta w}^{1-\gamma w}\left[A_m^{\alpha,\beta}(x,\gamma,\delta,w)A_m^{\alpha,\beta}(x,\gamma,\delta,w)\psi(x,\alpha,\beta,\gamma,\delta,w)\right]=0\ ;\ n<m.$$

Hence

(3.16) $$\sum_{x=-1+\delta w}^{1-\gamma w}\left[A_n^{\alpha,\beta}(x,\gamma,\delta,w)A_n^{\alpha,\beta}(x,\gamma,\delta,w)\psi(x,\alpha,\beta,\gamma,\delta,w)\right]=0;m\neq n,$$

which establishes the orthogonality of our polynomial set.

Appeals to (3.3), (3.11) with $m=n$ and (3.4), yield

(3.17) $$\sum_{x=1+\delta w}^{1-\gamma w}\left[\left\{A_n^{\alpha,\beta}(x,\gamma,\delta,w)\right\}^2\Psi(x,\alpha,\beta,\gamma,\delta,w)\right]$$

$$= \sum_{x=1+\delta w}^{1-\gamma w} \left[\left\{ \frac{(1+\alpha+\beta)_{2n}}{n!2^n (1+\alpha+\beta)_n} \left(\beta+\delta-\frac{x+1}{w} \right)_n + \prod_{n-1}(x) \right\} A_n^{\alpha,\beta}(x,\gamma,\delta,w) \Psi(x,\alpha,\beta,\gamma,\delta,w) \right]$$

$$= \frac{(1+\alpha+\beta)_{2n}(-w)^n}{n!2^n(1+\alpha+\beta)_n} \frac{1}{2^n w^n} \sum_{z=-1+\delta w}^{1-\gamma w} A_0^{\alpha,\beta}(x+nw,\gamma,\delta,w)\psi(x+nw,\alpha+m,\beta+n,\gamma,\delta,w)$$

$$= \frac{(-1)^n(1+\alpha+\beta)_{2n}}{n!2^{2n}(1+\alpha+\beta)_n} \sum_{z=-1+\delta w}^{1-\gamma w} \left(x-1+\overline{\gamma+n}\,w\right)^{\left[\overline{\alpha+n}\,w\right]} \left(x+1-\overline{\delta-n}\,w\right)^{\left[\overline{\beta+n}\,w\right]}.$$

Verification

Equation (3.17), when $w \to 0$, yields :

$$\int_{-1}^{1} (x-1)^{\alpha} (x+1)^{\beta} \left\{ P_n^{\alpha,\beta}(x) \right\}^2 dx = \frac{(-1)^n (1+\alpha+\beta)_{2n}}{n!2^{2n}(1+\alpha+\beta)_n} \int_{-1}^{1} (x-1)^{\alpha+n} (x+1)^{\beta+n} dx,$$

(3.18) $$\int_{-1}^{1} (1-x)^{\alpha} (1+x)^{\beta} \left\{ P_n^{\alpha,\beta}(x) \right\}^2 dx = \frac{(1+\alpha+\beta)_{2n}}{n!2^{2n}(1+\alpha+\beta)_n} \int_{-1}^{1} (1-x)^{\alpha+n} (1+x)^{\beta+n} dx$$

$$= \frac{2^{\alpha+\beta+1}\Gamma(1+\alpha+n)\Gamma(1+\beta+n)}{n!(1+\alpha+\beta+2n)\Gamma(1+\alpha+\beta+n)},$$

which stands verified from the known result for the Jacobi polynomials. Hence the orthogonality property of our polynomial set with $\lambda = 1 = \mu$, too, stands verified.

4. Another Orthogonality. Equation (2.1) with $\lambda = 2$ and $\mu = 0$ again makes our polynomial precisely of degree n in x, whose coefficient is

(4.1) $$\frac{(1+\alpha+n)_n w^n}{n!2^n} {}_2F_1\left(\begin{matrix} -n,-\beta \\ 1+\alpha+n \end{matrix} ;1 \right) = \frac{(1+\alpha+\beta)_{2n} w^n}{n!2^n(1+\alpha+\beta)_n}.$$

Therefore

(4.2) $$A_n^{\alpha+n,\,\beta-n}(x,\gamma,\delta,w) = \frac{(1+\alpha+\beta)_{2n}(-w)^n}{n!2^n(1+\alpha+\beta)_n} \left(\alpha-\gamma-\frac{x-1}{w} \right)_n + \Pi_{n-1}(x),$$

where $\prod_{n-1}(x)$ is a polynomial of degree n–1 in x.

Again taking the Jump function as in (3.4) and proceeding almost in a similar way as in §3, we get

$$(4.4)\quad \left[\underset{w}{\overset{-1}{\Delta}}{}_x \left\{ \left(\alpha-\gamma-\frac{x-1}{w} \right)_m A_n^{\alpha+n,\,\beta-n}(x,\gamma,\delta,w)\,\psi(x,\alpha,\beta,\gamma,\delta,w) \right\} \right]_{x=-1+\delta w}^{1-\gamma w}$$

$$=\frac{m}{2nw}\left[\underset{w}{\overset{-1}{\Delta}}{}_x \left\{ \left(\alpha-\gamma-\frac{x-1}{w} \right)_{m-1} A_{n-1}^{\alpha+n+1,\,\beta-n+1}(x+w,\gamma,\delta,w)\,\psi(x+w,\alpha+2,\beta,\gamma,\delta,w) \right\} \right]_{x=-1+\delta w}^{1-\gamma w}$$

$$=\frac{m!(n-m)!}{n!2^m w^m}\sum_{x=-1+\delta w}^{1-\gamma w} A_{n-m}^{\alpha+n+m,\,\beta-n+m}(x+mw,\gamma,\delta,w)\,\psi(x+mw,\alpha+2m,\beta,\gamma,\delta,w),$$

by continuing the process of (4.3) m-times in all.

Hence finally, we arrive at

$$(4.5)\quad \sum_{x=-1+\delta w}^{1-\gamma w}\left[A_n^{\alpha+n,\,\beta-n}(x,\gamma,\delta,w)\,A_m^{\alpha+m,\,\beta-m}(x,\gamma,\delta,w)\,\psi(x,\alpha,\beta,\gamma,\delta,w) \right]=0\ ;\quad m\neq n,$$

which establishes the orthogonality property of our polynomial set with $\lambda=2$, $\mu=0$.

Making on appeal to (4.2), (4.4) with $m=n$ and (3.4), we finally arrive at

$$(4.6)\quad \sum_{x=-1+\delta w}^{1-\gamma w}\left[\left\{ A_n^{\alpha+n,\,\beta-n}(x,\gamma,\delta,w) \right\}^2 \psi(x,\alpha,\beta,\gamma,\delta,w) \right]$$

$$=\frac{(-1)^n(1+\alpha+\beta)_{2n}}{n!a^{2n}(1+\alpha+\beta)_n}\sum_{x=-1+\delta w}^{1-\gamma w}\left[\left(x-1+\overline{\gamma+n}\ w \right)^{[\overline{\alpha+2n}\ w]} \left(x+1-\overline{\delta-n}\ w \right)^{[\beta w]} \right].$$

Special Case. Equation (4.6) with $w\to 0$, finally yields

$$(4.7)\quad \int_{-1}^{1}(1-x)^{\alpha+2n}(1+x)^{\beta-2n}\left\{ P_n^{\alpha+n,\,\beta-n}(x) \right\}^2 dx=\frac{2^{\alpha+\beta+1}(-1)^n\,\Gamma(1+\beta)\Gamma(1+\alpha+2n)}{n!(1+\alpha+\beta+2n)\Gamma(1+\alpha+\beta+n)}.$$

In the absence (in our knowledge) of such a result, we can not confirm the correctness of this orthogonality property, but however, with the use of this property, we, in § 8, derive an expansion formula, which being verified, testifies the eorreetness of our orthogonality property too.

5. Pseudo Orthogonality.

Definition. Let $\phi_n(x)$ be a simple set of real polynomials. If there exist discrete variables $x_i=x+iw(i=0,1,2,...,w-1)$ and Jump functions $J(x_i,n)$ and $J(x_i,m)$ for each x_i such that each $J(x_i,n)$ and $J(x_i,m$ is positive and $\Sigma_i J(x_i,m)$ and $\Sigma_i J(x_i,n)$ are finite and if

(5.1) $$\sum_{i=0}^{n} J(x_i,n)\phi_n(x_i)\phi_m(x_i)=0\ ;\ \ n>m$$

and

(5.2) $$\sum_{i=0}^{N} J(x_i,m)\phi_n(x_i)\phi_m(x_i)=0\ ;\ \ n<m\,.$$

we say that the polynomial set $\phi_n(x_i)$ are Pseudo orthogonal with respect to the Jump funetions $J(x_i,m)$ and $J(x_i,n)$ for each x_i. As we have taken $J(x_i,n)>0$ and $\phi_n(x_i)$ real, it follows that

(5.3) $$\sum_{i=0}^{N} J(x_i,n)\{\phi_n(x_i)\}^2 \neq 0.$$

6. Pseudo Orthogonality of Polynomial Set (2.4). Making an appeal to (2.4), in almost a similar way as in §3, as it is a polynomial again precisely of degree n in x, one can arrive at the following results:

(6.1) $$A_n^{\alpha-n,\beta-n}(x,\gamma,\delta,w)=\frac{(-w)^n(-\alpha-\beta)_n}{n!2^n}\left(1-\beta-\delta+\frac{x+1}{w}\right)_n+\prod_{n-1}(x),$$

where $\prod_{n-1}(x)$ is a polynomial of degree n–1 in x. we take the Jump function as :

(6.2) $$\psi(x,\alpha-n,\beta-n,\gamma,\delta,w)=(x-1+\gamma w)^{\left[\overline{\alpha-n}\ w\right]}(x+1-\delta w)^{\left[\overline{\beta-n}\ w\right]}$$

(6.3) $$\sum_{x=-1+\delta w}^{1-\gamma w}\left[A_n^{\alpha-n,\beta-n}(x,\gamma,\delta,w)\psi(x,\alpha-n,\beta-n,\gamma,\delta,w)\right]=0\,.$$

Therefore,

(6.4) $$\sum_{x=-1+\delta w}^{1-\gamma w}\left[\left(1-\beta-\delta+\frac{x+1}{w}\right)_m A_n^{\alpha-n,\beta-n}(x,\gamma,\delta,w)\psi(x,\alpha-n,\beta-n,\gamma,\delta,w)\right]$$

$$=-\frac{m}{2nw}\sum_{x=-1+\delta w}^{1-\gamma w}\left[\left(2-\beta-\delta+\frac{x+1}{w}\right)_{m-1} A_{n-1}^{\alpha-n+1,\beta-n+1}(x+w,\gamma,\delta,\omega)\psi(x+w,\alpha-n+1,\beta-n+1,\gamma,\delta,w)\right]$$

(6.5) $$=\frac{(-1)^m\,m!(n-m)!}{n!2^m w^m}$$

$$\sum_{x=-1+\delta w}^{1-\gamma w}\left[A_{n-m}^{\alpha-n+m,\beta-n+m}(x+mw,\gamma,\delta,w)\psi(x+mw,\alpha-n+m,\beta-n+m,\gamma,\delta,w)\right].$$

Hence

$$(6.6)\qquad \sum_{x=-1+\delta w}^{1-\gamma w}\left[A_m^{\alpha-m,\beta-m}(x,\gamma,\delta,w)A_n^{\alpha-n,\beta-n}(x,\gamma,\delta,w)\psi(x,\alpha-n,\beta-n,\gamma,\delta,w)\right]=0;\ n>m$$

and

$$(6.7)\qquad \sum_{x=-1+\delta w}^{1-\gamma w}\left[A_n^{\alpha-n,\beta-n}(x,\gamma,\delta,w)A_m^{\alpha-m,\beta-m}(x,\gamma,\delta,w)\psi(x,\alpha-m,\beta-m,\gamma,\delta,w)\right]=0\ ;\ n<m.$$

Equations (6.6) and (6.7) together imply the Pseudo orthogonelity of our polynomial set defined by (2.4).

Again, we have

$$(6.8)\qquad \sum_{x=-1+\delta w}^{1-\gamma w}\left[\left\{A_n^{\alpha-n,\beta-n}(x,\gamma,\delta,w)\right\}^2\psi(x,\alpha-n,\beta-n,\gamma,\delta,w)\right]$$

$$=\frac{(-\alpha-\beta)_n}{n!2^{2n}}\sum_{x=-1+\delta w}^{1-\gamma w}\left[\left(x-1+\overline{\gamma+n}\ w\right)^{[\alpha w]}\left(x+1-\overline{\delta-n}\ w\right)^{[\beta w]}\right].$$

Verification. Equation (6.8), when $w\to 0$, yields a known result

$$(6.9)\qquad \int_{-1}^{1}(1-x)^{\alpha-n}(1+x)^{\beta-n}\left\{P_n^{\alpha-n,\beta-n}(x)\right\}^2dx=\frac{2^{\alpha+\beta-2n+1}\Gamma(1+\alpha)\Gamma(1+\beta)}{n!\Gamma(1+\alpha+\beta)\Gamma(1+\alpha+\beta-n)},$$

thus making our Pseudo orthogonality stand verified.

7. Expansion of $\left(\beta+\delta-\frac{x+1}{w}\right)_n(-w)^n$ in a Series of Our Polynomial Set. Let

$$(7.1)\qquad (-w)^n\left(\beta+\delta-\frac{x+1}{w}\right)_n=\sum_{k=0}^{n}C_kA_k^{\alpha,\beta}(x,\gamma,\delta,w).$$

Multiplying both the sides by $A_\nu^{\alpha,\beta}(x,\gamma,\delta,w)\psi(x,\alpha,\beta,\gamma,\delta,w)$ and summing up from $-1+\delta w$ to $1-\gamma w$, we have

$$(7.2)\qquad \sum_{x=-1+\delta w}^{1-\gamma w}\left[(-w)^n\left(\beta+\delta-\frac{x+1}{w}\right)_n A_\nu^{\alpha,\beta}(x,\gamma,\delta,w)\psi(x,\alpha,\beta,\gamma,\delta,w)\right]$$

$$=C_\nu \sum_{x=-1+\delta w}^{1-\gamma w}\left[\left\{A_\nu^{\alpha,\beta}(x,\gamma,\delta,w)\right\}^2 \psi(x,\alpha,\beta,\gamma,\delta,w)\right]$$

$$=C_\nu \frac{(1+\alpha+\beta)_{2\gamma}(-1)^\gamma}{\nu!2^{2\nu}(1+\alpha+\beta)_\nu}\sum_{x=-1+\delta w}^{1-\gamma w}\left[\left(x-1+\overline{\gamma+\nu}\,\omega\right)^{[\overline{\alpha+\gamma}\,w]}\left(x+1-\overline{\delta-\nu}w\right)^{[\overline{\beta+\nu}\,w]}\right] \text{(by 3.17)}$$

Again from (3.10), we derive

$$(7.3)\qquad \sum_{x=-1+\delta w}^{1-\gamma w}\left[(-w)^n\left(\beta+\delta-\frac{x+1}{w}\right)_n A_\nu^{\alpha,\beta}(x,\gamma,\delta,w)\psi(x,\alpha,\beta,\gamma,\delta,w)\right]$$

$$=\frac{(-w)^n n!}{2\nu w}\sum_{x=-1+\delta w}^{1-\gamma w}\left[\left(\beta+\delta-\frac{x+1}{w}\right)_{n-1} A_{\nu-1}^{\alpha+1,\beta+1}(x+w,\gamma,\delta,w)\psi(x+w,\alpha+1,\beta+1,\gamma,\delta,w)\right]$$

$$=\frac{(-w)^n n!}{\nu!2^\nu w^\nu(n-\nu)!}\sum_{x=-1+\delta w}^{1-\gamma w}\left[\left(\beta+\delta-\frac{x+1}{w}\right)_{n-\nu} A_{n-\nu}^{\alpha+\nu,\beta+\nu}(x+\nu w,\gamma,\delta,w)\right.$$

$$\left.\psi(x+\nu w,\alpha+\nu,\beta+\nu,\gamma,\delta,w)\right]$$

$$=\frac{n!(-1)^n w^{n-\nu}}{\nu!2^\nu(n-\nu)!}\sum_{x=1+\delta w}^{1-\gamma w}\left[\left(\beta+\delta-\frac{x+1}{w}\right)_{n-\nu}\left(x-1+\overline{\gamma+\nu}\,w\right)^{[\overline{\alpha+\nu}\,w]}\left(x+1-\overline{\delta-\nu}\,w\right)^{[\overline{\beta+\nu}\,w]}\right],$$

(by continuing the process in all $\nu-$ times and then using (3.4)).

Hence

$$(7.4)\qquad C_\nu=\frac{n!(-1)^{n-\nu}w^{n-\nu}}{(n-\nu)!}$$

$$\frac{2^\nu(1+\alpha+\beta)_\nu}{(1+\alpha+\beta)_{2\nu}}\frac{\sum_{x=-1+\delta w}^{1-\gamma w}\left[\left(\beta+\delta-\frac{x+1}{w}\right)_{n-\nu}\left(x-1+\overline{\gamma+\nu}\,w\right)^{[\overline{\alpha+\nu}\,w]}\left(x+1-\overline{\delta-\nu}\,w\right)^{[\overline{\beta+\nu}\,w]}\right]}{\sum_{x=-1+\delta w}^{1-\gamma w}\left[\left(x-1+\overline{\gamma+\nu}\,w\right)^{(\overline{\alpha+\nu}\,w)}\left(x+1-\overline{\delta-\nu}\,w\right)^{(\overline{\beta+\nu}\,w)}\right]}$$

Therefore,

$$(7.5)\qquad (-w)^n\left(\beta+\delta-\frac{x+1}{w}\right)_n=\sum_{k=0}^{n}\left[\frac{n!(-w)^{n-k}(1+\alpha+\beta)_k 2^k}{(n-k)!(1+\alpha+\beta)_{2k}}\right.$$

$$\left.\frac{\sum_{x=-1+\delta w}^{1-\gamma w}\left\{\left(\beta+\delta-\frac{x+1}{w}\right)_{n-\nu}\left(x-1+\overline{\gamma+\nu}\,w\right)^{\left[\overline{\alpha+\nu}\,w\right]}\left(x+1-\overline{\delta-\nu}\,w\right)^{\left[\overline{\beta+\nu}\,w\right]}\right\}}{\sum_{x=-1+\delta w}^{1-\nu w}\left\{\left(x-1+\overline{\gamma+\nu}\,w\right)^{\left[\overline{\alpha+\nu}\,w\right]}\left(x+1-\overline{\delta-\nu}\,w\right)^{\left[\overline{\beta+\nu}\,w\right]}\right\}}\right]A_k^{\alpha,\beta}(x,\gamma,\delta,w)$$

which is the required axpansion in terms of our polynosials.

Verification. Equation (7.5), when $w\to 0,$ gives the known result for the Jacobi polynomials, viz.

$$(7.6)\qquad (x+1)^n=(-2)^n(1+\beta)_n\sum_{k=0}^{n}\frac{(-n)_k(1+\alpha+\beta)_k(1+\alpha+\beta+2k)}{(1+\beta)_k(1+\alpha+\beta)_{n+k+1}}P_k^{\alpha,\beta}(x)\ ,$$

thus making (7.5) stand verified.

8. Another Expension. Supposing

$$(8.1)\qquad (-w)^n\left(\alpha-\gamma-\frac{x-1}{w}\right)_n=\sum_{k=0}^{n}C_kA_k^{\alpha+k,\beta-k}(x,\gamma,\delta,w)$$

and proceeding almost in the same way as that of the previous articles we finally arrive at the required expansion; viz.

$$(8.2)\qquad (-w)^n\left(\alpha-\gamma-\frac{x-1}{w}\right)_n=\sum_{k=0}^{n}\frac{2^k(1+\alpha+\beta)_k(-1)^{n-k}\,n!w^{n-k}A_k^{\alpha+k,\beta-k}(x,\gamma,\delta,w)}{(n-k)!(1+\alpha+\beta)_{2k}}$$

$$\left[\frac{\sum_{x=-1+\delta w}^{1-\gamma w}\left[\left(\alpha-\gamma-\frac{x-1}{w}\right)_{n-\nu}\left(x-1+\overline{\gamma+\nu}\,w\right)^{\left[\overline{\alpha+2\nu}\,w\right]}\left(x+1-\overline{\delta-\nu}\,w\right)^{[\beta w]}\right]}{\sum_{x=-1+\delta w}^{1-\gamma w}\left[\left(x-1+\overline{\gamma+\nu}\,w\right)^{\left[\overline{\alpha+2\nu}\,w\right]}\left(x+1-\overline{\delta-\nu}\,w\right)^{[\beta w]}\right]}\right].$$

Verification. Taking the limit as $w\to 0$ in (8.2), we obtain a known result for the modified Jacobi polynomials; viz.

$$(1-x)^n=\sum_{k=0}^{n}\frac{2^n(1+\alpha+k)_n(-n)_k(1+\alpha+\beta+2k)(1+\alpha+\beta)_k}{(1+\alpha+k)_k(1+\alpha+\beta)_{n+k+1}}P_k^{\alpha+k,\beta-k}(x).$$

Thus making our expansion (8.2) as well as the orthogonality in §4, stand verified.

9. Yet Another Expansion. Taking

$$(9.1)\qquad (-w)^n\left(1-\beta-\delta+\frac{x+1}{w}\right)_n=\sum_{k=0}^{n}C_kA_k^{\alpha-k,\beta-k}\ (x,\gamma,\delta,w)\ ,$$

and proceeding almost in a similar way to that of §7, we finally get the required expansion

$$(9.2)\qquad (-w)^n\left(1-\beta-\delta+\frac{x+1}{w}\right)_n=\sum_{k=0}^{n}\frac{(-w)^{n-k}\,n!2^k A_k^{\alpha-k,\,\beta-k}(x,\gamma,\delta,w)}{(n-k)!(-\alpha-\beta)_k}$$

$$\left[\frac{\sum_{x=-1+\delta w}^{1-\gamma w}\left[\left(1+k-\beta-\delta+\frac{x+1}{w}\right)_{n-k}\left(x-1+\overline{\gamma+k}\;w\right)^{[\alpha w]}\left(x+1-\overline{\delta-k}\;w\right)^{[\beta w]}\right]}{\sum_{x=-1+\delta w}^{1-\gamma w}\left[\left(x-1+\overline{\gamma+k}\;w\right)^{[\alpha w]}\left(x+1-\overline{\delta-k}\;w\right)^{[\beta w]}\right]}\right].$$

Verification. The equation (9.2), when $w\to 0$, gives

$$(9.3)\qquad (x+1)^n=\sum_{k=0}^{n}\frac{(-n)_k(1+\beta)_{n-k}2^n}{(-\alpha-\beta)_k(2+\alpha+\beta)_{n-k}}P_k^{\alpha-k,\beta-k}(x),$$

which tallies with a similar corresponding result for the Jacobi polynomials with α and β replaced respectively by α–k and β–k. Hence our expansion (9.2) stands verified.

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AN APPLICATION OF JACK'S LEMMA FOR THE MINIMUM POINT

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ABSTRACT

For the analytic $f(z)$ in the open unit disk , H. Shiraishi and S. Owa [*Stud. Univ. Babes-Bolyai Math.,* **55** (2010), 207-211] have proved a theorem for the minimum value of $|f(z)|$. In this paper, we discuss an application of this theorem and some corollaries.

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1. Introduction. Let be the open unit disk given by

$$=\{z : z \in \mathbb{C} \text{ and } |z|<1\}.$$

Also let $H[a_0,n]$ denote the class of functions $p(z)$ of the form :

$$p(z)=a_0\sum_{k=n}^{\infty}a_k z^k, \tag{1.1}$$

which are analytic in U for some $a_0 \in \mathbb{C}$ and a positive integer n (see[5]).
The basic tool in the proof of our results is the following lemma due to H. Shiraishi and S. Owa [4].

Lemma 1. *Let* $p(z)\in H[a_0,n]$ *with* $p(z)\neq 0$ *for all* $z\in$. *If there exists a point* $z\in$ *such that*

$$\min_{|z|\leq|z_0|}|p(z)|=|p(z_0)|, \tag{1.2}$$

then

$$\frac{z_0 p'(z_0)}{p(z_0)}=-m \tag{1.3}$$

and

$$\Re\left(\frac{zop''(z_0)}{p'(z_0)}\right)+1 \geqq -m, \tag{1.4}$$

where

$$m \geqq n\frac{|a_0 - p(z_0)|^2}{|a_0|^2-|p(z_0)|^2} \geqq n\frac{|a_0|-|p(z_0)|}{|a_0|+|p(z_0)|}. \tag{1.5}$$

2. Main Theorem. Applying Lemma 1, we derive

Theorem 1. *Let the function* $f(z)$ *given by*

$$f(z)=a_n z^n + a_{n+1}z^{n+l} + a_{n+l+1}z^{n+l+1}+\ldots(a_n, a_{n+l} \neq 0) \tag{2.1}$$

be analytic in and $f(z)\neq 0$ *for* $z\in U\setminus\{0\}$. *If there exists a point* $z_0 \in \setminus\{0\}$ *such that*

$$\min_{|z|\leqq|z_0|}|f(z)|=|f(z_0)|, \tag{2.2}$$

then

$$\frac{z_0 f'(z_0)}{f(z_0)}=n-m \tag{2.3}$$

and

$$(n-m)\left(\Re\left(\frac{z_0 f''(z_0)}{f'(z_0)}\right)+1\right) \leqq (n-m)^2, \tag{2.4}$$

where

$$m \geqq l\frac{|a_0 z_0^n - f(z_0)|^2}{|a_n z_0^n|^2-|f(z_0)|^2} \geqq l\frac{|a_n z_0^n|-|f(z_0)|}{|a_n z_0^n|+|f(z_0)|}. \tag{2.5}$$

Proof. We define the function $p(z)$ by

$$\begin{aligned} p(z) &= f(z)/z^n \\ &= a_n + a_l z^l + a_{l+1}z^{l+1}+\ldots . \end{aligned} \tag{2.6}$$

Then, $p(z)\in H[a_n, l]$ and $p(0)=a_n\neq 0$. Furthermore, by the assumtion of the theorem, $|p(z)|$ takes its minimum value at $z=z_0$ in the closed disk $|z| \leqq |z_0|$. It follows from this that

$$|p(z_0)|=\frac{|f(z_0)|}{|z_0|^n}=\frac{\min_{|z|\leqq|z_0|}|f(z)|}{|z_0|^n}=\min_{|z|\leqq|z_0|}|p(z)|. \tag{2.7}$$

Therefore, applying Lemma 1 to $p(z)$, we observe that

$$\frac{z_0 p'(z_0)}{p(z_0)} = \frac{z_0 f'(z_0)}{f(z_0)} - n = -m, \tag{2.8}$$

which shows (2.3) and

$$\Re\left(\frac{z_0 p''(z_0)}{p'(z_0)}\right) + 1 = \Re\left(-n-1+\frac{\frac{z_0 f''(z_0)}{f'(z_0)}+1-n}{1-n\frac{f(z_0)}{z_0 f'(z_0)}}\right)+1$$

$$= -\frac{n-m}{m}\left(\Re\left(\frac{z_0 f''(z_0)}{f'(z_0)}\right)+1-n\right)-n$$

$$\geqq -m. \tag{2.9}$$

which implies (2.4), where

$$m \geqq l\frac{|a_n - p(z_0)|^2}{|a_n|^2 - |p(z_0)|^2} = l\frac{|a_n z_0^n - f(z_0)|^2}{|a_n z_0^n|^2 - |f(z_0)|^2} \geqq l\frac{|a_n z_0^n| - |f(z_0)|}{|a_n z_0^n| + |f(z_0)|}. \tag{2.10}$$

This completes the assertion of Theorem 1.
Letting l=n in Theorem 1, we obtain
Corollary 1. *Let the function f(z) given by*

$$f(z) = a_n z^n + a_{2n} z^{2n} + a_{2n+1} z^{2n+1} + \ldots.(a_n, a_{2n} \neq 0) \tag{2.11}$$

be analytic in *and* $f(z) \neq 0$ *for* $z_0 \in \ \backslash\{0\}$ *If there exists a point* $z_0 \in \ /\{0\}$ *such that*

$$\min_{|z| \leqq |z_0|} |f(z)| = |f(z_0)|, \tag{2.12}$$

then

$$\frac{z_0 f'(z_0)}{f(z_0)} = n - m \leqq 0 \tag{2.13}$$

and

$$\Re\left(\frac{z_0 f''(z_0)}{f'(z_0)}\right) + 1 \geqq 1 - m, \tag{2.14}$$

where

$$m \geqq n \frac{\left|a_n z_0^n - f(z_0)\right|^2}{\left|a_n z_0^n\right|^2 - \left|f(z_0)\right|^2} \geqq n \frac{\left|a_n z_0^n\right| - \left|f(z_0)\right|}{\left|a_n z_0^n\right| + \left|f(z_0)\right|}. \tag{2.15}$$

Moreover, putting $n=1$ and $a_n=1$ in Theorem 1, we get the following corollary due to *M.* Nunokawa and S. Owa [3].

Corollary 2. *Let the function f(z) given by*

$$f(z) = z + a_{l+1} z^{l+1} + a_{l+2} z^{l+2} + \dots \qquad (a_{l+1} \neq 0) \tag{2.16}$$

be analytic in U and $f(z) \neq 0$ *for* $z \in U \setminus \{0\}$. *If there exists a point* $z_0 \in U \setminus \{0\}$ *such that*

$$\min_{|z| \leqq |z_0|} |f(z)| = |f(z_0)|, \tag{2.17}$$

then

$$\frac{z_0 f'(z_0)}{f(z_0)} = 1 - m \leqq 0 \tag{2.18}$$

and

$$\Re\left(\frac{z_0 f''(z_0)}{f'(z_0)}\right) + 1 \geqq 1 - m, \tag{2.19}$$

where

$$m \geqq l \frac{\left|z_0 - f(z_0)\right|^2}{\left|z_0\right|^2 - \left|f(z_0)\right|^2} \geqq l \frac{\left|z_0\right| - \left|f(z_0)\right|}{\left|z_0\right| + \left|f(z_0)\right|}. \tag{2.20}$$

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NEW COEFFICIENT ESTIMATES FOR STARLIKE AND CONVEX FUNCTIONS OF ORDER α

By

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ABSTRACT

For starlike functions and convex functions in the open unit disk , Some estimats for their Taylor-Maclaurin cofficients due to M.S.Robertson [*Ann. of Math.* (*Ser.* 1) **37** (1936), 374-408] are well known. In the present paper, assuming that the coefficient estimates of these classes of functious depend upon the second coefficient $|a_2|$ of these functions, we discuss and derive new coefficient estimates depending upon the second coefficient $|a_2|$ for starlike functions and convex functions of order α in .

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Key Words and Phrases : Univalent functions; Starlike functions; Convex functions; Bieberbach conjecture (or de Branges' theorem); Taylor-Maclaurin coefficients; Coefficient estimates; Coefficient bounds.

1. Introduction, Definitions and Preliminaries. Let be the class of functions $f(z)$ of the form

(1.1) $$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are analytic in the open unit disk

$$U = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}.$$

Let denote the subclass of consisting of all univalent functions $f(z)$ in . Also let $\delta^*(\alpha)$ denote the subclass of consisting of $f(z)$ which satisfy the following inequality:

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha (z \in U) \tag{1.2}$$

for some real number α with $0 \leqq \alpha < 1$. Furthermore, we denote by (α) the subclass of consisting of functions $f(z)$ satisfying $zf'(z) \in {}^*(\alpha)$. We say that $*(0) =$ $*$ and $(0) =$. The classes $^*(\alpha)$ and (α) were intoduced by Robertson [5] (see also [6]). We alsonote that

$$f(z) = \frac{z}{(1-z)^{2(1-\alpha)}} = z + \sum_{n=2}^{\infty} \left(\frac{\prod_{k=2}^{n} (k - 2\alpha)}{(n-1)!} \right) z^n \tag{1.3}$$

is the extremal function for the class $^*(\alpha)$ and

$$f(z) = \frac{1-(1-z)^{2\alpha-1}}{2\alpha - 1} = z + \sum_{n=2}^{\infty} \left(\frac{\prod_{k=2}^{n} (k - 2\alpha)}{n!} \right) z^n \tag{1.4}$$

is the extremal function for the class (α) This means that, if $f(z) \in$ $*(\alpha)$ then

$$|a_n| \leqq \frac{\prod_{k=2}^{n} (k - 2\alpha)}{(n-1)!} (n = 2,3,4,...), \tag{1.5}$$

and that if $f(z) \in K(\alpha)$, then

$$|a_n| \leqq \frac{\prod_{k=2}^{n} (k - 2\alpha)}{(n!)} (n = 2,3,4,...). \tag{1.6}$$

The celebrated Bieberbach conjecture (now de Branges'theorem) states that, if

$f(z) \in$, then

(1.7) $$|a_n| \leqq n(n = 2,3,4,....).$$

Equality holds true for the Koebe function

(1.8) $$f(z) = \frac{z}{(1-z)^2}$$

and its rotation. This Bieberbach conjecture was given by Bieberbach [2] and proved by de Branges [4]. On the other hand, Ahlfors [1] has shown that, if $f(z) \in$, then

(1.9) $$|a_4| \leqq \frac{4\sqrt{11+|a_2|^2}}{\sqrt{15}}.$$

This gives us that

$$|a_4| \leqq 4 \text{ when } |a_2| = 2.$$

In view of the above considerations, it may be of interest to discuss here and derive the coefficient bounds by assuming that

(1.10) $$|a_n| \leqq B(n, \alpha, |a_2|) \qquad (n = 3,4,5,...)$$

for all functions $f(z) \in \ {}^*(\alpha)$ or for all functions $f(z) \in K(\alpha)$.

2. Coefficient Bounds and Coefficient Estimates. To discuss our problems, we need to introduce the Carathéodory functions

(2.1) $$p(z) = 1 + p_1 z + p_2 z^2 + \dots,$$

which are analytic in and satisfy

(2.2) $$\Re\{p(z)\} > 0 (z \in \ ).$$

We denote by the class of all Carathéodory functions *p(z)*. It is well-known that if $p(z) \in$, then

(2.3) $$|p_n| \leqq 2(n = 1,2,3,....)$$

and the equality holds true for

(2.4) $$p(z) = \frac{1+z}{1-z}(\text{see}[3]).$$

By applying the Carathéodory functions, we derive the following result.

Theorem 1. *If* $f(z) \in \ {}^*(\alpha)$, *then*

(2.5) $$|a_n| \leqq \frac{2(1-\alpha)(1+|a_2|)}{n-1} \prod_{k=2}^{n-2}\left(1+\frac{2(1-\alpha)}{k}\right) \quad (n=4,5,6,....)$$

with

(2.6) $$|a_2| \leqq 2(1-\alpha)$$

and

(2.7) $$|a_2| \leqq (1-\alpha)(1+|a_2|).$$

Proof. Let us define the function *p(z)* by

(2.8) $$p(z)=\frac{zf'(z)/f(z)-\alpha}{1-\alpha}=1+p_1z+p_2z^2+\ldots$$

for $f(z)\in \ *(a)$. Then, $p(z)\in$ because $p(z)$ is analytic in , $p(0)$ and

$\Re p(z)>0(z\in \ )$. It follows from (2.8) that

(2.9) $$(n-1)a_n=(1-\alpha)(p_n-1+a_2p_{n-3}+\ldots+a_{n-1}p_1)$$

that is, that

(2.10) $$|a_n| \leqq \frac{2(1-\alpha)}{n-1}(1+|a_2|+|a_3|+\ldots+|a_{n-1}|).$$

If $n=2$ we have

(2.11) $$|a_n| \leqq 2(1-\alpha).$$

If $n=3$ then (2.10) leads us that

(2.12) $$|a_3| \leqq (1-\alpha)(1+|a_2|).$$

Considering the case when $n=4$, we have

(2.13) $$|a_4| \leqq \frac{2(1-\alpha)}{3}(1+|a_2|+|a_3|)$$

$$\leqq \frac{2(1-\alpha)(1+|a_2|)}{3}(2-\alpha).$$

Thus, (2.5) holds true for $n=4$. Now, we suppose that (2.5) holds true for $n=j\geqq 4$. Then we find that

(2.14) $$|a_{j+1}| \leqq \frac{2(1-\alpha)}{j}(1+|a_2|+|a_3|+\ldots+|a_j|)$$

$$\leqq \frac{2(1-\alpha)}{j}\left\{(1+|a_2|)+(1-\alpha)(1+|a_2|)+\frac{2(1-\alpha)(1+|a_2|)}{3}(2-\alpha)\right.$$

$$+\frac{2(1-\alpha)(1+|a_2|)}{4}(2-\alpha)\left(1+\frac{2(1-\alpha)}{3}\right)+....+\frac{2(1-\alpha)(1+|a_2|)}{j-1}\prod_{k=2}^{j-2}\left(1+\frac{2(1-\alpha)}{k}\right)\Bigg\}$$

$$=\frac{2(1-\alpha)(1+|a_2|)}{j}\prod_{k=2}^{j-1}\left(1+\frac{2(1-\alpha)}{k}\right).$$

Therefore, (2.5) is true for $n=j+1$. Applying the principle of mathematical inducation, we thus complete the proof of Theorem 1.

Remark 1. If $|a_2|=2(1-\alpha)$ in Theorem 1, then (2.5) becomes (1.5).

Corollary 1. *If* $f(z)\in\ \ *$, *then*

(2.15) $$|a_n| \leqq \frac{2(1+|a_2|)}{n-1}\prod_{k=2}^{n-2}\frac{k+2}{k} \qquad (n=4,5,6,...)$$

with $|a_2| \leqq 2$ *and* $|a_3| \leqq 1+|a_2|$.

We next derive Theorem 2 below.

Theorem 2. *If* $f(z)\in\ \ (\alpha)$, *then*

(2.16) $$|a_n| \leqq \frac{2(1-\alpha)(1+|a_2|)}{n(n-1)}\prod_{k=2}^{n-2}\left(1+\frac{2(1-\alpha)}{k}\right) \qquad (n=4,5,6,...)$$

with

(2.17) $$|a_2| \leqq 1-\alpha$$

and

(2.18 $$|a_3| \leqq \frac{(1-\alpha)(1+|a_2|)}{3}.$$

Proof. From the definitions of the function classes $\ \ *(\alpha)$ and $\ \ (\alpha)$ we know that $f(z)\in\ \ (\alpha)$ if and only if $zf'(z)\in\ \ *(\alpha)$. This implies that

$$n|a_n| \leqq \frac{2(1-\alpha)(1+|a_2|)}{n-1}\prod_{k=2}^{n-2}\left(1+\frac{2(1-\alpha)}{k}\right) \qquad (n=4,5,6,...)$$

with

$$|a_2| \leqq 1-\alpha \text{ and } |a_3| \leqq \frac{(1-\alpha)(1+|a_2|)}{3}.$$

This evidently completes the proof of Theorem 2.

Remark 2. If we take $|a_2|=1-\alpha$ in Theorem 2, then (2.16) becomes (1.6).

Corollary 2. *If* $f(z) \in$ *, then*

$$|a_n| \leqq \frac{2(1+|a_2|)}{n(n-1)} \prod_{k=2}^{n-2} \left(\frac{k+2}{k} \right) \qquad (n = 4,5,6,\ldots) \tag{2.19}$$

with

$$|a_2| = 1 \ \textit{and} \ |a_3| \leqq \frac{1+|a_2|}{3}.$$

3. Concluding Remarks and Observations. In our present investigation, we have discussed and derived new estmates for the Taylor-Maclaurin coefficients of functions belonging to the familiar classes $^*(\alpha)$ and (α) of starlike functions of order α in the open unit disk and convex functions of order α in the open unit disk , respectively. We have assumed that the coefficient estimates of functions in each of these classes depend upon the second coefficient $|a_2|$ of these functions. We have also indicated several corollaries and consequences of our main results.

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COEFFICIENT ESTIMATES FOR A CERTAIN CLASS OF ANALYTIC FUNCTIONS INVOLVING THE ARGUMENTS OF THEIR DERIVATIVES

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ABSTRACT

For analytic functions $f(z)$ in the open unit disk U,which are normalized by

$$f(0)=f'(0)-1=0,$$

We address the problem of finding the coefficient bunds of $f(z)$ for the case when the origin is included in the iage of the disk U by their dervative $f'(z)$

2010 Mathematics Subject Classification : Primary 30C45,30C50.
Key Words and Phrases : Carathéodory function; Argument property; Coefficient bounds.

1. Introduction and Preliminaries. Let denote the class of functions $f(z)$ of the form:

$$f(z)=z+\sum_{n=2}^{\infty}a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk

$$=\{z : z\in\mathbb{C} \text{ and } |z|<1\}. \tag{1.2}$$

Also let $\mathcal{P}$ be the class of functions $p(z)$ of the form:

(1.3) $$p(z) = 1 + \sum_{k=1}^{\infty} c_k z^k,$$

which are analytic in and satisfy the following condition:

(1.4) $$\Re\{p(z)\} > 0 (z \in \ ).$$

The function $p(z) \in$ is said to be a Carathodory function. The following lemma is a well-known result for Caratheodory functions (see also [5]).

Lemma 1. *If* $p(z) \in$ *, then the following coefficient estimates hold true:*

(1.5) $$|c_k| \leqq 2 \qquad (k = 1, 2, 3, ...) .$$

The result is sharp for the function p(z) given by

(1.6) $$p(z) = \frac{1+z}{1-z} = 1 + \sum_{k=1}^{\infty} 2z^k.$$

In this paper, we discuss coefficient estimates of the functions $f(z) \in$ satisfying $f'(z_0) = 0$ for some $z_0 \in$. For such functions, we easily see that there are *some* (but, Presumably, not all) Points $z \in$ such that $f'(z) < 0$. Therefore, we define the set Ω as follows:

(1.7) $$\Omega = \{z : z \in \mathbb{C}, |z| = 1 \text{ and } f'(z) < 0 (f(z) \in \ )\}.$$

Then we say that $f(z) \in \ (\delta_1, \delta_2; r)$ if $f(z) \in$ satisfies the following condition:

(1.8) $$\sup\{\arg[f'(z) + r]\} = \delta_1 \text{ and } \inf\{\arg[f'(z) + r]\} = \delta_2 \quad (z \in \ )$$

for some δ_1 and δ_2 $(-\pi < \delta_2 < 0 < \delta_1 < \pi)$, where

(1.9) $$\min_{z\in\Omega}\{f'(z)\} = -r \qquad (r > 0).$$

For the case when the function $f(z) \in$ satisfies the following inequality:

$$f'(z) \neq 0 \qquad (z \in \ ),$$

we have earlier obtained the results for the class (δ_1, δ_2) in [2].

Remark 1. For a function $f(z) \in \ (\delta_1, \delta_2; r)$, supposing that

(1.10) $$p(z) = \frac{e^{-i\varphi}[f'(z) + r]^{1/\varphi} + i(1+r)^{1/\varphi}\sin\varphi}{(1+r)^{1/\varphi}\cos\varphi},$$

where $\varphi = \dfrac{(\delta_1+\delta_2)\pi}{2(\delta_1-\delta_2)}$ and $\psi = \dfrac{\delta_1-\delta_2}{\pi}$,

we see that $p(z)$ is a member of the class $\mathcal{P}$.
We now let

$$[f'(z)+r]^{1/\varphi} = (1-r)^{1/\varphi} + \sum_{k=1}^{\infty} b_k z^k, \tag{1.11}$$

for a function $f(z)\in$ .Then we have the following theorem b virtue of Lemma 1.

Theorem 1. *If the representation* (1.11) *for functions* $f(z)\in \ (\delta_1,\delta_2;r)$ *is obtained, then the following coefficient bounds holds true:*

$$|b_k| \le 2(1+r)^{1/\psi}\cos\varphi \quad (k=1,2,3,\dots), \tag{1.12}$$

where $\varphi = \dfrac{(\delta_1+\delta_2)\pi}{2(\delta_1-\delta_2)}$ *and* $\psi = \dfrac{\delta_1-\delta_2}{\pi}$.

Equality holds true for the function f(z) given by

$$|f'(z)+r|^{1/\varphi} = (1+r)^{1/\varphi}\left(\frac{1+e^{i2\varphi}z}{1-z}\right). \tag{1.13}$$

The proof of Theorem 1 is omitted here, because we can find similar results in (for example) [2] and [6].

2. A Set of Main Results. Unless otherwise mentioned, we shall assume that

$$\varphi = \frac{(\delta_1+\delta_2)\pi}{2(\delta_1-\delta_2)} \quad \text{and} \quad \psi = \frac{\delta_1-\delta_2}{\pi},$$

in this paper. Our first main result is contained in the following theorem.

Theorem 2. *If* $f(z)\in \ (\delta_1,\delta_2;r)$, *then the coefficients of f(z) are written as follows:*

$$a_n = \frac{1}{n}\sum_{m=1}^{n-1}\binom{\psi}{m}(1+r)^{1-m/\phi}\left(\sum_{h+l_2+\dots+l_m=n-1} b_{l_1}b_{l_2}\dots b_{l_m}\right) \quad (n=2,3,4,\dots), \tag{2.1}$$

where $l_1,l_2,\dots,l_m \in \mathbb{N} = \{1,2,3,\dots\}$

and the coefficients b_k *are given by* (1.11).

Proof. The equation (1.11) readily yields

(2.2) $$f'(z)=1+\sum_{n=2}^{\infty} na_n z^{n-1} = -r + \left((1+r)^{1/\varphi} + \sum_{k=1}^{\infty} b_k z^k \right)^{\psi}$$

$$= -r + (1+r)\left(1+\sum_{k=1}^{\infty} (1+r)^{-1/\phi} b_k z^k \right)^{\psi}$$

$$= 1+\sum_{m=1}^{\infty} \binom{\psi}{m} \left(\sum_{k=1}^{\infty} (1+r)^{1-1/\psi} b_k z^k \right)^{m}.$$

Checking the coefficients of z^{n-1} on the right-hand side, we deduce that

(2.3) $$1+\sum_{n=2}^{\infty} na_n z^{n-1} = 1+\sum_{m=2}^{\infty}\left[\sum_{m=1}^{n-1} (1+r)^{1-m/\psi} \binom{\psi}{m} \left(\sum_{l_1+l_2+....+l_m=n-1} b_{l_1} b_{l_2} ... b_{l_m} \right)\right] z^{n-1},$$

which completes the proof of Theorem 2.

We obtain the following coefficient bounds for the class $(\delta_1,\delta_2;r)$ through the use of Theorem 1 and Theorem 2.

Theorem 3. *If* $f(z)\in$ $(\delta_1,\delta_2;r)$, *then*

(2.4) $$|a_n| \leqq \frac{1}{n}\sum_{m=1}^{n-1}\left[\binom{n-2}{m-1} \frac{2^m}{m!} \left(\prod_{j=0}^{m-1} |j-\psi| \right)(1+r)\cos^m \varphi \right] \qquad (n=2,3,4,...),$$

where

(2.5) $$\binom{n-2}{m-1} = \begin{cases} 1 & (m=1) \\ \dfrac{(n-2)(n-3)...(n-m)}{(m-1)!} & m=2,3,4,... \end{cases}$$

Proof. The expression (2.1) and the triangle inequality lead us to the following inequality:

(2.6) $$|a_n| \leqq \frac{1}{n}\sum_{m=1}^{n-1} \left|\binom{\psi}{m}\right| (1+r)^{1-m/\psi} \left(\sum_{l_1+l_2+....+l_m=n-1} |b_{l_1}||b_{l_2}|...|b_{l_m}| \right).$$

Moreover, taking notice of the inequality (1.12) and counting the number of partitions of n-1 by m natural numbers, we see that

$$|a_n| \leqq \frac{1}{n}\sum_{m=1}^{n-1} \frac{|\psi|.|\psi-1|...|\psi-m+1|}{m!} 2^m (1+r)\cos^m \varphi \left(\sum_{l_1+l_2+....+l_m=n-1} 1 \right)$$

$$= \frac{1}{n} \sum_{m=1}^{n-1} \left[\binom{n-2}{m-1} \frac{2^m}{m!} \left(\prod_{j=0}^{m-1} |j - \psi| \right) (1+r) \cos^m \varphi \right].$$

By setting

(2.7) $$\delta_1 = \delta \text{ and } \delta_2 = \delta - \pi \qquad (0 < \delta < \pi)$$

in Theorem 3, we see that

$$\varphi = \delta - \frac{\pi}{2} \quad \text{and} \quad \psi = 1.$$

Hence we deduce the following corollary.

Corollary 1. *If* $f(z) \in \ _{\delta}r := \ (\delta, \delta - \pi; r)$, *then*

(2.8) $$|a_n| \leqq \frac{2}{n}(1+r)\sin\delta \, (n = 2, 3, 4, \ldots.).$$

The result is sharp for the function $f(z)$ *given by*

(2.9) $$f(z) = \left[e^{i2\delta}(1+r) - r \right] z - \left(1 - e^{i2\delta}\right)(1+r)\log(1-z) = z - \sum_{n=2}^{\infty} \frac{2(1+r)i\left(e^{i\delta}\sin\delta\right)}{n} z^n.$$

Proof. The coefficient inequality (2.8) is readily obtained by Teorem 3. To guarantree the sharpness, it is sufficient to consider the function *f*(*z*) given by (1.13) with

$$\varphi = \delta - \frac{\pi}{2} \quad \text{and} \quad \psi = 1.$$

Indeed, in this case, we define the function *P*(*z*) given by

(2.10) $$P(z) = \frac{e^{-i\delta} - e^{i\delta} z}{1 - z} \qquad (z \in \ )$$

and note that

(2.11) $$f'(z) + r = (1+r) e^{i\delta} P(z).$$

Then it follows that

(2.12) $$\Re\left(e^{i(\pi/2 - \delta)} \left[f'(z) + r \right] \right) = (1+r)\Re\{iP(z)\} = -(1+r)\Im\{P(z)\} > 0 \qquad (z \in \ ),$$

which means that

(2.13) $$\sup\left\{ \arg\left[f'(z) + r \right] \right\} = \delta \text{ and } \inf\left\{ \arg\left[f'(z) + r \right] \right\} = \delta - \pi \qquad (z \in \ ).$$

Therefore, we have $f(z) \in \ _{\delta}(r)$ and

(2.14) $$|a_n| = \left| -\frac{2(1+r)i\left(e^{i\delta}\sin\delta\right)}{n} \right| = \frac{2}{n}(1+r)\sin\delta.$$

By assuming that there exist some (but, presumably, not all) points in U such that

$$\frac{f(z)}{z} < 0,$$

we next define the set Ω_0 as follows:

(2.15) $$\Omega_0 = \left\{ z : z \in \mathbb{C}, |z| = 1 \text{ and } \frac{f(z)}{z} < 0 \right\},$$

for each $f(z) \in$. Then we say that $f(z) \in Q(\delta_1, \delta_2; r)$ if $f(z) \in$ satisfies the following conditions:

(2.16) $$\sup\left\{ \arg\left[\frac{f(z)}{z} + r \right] \right\} = \delta_1 \quad \text{and} \quad \inf\left\{ \arg\left[\frac{f(z)}{z} + r \right] \right\} = \delta_2 \qquad (z \in \ ).$$

for some real numbers δ_1 and δ_2 $(-\pi < \delta_2 < \delta_1 < \pi)$, where

(2.17) $$\min_{z\in\Omega_0} \left\{ \frac{f(z)}{z} \right\} = -r\,(r > 0).$$

In Particular, when

$$\delta_1 = \delta \text{ and } \delta_2 = \delta - \pi \qquad (0 < \delta < \pi),$$

we write

$$Q_\delta(r) = Q(\delta, \delta - \pi; r)$$

and we know the next relation between $(\delta_1, \delta_2;)$ and $Q(\delta_1, \delta_2; r)$.

Remark 2. The following equivalence holds true:

(2.18) $$f(z) \in Q(\delta_1, \delta_2; r) \Leftrightarrow \int_0^z \frac{f(\xi)}{\xi} d\xi = z + \sum_{n=2}^{\infty} \frac{a_n}{n} z^n \in R(\delta_1, \delta_2; r).$$

In view of Remark 2 and Theorem 3, we easily obtain the following results:

Theorem 4. If $f(z) \in Q(\delta_1, \delta_2; r)$, then

(2.19) $$|a_n| \leq \frac{1}{n} \sum_{m=1}^{n-1} \left[\binom{n-2}{m-1} \frac{2^m}{m!} \left(\prod_{j=0}^{m-1} |j - \psi| \right) (1+r)\cos^m \varphi \right] \qquad (n = 2, 3, 4, \ldots).$$

Corollary 2. *If* $f(z) \in Q_\delta(r)$, then

(2.20) $$|a_n| \leqq 2(1+r)\sin\delta \qquad (n = 2,3,4,\ldots).$$

The result is sharp for the function *f*(*z*) given by

(2.21) $$f(z) = \frac{z - \left[1 + 2(1+r)i\left(e^{i\delta}\sin\delta\right)\right]z^2}{1-z} = z - \sum_{n=2}^{\infty} 2(1+r)i\left(e^{i\delta}\sin\delta\right)z^n.$$

3. Concluding Remarks and Observations. In our present investigation, we have successfully addressed the problem of finding the co-efficient estimates of analytic function *f*(*z*) in the open unit disk , which are normalized by

$$f(0) = f'(0) - 1 = 0.$$

We have considered the case when the origin is included in the image of the disk by their derivative $f'(z)$ Our main results (Theorem 2, Theorem 3 and Theorem 4) and their consequences (Corollary 1 and Corollary 2) are capable of providing the solutions to the problem of finding the coefficient estimates in several different situations.

The derivations presented in this paper are based, in part, upon some interesting and useful properties of analytic functions belonging to the familiar Caratheodory class $\mathcal{P}$.

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ON THE MATHEMATICAL ASPECTS OF THE QUARK MIXING MATRIX:THE EXPONENTIAL PARAMETERIZATION

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ABSTRACT

The Exponential form of the quark mixing matrix has been proved to exhibit interesting mathematical properties, which greatly simplify the relevant perturbative expansion and its extension to a larger number of quark generation. The underlying algebraic structure naturally incorporates the Cabibbo structure and the hierarchical features of the Wolfenstein form. We extend our results to the neutrino mixing and introduce an exponential generator of the tribimaximal matrix.

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1. Introduction. The quark mixing matrix can be written in different ways, any of the proposed forms displays nice features and disadvantages. Whatever form one uses, four arbitrary parameters and the assumption of its unitarity are necessary to get physically meaningful results. The models can be roughly grouped in two categories, the first inspired to Euler like rotation matrices, the second, containing explicit hierarchical features, employs an expansion, around the unit matrix, in term of some key parameters. The original Kobayashi and Maskawa matrix[1] had been written in terms of three mixing angles $\theta_{1,2,3}$ and one CP violating phase δ. In this parameterization the first family decouples from the others in the limit $\theta_1 \to 0$. The particle data group[2] chooses a form in which the CP violating term is appended to the matrix entries responsible for the coupling of the first and third generations of quark masseigenstates. Finally Wolfenstein[3] has proposed a matrix emerging from a kind of perturbative expansion in terms of the Cabibbo coupling parameter $\lambda \cong 0.22^4$. A third model, bridging between the refs.[1,2] and ref.[3], is based on the so called exponential parameterization, which emerges from the request of unitarity, automatically satisfied by setting[5]

(1)
$$\widehat{V} = e^{\hat{A}},$$
$$\widehat{A}^{\dagger} = -A\cdot$$

The second condition in Eq. (1 equation.1.1), expressing the anti-hermiticity of the matrix, is ensured by the following specific choice

(2)
$$\widehat{A}^{\dagger} = \begin{pmatrix} 0 & \Lambda_1 & \Lambda_3 \\ -\Lambda_1 & 0 & \Lambda_2 \\ -\Lambda_3^* & -\Lambda_2 & 0 \end{pmatrix}.$$

The vanishing of the diagonal entries secures that the matrix $\widehat{V}$ be unimodular[1]. It would be sufficient to have a matrix with null trace, but for practical reasons we use the form(2). The sub-labels 1,2,3 determine the mixing *d–s*, *s–b*, *d–b* respectively, all the entrics, except Λ_3, are real. In the spirit of Wolfeinstein criteria, we use the Cabibbo strength λ as key parameter and make the following identifications[6]

(3)
$$\Lambda_1 = \lambda,$$
$$\Lambda_2 = y\lambda^2,$$
$$\Lambda_3 = x\lambda^3 e^{i\delta},$$

containing an implicit hierarchical assumption on the coupling between the different quark families. The vanishing of the *x,y* coefficients allows the decoupling from the *b* sector reducing the matrix to the *s–d* Cabibbo mixing, namely

(4)
$$\widehat{V}_{(x,y)\to 0} = e^{\begin{pmatrix} 0 & \lambda & 0 \\ -\lambda & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}} = \begin{pmatrix} \cos(\lambda) & \sin(\lambda) & 0 \\ -\sin(\lambda) & \cos(\lambda) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

It is also to be stressed that $\widehat{V}_{\lambda\to 0} = \hat{}$ which means that the parameterization in (3equation. 1.3) contains the assumption that the vanishing of the Cabibbo parameter determines the decoupling of the entire quark matrix. The phase δ is associated, as in the particle data group choice, with the smallest coupling term. In this paper we will see how the exponential parameterization yields a flexible tool to analyse the quark mixing phenomenology and the relevant consequences.

2. The Matrix a and The Wolfenstein Parameterization. We will prove that the quark mixing matrix written as in Eq. (1equation.1.1) naturally contains the Wolfenstein Parameterization and the Euler like forms as well. By Keeping the expansion of the exponential in Eq. (1equation.1.1) up to third order in λ, namely

(5) $$\widehat{V}=\hat{1}+\widehat{A}+\frac{\widehat{A}^2}{2!}+\frac{\widehat{A}^3}{3!}+\frac{\widehat{A}^4}{4!}+o\left(\lambda^5\right),$$

we obtain the mixing matrix in the form

(6) $$\widehat{V}\cong\begin{pmatrix} 1-\frac{\lambda^2}{2}+\frac{\lambda^4}{4!} & \lambda-\frac{\lambda^3}{3!} & AF\lambda^3 \\ -\lambda+\frac{\lambda^3}{3!} & 1-\frac{\lambda^2}{2}+\frac{\lambda^4}{4!}-\frac{\left(A\lambda^2\right)^2}{2} & \frac{AB\lambda^2}{2} \\ AG\lambda^3 & \frac{AC\lambda^2}{2} & 1-\frac{A^2\lambda^4}{2} \end{pmatrix},$$

$$A=y, F=p-i\eta,\ G=1-\rho-i\eta, B=2-\lambda^2\left(\rho-1/6-i\eta\right),$$

$$B+C=-2\lambda^2\left(\rho-1/2\right), \rho=x/y\cos\left(\delta\right)+1/2, \eta=-x/y\sin\left(\delta\right).$$

Eq. (6equation.2.6) is recognized as a Wolfenstein-type parameterization, the Taylor expansion at higher order can provide more accurate expansion in the Cabibbo coupling parameter, as we will see in the following. The expansion at the third order allows a one to one correspondence between the Wolfenstein parameters and those of the matrix $\widehat{A}$, which can be written in the form

(7) $$\widehat{A}=\begin{pmatrix} 0 & \lambda & A\lambda^3\left(\rho-i\eta-1/2\right) \\ -\lambda & 0 & A\lambda^2 \\ -A\lambda^3\left(\rho+i\eta-1/2\right) & -A\lambda^2 & 0 \end{pmatrix}.$$

Using for A,ρ,η the following values, close to those given in the literature[7]:

$$\lambda=0.2272\pm0.0010,\quad A=0.818^{+0.007}_{-0.017}$$

$$\rho=0.221^{+0.064}_{-0.028},\quad \eta=0.340^{+0.017}_{-0.045}$$

We find for x and δ the following values

$$x=-0.359^{+0.049}_{-0.52},\quad \delta=0.883,^{+0.145}_{-0.118}$$

and we get for the mixing matrix[2]

(8)
$$|A| = \begin{pmatrix} 0.97429 & 0.22523 & 3.86\cdot 10^{-3} \\ 0.22512 & 0.97341 & 0.04215 \\ 8.10\cdot 10^{-3} & 0.04154 & 0.99910 \end{pmatrix}$$

in good agreement with the values reported in[7].
Higher order expansions will be considered in the forthcoming sections.

3. The Geometrical Meaning of The Exponential Parameterization and The Euler Like Forms. We have so far proved that the exponential parameterization of the mixing matrix has some nice features which makes its use quite interesting. Before going further let us speculate on the geometrical (physical) meaning of the matrix $\widehat{A}$, which can be understood as a kind of Hamiltonian ruling the process of quark mixing. We introduce, therefore, the Schroedinger equation

(9)
$$i\partial_\tau \underline{\psi} = \widehat{H}\underline{\psi}$$

where $\underline{\psi}\big|_{\tau=0}$ are the quark mass eigenstates, and

(10)
$$\widehat{H} \propto i\widehat{A}\cdot$$

Within such a picture the matrix $\widehat{V}$ is the evolution operator associated with Eq. (9equation.3.9). In the case of vanishing CP phase $\delta \to 0$, the Hamiltonian in (10equation.3.10) can be written in terms of SO(3) generators, namely

(11)
$$\widehat{H} = \lambda R_1 + y\lambda^2 R_2 + x\lambda^3 R_3$$

with

$$R_1 = i\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, R_2 = i\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, R_3 = i\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

The Schroedinger equation (9equation.3.9) can, accordingly, be viewed as a vector equation of the type

[2] This result has been obtained by expanding the matrix at any arbitrary order, namely $\tilde{V} = \sum_0^N \frac{A^n}{n!}$ and by keeping N=50. We have not included the errors deriving from the experimental and systematic uncertainties, the relevant analysis would require extreme care for fitting the data and such an effeort is out of the purposes of the present note.

(12) $$\partial_\tau \vec{Q} = \vec{\Omega} \times \vec{Q} \quad \text{with} \quad \vec{\Omega} \equiv \lambda\left(-y\lambda, x\lambda^2, -1\right) ,$$

where $\vec{Q} \equiv (\psi_1, \psi_2, \psi_3)$ is the vector associated with the quark field. The problem of the quark mixing is therefore understood as a rotation, induced by an Euler-like torque equation. The torque vector $\vec{\Omega}$ is reported in FIg. (1 The Quark mixing torque vector figure.1) along with the role played by each vector compnent. The quark mizing matrix can be written

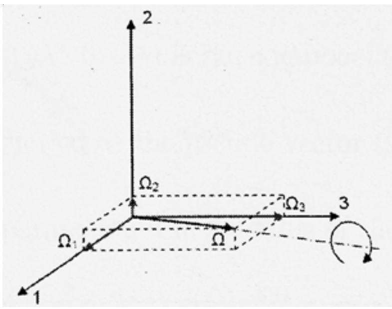

FIG. 1. *The Quark mixing torque vector*

using the Cayley Hamilton theorem[8] (see Sect. IV section*. 6) as

(13) $$\hat{V} = e^{\tau \hat{A}}\Big|_{\tau=1} = \hat{2} + \sin c\left(\left|\vec{\Omega}\right|\right)\hat{A} + \frac{1}{2}\left(\sin C\left(\left|\vec{\Omega}\right|/2\right)\right)^2 \hat{A}^2 ,$$

$$\left|\vec{\Omega}\right| = \lambda\sqrt{1 + y^2\lambda^2 + x^2\lambda^4}, \sin c(\alpha) = \frac{\sin(\alpha)}{\alpha} .$$

Moreover from Eq. (12equation.3.12) the action of the mixing matrix on the initial vector $\vec{Q}$ can be specified through the following Rodriguez rotation[9]

(14) $$\vec{Q} = \cos\left(\left|\vec{\Omega}\right|\right)\vec{Q}_0 + \sin\left(\left|\vec{\Omega}\right|\right)\vec{n} \times \vec{Q}_0 + 1\left(1 - \cos\left(\left|\vec{\Omega}\right|\right)\right)\left(\vec{n} \cdot \vec{Q}_0\right) \cdot \vec{n} ,$$

$$\text{with} \quad \vec{n} = \vec{\Omega}/\left|\vec{\Omega}\right| .$$

The geometrical interpretaion is less obvious if we include the CP violating term. We assume Eq. (11equation.3.11) to be still valid and with a slight abuse of the notation write

(15) $$\vec{\Omega} \equiv \vec{\Omega}_1 + i\vec{\Omega}_2 ,$$

$$\vec{\Omega}_1 \equiv \left(-y\lambda^2, x\lambda^3 \cos(\delta), -\lambda\right),$$

$$\vec{\Omega}_2 = \left(0, x\lambda^3 \sin(\delta), 0\right).$$

This assumption contains the bare essence of CP violation from a geometrical point of view. The vector $\vec{\Omega}$ splits into a real and imaginary part, as shown in Fig. (2The real (a) and the imaginary part (b) of the torque vector Ω figure.2) where the second component of the torque vector is composed by two subcomponents:

a): the coupling vector $\vec{\Omega}_{1,3} \equiv \left(y\lambda^2, 0, -\lambda\right)$ is the component of the vector in the 1-3 plane;

b): the CP violating sector is viewed as the pseudo vector $\left(\Omega_{1,3,}\, \mathrm{Im}(\Omega_2), \mathrm{Re}(\Omega_2)\right)$.

In terms of the Wolfenstein parameters the modulus of the Torque vector can be written as

(16) $$\left|\vec{\Omega}\right| = \left[\lambda^2 + \left(A\lambda^2\right)^2 + \left[(\rho - 1/2)^2 + \eta^2\right]\left(A\lambda^3\right)^2\right]^{1/2}$$

or as

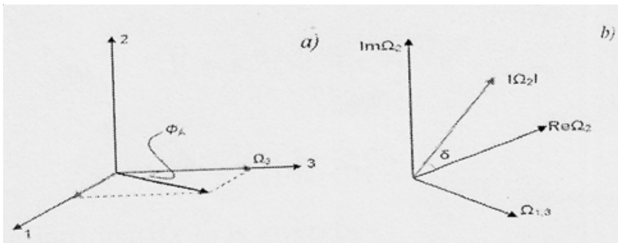

FIG. 2. *The real (a) and the imaginary part (b) of the torque vector*

(17) $$\left|\vec{\Omega}\right| = \left[\lambda^2\left(1 + \tan^2(\phi_A)\right) + \lambda^3(\rho - 1/2)^2\left[1 + \tan^2(\delta)\right]\tan^2(\phi_A)\right]^{1/2},$$

$$\tan(\phi_A) = A\lambda, \quad \tan(\delta) = \eta/\left[(\rho - 1/2)^2\right]^{1/2},$$

where ϕ_A and δ are indicated in Figs. (2 The real (a) and the imaginary part (b) of the torque vector Ω figure.2).

The angle ϕ_A lies in the (1,3) sector and specifies the directions of the $\vec{\Omega}$ vector components in this plane. We visualize the geometric content of our problem as indicated in the second of Figs. (2The real(a) and the imaginary part (b) of the torque vector figure.2), in which the complex vector component lying along the directon of the axis 2 is split into an imaginary and a real part.

In more rigorous mathematical terms we can illustrate the above procedure as it follows. We first note that

(18) $$\widehat{A} = \widehat{A}_1 + \widehat{A}_2\ ,$$

with

$$\widehat{A}_2 = \begin{pmatrix} 0 & 0 & \Lambda_3 \\ 0 & 0 & 0 \\ -\Lambda_3^* & 0 & 0 \end{pmatrix}, \quad \widehat{A}_1 = \begin{pmatrix} 0 & \Lambda_1 & 0 \\ -\Lambda_1 & 0 & \Lambda_2 \\ 0 & -\Lambda_2 & 0 \end{pmatrix}.$$

The matrices labelled with 2, 1 are not commuting each other, therefore we have at the first order in the Zassenhaus disentanglement formula[3]

(19)
$$\widehat{V} = e^{\hat{A}_2+\hat{A}_1} \cong e^{\hat{A}_2} e^{\hat{A}_1} e^{\hat{C}} \ ,$$

$$\widehat{C} = -\frac{1}{2}\left[\widehat{A}_2, \widehat{A}_1\right],$$

where the (anti-hermitian) matrix $\widehat{C}$ is given by

(20)
$$\widehat{C} = -\frac{1}{2}\begin{pmatrix} 0 & -\Lambda_3\Lambda_2 & 0 \\ \Lambda_3^*\Lambda_2 & 0 & \Lambda_1\Lambda_3 \\ 0 & -A_1 A_3^* & 0 \end{pmatrix} = -\frac{\lambda^4 x}{2}\begin{pmatrix} 0 & \lambda y e^{-\iota\delta} & 0 \\ \lambda y e^{-\iota\delta} & 0 & e^{-\iota\delta} \\ 0 & e^{-\iota\delta} & 0 \end{pmatrix}.$$

Neglecting the matrix $\widehat{C}$, which is of the order $o(\lambda^4)$, we find that the CKM matrix can be expressed as

(21)
$$\widehat{V} \cong e^{\hat{A}_2} e^{\hat{A}_1},$$

with

(22)
$$e^{\hat{A}_2} = \widehat{V}_2 = \begin{pmatrix} \cos(|\Lambda_3|) & 0 & \frac{\Lambda_3}{|\Lambda_3|}\sin(|\Lambda_3|) \\ 0 & 1 & 0 \\ -\frac{\Lambda_3^*}{|\Lambda_3|}\sin(|\Lambda_3|) & 0 & \cos(|\Lambda_3|) \end{pmatrix},$$

and the use of the Cayley Hamilton theorem allows the following (exact) form of the Second exponential

$$e^{\hat{A}_1} = \widehat{V}_1 = C_0\hat{2} + C_1\widehat{A}_1 + C_2\widehat{A}_1^2,$$

[3]The Zassenhaus formula writes $e^{\hat{A}+\hat{B}} = e^{\hat{A}} - e^{\hat{B}}\prod_{m=1}^{\infty} e^{\hat{C}_m}$, where the operators $\hat{C}_m$ are given in terms of successive commutators, the first two being $\hat{C}_1 = -\frac{1}{2}\left[\hat{A},\hat{B}\right], \hat{C}_2 = \frac{1}{3}\left[\hat{A},\hat{B}\right] + \frac{1}{6}\left[\hat{A};\left(\hat{A},\hat{B}\right)\right]$.

$$
\begin{pmatrix} C_0 \\ C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} |\Lambda_{1,2}|^2 & 0 & 0 \\ -\Upsilon & \Upsilon & \Upsilon \\ i & 1-i & -(1+i) \end{pmatrix} \begin{pmatrix} \dfrac{1}{|\Lambda_{1,2}|^2} \\ \dfrac{e^{i|\Lambda_{1,2}|}}{2|\Lambda_{1,2}|^2} \\ \dfrac{e^{-i|\Lambda_{1,2}|}}{2|\Lambda_{1,2}|^2} \end{pmatrix}, \tag{23}
$$

$$
\Upsilon = |\Lambda_{1,2}|(1+i), \quad |\Lambda_{1,2}| = \sqrt{\Lambda_1^2 + \Lambda_2^2}\,.
$$

The above formulae are a restatement of the tentative geometrical picture of Fig. (2The real (a) and the imaginary part (b) of the torque vector figure.2) The nave disentanglement has reduced the CKM generation to the produt of two matrices, $\hat{V}_1$ accounting for the mixing, induced by the vector $\vec{\Omega}_{1,3}$, and $\hat{V}_2$ specifying a complex rotation, responsible for the CP violating contributions.

The matirix (21equation.3.21) is an approximation of the exponential form at the order $o(\lambda^4)$, but it is not equivalent to Wolfenstein matrix. The matrix (21equation.3.21), albeit an approximation, since we have neglected higher order commutators, is unitary at any order in the coupling parameter, while $\hat{V}_W \hat{V}_W^\dagger = -\hat{1} + o(\lambda^4)$ (where $\hat{V}_W$ is the matrix (6equation.2.6)). We have stressed that the simple picture in terms of Euler rotation is hamplered by the presence of a complex term, the $\hat{V}$ matrix cannot be written in terms of the generators of rotations and indeed we find

$$
\hat{V} = e^{-\iota\left(\lambda \hat{R}_1 + y\lambda^2 \hat{R}_2 + x\lambda^3 \hat{T}\right)} \quad \text{with } \hat{T} = -i \begin{pmatrix} 0 & 0 & e^{\iota\delta} \\ 0 & 0 & 0 \\ -e^{-\iota\delta} & 0 & 0 \end{pmatrix}. \tag{24}
$$

The $\hat{T}$ matrix does not belong to SO(3) and the quark mixing matrix, written as the product of the exponential matrix correct up to the order (λ^4) is

$$
\hat{V} = e^{-\iota x\lambda^3 \hat{T}} e^{-\iota y\lambda^2 \hat{R}_2} e^{-\iota\lambda^3 \hat{R}_1} + o(\lambda^4), \tag{25}
$$

$$\widehat{V} \cong e^{\begin{pmatrix} 0 & 0 & x\lambda^3 e^{\imath\delta} \\ 0 & 0 & 0 \\ -x\lambda^3 e^{-\imath\delta} & 0 & 0 \end{pmatrix}} e^{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & y\lambda^2 \\ 0 & -y\lambda^2 & 0 \end{pmatrix}} e^{\begin{pmatrix} 0 & \lambda & 0 \\ -\lambda & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}} =$$

$$= \begin{pmatrix} C(x\lambda^3) & 0 & e^{\imath\delta} S(x\lambda^3) \\ 0 & 1 & 0 \\ -e^{-\imath\delta} S(x\lambda^3) & 0 & C(x\lambda^3) \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & C(y\lambda^2) & S(y\lambda^2) \\ 0 & -S(y\lambda^2) & C(y\lambda^2) \end{pmatrix} \begin{pmatrix} C(\lambda) & S(\lambda) & 0 \\ -S(\lambda) & C(\lambda) & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$C(\phi) = \cos(\phi), S(\phi) = \sin(\phi)$$

and displays the largely well-known feature that the mixing angles are proportional to the Cabibbo coupling parameter according to

(26) $$\vartheta_{1,3} \propto \lambda^3 \cong \sqrt{\frac{m_d}{m_b}}, \vartheta_{2,3} \propto \lambda^2 \cong \sqrt{\frac{m_s}{m_b}}, \vartheta_{1,2} \propto \lambda \cong \sqrt{\frac{m_d}{m_s}}.$$

Furthermore, in full agreement with the particle data group paradigm, we get

(27) $$e^{\begin{pmatrix} 0 & 0 & x\lambda^3 e^{\imath\delta} \\ 0 & 0 & 0 \\ -x\lambda^3 e^{-\imath\delta} & 0 & 0 \end{pmatrix}} = \widehat{U}_\delta^\dagger e^{-\imath x\lambda^3 \widehat{R}_3} \widehat{U}_\delta,$$

$$\widehat{U}_\delta = e^{\imath\hat{\delta}}, \text{ with } \hat{\delta} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \delta \end{pmatrix}, \widehat{R}_3 = i \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

It is also interesting to note that the $\widehat{T}$ matrix can be written as

(28) $$\widehat{T} = -i\cos(\delta)\widehat{R}_3 + i\sin(\delta)\widehat{S}_3 \text{ with } \widehat{S}_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

The naive disentanglement (order $o(\lambda^4)$)

(29) $$\widehat{V} \cong e^{-\imath\widehat{R}} e^{\imath x\lambda^3 \sin(\delta)\widehat{S}_3},$$

$$\widehat{R} = \lambda\widehat{R}_1 + y\lambda^2\widehat{R}_2 + x\lambda^3\cos(\delta)\widehat{R}_3$$

corresponds to the product of two matrices, namely

(30) $$\widehat{V} \cong \widehat{V}_R \widehat{V}_1 \text{ with } \widehat{V}_R = e^{\widehat{R}},$$

$$\hat{V}_1 = \begin{pmatrix} C\left(x\lambda^3 \sin(\delta)\right) & 0 & iS\left(x\lambda^3 \sin(\delta)\right) \\ 0 & 1 & 0 \\ iS\left(x\lambda^3 \sin(\delta)\right) & 0 & C\left(x\lambda^3 \sin(\delta)\right) \end{pmatrix} = \begin{pmatrix} C\left(A\lambda^3\eta\right) & 0 & iS\left(A\lambda^3\eta\right) \\ 0 & 1 & 0 \\ iS\left(A\lambda^3\eta\right) & 0 & C\left(A\lambda^3\eta\right) \end{pmatrix} \quad (30)$$

and $\hat{V}_R$ can be written as Eq. (13equation.3.13) with

$$\vec{\Omega} \equiv \lambda\left(-y\lambda, x\lambda^2 \cos(\delta), -1\right). \quad (31)$$

The matrix $\hat{V}_1$ mixes the first and third quark generation mass eigenstates and is responsible for the CP violation. It is a pseudo rotation matrix and is generated by a matrix whose determinant is the Jarlskog invariant[11], discussed in the forthcoming section.
We have so far shown that the exponential parameterization implicitly contains Wolfenstein and Euler type forms, in the following sections we will dwell on its further advantages.

4. The Cayley Hamilton Theorem and The Quark Mixing Matrix.
The exponential matrix (1equation.1.1) can be treated in different ways.
We have already shown that the use of a Taylor expansion leads to a Wolfenstein form, which preserves the unitarity of $\hat{V}$ at the expansion order (the mixing matrix in Eq. (6equation.2.6) is unitary at the order $o\left(\lambda^4\right)$).

The method of the exponential disentanglement can be used too and such a procedure allows an interesting geometrical picture of the mixing dynamics and albeit an approximation in the Cabibbo coupling parameter, the mixing matrix written as in Eq. (17equation.3.17) preserves the unitarity at any order in λ, as discussed more accurately in the concluding remarks.

The matrix $\hat{V}$ can, however, be written in an exact form using the Cayley Hamilton theorem, by setting

$$\hat{V} = C_0 \hat{1} + C_1 \hat{A} + C_2 \hat{A}^2, \quad (32)$$

where

$$e^{\varepsilon_j} = C_0 + \varepsilon_j C_1 + \varepsilon_j^2 C_2, \text{ with } \quad j = 1,2,3 \quad (33)$$

with ε_j being the roots associated with the characteristic equation of the matrix $\hat{A}$, namely

$$\varepsilon_j^3 + \left|\vec{\Omega}\right|^2 \varepsilon_j + i\Delta = 0 \quad (34)$$

where

$$\Delta = 2xy\lambda^6 \sin(\delta) = -2A\eta\lambda^6 .$$

(35) $$\left|\vec{\Omega}\right| = \lambda\sqrt{1+y^2\lambda^2+x^2\lambda^4} = \lambda\sqrt{1+(A\lambda)^2+\left(A^2\lambda^4\right)\left[\left(\rho-\frac{1}{2}\right)^2+\eta^2\right]} ,$$

$i\Delta$ is the determinant of the matrix $\hat{A}$.

A little bit of algebra yields to define the $C_i\,(i=0,1,2)$ coefficients as the product of two matrices

(36) $$\begin{pmatrix} C_0 \\ C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} \varepsilon_2\varepsilon_3 & \varepsilon_1\varepsilon_3 & \varepsilon_1\varepsilon_2 \\ -(\varepsilon_2+\varepsilon_3) & -(\varepsilon_1+\varepsilon_3) & -(\varepsilon_1+\varepsilon_2) \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \dfrac{e^{\varepsilon_1}}{(\varepsilon_2-\varepsilon_1)(\varepsilon_3-\varepsilon_1)} \\ \dfrac{e^{\varepsilon_2}}{(\varepsilon_1-\varepsilon_2)(\varepsilon_3-\varepsilon_2)} \\ \dfrac{e^{\varepsilon_3}}{(\varepsilon_1-\varepsilon_3)(\varepsilon_2-\varepsilon_3)} \end{pmatrix} .$$

Eq. (23equation.3.23) (along with Eqs. (27equation.3.27)) is the most general form of the quark mixing matrix which can be derived from an exponential parameterization, it is exact but not easy to remember.

Let us now given an idea of the orders of the numerical values characterizing the various quantities entering the above equations. The use of the previously quoted values for the Wolfenstein parameters lead to the following evaluations for the solution of Eq. (34equation.4.34)

(37) $\varepsilon_1 \cong -0.23171i,\ \varepsilon_2 \cong -0.00117i,\ \varepsilon_2 \cong -0.23054i.$

It is worth stressing that the matrix $\hat{D}$ provides the diagonal forrms of either $\hat{V}$ and $\hat{A}$. It follows therefore that the two matrices have the same eigenvectors. They can be determined using $\hat{A}$ instead of $\hat{V}$, because the procedure is significantly simpler. We find that the eigenvalues are in the form

(38) $$|j> = \begin{pmatrix} 1 \\ -\varepsilon_j - xy\lambda^5 e^{-\iota\delta} \\ -y\lambda^3 - \varepsilon_j x\lambda^3 e^{-\iota\delta} \end{pmatrix} .$$

It is worth mentioning the companion matrix associated with the characteristic equation (30)[12], which writes

(39)
$$C_A = \begin{pmatrix} 0 & 0 & -\varepsilon_1\varepsilon_2\varepsilon_3 \\ 1 & 0 & -(\varepsilon_1\varepsilon_2+\varepsilon_2\varepsilon_3+\varepsilon_1\varepsilon_3) \\ 0 & 1 & -(\varepsilon_1+\varepsilon_2+\varepsilon_3) \end{pmatrix}.$$

It is accordingly expressed in terms of three invariants[4], namely

(40)
$$\varepsilon_1\varepsilon_2\varepsilon_3 = -2ix\lambda^6\sin(\delta),$$
$$\varepsilon_1^2+\varepsilon_2^2+\varepsilon_3^2 = \left|\vec{\Omega}\right|^2,$$
$$\varepsilon_1+\varepsilon_2+\varepsilon_3 = 0,$$

the first of which is the Jarlskog invariant, a measure of the amount of CP violations, emerging in quite a natural way in the present analysis.

5. Concluding Remarks. We have shown that the exponential parameterization interpolates between Wolfestein and Euler like forms and could provide a useful and flexible tool of analysis. Its approxima-tions in terms of the Cabibbo coupling can be either expressed as Taylor expansions or as unitarity preserving forms bassed on the Zassenhaus formula.

The Taylor expansion does not meet too much aesthetical criteria, but it can usefully be exploited to get higher order approximations of Wolfenstein type parameterizations an example is shown below, where we report the nave expansion of the exponential matrix up to the order $o(\lambda^7)$.

(41)
$$\hat{V} = \begin{pmatrix} C(\lambda)+\frac{A^2\lambda^6}{4!}\Phi & S(\lambda)-\frac{A^2\lambda^5}{2}\left(\prod^*-\frac{1}{6}\right) & A\lambda^3\left[\prod^*-\frac{\lambda^2}{6}\left(\prod^*-\frac{1}{4}\right)\right] \\ -S(\lambda)-\frac{A^2\lambda^5}{2}\left(\prod-\frac{5}{6}\right) & C(\lambda)-(A\lambda^2)^2\left[\frac{1}{2}-\frac{\lambda^2}{3}\left(\frac{1}{4}-i\eta\right)\right] & S(A\lambda^2)+\frac{A\lambda^4}{4!}\left(\lambda^2\left(\prod^*-\frac{3}{10}\right)-12\left(\prod^*-\frac{1}{3}\right)\right) \\ -A\lambda^3\left[\left(\prod-1\right)-\frac{\lambda^2}{6}\left(\prod-\frac{3}{4}\right)\right] & -S(A\lambda^2)+\frac{A\lambda^4}{4!}\left(\lambda^2\left(\prod-\frac{7}{10}\right)-12\left(\prod-\frac{2}{3}\right)\right) & 1-\frac{(A\lambda^2)^2}{2}+\frac{A^2\lambda^6}{4!}\Phi \end{pmatrix}$$

[4] A 3×3 matix has three invariants given by its determinant, its trace and by the sum of the determinants of its minors

$$\Phi = -12\left(\rho^2+\eta^2\right) - 8i\eta + 12\rho - 2; \prod = \rho + i\eta; \prod{}^* = \rho - i\eta,$$

where $C(\lambda)$, $S(\lambda)$ in the above matrix denote the expansion of cosine and sine up to the order $o\left(\lambda^7\right)$.

We have reported the matrix (41equation.5.41) for comparison purposes with other forms available in literature. The accuracy of this last matrix is one part over 10^9 and can therefore considered exact for any expansion purposes. The extension of the CKM matrix to higher dimensions by the use of the exponential matrix method is not complicated. In the case of four quark generations, we define the matrix containing 2 CP violating phases, appended to the smallest coupling terms. We have furthermore assumed that the coupling strengths to the fourth family be of the order

$\lambda^{3+n}, n = 1,2,3.$

$$A = \begin{pmatrix} 0 & \lambda & e^{\iota\delta_1} x\lambda^3 & e^{\iota\delta_2} z\lambda^6 \\ -\lambda & 0 & y\lambda^2 & p\lambda^5 \\ -e^{-\iota\delta_1} x\lambda^3 & -y\lambda^2 & 0 & u\lambda^4 \\ -e^{-\iota\delta_2} z\lambda^6 & -p\lambda^5 & -u\lambda^4 & 0 \end{pmatrix}. \tag{42}$$

The relevant Wolfenstein like approximation of the mixing matrix reads

$$\widehat{V} = \left(\begin{matrix} C(\lambda) + \frac{A^2\lambda^6}{4!}\Phi & S(\lambda) - \frac{A^2\lambda^5}{2}\left(\prod{}_1^* - \frac{1}{6}\right) \\ -S(\lambda) - \frac{A^2\lambda^5}{2}\left(\prod{}_1 - \frac{5}{6}\right) & C(\lambda) - \left(A\lambda^2\right)^2\left[\frac{1}{2} - \frac{\lambda^2}{3}\left(\frac{1}{4} - i\eta\right)\right] \\ -A\lambda^3\left[\left(\prod{}_1 - 1\right) - \frac{\lambda^2}{6}\left(\prod{}_1 - \frac{3}{4}\right)\right] & -S\left(A\lambda^2\right) + \frac{A\lambda^4}{4!}\left(\lambda^2\left(\prod{}_1 - \frac{7}{10}\right) - 12\left(\prod{}_1 - \frac{2}{3}\right)\right) \\ -\lambda^6 p\left(\prod{}_2 - 1\right) & -\lambda^5\left(-p + Au\frac{\lambda}{2}\right) \end{matrix} \right. \tag{43}$$

$$\left. \begin{matrix} A\lambda^3\left[\prod{}_1^* - \frac{\lambda^2}{6}\left(\prod{}_1^* - \frac{1}{4}\right)\right] & -\prod{}_2^* p\lambda^6 \\ S\left(A\lambda^2\right) + \frac{A\lambda^4}{4!}\left(\lambda^2\left(\prod{}_1^* - \frac{3}{10}\right) - 12\left(\prod{}_1^* - \frac{1}{3}\right)\right) & \lambda^5\left(p + Au\frac{\lambda}{2}\right) \\ 1 - \frac{\left(A\lambda^2\right)^2}{2} + \frac{A^2\lambda^6}{4!}\Phi & u\lambda^4 \\ -u\lambda^4 & 1 \end{matrix} \right),$$

$\Phi_1 = \rho_1 + i\eta_1$ with $\rho_1 = x/y\cos(\delta_1) + 1/2, \quad \eta_1 = -x/y\sin(\delta_1),$

$\Phi_2 = \rho_2 + i\eta_2$ with $\rho_2 = z/p\cos(\delta_2) + 1/2, \quad \eta_2 = -z/p\sin(\delta_2),$

$\Phi_j^* = \rho_j - i\eta_j j = 1,2$.

Furtheremore the invariants (4 for a 4×4 matrix), obtained directly from (42equation.5.42) read

$$J_2 = \lambda^2 f(1, y\lambda, x\lambda^2) + \lambda^8 f(u, p\lambda, z\lambda^2) f(a,b,c) = a^2 + b^2 + c^2$$

$$J_3 = 2i\lambda^6\left[xy\sin(\delta_1) + zp\lambda^6\sin(\delta_2)\right] = xy\lambda^6\left(e^{i\delta_1} - e^{-i\delta_1}\right) + zp\lambda^{12}\left(e^{i\delta_2} - e^{-i\delta_2}\right)$$

$$J_4 = u^2\lambda^{10} + \left[2uyz\cos(\delta_2) - 2pux\cos(\delta_1)\right]\lambda^{1/3}$$
$$+\left[p^2x^2 - px^2ye^{i(\delta_2-\delta_1)} - pxyze^{-i(\delta_2-\delta_1)} + xy^2z\right]\lambda^{16}$$

the first invariant, associated with the trace of A, is zero.

It is evident that the J_2 and J_3 invariants are just a generalization of those reported in Eq. (32equation.4.32) while the fourth is completely new being associated to the full determinant of the matrix. We have reported this example to show the flexibility of the method it is however evident that the detection of CP violating effects due to the new phase require an accuracy at least of the order λ^6.

Before concluding the paper we will address the problems associated with the exponential forms of the neutrino mixing matrix, which have also discussed in[13], where the leptonquark complementarity[14] has been reformulated by noting that the relevant rotation occur around axes forming an angle of 45^0. The present experimental data seem to favor the tribimaximal (TBM) form[15] therefore the neutrino mixing matrix reads

$$\hat{U} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix}. \tag{44}$$

If we assume that also this form is generated by an exponential matrix (with all real entries) according to

$$\hat{U} = e^{\hat{B}}; \quad \hat{B} = \begin{pmatrix} 0 & \alpha & \beta \\ -\alpha & 0 & \gamma \\ -\beta & -\gamma & 0 \end{pmatrix}. \tag{45}$$

We obtain the following correspondence between the entries of the $\hat{B}$ matrix and those of the TBM form

(46) $$\hat{B} = \alpha \begin{pmatrix} 0 & 1 & -\frac{1}{\sqrt{2}+1} \\ -1 & 0 & \frac{\sqrt{3}+\sqrt{2}}{\sqrt{2}+1} \\ \frac{1}{\sqrt{2}+1} & -\frac{\sqrt{3}+\sqrt{2}}{\sqrt{2}+1} & 0 \end{pmatrix}$$

$$\alpha = 2\frac{\sqrt{2\sqrt{2}+3}}{\sqrt{2\sqrt{2}+2\sqrt{6}+9}}\frac{1}{\sin\left[3-\left(\sqrt{2/3}+1/\sqrt{3}+1/\sqrt{2}\right)\right]^{1/2}/2}$$

The values of the entries of the TBM matrix do not allow the interpretation of the neutrino mixing matrix as an expansion around the unit, notwithstanding it is possible to get a better agreement with experimental by making an appropriate expansions around the matrix $\hat{B}$ and then around the TBM, as it will be shown in a dedicated paper.

In this paper we have provided an extensive account of the possibilities offered by the exponential form of the CKM matrix, which looks like a prototype from which all the other forms can be derived, we hope that our suggestions provide a useful tool in the relevant applications.

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COEFFICIENT ESTIMATES FOR CERTAIN ANALYTIC FUNCTION CLASSES CONCERNED WITH THE PRINCIPLE OF SUBORDINATION

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ABSTRACT

By using a method of the proof of a certain coefficient inequality which was discussed by J. Zamorski [*Ann. Polon. Math.* **9** (1961), 265-273], the author investigates the coefficient estimates for functions in several analytic function classes which are concerned with the principle of subordination.

2010 Mathematics Subject Classification : Primary 30C45
Key Words and Phrases : Analytic functions; Starlike functions; Convex functions; Principle of subordination; Coefficient estimate

1. Introduction. Let denote the class of functions $f(z)$ of the form:

(1.1) $$f(z)=z+\sum_{n=2}^{\infty}a_n z^n,$$

which are analytic in the open unit disk

(1.2) $$=\{z:z\in\mathbb{C} \text{ and } |z|<1\}.$$

If $f(z)\in$ satisfies the following inequality:

$$\Re\left(\frac{zf''(z)}{f'(z)}\right)>\alpha \qquad (z\in \quad)$$

for some real number α with $0\leq\alpha<1$, then $f(z)$ is said to be starlike of order α in . This class is denoted by $^*(\alpha)$. Similarly, we say that $f(z)$ belongs to the class (α) of convex functions of order α in if $f(z)\in$ satisfies the following inequality:

$$\Re\left(1+\frac{zf''(z)}{f'(z)}\right)>\alpha \qquad (z\in \quad)$$

for some real number α with $0\leqq\alpha<1$. The classes $*(\alpha)$ and (α) were introduced by Robertson [5].

Let $p(z)$ and $q(z)$ be analytic in . Then the function $p(z)$ is said to be subordinate to $q(z)$ in , written as follows :

$$p(z) \prec q(z) \qquad (z \in \ ), \tag{1.1}$$

if there exists a function $w(z)$, which is analytic in with $w(0)=0$ and $|w(z)|<1$ $(z \in \ )$ such that

$$p(z) = q(w(z)) \qquad (z \in \ ).$$

From the definition of the principle of subordination between analytic functions, it is easy to show that the subordination (1.1) implies that

$$p(0) = q(0) \text{ and } p(U) \subset q(U) \tag{1.2}$$

In particular, if $q(z)$ is univalent in , then the subordination (1.1) is equivalent to the conditon (1.2).

For some real number A and B with $-1 \leqq B < A \leqq 1$, Janowski [1] investigated the following linear transformation:

$$p(z) = \frac{1+Az}{1+Bz} \qquad (z \in \ ),$$

which is analytic and univalent in . This funciton $p(z)$ is called the Janowski funciton. Moreover, as a generalization of the Janowski functions, Kuroki and Owa [2] discussed the Janowski functions for some complex parameters A and B which satisfy the following conditions:

$$A \neq B, \ |B| \leqq 1 \text{ and } |A-B| + |A+B| \leqq 2. \tag{1.3}$$

We note that the Janowski function defined by the conditions (1.3) is analytic and univalent in and has a positive real part in (see [2]; see also [3]).

For some complex numbers A and B with $A \neq B$ and $|B| \leqq 1$, we consider the subclasses $*(A,B)$ and (A,B) of the normalized analytic function class as follows:

$$*(A,B) = \left\{ f : f(z) \in \quad \text{and } \frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz} \quad (z \in \ ) \right\}$$

and

$$(A,B) = \left\{ f : f(z) \in \quad \text{and } 1 + \frac{zf''(z)}{f'(z)} \prec \frac{1+Az}{1+Bz} \quad (z \in \ ) \right\}.$$

Then we can observe that

$*(1-2\alpha,-1)= \; *(\alpha)$ and $\;(1-2\alpha,-1)= \;(\alpha)$.

Example 1.

$$f(z)=\begin{cases}\dfrac{z}{(1+Bz)^{1-A/B}}=z+\sum_{n=2}^{\infty}\left(\prod_{k=1}^{n-1}\dfrac{A-kB}{k}\right)z^n\in \; *(A,B) & (A\neq 0;B\neq 0)\\ \dfrac{z}{1+Bz}=z+\sum_{n=2}^{\infty}(-B)^{n-1}z^n\in \; *(0,B) & A\neq 0;B\neq 0\\ ze^{Az}=z+\sum_{n=2}^{\infty}\dfrac{A^{n-1}}{(n-1)!}z^n\in \; *(A,0) & A\neq 0;B\neq 0\end{cases}\cdot \tag{1.4}$$

Example 2.

$$f(z)=\begin{cases}\dfrac{(1+Bz)^{A/B}-1}{A}=z+\sum_{n=2}^{\infty}\left(\prod_{k=1}^{n-1}\dfrac{A-kB}{k+1}\right)z^n\in \;(A,B) & (A\neq 0;\ B\neq 0)\\ \dfrac{1}{B}\log(1+Bz)=z+\sum_{n=2}^{\infty}(-B)^{n}z^n\in \;(0,B) & (A\neq 0;\ B\neq 0)\\ \dfrac{1}{A}\left(e^{Az}-1\right)=z+\sum_{n=2}^{\infty}\dfrac{A^{n-1}}{n!}z^n\in \;(A,0) & (A\neq 0;\ B=0)\end{cases}\cdot \tag{1.5}$$

Remark 1. For a function $f(z)\in \;$, it follows that

$$f(z)\in \;(A,B) \text{ if and only if } zf'(z)\in \; *(A,B).$$

Robertson [5] has proved the following well-known results :

Lemma 1. *If the funciton* $f(z)=z+\sum_{n=2}^{\infty}a_n z^n \; *\in(\alpha)$, *then*

$$|a_n| \leqq \frac{\prod_{j=2}^{n}(j-2\alpha)}{(n-1)!}(n=2,3,4,...)$$

Lemma 2. *If the funciton* $f(z)=z+\sum_{n=2}^{\infty}a_n z^n \;\in(\alpha)$, *then*

$$|a_n| \leqq \frac{\prod_{j=2}^{n}(j-2\alpha)}{n!}(n=2,3,4,...)$$

2. Coefficient Estimates. In this section, by using a certain method of the proof of a coefficient inequality which was discussed by Zamorski [6] (see also MacGregor [4]), we first consider sharp bounds on the coefficients for $f(z)\in \; *(A,B)$.

Theorem 1. *If the function* $f(z)=z+\sum_{n=2}^{\infty}a_nz^n \in \quad ^*(A,B)$, *then*

$$|a_n|\leq\prod_{k=1}^{n-1}\frac{|A-kB|}{k} \qquad (n=2,3,4,...) \tag{2.1}$$

with equality for the function f(z) given in (1.4) *and its rotation.*

Proof. Since $f(z)\in \quad ^*(A,B)$, we have

$$\frac{zf'(z)}{f(z)}\prec\frac{1+Az}{1+Bz} \qquad (z\in \quad).$$

From the defintion of the principle of subordination between analytic functions, there exists an analytic function $w(z)$ in , with $w(0)=0$ and $|w(z)|<1 \ (z\in \quad)$, such that

$$\frac{zf'(z)}{f(z)}=\frac{1+Aw(z)}{1+Bw(z)} \qquad (z\in \quad) \tag{2.2}$$

It follows from the equality (2.2) that

$$zf'(z)-f(z)=w(z)\big(Af(z)-Bzf'(z)\big),$$

which implies that

$$\sum_{k=1}^{\infty}(k-1)a_kz^k=w(z)\sum_{k=1}^{\infty}(A-kB)a_kz^k, \tag{2.3}$$

where $a_1=1$. The equality (2.3) can be written as follows:

$$\sum_{k=1}^{n}(k-1)a_kz^k+\sum_{k=n+1}^{\infty}c_kz^k=w(z)\sum_{k=1}^{n-1}(A-kB)a_kz^k, \tag{2.4}$$

where $c_k\ (k=n+1,n+2,n+3,...)$ are some numbers. Since the equality (2.4) has the form $F(z)=w(z)G(z)$, where $|w(z)|<1 (z\in \quad)$, it follows that

$$\frac{1}{2\pi}\int_0^{2\pi}\left|F\left(re^{i\theta}\right)\right|^2 d\theta\leqq\frac{1}{2\pi}\int_0^{2\pi}\left|G\left(re^{i\theta}\right)\right|^2 d\theta \tag{2.5}$$

for each r with $0<r<1$. Expressing the inequality (2.5) in terms of the coefficients in the equality (2.4), we obtain the following inequality:

$$\sum_{k=1}^{n}(k-1)^2|a_k|^2r^{2k}+\sum_{k=n+1}^{\infty}|c_k|^2r^{2k}\leqq\sum_{k=1}^{n-1}|A-kB|^2|a_k|^2r^{2k}. \tag{2.6}$$

In particular, the inequality (2.6) implies that

$$\sum_{k=1}^{n}(k-1)^2|a_k|^2 r^{2k} \leqq \sum_{k=1}^{n-1}|A-kB|^2|a_k|^2 r^{2k}. \tag{2.7}$$

By letting $r \to 1$ in the inequality (2.7), we conclude that

$$\sum_{k=1}^{n}(k-1)^2|a_k|^2 \leqq \sum_{k=1}^{n-1}|A-kB|^2|a_k|^2,$$

which is equivalent to

$$|a_n|^2 \leqq \sum_{k=1}^{n-1}\frac{|A-kB|^2-(k-1)^2}{(n-1)^2}|a_k|^2.$$

To prove the assertion of Theorem 1, we need to show that

$$X_n \equiv \sum_{k=1}^{n-1}\frac{|A-kB|^2-(k-1)^2}{(n-1)^2}|a_k|^2 \leqq \prod_{k=1}^{n-1}\frac{|A-kB|^2}{k^2} \quad (n=2,3,4,....)\ . \tag{2.8}$$

We now use the principle of mathematical induction for the proof of the inequality (2.8). Since

$$X_2 = |A-B|^2|a_1|^2 = |A-B|^2,$$

it is clear that the assertion holds true for $n=2$.
We assume that the proposition is true for $n=m$. Some calculation gives us that

$$\begin{aligned}
X_{m+1} &= \sum_{k=1}^{m}\frac{|A-kB|^2-(k-1)^2}{m^2}|a_k|^2 \\
&= \frac{(m-1)^2}{m^2}X_m + \frac{|A-mB|^2-(m-1)^2}{m^2}|a_m|^2 \\
&\leqq \frac{(m-1)^2}{m^2}X_m + \frac{|A-mB|^2-(m-1)^2}{m^2}X_m = \frac{|A-mB|^2}{m^2}X_m \\
&\leqq \frac{|A-mB|^2}{m^2}\prod_{k=1}^{m-1}\frac{|A-kB|^2}{k^2} = \prod_{k=1}^{m}\frac{|A-kB|^2}{k^2}.
\end{aligned}$$

This implies that the inequality (2.8) is true for $n=m+1$.
By the principle fo mathematical induction, we have thus proved that

$$|a_n|^2 \leqq \prod_{k=1}^{n-1}\frac{|A-mB|^2}{k^2} \quad (n=2,3,4,...)$$

which means the inequality (2.1). In addition, we can obtain an equality in the inequality (2.1) for the coefficients of the function $f(z)$ given in (1.4) and its rotaion. This completes the proof o f Theorem 1.

Remark 2. Letting

$$A = 1 - 2\alpha \ (0 \leqq \alpha < 1) \text{ and } B = -1$$

in Theorem 1, we obtain the coefficient estimates for $f(z) \in \quad ^*(\alpha)$ in Lemma 1.

Applying the relation in Remark 1, we deduce sharp coefficient estimates for $f(z) \in \quad (A,B)$ as follows

Theorem 2. *If the function* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \quad (A,B)$, *then*

$$|a_n| \leq \prod_{k=1}^{n-1} \frac{|A - kB|}{k+1} (n = 2,3,4,\ldots) \tag{2.9}$$

with equality for the function f(z) given in (1.5) and its rotation.

Proof. According to Remark 1, $f(z) \in \quad (A,B)$ if and only if

$$zf'(z) = z + \sum_{n=2}^{\infty} n a_n z^n \in \quad ^*(A,B).$$

Then, by Theorem 1, we can find the inequality (2.9). Moreover, we obtain an equality in the inequality (2.9) for the coefficients of the function $f(z)$ given in (1.5) and its rotaion. This completes the proof of Theorem 2.

Remark 3. Taking

$$A = 1 - 2\alpha \ (0 \leqq \alpha < 1) \text{ and } B = -1$$

in Theorem 2, we get the coefficient estimates for $fz \in \quad (\alpha)$ in Lemma 2.

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THE DIRAC FACTORIZATION METHOD AND THE HARMONIC OSCILLATOR

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ABSTRACT

We apply the Dirac factorization method to the nonrelativistic harmonic oscilllator and, more in general, to Hamiltonians with a generic potential. It is shown that this procedure naturally leads to a supersymmetric formulation of the problems under study. It is also speculated on the physical meaning underlying this method and it is suggested that the vacuum field fluctuations can be viewed as the spontaneous emission of the associated two-level system, whose quantization is due to the noncommuting nature of the harmonic oscillator canonical variables.

2010 Mathematics Subject Classification : 47A68, 47B36.

Keywords and Phrases : Dirac factorization method, Harmonic oscillator, Hamiltonian.

1. Introduction. As stressed in [1], spin systems (i.e., any system with only two energy levels) and harmonic oscillators comprise two archetypes in quantum mechanics. Recent experiments [2] are going deeper in their peculiar nature and in this paper we show that two-level spin systems and oscillator states are naturally entangled in a supersymmetric framework. It will be shown, starting from fairly simple mathematical considerations, that such a realization involves nothing but that the Dirac factorization method.

The Hamiltonian of a harmonic oscillator can be written in terms of creation-annihilation operators according to the identity[1]

$$H = \left(p^2 + q^2\right)/2 = a^+ a^- + 1/2 \tag{1.1}$$

where

$$a^{\pm} = \left(q \mp ip\right)/\sqrt{2} \tag{1.2}$$

[1] In the following we use the natural units $h=c=1$ and assume, for simplicity, $m=1$ and $w=1$.

and

$$[q,p]=i\ ,\qquad\qquad [a^-,a^+]=1. \tag{1.3}$$

The Pauli matrices [1]

$$\sigma_1=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\quad \sigma_2=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix},\quad \sigma_3=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \tag{1.4}$$

are the realization of the generators of a Clifford algebra and satisfy the identities

$$[\sigma_j,\sigma_k]=2i\in_{jkm}\sigma_m,\quad \{\sigma_j,\sigma_k\}=2\delta_{jk}, \tag{1.5}$$

which allow us to rewrite the sum of the squares of two operators as follows

$$A^2+B^2=(A\sigma_j+B\sigma_k)^2-i\in_{jkl}[A,B]\sigma_l\qquad (j\neq k). \tag{1.6}$$

This identity will be referred as *Dirac factorization method.* It was the breakthrough paving the way to the relativistic Dirac equation [3]. More recently [4], an analogous procedure has been applied to get a unified view of the theory of relativistic wave equations, using identity (1.6) as a tool to provide a different formulation of the definition of fractional operators and derivatives. We will exploit and further expand the formalism developed in [4] by applying the Dirac factorization method to obtain alternative forms of the harmonic oscillator Hamiltonian. This technique leads to the introduction of a set of operators which are not the ordinary creation-annihilation pairs and are recognized as supercharges. THe proposed procedure brings naturally to a supersymmetric formulation of the harmonic oscillator Hamiltonian and it is easily generalized to more complicated Hamiltonian forms. THe method suggests that oscillator-like Hamiltonians are equipped, through Dirac factorization method, with a two-level structure providing a fairly transparent understanding of the physical role played by the algebraic grading.

2. Dirac Factorization and the Harmonic Oscillator. To proceed in the Dircac factorization of the Hamiltonian (1.1), let us introduce the following combination of matrices and differential opereators

$$\sum=\frac{1}{\sqrt{2}}(q\sigma_1-p\sigma_2)=\begin{pmatrix}0 & a^-\\ a^+ & 0\end{pmatrix} \tag{2.1}$$

that, according to eqs. (1.5), and as a consequence of the non-commuting nature of the operators q and p, yields

$$\sum{}^2=\frac{1}{2}\{(q^2+p^2)-i\sigma_1\sigma_2\}, \tag{2.2}$$

and, thus, the Hamiltonian (1.1) can be rewritten as

$$H = \Sigma^2 - \frac{1}{2}\sigma_3 = \begin{pmatrix} H_+ - 1/2 & 0 \\ 0 & H_- + 1/2 \end{pmatrix} \tag{2.3}$$

with

$$H_+ = a^- a^+, \qquad H_- = a^+ a^-. \tag{2.4}$$

This form of the harmonic oscillator Hamiltonian can be interpreted in terms of supersymmetric Quantum Mechanics (SUSY QM) [5] because, as shown by Eq. (2.7) below, the operator Σ can be understood as the sum of the supercharge operators associated with this spocific problem. Let us remark that from the point of view of recovering the harmonic oscillator Hamiltonian, the expression (2.1) it's not the only one that can be adopted. However, it's easy to show that the general form

$$\tilde{\Sigma} = \frac{1}{\sqrt{\alpha^2 + \beta^2}}\left(\alpha\sigma_j + \beta\sigma_k\right) \qquad (j \neq k), \tag{2.5}$$

can always be reduced to the form (2.1) by means of a unitary transformation

$$u = \exp\left\{ i \sum_{m=1}^{3} a_m \sigma_m \right\}. \tag{2.6}$$

The term proportional to σ_3 in Eq. (2.3) is linked to the vacuum field fluctuations. In the usual treatment of SUSY QM the ground state is associated only with the operator H_-, while the present formalism suggests that the vacuum field fluctuations can be interpreted as a contribution emerging from a kind of population inversion. We note indeed that, apart from its mathematical role, the operator Σ is amenable for a transparent interpretation in physical terms. The joint use of the properties of Pauli matrices and ladder operators allows us to cast Eq. (2.1) in the form

$$\Sigma = a^- \sigma_+ + a^+ \sigma_-, \tag{2.7}$$

with

$$\sigma_+ = (\sigma_1 + i\sigma_2)/2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma_- = (\sigma_1 - i\sigma_2)/2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{2.8}$$

The operator (2.7) is formally equivalent to the interaction potential in Jaynes-Cummings Hamiltonian [6], describing the interaction of a quantized field with a two-level system, whose energies differ by the characteristic gap of the harmonic oscillator spectrum[2]. We can therefore say that the use of the Dirac factorization method and the quantized nature of the Hamiltonian operator induces, in quite a natural way, an analogous quantized two level-structure. This, in turn, can be interpreted as the physical origin of the graded algebraic nature of the model we are developing.

The algebraic structure, underlying the Hamiltonian (2.3), requires the embedding of bosonic $\left(a^{\pm}\right)$ and fermionic (Pauli matrices) operators which close up to form a super-algebra. Following [7-9], we introduce the operators

$$U_{\pm} = a^{\pm}\sigma_{\pm}\ , \qquad V_{\pm} = a^{\pm}\sigma_{\pm} \tag{2.9}$$

that satisfy the commutation relations

$$\begin{aligned} &[U_+,U_-] = H\sigma_3 - 1/2\,, && [V_+,V_-] = H\sigma_3 - 1/2 \\ &[U_{\pm},V_{\pm}] = \pm\sigma_{\pm}^2\,, && [U_{\pm},V_{\pm}] = \pm 2K_{\pm}\sigma_3\,, \end{aligned} \tag{2.10}$$

where we have introduced the two operators

$$K_{\pm} = \left(a^{\pm}\right)^2/2\,, \tag{2.11}$$

that, together the Hamiltonian operator, generate the SU (1,1) algebra:

$$[H,K_{\pm}] = \pm K_{\pm} \qquad [K_+,K_-] = -H \tag{2.12}$$

and allows us to recover the full structure of the ortho-symplectic algebra osp (1|2) with the remaining commutation brackets (the not mentioned brackets are zero)

$$[K_{\pm}U_{\pm}] = \mp V_{\mp} \qquad [K_{\pm},V_{\pm}] = \mp U_{\mp}\,. \tag{2.13}$$

As anticipated, the operators $V_{\mp}$ play the role of supercharges of the system. Indeed, from Eq. (2.7), we can write $\Sigma = V_+ + V_-$, and, since $V_{\pm}^2 = 0$, the supersymmetric part of the Hamiltonian is given by $\Sigma^2 = \{V_+,V_-\}$.   .

By using the previous identities, it's easy to show that the Heisenberg equation of motion for the operators σ_k are:

$$\dot{\sigma}_1 = \dot{\sigma}_2 = 0 \qquad , \qquad \dot{\sigma}_3 = 2\{q,p\} = 4i\left(K_+ - K_-\right), \tag{2.14}$$

from which it's easy to show that

$$\sigma_3(t) \propto \sin^2\left(\sqrt{2}t\right), \tag{2.15}$$

that represents the vacuum field spontancous emission. As for the time evolution of the operator Σ we obtain

$$\ddot{\Sigma} = -\Sigma \tag{2.16}$$

i.e., a simple harmonic oscillation, physically associated with the emission and the absorption of a photon.

We have so far shown that the Dirac factorization method dresses the harmonic oscillator with a spin-like (two-level structure), yielding a graded Lie algebraic structure. It is therefore natural to ask about the real physical meaning of such a superimposed structure. The main representative of the fermionic structure is the vector $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ which may be considered as rather artificial. However, apart from the fact that it emerges in a very natural way from the mathematical procedure, its role is physically understandable and should be interpreted as that of a quantized two-level structure induced by the noncommuting nature of the q and p variables. More in general, we can view vacuum as an ensemble of coupled two-level systems continuously emitting and absorbing a photon.

Before closing the section, we briefly show how the Dirac factorization proceeds in the case of the slightly more complicated Hamiltonian

$$H = \left[p^2 + f(q) \right] / 2 . \tag{2.17}$$

If, in analogy with Eq. (2.1), we introduce the operator

$$\Upsilon = \frac{1}{\sqrt{2}} \left(\sqrt{f(q)}\ \sigma_1 - p\sigma_2 \right) = \begin{pmatrix} 0 & A^- \\ A^+ & 0 \end{pmatrix}, \tag{2.18}$$

with

$$A^{\pm} = \frac{1}{\sqrt{2}} \left(\sqrt{f(q)} \mp ip \right), \tag{2.19}$$

we can write

$$H = \Upsilon^2 - \frac{1}{4} \frac{f'(q)}{\sqrt{f(q)}} \sigma_3 = \begin{pmatrix} A^- A^+ - \frac{1}{4} \frac{f'(q)}{\sqrt{f(q)}} & 0 \\ 0 & A^+ A^- + \frac{1}{4} \frac{f'(q)}{\sqrt{f(q)}} \end{pmatrix}. \tag{2.20}$$

Also in this case

$$\nu_- = \begin{pmatrix} 0 & A^- \\ 0 & 0 \end{pmatrix}, \qquad \nu_+ = \begin{pmatrix} 0 & 0 \\ A^+ & 0 \end{pmatrix}. \tag{2.21}$$

are recognized as super-charges associated with the Hamiltonian (2.17), and the quantities

$$f_{\pm} = f(q) \pm \frac{1}{4} \frac{f'(q)}{\sqrt{f(q)}} \tag{2.22}$$

are interpreted as super-partners potentials. The example which follows has a twofold motivation i.e., an explicit application of the techniques we have developed and the use of nonstandard special functions emerging from the analysis of the problem under study. We consider the case of the quartic oscillator, i.e. $f(q) = \lambda q^4$, for which the ladder operators are

$$A^{\pm} = \frac{1}{\sqrt{2}}\left(\sqrt{\lambda} q^2 \mp ip\right), \tag{2.23}$$

with

$$\left[A^-, A^+\right] = 2\sqrt{\lambda}\, q. \tag{2.24}$$

The states can be defined as follows

$$\varphi_n = \frac{1}{\sqrt{n!}}\left(A^+\right)^n |0\rangle, \tag{2.25}$$

where, from the condition $A^-|0\rangle = 0$, for the vacuum it is possibile to deduce the following expression

$$|0\rangle \propto \exp\left(-\frac{\sqrt{\lambda}}{3} q^3\right), \qquad (q > 0). \tag{2.26}$$

The explicit form of the functions (2.25) can be obtained by using the identities (see Appendix)

$$\left(A^+\right)^n = \frac{1}{\sqrt{2^n}} \sum_{k=0}^{n} \binom{n}{k} (-1)^k H_{n-k}^{(3)}\left(\sqrt{\lambda} q^2, \sqrt{\lambda} q, \frac{\sqrt{\lambda}}{3}\right) \partial_q^k, \tag{2.27}$$

$$\partial_q^k e^{-\sqrt{\lambda} q^3/3} = H_k^{(3)}\left(\sqrt{\lambda} q^2, \sqrt{\lambda}\, q, \frac{\sqrt{\lambda}}{3}\right) e^{-\sqrt{\lambda} q^3/3}, \tag{2.28}$$

where $H_k^{(3)}$ are the third-order Hermite polynomials whose definition is given in Appendix. The use of the addition theorem

$$\sum_{k=0}^{n} \binom{n}{k} H_{n-k}^{(3)}(x_1, x_2, x_3) H_k^{(3)}(y_1, y_2, y_3) = H_n^{(3)}(x_1 + y_1, x_2 + y_2, x_3 + y_3), \tag{2.29}$$

and taking into account that $(-1)^k H_k^{(3)}(x, y, z) = H_k^{(3)}(-x, y, -z)$, finally yields

$$\varphi_n \propto \frac{1}{\sqrt{2^n n!}} H_n^{(3)}\left(2\sqrt{\lambda} q^2, 0, \frac{2}{3}\sqrt{\lambda}\right) e^{-\sqrt{\lambda} q^3/3}. \tag{2.30}$$

The functions φ_n are significantly different from the ordinary harmonic oscillator functions, and the discussion of their properties is out of the scope of this paper.

In the forthcoming section we will consider more physical examples yielding further elements supporting the physical reality of the mathematical devices discussed so far.

3. Physical Examples. We have mentioned that the quantized nature of the spin-like structure introduced in the previous sections reflects the quantum nature of q and p variables. The same happens with the so-called Landau states, emerging in the quantum analysis of the motion of a particle with electric charge e in a classical magnetic field of intensity B[1].

By identifying the z-axis of the reference frame with the direction of the field, the associated vector potential writes $\vec{A} = (0, Bx, 0)$. This choice is not unique but it ensures that the magnetic field is orthogonal to the plane of motion, i.e., $\vec{p} = (p_x, p_y, 0)$. The relativistic Hamiltonian operator ruling such a process can be written as[3]

$$H = c\sqrt{\left(\vec{p} - e\vec{A}\right)^2 + (mc)^2} \quad = c\sqrt{p_x^2 + p_y^2 - 2eBxp_y + (eBx)^2 + (mc)^2}\ . \tag{3.1}$$

Since $\left[p_y, H\right] = 0$ the operatior p_y can be replaced by its eigenvalue $\hbar k_y$ and the Hamiltonian can be rewritten as

$$H = c\sqrt{p_x^2 + m^2 w_c^2 X^2 + \left(\hbar k_y\right)^2 - m w_c^2 x_B^2 + (mc)^2}\ , \tag{3.2}$$

with

$$w_c = \frac{|e|B}{mc}, \qquad x_B = \frac{\hbar k_y}{m w_c}, \qquad X = x - x_B.$$

By introducing the operator

$$W = p_x \sigma_1 + m w_c X \sigma_2 + \sqrt{\left(\hbar k_y\right)^2 - m w_c^2 x_B^2 + (mc)^2}\,\sigma_3 \tag{3.3}$$

it's easy to show that

$$H = c\sqrt{w^2 - m\hbar w_c \sigma_3}\ . \tag{3.4}$$

In the common experimental situations, the term proportional to σ_3 can be neglected, since, for values of the magnetic field around 0.1 T (or even larger) it is of the order of meV, while the term cW varies in the region of hundreds of keV. Under this approximation, the study of the relativistic Landau states reduces to a Jaynes-Cummings problem. The Hamiltonian (3.4) describes the dynamics of a

[3] In this section we restore the mass m of the particle and the constants h and e.

two-level system with a level spacing fixed by the strength of the magnetic field. The supersymmetric nature of the Jaynes-Cummings model has been discussed elsewhere [6] and we will not dwell on it. Here we note that a physical realization of such a system is given, for example, by a Free Electron Laser (FEL) source where a relativistic beam of electrons propagates inside an axial magnetic field [10]. The possibility of using such a point of view to consturuct a laser-like theory for the FEL-like devices has been partially considered in [11], and will be the topic of a future speculation.

Before closing this section we will reconsider a point partially touched in [4], and in previous sections, concerning the evolution of the spin-like system associated with the harmonic oscillator via the Dirac factorization method. To simplify the problem we will refer to the relativistic 1-dimensional Hamiltonian

$$H = c\sqrt{p^2 + (mc)^2}\ , \tag{3.5}$$

that, for example, can be factorized as follows

$$H = c(p\sigma_1 + mc\sigma_3)\,. \tag{3.6}$$

The physical meaning of the above Hamiltonian has been discussed in [4], where it has been stressed that is should not be confused with the Dirac Hamiltonian nor with the Pauli counterpart. Here, we will use it as a toy model to get a further support to the previous speculations.

The equations of mtion for the vector $\vec{\sigma}$ are

$$\frac{d}{dt}\vec{\sigma} = \vec{\Omega}\times\vec{\sigma} \qquad\qquad \vec{\Omega} = \frac{1}{\hbar}\left(pc, 0, mc^2\right) \tag{3.7}$$

This equation deseribes a purely quantum motion (with no classical counterpart) which should be understood as a kind of *zitterbewegung* [4,12] (a trembling mottion due to the interence between negatiove and positive states contained in the Hamiltonian (3.6)). Even though there is not any explicit presence of ladder operatiors, the Hamiltonian (3.6) contains a hidden two-level structure and a supersyammetric underlying algebra. In fact, by expressing the momentum in terms of the ladder operators, we get ($g = c/\sqrt{2\hbar}$)

$$H = mc^2\sigma_3 + i\hbar g\left(a^+ + a^-\right)\left(\sigma_+ - \sigma_-\right), \tag{3.8}$$

which writes as a Jaynes-Cummings Hamiltonian, but without the rotating wave approxi-mation assumption.

4. Concluding Remarks. As it is shown in Sec. 2, the method of factorization exhibits, in a fairly natural way, all the essential features of SUSY QM :i) the vacuum field energy is factorized out; it) two potentials with isospectral properties are recovered. But, as often happens, nothing is really new and the procedures leading to super-potentials traces back to methods known well before to the birth of quantum mechanics itself [13, 14][4]. In order to appreciate this point, let us consider the following second-order differential equation:

$$z''(x)+\mu_-(x)z(x)=0. \tag{4.1}$$

If the "potential" $\mu_-(x)$ is expressible as

$$\mu_-(x)=-\frac{1}{4}\phi^2(x)-\frac{1}{2}\phi'(x), \tag{4.2}$$

i.e., is solution of a Riccati eqation, the solution of Eq. (4.1) can written as

$$z(x)=\exp\left\{\frac{1}{2}\int^x d\xi\phi(\xi)\right\}u_-(x) \tag{4.3}$$

with the function $u_-(x)$ satisfying the differential equation

$$u''_-(x)+\phi(x)u'_-(x)=0. \tag{4.4}$$

Therefore, we get

$$z(x)=\exp\left\{\frac{1}{2}\int^x d\xi\phi(\xi)\right\}\int^x d\eta\exp\left\{-\int^\eta d\nu\phi(\nu)\right\}. \tag{4.5}$$

Furthermore once $\phi(x)$ is fixed, we can define a second "potential"

$$\mu_+(x)=-\frac{1}{4}\phi^2(x)+\frac{1}{2}\phi'(x) \tag{4.6}$$

(the super-partner of $u_-(x)$, according to the present terminology) which specifies the solutions of a second differential equation, obtained from the first by just replacing $\phi(x)$ with $-\phi(x)$.

The outlined procedure is essentially the Liouville method to reduce a second-order differential equation to its standard form[5]. However, it is remarkable that all the features concerning the supersymmetry are recoverd in a very natural way by just appliying the Dirac factorization method, which leads to the two-level Hamiltonian (2.17) with the role of the vacuum field factorized out from the very

[4] In particular, in [13] the problem has been treated by showing that a second-order differential equation with non-constant coefficients can be written in terms of bio-orthogonal partner solutions, a posteriori recognized as supper symmetric components.

[5] We remind that, given the differential equation $y''+a(x)y'(x)+b(x)y(x)=0$, the Liouville transformation $y(x)=\exp\left\{-1/2\int^x d\xi a(\xi)\right\}v(x)$ reducess the initial equation to $v''(x)+c(x)\ v(x)=0$ where $c(x)=b(x)-\frac{1}{4}\left[a^2(x)+2a'(x)\right]$

beginning.

Let us come back to the operator Υ defined in Eq. (2.18). Also in this case, we can write

$$\Upsilon = A^{-}\sigma_{+} + A^{+}\sigma_{-}, \tag{4.7}$$

and, therefore, it can be interpreted as the interaction between a two-level system and the bosonic field defined by ladder operators given in Eq. (2.19).

More in general, we can conjecture that the Jaynes-Cummings model can be generalized by the following interaction Hamiltonian

$$H_{JC} = \Upsilon - \frac{1}{2}w(q)\sigma_3, \quad \left(w(q) \propto \frac{f'(q)}{\sqrt{f(q)}} \right). \tag{4.8}$$

As a consequence of the more complicated commutation relations involved, the algebraic nature of this operator is less direct than in the case of the ordinary J-C operator. However, the study of quantum states ruled by the interaction Hamiltonian (4.8) can be done using the evolution operator associated with the Hamiltonian The properties of this operator can be discussed by using the methods developed in the past to treat evolution problems in quantum mechanics (see [15, 16]). For example, using the so-called symmetric-split decomposition method [17], the evolution operator can be approximated as follows

$$\psi(t) = \prod_{j=1}^{N} U_j \psi(0) \tag{4.9}$$

with

$$U_j = e^{-it_j\Upsilon/2} e^{-it_j w(q)\sigma_3} e^{-it_j,\Upsilon/2} + O\left(t_j^3\right). \tag{4.10}$$

The use of standard identities for exponential operator (see [16]), along with the iterated application of the evolution operator in Eq. (4.9), yields an efficient way of calculationg the evolution of these quantum states.

In this paper we have touched different topics, whose underlying leitmotif is the Dirac factorization. We have discussed the relativistic quantum mechanics, the zitterbewegung, the Jaynes-Cummings model, the relativistic Landau levels, and the Dirac oscillator (even though not explicitly mentioned). Most of these effects are difficult to be subjected to experimental investigation. For example, the electron zitterbewegung exhibits oscillations at very large frequencies ($\sim 10^{21}$ Hz) not accessible to currently available experimental techniques. Such drawback is overcome in the context referred to as quantum simulation [18], where the trembling motion may occur, for example, in crystalline solids when their band structure is represented by a two-band model reminiscent of the one-dimensional Dirac equation [19]. The explicit realization of the simullation [18, 20] has provided strong indications for the existence of such a genuine quantum behavior. Further experiments using quantum simulator techniques are suggested to test other quantum paradoxes like the Klein paradox [21].

We believe that the topics treated in this paper can be framed within the context of the quantum phenomenology which can be experimentally tested using quantum simulators. A more specific analysis in this direction will be developed elsewhere.

Appendix A. In section IV we have used the higher-order Hermite polynomials to express the ordered form of the operator $O_n = \left(\partial_x + P(x)\right)^n$. These polynomials are defined through the generating function

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} H_n^{(m)}(x_1, x_2, \ldots, x_m) = \exp\left\{\sum_{k=1}^{m} x_k t^k\right\} \tag{A1}$$

and can be construeted recursively according to formula

$$H_n^{(m)}(x_1, x_2, \ldots, x_m) = n! \sum_{k=0}^{[n/m]} \frac{x_m^k}{k!(n-mk)!} H_{n-mk}^{(m-1)}(x_1, x_2, \ldots, x_{m-1}). \tag{A2}$$

By using the generating function methods, we can define the following operator

$$E(x,t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} O_n = \exp\left\{t\left[\partial_x + P(x)\right]\right\} \tag{A3}$$

that satisfies the differential equation

$$\partial_t E(x,t) = \left[\partial_x + P(x)\right] E(x,t), \qquad E(x,0) = 1, \tag{A4}$$

whose solution is

$$E(x,t) = \exp\left\{t\partial_x + \int_0^t dt' P(x-t')\right\} = \exp\left\{\int_0^t dt' P(x+t-t')\right\} e^{t\partial_x}. \tag{A5}$$

In the case $P(x) = \alpha x^2$, performing the integral and using the Eq. (A1), we easily obtain

$$E(x,t) = \sum_{k=0}^{\infty} \frac{t^n}{k!} H_k^{(3)}\left(\alpha x^2, \alpha x, \frac{\alpha}{3}\right) e^{t\partial_x}, \tag{A6}$$

and, thus,

$$O_n = \sum_{k=0}^{n} \binom{n}{k} H_{n-k}^{(3)}\left(\alpha x^2, \alpha x, \frac{\alpha}{3}\right) \partial_x^k. \tag{A7}$$

As for the successive derivatives of the function e^{-x^3}, by applying the generating function method we get

$$\sum_{n=0}^{\infty}\frac{t^n}{n!}\partial_x^n e^{-x^3}=e^{t\partial_x}e^{-x^3}=e^{-(x+t)^3}=\sum_{n}^{\infty}\frac{t^n}{n!}H_n^{(3)}\left(-3x^2,-3x,-1\right)e^{-x^3} \tag{A8}$$

and, therefore, from the comparison of the same powers of *t,*the second identity in (2.28).

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CONVEXITY AND UNIVALENCE CONDITIONS FOR CERTAIN INTEGRAL OPERATORS

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ABSTRACT

The purpose of the present paper is to give sufficient conditions for certain integral operators to be convex and uvicalent in the open unit disk.

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1. Introduction. Let be the class of function of the form :

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n ,$$

which are analytic in the open unit disk

$$:= \{z : z \in \mathbb{C} \text{ and } |z| < 1\}.$$

We denote by S the subclass of consisting of functions $f \in$ which are univalent in U. A function $f \in$ is said to be starlike of order $\delta (0 \le \delta < 1)$ if it satisfies

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \delta (z \in \quad).$$

Also, we say that a function $f \in$ is said to be convex of order $\delta (0 \le \delta < 1)$ if it satisfies

$$\mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \delta (z \in \quad).$$

We denote by $*(\delta)$ and (δ) the classes of functions that starlike of order δ and convex of order δ in , respectively.

Silverman [11] investigated an expression involving the quotient of the analytic representation of convex and starlike functions. Precisely, for $0 < \mu \le 1$, he consid-ered the class

$$_{\mu} = \left\{ f \in \quad : \left| \frac{1 + zf''(z)/f'(z)}{zf'(z)/f(z)} - 1 \right| < \mu, z \in \quad \right\},$$

and proved that

$$_{\mu} \subset \quad ^{*}\left(2/\left(1+\sqrt{1+8\mu}\right)\right).$$

Moreover, Tuneski [12] proved that if $f = \quad_{\mu} (0 < \mu < 1)$, then

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| < \frac{\mu}{1-\mu} (z \in U). \tag{1.1}$$

For the parameter $\alpha_i, \beta_i \in \mathbb{C}$ for all $i \in \mathbb{N} = \{1,2,\ldots,n\}$ and $\gamma \in \mathbb{C}$ with $\mathrm{Re}\{\gamma\} > 0$, we define the integral operator $_{\gamma}^{\alpha_i,\beta_i} \quad ^{2n} \rightarrow \quad$ as follows :

$$_{\gamma}^{\alpha_i,\beta_i}(f_1,\ldots,f_{ni} g_1,\ldots,g_n)(z) := \left\{ \int_0^z \gamma t^{\gamma-1} \prod_{i=1}^{n} \left(f_i'(t)\right)^{\alpha_i} \left(\frac{g_i(t)}{t} \right)^{\beta_i} dt \right\}^{1/\gamma}. \tag{1.2}$$

We note that for some special real or complex parameters α_i, β_i and γ, the integral operator $_{\gamma}^{\alpha_i,\beta_i}(f_1,\ldots,f_n; g_1,\ldots,g_n)$ defined by (1.2) have been extensively studied by many authors [1-6, 8, 10]. In the present paper, we determined the order of convexity of the integral operator $_{\gamma}^{\alpha_i,\beta_i}(f_1,\ldots,f_{ni}; g_1,\ldots,g_n)$. Furthermore, we derive sufficient conditions for the operator $_{\gamma}^{\alpha_i,\beta_i}(f_1,\ldots,f_{ni}; g_1,\ldots,g_n)$ to be univalent in .

The following lemmas will be required in our present investigation.

Lemma 1.1 [7]. *Let* $\eta \in \mathbb{C}$ *with* $\mathrm{Re}\{\eta\} > 0$. *If* $f \in \quad$ *satisfies*

$$\frac{1 - |z|^{2\mathrm{Re}}\{\eta\}}{\mathrm{Re}\{\eta\}} \left| \frac{zf''(z)}{f'(z)} \right| \le 1 \qquad (z \in \quad),$$

then the integral operator

$$F_{\eta}(z) = \left\{ \eta \int_0^z t^{n-1} f'(t) dt \right\}^{1/\eta}$$

is in the class .

Lemma 1.2 [9]. *Let* $\zeta \in \mathbb{C}$ *with* $\mathrm{Re}\{\zeta\} > 0$ *and* $c \in \mathbb{C}$ *with* $|c| \le 1$ *and* $c \neq -1$. *If* $f \in$ *satisfies*

$$\left| c|z|^{2\zeta} + \left(1-|z|^{2\zeta}\right)\frac{zf''(z)}{\zeta f'(z)} \right| \le 1 \qquad (z \in \quad),$$

then the integral operator

$$F_{\zeta}(z) = \left\{ \zeta \int_0^z t^{\zeta-1} f'(t)\,dt \right\}^{1/\zeta}$$

is in the class .

2. Convexity of $I_1^{\alpha_i,\beta_i}\left(f_1,\ldots,f_{ni};g_1,\ldots,g_n\right)$

Firstly, we begin by investigating the order of convexity of the integral operator $\ {}_{\gamma}^{\alpha_i,\beta_i}\left(f_1,\ldots,f_{ni};g_1,\ldots,g_n\right)$ defined by (1.2) with $\gamma = 1$.

Theorem 2.1. *Let* $f_i, g_i \in \ _{\mu_i}$ *for all* $i \in \mathbb{N}$ *and satisfy*

$$0 < \sum_{i=1}^{n} \left(2|\alpha_i| + |\beta_i|\right)\frac{\mu_i}{1-\mu_i} \le 1 \quad \left(\alpha_i, \beta_i \in \mathbb{C}; 0 < \mu_i < 1; i \in \mathbb{N}\right).$$

Then the integral operator defined by

$$h(z) := I_{\gamma}^{\alpha_i,\beta_i}\left(f_1,\ldots,f_{ni};g_1,\ldots,g_n\right)(z) = \int_0^z \prod_{i=1}^{n} \left(f_i'(t)\right)^{\alpha_i} \left(\frac{g_i(t)}{t}\right)^{\beta_i} dt \tag{2.1}$$

is convex of order δ *given by*

$$\delta = 1 - \sum_{i=1}^{n} \left(2|\alpha_i| + |\beta_i|\right)\frac{\mu_i}{1-\mu_i}.$$

Proof. Form (2.1), it is easy to see that

$$h'(z) = \prod_{i=1}^{n} \left(f_i'(z)\right)^{\alpha_i} \left(\frac{g_i(z)}{z}\right)^{\beta_i} \ (z \in U) \tag{2.2}$$

and

$$h(0) = h'(0) - 1 = 0.$$

Differentiating both sides of (2.2) logarithmically, we obtain

$$\frac{zh''(z)}{h'(z)} = \sum_{i=1}^{n} \alpha_i \left(\frac{zf_i''(z)}{f_i'(z)}\right) + \sum_{i=1}^{n} \beta_i \left(\frac{zg_i'(z)}{g_i(z)} - 1\right). \tag{2.3}$$

Hence from the definition of $G_{\mu i}$ and (1.1), we have

$$\left|\frac{zh''(z)}{h'(z)}\right| \le \sum_{i=1}^{n} |\alpha_i| \left(\mu_i \left|\frac{zf'(z)}{f_i(z)}\right| + \left|\frac{zf_i'(z)}{f_i(z)} - 1\right| \right) + \sum_{i=1}^{n} |\beta_i| \left|\frac{zg_i'(z)}{g_i(z)} - 1\right|$$

$$\le \sum_{i=1}^{n}|\alpha_i|\left[\mu_i\left(\frac{\mu_i}{1-\mu_i}+1\right)+\frac{\mu_i}{1-\mu_i}\right]+\sum_{i=1}^{n}|\beta_i|\left(\frac{\mu_i}{1-\mu_i}\right)$$

$$=\sum_{i=1}^{n}|\alpha_i|\frac{2\mu_i}{1-\mu_i}+\sum_{i=1}^{n}|\beta_i|\frac{\mu_i}{1-\mu_i}$$

$$=\sum_{i=1}^{n}\left(2|\alpha_i|+|\beta_i|\right)\frac{\mu_i}{1-\mu_i}$$

$$=1-\delta. \tag{2.4}$$

Terefore, the function h is convex of order

$$\delta=1-\sum_{i=1}^{n}\left(2|\alpha_i|+|\beta_i|\right)\frac{\mu_i}{1-\mu_i},$$

which completes the proof of Theorem 2.1.

If we take $n=1,\alpha_1=\alpha,\beta_1=\beta$ and $f_1=f,g_1=g$ in Teorem 2.1, we obtain the following result.

Corollary 2.1. *Let* $f,g\in G_\mu$ *and satisfy*

$$0<\left(2|\alpha|+|\beta|\right)\frac{\mu}{1-\mu}\le 1 \qquad \left(\alpha,\beta\in\mathbb{C};0<\mu<1\right)$$

Then the integral operator $I_1^{\alpha,\beta}(f;g)$ *defined by* (1.2) *with* n=1 *is convex order of* δ *given by*

$$\delta=1-\left(2|\alpha|+|\beta|\right)\frac{\mu}{1-\mu}.$$

3. Univalency of $\ _{\gamma}^{\alpha_i,\beta_i}(f_1,...,f_n;g_1,...,g_n)$**.** Next, applying Lemma 1.1 and Lemma 1.2, we obtain some sufficient conditions for the integral operator $I_\gamma^{\alpha_i,\beta_i}(f_1,...,f_n;g_1,...,g_n)$ defined by (1.2) to be univalent in .

Theorem 3.1. Let $\gamma\in\mathbb{C}$ with

$$\operatorname{Re}\{\gamma\}\ge\sum_{i=1}^{n}\left(2|\alpha_i|+|\beta_i|\right)\frac{\mu_i}{1-\mu_i}\left(\alpha_i,\beta_i\in\mathbb{C};0<\mu_i<1;i\in\mathbb{N}\right). \tag{3.1}$$

If $f_i,g_i\in G_{\mu_i}$ *for all* $i\in\mathbb{N}$, *then the integral operator*

$I_\gamma^{\alpha_i,\beta_i}(f_1,...,f_n;g_1,...,g_n)$ defined by (1.2) is univalent in .

Proof. Let us define the function h as in Theorem 2.1. Then we have (2.3). By using the same method as in (2.4) and the assumption (3.1), we obtain

$$\frac{1-|z|^{2\operatorname{Re}\{\gamma\}}}{\operatorname{Re}\{\gamma\}}\left|\frac{zh''(z)}{h'(z)}\right| \le \frac{1-|z|^{2\operatorname{Re}\{\gamma\}}}{\operatorname{Re}\{\gamma\}}\sum_{i=1}^{n}\left(2|\alpha_i|+|\beta_i|\right)\frac{\mu_i}{1-\mu_i}$$

$$\le \frac{1}{\operatorname{Re}\{\gamma\}}\sum_{i=1}^{n}\left(2|\alpha_i|+|\beta_i|\right)\frac{\mu_i}{1-\mu_i}$$

$$\le 1 \cdot$$

Therefore, by applying Lemma 1.1 for the function h, we prove that the integral operator $I_{\gamma}^{\alpha_i,\beta_i}\left(f_1,\dots,f_n;g_1,\dots,g_n\right)$ is univalent in .

Theorem 3.2. Let $c\in\mathbb{C}$ be such that

$$|c| \le 1-\frac{1}{\operatorname{Re}\{\zeta\}}\sum_{i=1}^{n}\left(2|\alpha_i|+|\beta_i|\right)\frac{\mu_i}{1-\mu_i},\left(\alpha_i,\beta_i\in\mathbb{C};0<\mu_i<1;i\in\mathbb{N}\right), \tag{3.2}$$

where $\zeta\in\mathbb{C}$ *with*

$$\operatorname{Re}\{\zeta\} \ge \sum_{i=1}^{n}\left(2|\alpha_i|+|\beta_i|\right)\frac{\mu_i}{1-\mu_i},\left(\alpha_i,\beta_i\in\mathbb{C}\ ;\ 0<\mu_i<1;i\in\mathbb{N}\right). \tag{3.3}$$

If $f_i,g_i\in G_{\mu_i}$ *for all* $i\in\mathbb{N}$, *then integral oprator* ${}_{\gamma}^{\alpha_i,\beta_i}\left(f_1,\dots,f_n;g_1,\dots,g_n\right)$ *defined by (1.2) is univalent in* .

Proof. Let us define the function h as in Theorem 2.1. Then from (2.4), (3.2) and (3.3), we have

$$\left|c|z|^{2\zeta}+\left(1-|z|^{2\zeta}\right)\frac{zh'(z)}{\zeta h'(z)}\right| \le \left|c|z|^{2\zeta}+\frac{1-|z|^{2\zeta}}{\zeta}\sum_{i=1}^{n}\left(2|\alpha_i|+|\beta_i|\right)\frac{\mu_i}{1-\mu_i}\right|$$

$$\le |c|+\frac{1}{\operatorname{Re}\{\zeta\}}\sum_{i=1}^{n}\left(2|\alpha_i|+|\beta_i|\right)\frac{\mu_i}{1-\mu_i}$$

$$\le 1 \cdot$$

Therefore, by applying Lemma 1.2 for the function h, we conclude that the integral operator ${}_{\gamma}^{\alpha_i,\beta_i}\left(f_1,\dots,f_n;g_1,\dots,g_n\right)$ defined by (1.2) is univalent in .

If we let $n=1,\alpha_1=\alpha,\beta_1=\beta$ and $f_1=f,g_1=g$ in Theorem 3.1 and Theorem 3.2, respectively, we have the following two corollaries.

Corollary 3.2. Let $f,g\in G_{\mu}$ and $\gamma\in\mathbb{C}$ with

$$\operatorname{Re}\{\gamma\} \ge \sum_{i=1}^{n}\left(2|\alpha_i|+|\beta_i|\right)\frac{\mu}{1-\mu}\left(\alpha,\beta\in\mathbb{C};0<\mu<1\right)$$

Then the integral operator $_{\gamma}^{\alpha_i,\beta_i}$ defined by (1.2) with $n = 1$ is univalent in .

Corollary 3.3. *Let* $f, g \in G_\mu$ *and* $c \in \mathbb{C}$ *such that*

$$|c| \leq 1 - \frac{1}{\operatorname{Re}\{\gamma\}}(2|\alpha| + |\beta|)\frac{\mu}{1-\mu}(\alpha, \beta \in \mathbb{C}; 0 < \mu < 1)$$

where $\zeta \in \mathbb{C}$ *with*

$$\operatorname{Re}\{\gamma\} \geq (2|\alpha| + |\beta|)\frac{\mu}{1-\mu}(\alpha, \beta \in \mathbb{C}\ ;\ 0 < \mu < 1)$$

Then the integral operator $_{\gamma}^{\alpha_i,\beta_i}$ defined by (1.2) with $n = 1$ is univalent in .

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ON CLASSES OF HARMONIC UNIVALENT FUNCTIONS DEFINED BY FRACTIONAL DIFFERNTIAL OPERATOR

By

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ABSTRACT

In this paper classes of harmonic univalent functions are defined by using a linear fractional operator. Coefficient inequalities which lead to distortion bounds, extreme points and convex combination are obtained. Also, convolution condition and integral operator of such functions are considered. Finally, multipliers sequences are provided to transit through the classes.

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1. Introduction. Let U denote the open unit and S_H denote the class of functions which are complex-valued, harmonic, univalent, sense-preserving in U normalized by $f(0)=f_z(0)-1=0$. Each $f\in S_H$ can be expressed as $f=h+\overline{g}$, where h and g are analytic in U. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and sense-preserving in U is that $|h'(z)|>|g'(z)|$ in U (see [4]). Thus for $f=h+\overline{g}\in S_H$, we may write

(1) $$h(z)=z+\sum_{k=2}^{\infty}a_kz^k,\ g(z)=\sum_{k=1}^{\infty}b_kz^k, |b_1|<1.$$

Note that S_H reduces to S, the class of normalized analytic univalent functions if the co-analytic part of $f=h+\overline{g}$ is identically zero.

For $f=h+\overline{g}$ be given by (1), Al-Khal and Al-Kharsani [1] defined the linear operator $\Omega^\alpha\,(0\le\alpha<1)$ of f by

(2) $$\Omega^\alpha f(z)=\Omega^\alpha h(z)+\overline{\Omega^\alpha g(z)},$$

where

$$\Omega^\alpha h(z):=\Gamma(2-\lambda)z^\alpha D_z^\alpha h(z)$$

$$= \sum_{k=0}^{\infty} \frac{\Gamma(k+2)\Gamma(2-\alpha)}{\Gamma(k+2-\alpha)} a_{k+1} z^{k+1}, a_1 = 1;$$

$$\Omega^{\alpha} g(z) := \Gamma(2-\alpha) z^{\alpha} D_z^{\alpha} g(z)$$

$$= \sum_{k=0}^{\infty} \frac{\Gamma(k+2)\Gamma(2-\alpha)}{\Gamma(k+2-\alpha)} a_{k+1} z^{k+1}, b_1 = 0 ;$$

and D_z^{α} is the fractional derivative of order α [7] defined by

$$D_z^{\alpha}(F(z)) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz} \int_0^z \frac{F(\xi)}{(z-\xi)^{\alpha}} d\xi, \quad 0 \le \alpha < 1,$$

where F is an analytic function in a simply connected domain of the z-plane containing the origin and the multiplicity of $(z-\xi)^{-\alpha}$ is removed by requiring $\log(z-\xi)$ to be real when $z-\xi > 0$.

When $f(z)$ is an analytic function in (2), we get the linear operator Ω^{α} defined by Owa and Srivastava [8]. Also, Al-Oboudi and Al-Amoudi [2] defined the linear fractional differential operator $D_{\lambda}^{n,\alpha}\left(n \in N_0 = \{0,1,2,....,\}, 0 \le \alpha < 1, \lambda \ge 0\right)$ on the analytic function $f(z)$ as follows

$$(3) \qquad \begin{cases} D_{\lambda}^{0,0} f(z) = f(z) \\ D_{\lambda}^{1,0} f(z) = (1-\lambda)\Omega^{\alpha} f(z) + \lambda z\left(\Omega^{\alpha} f(z)\right)' = D_{\lambda}^{\alpha} f(z) \\ D_{\lambda}^{2,\alpha} f(z) = D_{\lambda}^{\alpha}\left(D_{\lambda}^{1,\alpha} f(z)\right), \\ \vdots \\ D_{\lambda}^{n,\alpha} f(z) = D_{\lambda}^{\alpha}\left(D_{\lambda}^{n-1,\alpha} f(z)\right). \end{cases}$$

Using definitions (2) and (3), we define the modified fractional differential operator $D_{\lambda}^{n,\alpha}$ for a function f in S_H by

$$(4) \qquad D_{\lambda}^{n,\alpha} f(z) = D_{\lambda}^{n,\alpha} h(z) + \overline{D_{\lambda}^{n,\alpha} g(z)}$$

where

$$D_{\lambda}^{n,\alpha} h(z) = z + \sum_{k=2}^{\infty} \psi_{k,n}(\alpha,\lambda)\left(1+(k-1)\lambda\right)^n a_k z^k,$$

$$D_{\lambda}^{n,\alpha} g(z) = \sum_{k=2}^{\infty} \psi_{k,n}(\alpha,\lambda)\left(1+(k-1)\lambda\right)^n b_k z^k, \qquad |b_1| < 1$$

and

$$\psi_{k,n}(\alpha,\lambda)=\left[\frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)}\right]^n.$$

When $\alpha=0$, we get the generalized Salgean operator [11]. If $f(z)$ is an analytic function and $\alpha=0$, we get Al-Oboudi differential operator [3]. When $\alpha=0$, and $\lambda=1$, we get Salgean differential operator [10] and when $n=1$ and $\lambda=0$, we get Owa-Srivastava fractional differential operator [8].

For $m\in N, n\in N_0, m>n, \lambda\geq 0, \mu\geq 0, \eta\geq 0, 0\leq\beta<1, 0\leq\alpha<1$, and $z\in U$, we get $F_H^\alpha(m,n;\lambda,\mu,\eta,\beta)$ be the class of harmonic function $f=h+\overline{g}$, where h and g are of the form (1) such that

$$\operatorname{Re}\left\{\frac{D_\eta\left(D_\lambda^{m,\alpha}f(z)\right)}{D_\mu\left(D_\lambda^{n,\alpha}f(z)\right)}\right\}>\beta,$$

where $D_\lambda^{m,\alpha}f$ and $D_\lambda^{n,\alpha}f$ are defined by (4).

We further denote by $\overline{F}_H^\alpha(m,n;\lambda,\mu,\eta,\beta)$ the subclass of $F_H^\alpha(m,n;\lambda,\mu,\eta,\beta)$ such that the function h and g in $f=h+\overline{g}$ are of the form

(5) $$h(z)=z-\sum_{k=2}^{\infty}|a_k|z^k,\quad g(z)=-\sum_{k=1}^{\infty}|b_k|z^k.$$

Remark. The class $\overline{F}_H^\alpha(m,n;\lambda,\mu,\eta,\beta)$ are generalized of the classes

(i) $\overline{F}_H^0(2,1;2,0,0,\beta)$ and $\overline{F}_H^0(n+1,1;1,0,0,\beta)$ studied by Jahangiri et al. [5].

(ii) $\overline{F}_H^0(1,0;1,0,0,\beta)$ studied by Jahangiri [6].

(iii) $\overline{F}_H^0(m,n;1,0,0,\beta)$ studied by Yalcin [12].

(iv) $\overline{F}_H^0(1,0;1,\mu,0,\beta)$ studied Öztrük et al. [9].

2. Main Results. First we give the sufficient coefficient bound for function in the class $F_H^\alpha(m,n;\lambda,\mu,\eta,\beta)$.

Theorem 1. *Let $f=h+\overline{g}$ where h and g are given by (1). If*

(6) $$\sum_{k=1}^{\infty}\frac{(1+(k-1)\eta)(1+(k-1)\lambda)^m\psi_{k,m}(\alpha,\lambda)-\beta(1+(k-1)\mu)(1+(k-1)\lambda)^m\psi_{k,m}(\alpha,\lambda)}{1-\beta}$$

$$\left(a_k|+|b_k|\right)\leq 2,$$

where $a_1 = 1, m \in N, n \in N_0,$ $m > n, \lambda \geq 0, \mu \geq 0, \eta \geq 0, 0 \leq \beta < 1, 0 \leq \alpha < 1$ *and* $\Psi_{k,n}(\alpha,\lambda) = \left[\frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)}\right]^n$, then f is a sense-preserving harmonic univalent function in U and $f \in F_H^{\alpha}(m,n;\lambda,\mu,\eta,\beta)$.

Proof. For $|z_1| \leq |z_2| < 1$, we have by (6)

$$|f(z_1) - f(z_2)| \geq |h(z_1) - h(z_2)| - |g(z_1) - g(z_2)|$$

$$\geq |z_1 - z_2| \left(1 - \sum_{k=2}^{\infty} k|a_k||z_2|^{k-1} - \sum_{k=1}^{\infty} k|b_k||z_2|^{k-1}\right)$$

$$> |z_1 - z_2| \left[1 - |z_2| \left(\sum_{k=2}^{\infty} k|a_k| - \sum_{k=1}^{\infty} k|b_k|\right)\right]$$

$$\geq |z_1 - z_2| \left[1 - \left(\sum_{k=2}^{\infty} k\left(|a_k| + |b_k|\right) + |b_1|\right)\right]$$

$$\geq |z_1 - z_2| \left[1 - \left(\sum_{k=2}^{\infty} \left[\left(1+(k-1)\eta\right)\left(1+(k-1)\lambda\right)^m \Psi_{k,m}(\alpha,\lambda)\right.\right.\right.$$

$$\left.\left.\left. -\beta\left(1+(k-1)\mu\right)\left(1+(k-1)\lambda\right)^n \Psi_{k,n}(\alpha,\lambda)\right]\left(|a_k| + |b_k|\right) + |b_1|\right)\right]$$

$$\geq |z_1 - z_2| \left[1 - \left(1 - \beta - |b_1| + |b_1|\right)\right]$$

$$= \beta|z_1 - z_2| \geq 0 \text{ by (6).}$$

Hence, f is univalent in U.
f is sense-preserving in U. This is because

$$|h'(z)| \geq 1 - \sum_{k=2}^{\infty} k|a_k||z|^{k-1}$$

$$> 1 - \sum_{k=2}^{\infty} \left[\left(1+(k-1)\eta\right)\left(1+(k-1)\lambda\right)^m \Psi_{k,m}(\alpha,\lambda)\right.$$

$$\left. -\beta\left(1+(k-1)\mu\right)\left(1+(k-1)\lambda\right)^n \Psi_{k,n}(\alpha,\lambda)\right]|a_k|$$

$$\geq \beta + \sum_{k=1}^{\infty} \left[\left(1+(k-1)\eta\right)\left(1+(k-1)\lambda\right)^m \Psi_{k,m}(\alpha,\lambda)\right.$$

$$\left. -\beta\left(1+(k-1)\mu\right)\left(1+(k-1)\lambda\right)^n \Psi_{k,n}(\alpha,\lambda)\right]|b_k|$$

$$> \sum_{k=1}^{\infty} k|b_k||z|^{k-1}$$

$$\geq |g'(z)|.$$

Using the fact that $\mathrm{Re}\,w \geq \beta$ if and only if $|1-\beta+w| \geq |1+\beta-w|$, it suffices to show that

$$\text{(7)} \qquad \left|(1-\beta)D_\mu\left(D_\lambda^{n,\alpha}f(z)\right)+D_\eta\left(D_\lambda^{m,\alpha}f(z)\right)\right| - \left|(1+\beta)D_\mu\left(D_\lambda^{n,\alpha}f(z)\right)-D_\eta\left(D_\lambda^{m,\alpha}f(z)\right)\right| \geq 0.$$

Substituting for $D_\lambda^{n,\alpha}f(z)$ and $D_\lambda^{m,\alpha}f(z)$ in (7) yields

$$\Bigg|(2-\beta)z+\sum_{k=2}^{\infty}\Big[\big(1+(k-1)\mu\big)\big(1+(k-1)\lambda\big)^n\,\Psi_{k,n}(\alpha,\lambda)(1-\beta)$$

$$+\big(1+(k-1)\eta\big)\big(1+(k-1)\lambda\big)^m\,\Psi_{k,m}(\alpha,\lambda)\Big]a_k z^k$$

$$+\sum_{k=1}^{\infty}\Big[\big(1+(k-1)\mu\big)\big(1+(k-1)\lambda\big)^n\Psi_{k,n}(\alpha,\lambda)(1-\beta)$$

$$+\big(1+(k-1)\eta\big)\big(1+(k-1)\lambda\big)^m\,\Psi_{k,m}(\alpha,\lambda)\Big]\overline{b_k z^k}\Bigg|$$

$$-\Bigg|\beta z+\sum_{k=2}^{\infty}\Big[\big(1+(k-1)\eta\big)\big(1+(k-1)\lambda\big)^m\,\Psi_{k,m}(\alpha,\lambda)$$

$$-\big(1+(k-1)\mu\big)\big(1+(k-1)\lambda\big)^n\,\Psi_{k,m}(\alpha,\lambda)(1+\beta)\Big]a_k z^k$$

$$-\sum_{k=1}^{\infty}\Big[\big(1+(k-1)\eta\big)\big(1+(k-1)\lambda\big)^m\Psi_{k,m}(\alpha,\lambda)-\big(1+(k-1)\mu\big)\big(1+(k-1)\lambda\big)^n$$

$$\Psi_{k,n}(\alpha,\lambda)(1+\beta)\overline{b_k z^k}\Bigg|$$

$$\geq 2(1-\beta)|z|-2\sum_{k=2}^{\infty}\Big[\big(1+(k-1)\eta\big)\big(1+(k-1)\lambda\big)^m\,\Psi_{k,m}(\alpha,\lambda)$$

$$-\beta\big(1+(k-1)\mu\big)\big(1+(k-1)\lambda\big)^n\,\Psi_{k,n}(\alpha,\lambda)\Big]|a_k||z|^k$$

$$-2\sum_{k=1}^{\infty}\Big[\big(1+(k-1)\eta\big)\big(1+(k-1)\lambda\big)^m\Psi_{k,m}(\alpha,\lambda)$$

$$-\beta\big(1+(k-1)\mu\big)\big(1+(k-1)\lambda\big)^n\,\Psi_{k,n}(\alpha,\lambda)\Big]|b_k||z|^k$$

$$=2(1-\beta)|z|\left\{1-\sum_{k=2}^{\infty}\frac{(1+(k-1)\eta)(1+(k-1)\lambda)^{n}\Psi_{k,m}(\alpha,\lambda)-\beta(1+(k-1)\mu)(1+(k-1)\lambda)^{n}\Psi_{k,n}(\alpha,\lambda)}{1-\beta}|a_k||z|^{k-1}\right.$$

$$\left.-\sum_{k=1}^{\infty}\frac{(1+(k-1)\eta)(1+(k-1)\lambda)^{m}\Psi_{k,m}(\alpha,\lambda)-\beta(1+(k-1)\mu)(1+(k-1)\lambda)^{n}\Psi_{k,n}(\alpha,\lambda)}{1-\beta}|b_k||z|^{k-1}\right\}$$

$$>2(1-\beta)\left\{1-\left(\sum_{k=2}^{\infty}\frac{(1+(k-1)\eta)(1+(k-1)\lambda)^{m}\Psi_{k,m}(\alpha,\lambda)-\beta(1+(k-1)\mu)(1+(k-1)\lambda)^{n}\Psi_{k,n}(\alpha,\lambda)}{1-\beta}|a_k|\right.\right.$$

$$\left.\left.+\sum_{k=1}^{\infty}\frac{(1+(k-1)\eta)(1+(k-1)\lambda)^{m}\Psi_{k,m}(\alpha,\lambda)-\beta(1+(k-1)\mu)(1+(k+1)\lambda)^{n}\Psi_{k,n}(\alpha,\lambda)}{1-\beta}|b_k|\right)\right\}$$

This last expression is non-negative by (6) and so the proof is complete.

The harmonic univalent functions

(8) $f(z)=$

$$z+\sum_{k=2}^{\infty}\frac{1-\beta}{(1+(k-1)\eta)(1+(k-1)\lambda)^{m}\Psi_{k,m}(\alpha,\lambda)-\beta(1+(k-1)\mu)(1+(k-1)\lambda)^{n}\Psi_{k,n}(\alpha,\lambda)}x_kz^k$$

$$+\sum_{k=1}^{\infty}\frac{1-\beta}{(1+(k-1)\eta)(1+(k-1)\lambda)^{m}\Psi_{k,m}(\alpha,\lambda)-\beta(1+(k-1)\mu)(1+(k-1)\lambda)^{n}\Psi_{k,n}(\alpha,\lambda)}\overline{y_kz^k}$$

where $m\in N,n\in N_0,m>n$ and $\sum_{k=2}^{\infty}|x_k|+\sum_{k=1}^{\infty}|y_k|=1$ shows that the coefficient bound given by (6) is sharp. The functions of the form (8) are in $F_H^{\alpha}(m,n;\lambda,\mu,\eta,\beta)$ because

$$\sum_{k=1}^{\infty}\frac{(1+(k-1)\eta)(1+(k-1)\lambda)^{m}\Psi_{k,m}(\alpha,\lambda)-\beta(1+(k-1)\mu)(1+(k-1)\lambda)^{n}\Psi_{k,n}(\alpha,\lambda)}{1-\beta}(|a_k|+|b_k|)$$

$$=1+\sum_{k=2}^{\infty}|x_k|+\sum_{k=1}^{\infty}|y_k|=2.$$

Our next theorem shows that the condition (6) is also necessary for $f=h+\overline{g}$ given by (5).

Theorem 2. *Let* $f=h+\overline{g}$ *be given by (5). Then* $f\in\overline{F}_H^{\alpha}(m,n;\lambda,\mu,\eta,\beta)$ *if and only if*

(9) $$\sum_{k=1}^{\infty}\Big[(1+(k-1)\eta)(1+(k-1)\lambda)^{m}\Psi_{k,m}(\alpha,\lambda)-\beta(1+(k-1)\mu)(1+(k-1)\lambda)^{n}$$

$\Psi_{k,n}(\alpha,\lambda)\big]\left(|a_k|+|b_k|\right)\leq 2(1-\beta).$

Proof. Since $\bar{F}_H^{\alpha}(m,n;\lambda,\mu,\eta,\beta)\subset F_H^{\alpha}(m,n;\lambda,\mu,\eta,\beta)$, we only need to prove the "only if" part of the theorem. To this end, for functions $f\in\bar{F}_H^{\alpha}(m,n;\lambda,\mu,\eta,\beta)$, we notice that the condition $\operatorname{Re}\left\{\left(D_\eta\left(D_\lambda^{m,\alpha}f(z)\right)\right)/\left(D_\mu\left(D_\lambda^{n,\alpha}f(z)\right)\right)\right\}>\beta$ is equivalent to

$$(10)\quad \operatorname{Re}\Bigg\{\Bigg[(1-\beta)z-\sum_{k=2}^{\infty}\Big(\big(1+(k-1)\eta\big)\big(1+(k-1)\lambda\big)^m\Psi_{k,m}(\alpha,\lambda)$$
$$-\beta\big(1+(k-1)\mu\big)\big(1+(k-1)\lambda\big)^n\Psi_{k,n}(\alpha,\lambda)\Big)a_kz^k$$
$$-\sum_{k=1}^{\infty}\Big(\big(1+(k-1)\eta\big)\big(1+(k-1)\lambda\big)^m\Psi_{k,m}(\alpha,\lambda)-\beta\big(1+(k-1)\mu\big)\big(1+(k-1)\lambda\big)^n$$
$$\Psi_{k,n}(\alpha,\lambda)\Big)b_k\bar{z}^k\Bigg]\Bigg/\Bigg[z-\sum_{k=2}^{\infty}\big(1+(k-1)\mu\big)\big(1+(k-1)\lambda\big)^n\Psi_{k,n}(\alpha,\lambda)a_kz^k$$
$$-\sum_{k=1}^{\infty}\big(1+(k-1)\mu\big)\big(1+(k-1)\lambda\big)^n\Psi_{k,n}(\alpha,\lambda)b_k\bar{z}^k\Bigg]\Bigg\}\geq 0.$$

The above condition must hold for all values in U. Upon choosing the values of z on the positive real axis where $0\leq z=r<1$, we must have

$$(11)\quad \Bigg\{\Bigg[(1-\beta)-\sum_{k=1}^{\infty}\Big(\big(1+(k-1)\eta\big)\big(1+(k-1)\lambda\big)^m\Psi_{k,m}(\alpha,\lambda)-\beta\big(1+(k-1)\mu\big)\big(1+(k-1)\lambda\big)^n$$
$$\Psi_{k,n}(\alpha,\lambda)\Big)a_kr^{k-1}-\sum_{k=1}^{\infty}\Big(\big(1+(k-1)\eta\big)\big(1+(k-1)\lambda\big)^m\Psi_{k,m}(\alpha,\lambda)-\beta\big(1+(k-1)\mu\big)$$
$$\big(1+(k-1)\lambda\big)^n\Psi_{k,n}(\alpha,\lambda)\Big)b_kr^{k-1}\Bigg]\Bigg/\Bigg[1-\sum_{k=2}^{\infty}\big(1+(k-1)\mu\big)\big(1+(k-1)\lambda\big)^n\Psi_{k,n}(\alpha,\lambda)a_kr^{k-1}$$
$$-\sum_{k=1}^{\infty}\big(1+(k-1)\mu\big)\big(1+(k-1)\lambda\big)^n\Psi_{k,n}(\alpha,\lambda)b_kr^{k-1}\Bigg]\Bigg\}\geq 0.$$

If condition (9) does not hold then the numerator in (11) is negative for r sufficiently close to 1. Hence there exist $z_0=r_0$ in (0,1) for which the quotient of (11) is negative. This contradicts the fact that $f\in\bar{F}_H^{\alpha}(m,n;\lambda,\mu,\eta,\beta)$ and so the proof is complete.

Theorem 3. *If* $f\in\bar{F}_H^{\alpha}(m,n;\lambda,\mu,\eta,\beta)$, *then for* $|z|=r<1$, *we have*

$$|f(z)| \le (1+|b_1|)r + \frac{1}{(1+\lambda)^n}\left(\frac{1-\beta}{(1+\eta)(1+\lambda)^{m-n}\Psi_{2,m}(\alpha,\lambda)-\beta(1+\mu)\Psi_{2,n}(\alpha,\lambda)}\right.$$

$$\left.-\frac{1-\beta}{(1+\eta)(1+\lambda)^{m-n}\Psi_{2,m}(\alpha,\lambda)-\beta(1+\mu)\Psi_{2,n}(\alpha,\lambda)}|b_1|\right)r^2$$

and

$$|f(z)| \ge (1-|b_1|)r - \frac{1}{(1+\lambda)^n}\left(\frac{1-\beta}{(1+\eta)(1+\lambda)^{m-n}\Psi_{2,m}(\alpha,\lambda)-\beta(1+\mu)\Psi_{2,n}(\alpha,\lambda)}\right.$$

$$\left.-\frac{1-\beta}{(1+\eta)(1+\lambda)^{m-n}\Psi_{2,m}(\alpha,\lambda)-\beta(1+\mu)\Psi_{2,n}(\alpha,\lambda)}|b_1|\right)r^2.$$

The bounds given in Theorem 3 for functions $f = h + \overline{g}$ of the form (5) also hold for functions of the form (1) if the coefficient condition (6) is satisfied.

The functions

$$f(z) = z - |b_1|\overline{z} - \frac{1}{(1+\lambda)^n}\left(\frac{1-\beta}{(1+\eta)(1+\lambda)^{m-n}\Psi_{2,m}(\alpha,\lambda)-\beta(1+\mu)\Psi_{2,n}(\alpha,\lambda)}\right.$$

$$\left.-\frac{1-\beta}{(1+\eta)(1+\lambda)^{m-n}\Psi_{2,m}(\alpha,\lambda)-\beta(1+\mu)\Psi_{2,n}(\alpha,\lambda)}|b_1|\right)\overline{z}^2$$

and

$$f(z) = z - |b_1|\overline{z} - \frac{1}{(1+\lambda)^n}\left(\frac{1-\beta}{(1+\eta)(1+\lambda)^{m-n}\Psi_{2,m}(\alpha,\lambda)-\beta(1+\mu)\Psi_{2,n}(\alpha,\lambda)}\right.$$

$$\left.-\frac{1-\beta}{(1+\eta)(1+\lambda)^{m-n}\Psi_{2,m}(\alpha,\lambda)-\beta(1+\mu)\Psi_{2,n}(\alpha,\lambda)}|b_1|\right)z^2$$

for $|b_1| \le 1$ show that the bounds given in Theorem 3 are sharp.

The following result follows from the left hand inequality in Teorem 3.

Corollary 4. *If* $f \in \overline{F}_H^{\alpha}(m,n;\lambda,\mu,\eta,\beta)$, *then*

$$\left\{w : |w| < \frac{(1+\eta)(1+\lambda)^m\Psi_{2,m}(\alpha,\lambda)-\beta(1+\mu)(1+\lambda)^n\Psi_{2,n}(\alpha,\lambda)-1+\beta}{(1+\eta)(1+\lambda)^m\Psi_{2,m}(\alpha,\lambda)-\beta(1+\mu)(1+\lambda)^n\Psi_{2,n}(\alpha,\lambda)}\right.$$

$$-\frac{(1+\eta)(1+\lambda)^m\Psi_{2,m}(\alpha,\lambda)-\beta(1+\mu)(1+\lambda)^n\Psi_{2,n}(\alpha,\lambda)-1+\beta}{(1+\eta)(1+\lambda)^m\Psi_{2,m}(\alpha,\lambda)-\beta(1+\mu)(1+\lambda)^n\Psi_{2,n}(\alpha,\lambda)}|b_1|\Bigg\}\subset f(U).$$

Next we determine the extreme points of the closed convex hulls of $\overline{F}_H^{\alpha}(m,n;\lambda,\mu,\eta,\beta)$ denoted by clco $\overline{F}_H^{\alpha}(m,n;\lambda,\mu,\eta,\beta)$.

Theorem 5. *Let f be given by (5). Then* $f\in\overline{F}_H^{\alpha}(m,n;\lambda,\mu,\eta,\beta)$ *if and only if*

(12) $$f(z)=\sum_{k=1}^{\infty}\left(x_k h_k(z)+y_k g_k(z)\right),$$

where

$h_1(z)=z, h_k(z)$

$$=z-\frac{1-\beta}{(1+(k-1)\eta)(1+(k-1)\lambda)^m\Psi_{k,m}(\alpha,\lambda)-\beta(1+(k-1)\mu)(1+(k-1)\lambda)^n\Psi_{k,n}(\alpha,\lambda)}z^k,$$

$k=2,3,..$

$g_k(z)$

$$=z-\frac{1-\beta}{(1+(k-1)\eta)(1+(k-1)\lambda)^m\Psi_{k,m}(\alpha,\lambda)-\beta(1+(k-1)\mu)(1+(k-1)\lambda)^n\Psi_{k,n}(\alpha,\lambda)}\overline{z}^k,$$

$k=1,2,3,...$

and

$$\sum_{k=1}^{\infty}(x_k+y_k)=1,\ \ x_k\ge 0,\ \ y_k\ge 0.$$

In particular, the extreme points of $\overline{F}_H^{\alpha}(m,n;\lambda,\mu,\eta,\beta)$ *are* $\{h_k\}$ *and* $\{g_k\}$.

For our next theorem, we need to define the convolution of two harmonic functions of the form

$$f(z)=z-\sum_{k=2}^{\infty}|a_k|z^k-\sum_{k=1}^{\infty}|b_k|\overline{z}^k \text{ and } g(z)=z-\sum_{k=2}^{\infty}|A_k|z^k-\sum_{k=1}^{\infty}|B_k|\overline{z}^k$$

as

(13) $$(f*F)(z)=f(z)*F(z)$$

$$=z-\sum_{k=2}^{\infty}|a_k||A_k|z^k-\sum_{k=1}^{\infty}|b_k||B_k|\overline{z}^k$$

to prove that the class $\overline{F}_H^{\alpha}(m,n;\lambda,\mu,\eta,\beta)$ is closed under convolution.

Theorem 6. *For* $0 \le \mu_1 \le \mu_2, 0 \le \eta_1 \le \eta_2,\ 0 \le \beta_1 \le \beta_2 < 1$. *Let* $f \in \bar{F}_H^\alpha(m,n;\lambda,\mu_2,\eta_2,\beta_2)$, $F \in \bar{F}_H^\alpha(m,n;\lambda,\mu_1,\eta_1,\beta_1)$ *Then the convolution* $f * F \in \bar{F}_H^\alpha(m,n;\lambda,\mu_2,\eta_2,\beta_2) \subset \bar{F}_H^\alpha(m,n;\lambda,\mu_1,\eta_1,\beta_1)$.

Theorem 7. *The Class* $\bar{F}_H^\alpha(m,n;\lambda,\mu,\eta,\beta)$ *is closed under convex combination.*

In the next theorem we obtain the radii of convexity for the class $\bar{F}_H^\alpha(m,n;\lambda,\mu,\eta,\beta)$.

Theorem 8. *If* $f \in \bar{F}_H^\alpha(m,n;\lambda,\mu,\eta,\beta)$ *then f is convex in the disc*

$$|z| < \min_k \left[\frac{1-\beta-|b_1|}{k}\right]^{1/(k-1)} \qquad k = 2,3,\ldots,|b_1| < 1-\beta.$$

Proof. Let $f \in \bar{F}_H^\alpha(m,n;\lambda,\mu,\eta,\beta)$ and let r be fixed such that $0 < r < 1$, then $r^{-1}f(rz) \in \bar{F}_H^\alpha(m,n;\lambda,\mu,\eta,\beta)$ and we have

$$\sum_{k=2}^{\infty} k^2\left(|a_k| + |b|_k\right) r^{k-1} = \sum_{k=2}^{\infty} k\left(|a_k| + |b_k|\right)\left(kr^{k-1}\right)$$

$$\le \sum_{k=2}^{\infty}\Big[\left(1+(k-1)\eta\right)\left(1+(k-1)\lambda\right)^m \Psi_{k,m}(\alpha,\lambda) - \beta\left(1+(k-1)\mu\right)\left(1+(k-1)\lambda\right)^n$$

$$\Psi_{k,n}(\alpha,k)\Big]\left(|a_k| + |b_k|\right)\left(kr^{k-1}\right)$$

$$\le 1-\beta-|b_1|$$

provided

$$\left\{kr^{k-1}\right\} \le 1-\beta-|b_1|,$$

which is true if

$$r \le \min_k \left[\frac{1-\beta-|b_1|}{k}\right]^{1/(k-1)}, \quad k = 2,3,\ldots$$

Theorem 9. *For* $c > -1, f \in \bar{F}_H^\alpha(m,n;\lambda,\mu,\eta,\beta)$. *Let*

$$F(z) = I(f(z)) = I(h) + \overline{I(g)},$$

where

$$I(h)=\frac{c+1}{z}\int_0^z t^{c-1}h(t)\,dt, I(g)=\frac{c+1}{z}\int_0^z t^{c-1}g(t)\,dt.$$

Then $F(z)\in\overline{F}_H^{\alpha}(m,n;\lambda,\mu,\eta,\beta)$.

Proof. From the representation of $F(z)$, it follows that

$$F(z)=\frac{c+1}{z^c}\left(\int_0^z t^{c-1}\left(t-\sum_{k=2}^{\infty}|a_k|t^k\right)dt+\overline{\int_0^z t^{c-1}\left(\sum_{k=1}^{\infty}|b_k|t^k\right)dt}\right)$$

$$=z-\sum_{k=2}^{\infty}A_k z^k+\sum_{k=1}B_k\overline{z}_k,$$

where $A_k=\frac{c+1}{c+k}|a_k|, B_k=\frac{c+1}{c+k}|b_k|$. Therefore,

$$\sum_{k=1}^{\infty}\Big[\big(1+(k-1)\eta\big)\big(1+(k-1)\lambda\big)^m\Psi_{k,m}(\alpha,\lambda)-\beta\big(1+(k-1)\mu\big)\big(1+(k-1)\lambda\big)^n$$

$$\Psi_{k,n}(\alpha,\lambda)\Big]\big(|A_k|+|B_k|\big)$$

$$=\sum_{k=1}^{\infty}\Big[\big(1+(k-1)\eta\big)\big((1+k-1)\lambda\big)^m\Psi_{k,m}(\alpha,\lambda)-\beta\big(1+(k-1)\mu\big)\big(1+(k-1)\lambda\big)^n$$

$$\Psi_{k,n}(\alpha,\lambda)\Big]\left(\frac{c+1}{c+k}\right)\big(|a_k|+|b_k|\big)$$

$$\le\sum_{k=1}^{\infty}\Big[\big(1+(k-1)\eta\big)\big(1+(k-1)\lambda\big)^m\Psi_{k,m}(\alpha,\lambda)-\beta\big(1+(k-1)\mu\big)\big(1+(k-1)\lambda\big)^n$$

$$\Psi_{k,n}(\alpha,\lambda)\Big]\big(|a_k|+|b_k|\big)$$

$$\le 2(1-\beta).$$

Since $f\in\overline{F}_H^{\alpha}(m,n;\lambda,\mu,\eta,\beta)$, therefore by Theorem 2, $F\in\overline{F}_H^{\alpha}(m,n;\lambda,\mu,\eta,\beta)$.

Finally, we provide multiplies sequences to transit through the classes.

Theorem 10. *Let* $f=h+\overline{g}$, *where h and g are given by (5). Then there exist multipliers sequences*

$$\{c_k\}=\{d_k\}=\frac{1}{\big(1+(k-1)\lambda\big)^r}\left[\frac{\Gamma(k+1-\alpha)}{\Gamma(k+1)\Gamma(2-\alpha)}\right]^r,\lambda\ge 0,r\in N,0\le\alpha<1$$

such that

$$\sum_{k=2}^{\infty}c_k|a_k|+\sum_{k=1}^{\infty}d_k|b_k|\le 1-\beta, d_1|b_1|<1.$$

The multipliers $\{c_k\}$*and* $\{d_k\}$ *provide a transition from the class* $\bar{F}_H^{\alpha}(m+r, n+r, \lambda, \mu, \eta, \beta)$ *to the class* $\bar{F}_H^{\alpha}(m, n; \lambda, \mu, \eta, \beta)$.

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SOME RESULTS ON HOMOGENEOUS GENERALIZED HYPERGEOMETRIC FUNCTION AND $\bar{H}$-FUNCTION

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ABSTRACT

The aim of this paper is to obtain certain relations between new homogeneous generalized hypergeometric function and various well known polynomials and functions defined by Mellin-Barnes contour integrals. These relations are very general in nature and consequently contain a large number of new and known relations as special cases.

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1. Introduction. The $\bar{H}$-function defined by Inayat-Hussain [4,5] as

$$\bar{H}_{P,Q}^{M,N}[z]=\bar{H}_{P,Q}^{M,N}\left[z\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right.\right]=\frac{1}{2\pi\omega}\int_{-i\infty}^{+i\infty}\phi(\xi)z^{\xi}d\xi \quad , \qquad (1.1)$$

where
$$\phi(\xi)=\frac{\prod_{j=1}^{M}\Gamma(b_j-\beta_j\xi)\prod_{j=1}^{N}\left\{\Gamma(1-a_j+\alpha_j\xi)\right\}^{A_j}}{\prod_{j=M+1}^{Q}\left\{\Gamma(1-b_j+\beta_j\xi)\right\}^{B_j}\prod_{j=N+1}^{P}\Gamma(a_j-\alpha_j\xi)} \quad . \qquad (1.2)$$

Here z may be real or complex but is not equal to zero and an empty product is interpreted as unity; P,Q,M and N are integers such that $1\le M\le Q$, $0\le N\le P$, $\alpha_j(j=1,...,P),\beta_j(j=1,...,Q)$ are complex numbers. The exponents $A_j(j=1,...,n)$ and

$B_j(j = M+1,...,Q)$ can take non-integer values, when these exponents take integers values, the $\bar{H}$-function reduces to familier H-function due to Fox [3]. Also due to Inayat-Hussain [5] as well as Rathi [8], it follows that

$$\bar{H}_{P,Q}^{M,N}[z] = o\left(|z|^f\right) \text{ for small } z,$$

$$\text{where } f = \underset{1\le j\le N}{Min.}\left\{\mathrm{Re}\left(b_j/\beta_j\right)\right\}. \tag{1.3}$$

Again due to Inayat-Hussain [5], we have

$$\bar{H}_{P,Q}^{M,N}[z] = o\left(|z|^g\right) \text{ for large } z,$$

$$\text{where } g = \underset{1\le j\le N}{Max.}\left[\mathrm{Re}\left\{A_j\left(a_j - 1\right)/a_j\right\}\right]. \tag{1.4}$$

The following sufficient conditions for the absolute convergence of the defining integral for $\bar{H}$-function given by equation (1.6) have been given by Buschman and Srivastava [2].

$$\theta = \sum_{j=1}^{M}|\beta_j| + \sum_{j=1}^{N}|A_j\alpha_j| - \sum_{j=M+1}^{Q}|B_j\beta_j| - \sum_{j=N+1}^{P}|\alpha_j| > 0, \tag{1.5}$$

and $|\arg z| < \theta\pi/2$. (1.6)

where θ is given by (1.5).

The homogeneous generalized hypergeometric function ${}_pB_q[\alpha_r;\beta_t;z]$ was defined by Basister A.W. [1] in 1967 as

$${}_pB_q[\alpha_r;\beta_t;z] = \sum_{n=0}^{\infty}\frac{(\alpha_1)_n\ldots(\alpha_p)_n}{(\beta_1)_n\ldots(\beta_q)_n}\Omega(\alpha_{p+n},\beta_{q+n},0)\frac{z^n}{n!} \quad , \tag{1.7}$$

where Ω is the generalized modified struve function (Basister [1],1967, p. 96) defined as

$$\Omega(a,c,z) = (2)^{-1-c}(\pi)^{-2}e^{-i\pi c}\Gamma(1-a)\Gamma(c)\Gamma(1+a-c)\times$$

$$e^{(1/2)z}\left[(1-e^{2\pi ia})\int_0^{(1+)}e^{(1/2)zu}(1+u)^{a-1}(1-u)^{c-a-1}du\right.$$

$$\left.+\left\{1-e^{2\pi ia(c-a)}\right\}\int_0^{(-1+)}e^{(1/2)zu}(1+u)^{a-1}(1-u)^{c-a-1}du\right]. \tag{1.8}$$

If $R(\beta_q) > R(\alpha_p)$, the series (1.7) converges for all z, if $p \leq q$, converges for $|z| < 1$, if $p=q+1$, diverges for all non zero z, if $p>q+1$. We shall take $2 \leq p \leq q+1$.

2. Required Results.

(i) $\int_0^1 C_n^\lambda(1-2y^2)(1-y^2)^{\lambda-1/2} y^{2\lambda+2r+2u} dy$

$$= \frac{\sqrt{\pi}}{2^{2\lambda}\Gamma(\lambda)} \frac{\Gamma(n+2\lambda)(-1)^n}{n!} \frac{\Gamma(u+r+1)\Gamma(u+r+\lambda+1/2)}{\Gamma(u+r-n+1)\Gamma(u+r+n+2\lambda+1)}, \tag{2.1.1}$$

where $\lambda + r + u > -1/2$ and $C_n^\lambda(x)$ is an Ultraspherical polynomial defined in [7] as

$C_n^\lambda(x) = \dfrac{(2\upsilon)_n p_n^{(\upsilon-1/2,\upsilon-1/2)}(x)}{(\upsilon+1/2)_n}$, where $p_n^{(\alpha,\alpha)}$ is the well known Jacobi polynomial also discussed in [7].

(ii) The following recurrence relations will be used in our investigations given in [7].

$$(n+1)C_{n+1}^\lambda(x) = 2(n+\lambda)xC_n^\lambda(x) - (n+2\lambda-1)C_{n-1}^\lambda(x) \;, \tag{2.2.1}$$

$$2\lambda(1-x^2)C_{n-1}^{\lambda+1}(x) = (n+2\lambda-1)C_{n-1}^\lambda(x) - nxC_n^\lambda(x)\,. \tag{2.2.2}$$

(iii) The following recurrence relations will be also used in our investigations given in [7].

$$\frac{1}{2}(2+\alpha+\beta+2n)(x+1)P_n^{(\alpha,\beta+1)}(x) = (n+1)P_{n+1}^{(\alpha,\beta)}(x) + (1+\beta+n)P_n^{(\alpha,\beta)}(x)\,, \tag{2.3.1}$$

$$(\alpha+\beta+2n)P_n^{(\alpha,\beta-1)}(x) = (\alpha+\beta+n)P_n^{(\alpha,\beta)}(x) + (\alpha+n)P_{n-1}^{(\alpha,\beta)}(x)\,, \tag{2.3.2}$$

$$(x+1)P_n^{(\alpha,\beta+1)}(x) + (1-x)P_n^{(\alpha+1,\beta)}(x) = 2P_n^{(\alpha,\beta)}(x) \;, \tag{2.3.3}$$

(iv) $$\int_0^1 p_n^{\alpha,\beta}(2y^2-1)(1-y^2)^\alpha y^{2\sigma+1} dy = \frac{1}{2}\frac{\Gamma(\sigma+1)\Gamma(\alpha+n+1)\Gamma(\sigma-\beta+1)}{\Gamma(\sigma-\beta-n+1)\Gamma(\alpha+\sigma+n+2)}\;, \tag{2.4.1}$$

where $P_n^{\alpha,\beta}(x)$ is Jacobi polynomial and $\{(2\sigma+1) > -1\}$.

Proof. By putting $x = 1-2y^2$ in the result of Gradshteyn and Ryzhik ([4], 4, p. 834) we get the result (2.1.1) after little simplification.

Also we put $2y^2-1 = x$ in the result of Gradshteyn and Ryzhik ([4], 3,P.847) we get the result (2.4.1) after little simplification.

3. Main Integrals.

First Integral.

$$\int_0^1 C_n^{\lambda}(1-2y^2)(1-y^2)^{\lambda-1/2}\, y^{2\lambda+2r}\; {}_pB_q\left[\alpha_s;\beta_t;(zy)^2\right]\times$$

$$\bar{H}_{P,Q}^{M,N}\left[(xy^2)^{\sigma}\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right.\right]\text{dy}$$

$$=\frac{\Gamma(n+2\lambda)\sqrt{\pi}(-1)^n}{4^{\lambda}n!\Gamma(\lambda)}\sum_{k=0}^{\infty}\frac{(\alpha_1)_k\ldots(\alpha_p)_k}{(\beta_1)_k\ldots(\beta_q)_k}\Omega(\alpha_{p+k},\beta_{q+k},0)\frac{z^{2k}}{k!}$$

$$\bar{H}_{P+2,Q+2}^{M,N+2}\left[(x)^{\sigma}\left|\begin{matrix}(-k-r,\sigma;1),(1/2-k-r-\lambda,\sigma;1),(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q},(-k-r+n,\sigma;1),(-k-r-n-2\lambda,\sigma;1)\end{matrix}\right.\right],\quad(3.1)$$

where the following conditions are satisfied

(i) $\bar{H}$-function must satisfy the conditions of convergence given by (1.3) to (1.6),

(ii) $R(\beta_q)>R(\alpha_p)$ for $0\le p\le q+1$,

(iii) $(\lambda+r)>-1/2$.

Second Integral.

$$\int_0^1 {}_pB_q\left[\alpha_s;\beta_t;(zy)^2\right](1-y^2)^{\lambda-1/2}y^{2r}\times\bar{H}_{P,Q}^{M,N}\left[(xy^2)^{\sigma}\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right.\right]\text{dy}$$

$$=\frac{\sqrt{\pi}}{\Gamma(\lambda)}\sum_{n=0}^{\infty}\frac{\Gamma(n+2\lambda)(-1)^n}{n!}\sum_{k=0}^{\infty}\frac{(\alpha_1)_k\ldots(\alpha_p)_k}{(\beta_1)_k\ldots(\beta_q)_k}\Omega(\alpha_{p+k},\beta_{q+k},0)\frac{z^{2k}}{k!}$$

$$\bar{H}_{P+2,Q+2}^{M,N+2}\left[(x)^{\sigma}\left|\begin{matrix}(-k-r,\sigma;1),(1/2-k-r-\lambda,\sigma;1),(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q},(-k-r+n,\sigma;1),(-k-r-n-2\lambda,\sigma;1)\end{matrix}\right.\right].\quad(3.2)$$

The conditions of convergence are as below

(i) $\bar{H}$-function must satisfy the conditions of convergence given by (1.3) to (1.6).

(ii) $R(\beta_q)>R(\alpha_p)$ for $0\le p\le q+1$

(iii) $r>-1/2$

Third Integral.

$$\int_0^1 {}_pB_q\left[\alpha_s;\beta_t;(zy)^2\right](1-y^2)^{-1/2}\,y^{2\lambda+2r}\times \bar{H}^{M,N}_{P,Q}\left[(xy^2)^{\sigma}\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right.\right]\mathrm{dy}$$

$$=\frac{\sqrt{\pi}}{\Gamma(\lambda)}\sum_{n=0}^{\infty}\frac{\Gamma(n+2\lambda)}{n!}\sum_{k=0}^{\infty}\frac{(\alpha_1)_k...(\alpha_p)_k}{(\beta_1)_k...(\beta_q)_k}\Omega(\alpha_{p+k},\beta_{q+k},0)\frac{z^{2k}}{k!}$$

$$\bar{H}^{M,N+2}_{P+2,Q+2}\left[(x)^{\sigma}\left|\begin{matrix}(-k-r,\sigma;1),(1/2-k-r-\lambda,\sigma;1),(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q},(-k-r+n,\sigma;1),(-k-r-n-2\lambda,\sigma;1)\end{matrix}\right.\right]\quad(3.3)$$

The following conditions of convergence must satisfied by the above integral:-

(i) $\bar{H}$-function must satisfy the conditions of convergence given by (1.3) to (1.6).

(ii) $R(\beta_q)>R(\alpha_p)$ for $0\le p\le q+1$

(iii) $(\lambda+r)>-1/2$

Proof. The integral (3.1) can be established by expressing the homogeneous generalized function ${}_pB_q\left[\alpha_s,\beta_t;(zy)^2\right]$ as given in (1.7) and the $\bar{H}$-function as defined by (1.1), and changing the order of integration, we get

$$\int_0^1 C_n^{\lambda}(1-2y^2)(1-y^2)^{\lambda-1/2}\,y^{2\lambda+2r}\,{}_pB_q\left[\alpha_r;\beta_t;(zy)^2\right]\times$$

$$\bar{H}^{M,N}_{P,Q}\left[(xy^2)^{\sigma}\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right.\right]\mathrm{dy}$$

$$=\sum_{k=0}^{\infty}\frac{(\alpha_1)_k...(\alpha_p)_k}{(\beta_1)_k...(\beta_q)_k}\Omega(\alpha_{p+k},\beta_{p+k},0)\frac{(z)^{2k}}{k!}\frac{1}{2\pi\omega}\times$$

$$\left\{\int_L\theta(s)\int_0^1 C_n^{\lambda}(1-2y^2)(1-y^2)^{\lambda-1/2}\,y^{2\lambda+2r+2k+2\sigma s}dy\right\}ds$$

Now using the result (2.1.1), we get (3.1) after little simplification.

To prove the integral (3.2) and (3.3) we use the following result

$$(1-2xh+h^2)^{-\lambda}=\sum_{n=0}^{\infty}C_n^{\lambda}(x)h^n$$

put $x=1-2y^2$ in the above result and then multiplying by

$$ {}_pB_q\left[\alpha_s;\beta_t;(zy)^2\right]\times y^{2\lambda+2r}(1-y^2)^{\lambda-1/2}\,\bar{H}^{M,N}_{P,Q}\left[(xy^2)^{\sigma}\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right.\right] $$

on both sides and then integrating between the limits 0 to 1 and using the result (2.1.1) we get the result obtained in (3.2) and (3.3).

4. Recurrence Relations. The following recurrence relations has been established

$$ (n+2\lambda)\sum_{k=0}^{\infty}\frac{(\alpha_1)_k\ldots(\alpha_p)_k}{(\beta_1)_k\ldots(\beta_q)_k}\Omega(\alpha_{p+k},\beta_{p+k},0)\frac{(z)^{2k}}{k!}\times $$

$$ \bar{H}^{M,N+2}_{P+2,Q+2}\left[(x)^{\sigma}\left|\begin{matrix}(-k-r,\sigma;1),(1/2-k-r-\lambda,\sigma;1),(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q},(-k-r+n,\sigma;1),(1-k-r-n-2\lambda-1,\sigma;1)\end{matrix}\right.\right] $$

$$ =-2(n+\lambda)\sum_{k=0}^{\infty}\frac{(\alpha_1)_k\ldots(\alpha_p)_k}{(\beta_1)_k\ldots(\beta_q)_k}\Omega(\alpha_{p+k},\beta_{q+k},0)\frac{z^{2k}}{k!} $$

$$ \bar{H}^{M,N+2}_{P+2,Q+2}\left[(x)^{\sigma}\left|\begin{matrix}(-k-r,\sigma;1),(-k-r-\lambda+1/2,\sigma;1),(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q},(-k-r+n,\sigma;1),(-k-r-n-2\lambda,\sigma;1)\end{matrix}\right.\right] $$

$$ +4(n+\lambda)\sum_{k=0}^{\infty}\frac{(\alpha_1)_k\ldots(\alpha_p)_k}{(\beta_1)_k\ldots(\beta_q)_k}\Omega(\alpha_{p+k},\beta_{q+k},0)\frac{z^{2k}}{k!} $$

$$ \bar{H}^{M,N+2}_{P+2,Q+2}\left[(x)^{\sigma}\left|\begin{matrix}(-k-r-1,\sigma;1),(-k-r-\lambda-1/2,\sigma;1),(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q},(-k-r+n-1,\sigma;1),(-k-r-n-2\lambda-1,\sigma;1)\end{matrix}\right.\right] $$

$$ -n\sum_{k=0}^{\infty}\frac{(\alpha_1)_k\ldots(\alpha_p)_k}{(\beta_1)_k\ldots(\beta_q)_k}\Omega(\alpha_{p+k},\beta_{q+k},0)\frac{z^{2k}}{k!} $$

$$ \bar{H}^{M,N+2}_{P+2,Q+2}\left[(x)^{\sigma}\left|\begin{matrix}(-k-r,\sigma;1),(-k-r-\lambda+1/2,\sigma;1),(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q},(-k-r+n-1,\sigma;1),(1-k-r-n-2\lambda,\sigma;1)\end{matrix}\right.\right] \quad (4.1) $$

$$ 2(n+2\lambda)\sum_{k=0}^{\infty}\frac{(\alpha_1)_k\ldots(\alpha_p)_k}{(\beta_1)_k\ldots(\beta_q)_k}\Omega(\alpha_{p+k},\beta_{q+k},0)\frac{z^{2k}}{k!} $$

$$ \bar{H}^{M,N+2}_{P+2,Q+2}\left[(x)^{\sigma}\left|\begin{matrix}(-k-r,\sigma;1),(-k-r-\lambda-1/2,\sigma;1),(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q},(-k-r+n-1,\sigma;1),(-k-r-n-2\lambda-1,\sigma;1)\end{matrix}\right.\right] $$

$$= \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \dots (\alpha_p)_k}{(\beta_1)_k \dots (\beta_q)_k} \Omega(\alpha_{p+k}, \beta_{q+k}, 0) \frac{z^{2k}}{k!}$$

$$\bar{H}^{M,N+2}_{P+2,Q+2}\left[(x)^{\sigma}\left|\begin{matrix}(-k-r,\sigma;1),(-k-r-\lambda+1/2,\sigma;1),(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q},(-k-r+n-1,\sigma;1),(1-k-r-n-2\lambda,\sigma;1)\end{matrix}\right.\right]$$

$$+\sum_{k=0}^{\infty} \frac{(\alpha_1)_k \dots (\alpha_p)_k}{(\beta_1)_k \dots (\beta_q)_k} \Omega(\alpha_{p+k}, \beta_{q+k}, 0) \frac{z^{2k}}{k!}$$

$$\bar{H}^{M,N+2}_{P+2,Q+2}\left[(x)^{\sigma}\left|\begin{matrix}(-k-r,\sigma;1),(-k-r-\lambda+1/2,\sigma;1),(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q},(-k-r+n,\sigma;1),(-k-r-n-2\lambda,\sigma;1)\end{matrix}\right.\right]$$

$$-2\sum_{k=0}^{\infty} \frac{(\alpha_1)_k \dots (\alpha_p)_k}{(\beta_1)_k \dots (\beta_q)_k} \Omega(\alpha_{p+k}, \beta_{q+k}, 0) \frac{z^{2k}}{k!} \times$$

$$\bar{H}^{M,N+2}_{P+2,Q+2}\left[(x)^{\sigma}\left|\begin{matrix}(-k-r-1,\sigma;1),(-k-r-\lambda-1/2,\sigma;1),(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q},(-k-r+n-1,\sigma;1),(-k-r-n-2\lambda-1,\sigma;1)\end{matrix}\right.\right] \quad (4.2)$$

Proof. To prove (4.1), we multiply the result (2.2.1) by

$$y^{2\lambda+2r}(1-y^2)^{\lambda-1/2}\ {}_pB_q\left[\alpha_s;\beta_t;(xy)^2\right]\bar{H}^{M,N}_{P,Q}\left[(xy^2)^{\sigma}\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right.\right]$$

and integrating between the limits 0 to 1 and using the result (2.1.1), we get the relation (4.1). Now we use the result (2.2.2) and proceeding the same manner as in the proof of (4.1), we get the relation (4.2).

5. Integral Involving Products of $\bar{H}$-Function, Jacobi Polynomial and Homogeneous Generalized Hypergeometric Function.

$$\int_0^1 p_n^{\alpha,\beta}(2y^2-1)(1-y^2)^{\alpha}y^{2\sigma+1}\ {}_pB_q\left[\alpha_u;\beta_v;(xy)^2\right]\times$$

$$\bar{H}^{M,N}_{P,Q}\left[(zy^2)^{r}\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right.\right]\mathrm{dy}$$

$$=\frac{\Gamma(\alpha+n+1)}{2}\sum_{k=0}^{\infty} \frac{(\alpha_1)_k \dots (\alpha_p)_k}{(\beta_1)_k \dots (\beta_q)_k} \Omega(\alpha_{p+k}, \beta_{q+k}, 0) \frac{x^{2k}}{k!} \times$$

$$\bar{H}^{M,N+2}_{P+2,Q+2}\left[(z)^r\left|\begin{matrix}(-\sigma-k,r;1),(-\sigma-k+\beta,r;1),(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q},(-\sigma-k+\beta+n,r;1),(-\alpha-\sigma-k-n-1,r;1)\end{matrix}\right.\right],(5.1)$$

where the following conditions are satisfied by the above integral

(i) $\bar{H}$-function must satisfy the conditions of convergence given by (1.3) to (1.6),

(ii) $R(\beta_q) > R(\alpha_p)$ for $0 \le p \le q+1$,

(iii) $(2\sigma+1) > -1$.

Proof. To establish (5.1), we express the homogeneous generalized hypergeometric function by (1.7) and then $\bar{H}$-function by (1.1) in the L.H.S. of the equation (5.1), we get

$$\int_0^1 p_n^{\alpha,\beta}(2y^2-1)(1-y^2)^{\alpha}y^{2\sigma+1}\,{}_pB_q\left[\alpha_u;\beta_v;(xy)^2\right]\times$$

$$\bar{H}^{M,N}_{P,Q}\left[(zy^2)^r\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right.\right]\mathrm{dy}$$

$$=\sum_{k=0}^{\infty}\frac{(\alpha_1)_k\ldots(\alpha_p)_k}{(\beta_1)_k\ldots(\beta_q)_k}\Omega(\alpha_{p+k},\beta_{p+k},0)\frac{(x)^{2k}}{k!}\frac{1}{2\pi\omega}\times$$

$$\left\{\int_L\theta(s)z^{rs}\int_0^1 p_n^{\alpha,\beta}(2y^2-1)(1-y^2)^{\alpha}y^{2\sigma+2rs+2k+1}dy\right\}ds$$

Now using the integral (2.3.4) we obtain the result given in (5.1)

6. Recurrence Relations of $\bar{H}$-function. The following recurrence relations have been established

$$(2+\alpha+\beta+2n)\sum_{k=0}^{\infty}\frac{(\alpha_1)_k\ldots(\alpha_p)_k}{(\beta_1)_k\ldots(\beta_q)_k}\Omega(\alpha_{p+k},\beta_{q+k},0)\frac{x^{2k}}{k!}\times$$

$$\bar{H}^{M,N+2}_{P+2,Q+2}\left[(z)^r\left|\begin{matrix}(-\sigma-k-1,r;1),(-\sigma-k+\beta,r;1),(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q},(-\sigma-k+\beta+n,r;1),(-\alpha-\sigma-k-n-2,r;1)\end{matrix}\right.\right]$$

$$=(n+1)(\alpha+n+1)\sum_{k=0}^{\infty}\frac{(\alpha_1)_k\ldots(\alpha_p)_k}{(\beta_1)_k\ldots(\beta_q)_k}\Omega(\alpha_{p+k},\beta_{q+k},0)\frac{x^{2k}}{k!}\times$$

$$\bar{H}^{M,N+2}_{P+2,Q+2}\left[(z)^r\left|\begin{matrix}(-\sigma-k,r;1),(-\sigma-k+\beta,r;1),(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q},(1-\sigma-k+\beta+n,r;1),(-\alpha-\sigma-k-n-2,r;1)\end{matrix}\right.\right]$$

$$+(1+\beta+n)\sum_{k=0}^{\infty}\frac{(\alpha_1)_k\ldots(\alpha_p)_k}{(\beta_1)_k\ldots(\beta_q)_k}\Omega(\alpha_{p+k},\beta_{q+k},0)\frac{x^{2k}}{k!}\times$$

$$\bar{H}_{P+2,Q+2}^{M,N+2}\left[(z)^r\left|\begin{matrix}(-\sigma-k,r;1),(-\sigma-k+\beta,r;1),(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q},(-\sigma-k+\beta+n,r;1),(-\alpha-\sigma-k-n-1,r;1)\end{matrix}\right.\right],(6.1)$$

$$(\alpha+\beta+2n)\sum_{k=0}^{\infty}\frac{(\alpha_1)_k\ldots(\alpha_p)_k}{(\beta_1)_k\ldots(\beta_q)_k}\Omega(\alpha_{p+k},\beta_{q+k},0)\frac{x^{2k}}{k!}\times$$

$$\bar{H}_{P+2,Q+2}^{M,N+2}\left[(z)^r\left|\begin{matrix}(-\sigma-k,r;1),(-\sigma-k+\beta-1,r;1),(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q},(-\sigma-k+\beta+n-1,r;1),(-\alpha-\sigma-k-n-1,r;1)\end{matrix}\right.\right]$$

$$=(\alpha+\beta+n)\sum_{k=0}^{\infty}\frac{(\alpha_1)_k\ldots(\alpha_p)_k}{(\beta_1)_k\ldots(\beta_q)_k}\Omega(\alpha_{p+k},\beta_{q+k},0)\frac{x^{2k}}{k!}\times$$

$$\bar{H}_{P+2,Q+2}^{M,N+2}\left[(z)^r\left|\begin{matrix}(-\sigma-k,r;1),(-\sigma-k+\beta,r;1),(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q},(-\sigma-k+\beta+n,r;1),(-\alpha-\sigma-k-n-1,r;1)\end{matrix}\right.\right]$$

$$+\sum_{k=0}^{\infty}\frac{(\alpha_1)_k\ldots(\alpha_p)_k}{(\beta_1)_k\ldots(\beta_q)_k}\Omega(\alpha_{p+k},\beta_{q+k},0)\frac{x^{2k}}{k!}\times$$

$$\bar{H}_{P+2,Q+2}^{M,N+2}\left[(z)^r\left|\begin{matrix}(-\sigma-k,r;1),(-\sigma-k+\beta,r;1),(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q},(-\sigma-k+\beta+n-1,r;1),(-\alpha-\sigma-k-n,r;1)\end{matrix}\right.\right],(6.2)$$

$$\sum_{k=0}^{\infty}\frac{(\alpha_1)_k\ldots(\alpha_p)_k}{(\beta_1)_k\ldots(\beta_q)_k}\Omega(\alpha_{p+k},\beta_{q+k},0)\frac{x^{2k}}{k!}\times$$

$$\bar{H}_{P+2,Q+2}^{M,N+2}\left[(z)^r\left|\begin{matrix}(-\sigma-k-1,r;1),(-\sigma-k+\beta,r;1),(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q},(-\sigma-k+\beta+n,r;1),(-\alpha-\sigma-k-n-2,r;1)\end{matrix}\right.\right]$$

$$+(\alpha+n+1)\sum_{k=0}^{\infty}\frac{(\alpha_1)_k\ldots(\alpha_p)_k}{(\beta_1)_k\ldots(\beta_q)_k}\Omega(\alpha_{p+k},\beta_{q+k},0)\frac{x^{2k}}{k!}\times$$

$$\bar{H}_{P+2,Q+2}^{M,N+2}\left[(z)^r\left|\begin{matrix}(-\sigma-k,r;1),(-\sigma-k+\beta,r;1),(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q},(-\sigma-k+\beta+n,r;1),(-\alpha-\sigma-k-n-2,r;1)\end{matrix}\right.\right]$$

$$=\sum_{k=0}^{\infty}\frac{(\alpha_1)_k\ldots(\alpha_p)_k}{(\beta_1)_k\ldots(\beta_q)_k}\Omega(\alpha_{p+k},\beta_{q+k},0)\frac{x^{2k}}{k!}\times$$

$$\bar{H}^{M,N+2}_{P+2,Q+2}\left[(z)^r\left|\begin{matrix}(-\sigma-k,r;1),(-\sigma-k+\beta,r;1),(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q},(-\sigma-k+\beta+n,r;1),(-\alpha-\sigma-k-n-1,r;1)\end{matrix}\right.\right].(6.3)$$

Special Cases. Taking the exponents $A_j=1(j=1,...,P)$ and $B_j=1(j=1,...,Q)$ in the integral (2.1) the $\bar{H}$-function reduces to well known Fox's H-function we get the result

$$\int_0^1 C_n^\lambda(1-2y^2)(1-y^2)^{\lambda-1/2}y^{2\lambda+2r}\,{}_pB_q\left[\alpha_r;\beta_t;(zy)^2\right]$$

$$\times\bar{H}^{M,N}_{P,Q}\left[(xy^2)^\sigma\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right.\right]dy$$

$$=\frac{\Gamma(n+2\lambda)\sqrt{\pi}(-1)^n}{4^\lambda n!\Gamma(\lambda)}\sum_{k=0}^{\infty}\frac{(\alpha_1)_k\ldots(\alpha_p)_k}{(\beta_1)_k\ldots(\beta_q)_k}\Omega(\alpha_{p+k},\beta_{q+k},0)\frac{z^{2k}}{k!}$$

$$H^{M,N+2}_{P+2,Q+2}\left[(x)^\sigma\left|\begin{matrix}(-k-r,\sigma),(1/2-k-r-\lambda,\sigma),(a_j,\alpha_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j)_{M+1,Q},(-k-r+n,\sigma),(-k-r-n,\sigma)\end{matrix}\right.\right]$$

Similarly taking the exponents $A_j=1(j=1,...,P)$ and $B_j=1(j=1,...,Q)$ in the other integrals and recurrence relations we get the results involving the well known Fox's H-function.

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LIVINGSTON PROBLEM FOR CLOSE-TO-CONVEX FUNCTIONS WITH FIXED SECOND COEFICIENT

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ABSTRACT

Let $F(z)=z+a_2z^2+a_3z^3+\dots$ be a function with fixed second coefficient a_2 and starlike of order β with respect to symmetric (or conjugate or symmetric conjugate) points. A sharp radius ρ is determined such that the function $f(z)\left(z^{1-m}\left(z^mF(z)\right)'\right)/(m+1)$ is also starlike of order β with respect to symmetric (or conjugate or symmetric conjugate) points. Furthermore, radius of convexity (or close-to-convexity) with respect to symmetric (or conjugate or symmetric conjugate) points of f is determined whenever F is convex (or close-to-convex) with respect to symmetric (or conjugate or symmetric conjugate) points.

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Key Words and Phrases : Starlike function, convex function, starlike function with respect to symmetric points, conjugate points, Livingston problem.

1. Introduction. Let be the class of functions of the form $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$ which are analytic in the open unit disk $=\{z\in\ :|z|<1\}$. Let be the subclass of consisting of univalent functions. For $0\le\alpha<1$, let $^*(\alpha)$ and (α) be the subclasses of consisting of starlike functions of order α and convex functions of order α, respectively, defined analytically by the equivalences

$$f\in\ ^*(\alpha)\Leftrightarrow \operatorname{Re}\left(\frac{zf'(z)}{g(z)}\right)>\alpha \text{ and } f\in\ (\alpha)\Leftrightarrow \operatorname{Re}\left(\frac{zf''(z)}{f'(z)}+1\right)>\alpha .$$

The classes $^*(\alpha)$ and (α) were introduced by Robertson [37] and they have been the subject of investigation by several authors subsequently. The classes $^*:=\ (0)$ and $:=\ (0)$ are the familiar classes of starlike and convex functions respectively. Let (α,β) be a susbclass of of close-to-convex functions of order

α and type β. A function

$$f \in \ (\alpha,\beta) \Leftrightarrow \operatorname{Re}\left(\frac{zf'(z)}{g(z)}\right) > \alpha, \quad 0 \leq \alpha < 1,$$

where $g \in \ ^*(\beta)$ for $0 \leq \beta < 1$. The class $\ := \ (0,0)$ is class of close-to-convex functions, and was introduced by Kaplan [22] in 1952.

In 1947, Robinson [38] proved that if F is only assumed to be univalent in , then $f(z) = (1/2)(zF(z))'$ is starlike for $|z| < 0.38$. He claimed that it is probable that f is univalent for $|z| < 1/2$. In 1965, Libera [28] proved that if f is in $\ ^*$ or , then the function

$$F(z) = \frac{2}{z}\int_0^z f(u)\,du,$$

also belongs to $\ ^*$ or . Subsequently Livingston [30] in 1966 considered the converse problem and proved Robinson conjecture by applying method of MacGregor [31] and obtained the sharp result that if F is in $\ ^*$ (resp. and), then the function

$$f(z) = \frac{1}{2}(zF(z))',$$

belongs to $\ ^*$ (resp. and) in $|z| < 1/2$. In 1969 Bernardi [11] proved that if $f \in$ is a member of , $\ ^*$ or then

$$F(z) = \frac{c+1}{z^c}\int_0^z t^{c-1} f(t)\,dt = z + \sum_{n=2}^{\infty} \frac{c+1}{c+n} a_n z^n, \quad c = 1,2,3...$$

is the member of the same class. In 1970, he [12] investigated the converse problem and obtained sharp radius results for the function

$$f(z) = \frac{1}{c+1} z^{1-c}\left(z^c F(z)\right)'$$

to belong to certain subclass of univalent functions. Various generalizations of the work of Libera [28] and Livingston [30] were studied in the past and we present below a brief review of the research on this topic.

In 1969, Padmanabhan [35] refined the results of Livingston [30] by proving that if $F \in \ ^*(\alpha)$ or (α) or (β,α) for $0 \leq \alpha < 1/2$ and $0 \leq \beta < 1$, then f is in same class of the same order. In 1970, Yoshikai [50] extended the result for the analytic function $F(z) = z + a_{n+1}z^{n+1} + \ldots$ belonging to $*$ (resp. and) and obtained the sharp radius $|z| < (1/n+1)^{1/n}$ for f to be in $*$ (resp. and). In 1971, Libera and Livingston [29] extended the results of Padmanabhan [35] to include the range of α when $1/2<\alpha<1$ and generalized by finding the sharp radius of the disk in which $f \in\ *(\beta)$ when $F \in\ *(\alpha), 0 \le \alpha \le \beta < 1$. In 1971, Singh and Goel [46] obtained the sharp radius of starlikeness for the function *f* if $F \in\ *(\beta)$. In 1973, Sizhuk [47] determined radius of starlikeness for the function *f* if $f \in\ *(\beta)$. In 1973, Sizhuk [47] determined radius for $f \in\ *(\beta)$ if $F \in\ *(\alpha)$ for $0 \le \alpha, \beta < 1$. In 1973, Al-Amiri [1] extended the result of Livingston [30] by fixing the second coefficient in the Taylor series expansion of the function *F*. In 1974, Al-Amiri [2] extended the result of Libera and Livingston [29], using the technique of Zomorovic [53] to include the complementary case $0 \le \beta < \alpha$. In the same year, Bajpai and Singh [6] also extended the result of Libera and Livingston [29] for all values of α, β in [0,1). Their technique was better than the technique given in [46] as it was also applicable for functions *f* of the form $f(z) = z + \sum_{k=1}^{\infty} a_{kn+1} z^{kn+1}, n \ge 1$. In the year 1974, Singh and Singh [45] determined radius of starlikeness and the radius of close-to-convexity of n^{th} partial sum of functions *f* when *F* is analytic, univalent and convex function with fixed second coefficient in . In 1975, Chandra and Singh [14] obtained the same result of Yoshikai [50]. In 1983, Owa [34] proved for functions of the form $F(z) = z - \sum_{n=2}^{\infty} a_n z^n$, if $F \in\ (\alpha)$ then $f \in\ *(\alpha)$ using the technique of Hadamard product.

Barnard [8] in 1975 answered the question whether 1/2 obtained from Koebe function is the smallest radius of starlikeness and hence the smallest radius of univalence for all functions $f(z) = (1/2)(zF(z))', F \in$. He showed by giving two examples that 1/2 is not the smallest radius of starlikeness for *f*, for all $F \in$. In 1983 Pearce [36] improved the radius of starlikeness on the univalence class from 0.38 (Robinson [38]) to 0.435. He also gave an example to support Barnard [8] result that radius of starlikeness for all $F \in$ is less than 1/2.

In 1970, Yoshikai [50] obtained the sharp radius for above *f* to be in $*$ (resp. and) if $F(z) = z + a_{n+1}z^{n+1} + \ldots$ belongs to $*$ (resp. and). In 1972, Bajpai et al. [7] extended the results of Bernardi [11,12] for the subclasses

$*(\alpha)$, (α) and (β,α) and obtained sharp results. These results contained the theorems of Padmanabhan [35] as special cases. In 1972, Yoshikai [51] generalized the result of Al-Amiri [2] by applying the technique of Zomorvoič [53] and obtained the radius for $f\in\ *(\beta)$ if $F\in\ *(\alpha), 0\le\alpha,\beta<1$. In 1973 Goel and Singh [17] generalized Bernardi [12] result for $c>0$. In 1981 Goel and Sohi [18] extended the result of [17] for functions meromorphic in the unit disk. In 1978, Barnard and Kellogg [9] verified result of Livingston [30] and Bernardi [11,12] using convolution operator technique [7].

In 1971/72, Goel [16] worked on analytic p-valent functions $f(z)=z^p+a_{p+1}z^{p+1}+\ldots, z\in$ and obtained sharp radius for f to be p-valently starlike (resp. p-valently convex and p-valently close-to-convex), if F is p-valently starlike (resp. p-valently convex and p-valently close-to-convex), when

$$f(z)=\frac{1}{p+1}\left(zF(z)\right)'.$$

In 1971, Calys [13] obtained the sharp radius of univalence and starlikeness of analytic functions $f\in$ in that satisfy

$$F(z)=\frac{2}{z}\int_0^z\frac{f(t)g(t)}{t}dt$$

where $F\in\ *$ and $g\in$ (or $g\in$ or $\operatorname{Re} g(z)/z>0$ in). In 1972, Nikolaeva and Repnina [33] found a sharp radius r for the function f to be in $*(\alpha)$, (α) or $(\alpha,0)\alpha\in[0,1]$ if F is in $*(\alpha)$, (α) or $(\alpha,0)$ when

$$f(z)=\lambda F(z)+\mu zF'(z) \text{ where } \lambda\ge 0,\mu>0,\lambda+\mu=1.$$

In 1986, Anh and Tuan [3] obtained radius of starlikeness of order $\beta, 0\le\beta<1$, for the above function f when $-\infty<\lambda<1$ and $F\in\ *(A,B)$, that is, $zF'(z)/F(z)\prec(1+Az)/(1+Bz)$. In 1987, Srivastava et al. [27] obtained radius of starlikeness and convexity of order β for the function f with the fixed second coefficient if $\lambda=a/(a+1)(a\ge 0)$ provided F is starlike of order α or convex of order α with fixed second coefficient. In 2003, Greiner and Roth [19] considered the problem of finding radius of convexity of the funciton f for $F\in\ , 0\le\lambda<1$.

In 1973, Singh [44] proved that for $\alpha, c\in$ and $f\in\ *$, the function F defined by

$$F(z)=\frac{\alpha+c}{z^c}\int_0^z t^{c-1}\left(f(t)\right)^\alpha dt$$

also belongs to $*$. He also obtained the sharp radius for $f\in\ *$ if $F\in\ *$ as generalization of [12]. Furthermore, he extended these results on the largest known subclass of , introduced by Bazilevič [10], that is, Bazilevic functions. In 1975, Karunakaran [23] obtained the radius for

$$F(z)=\frac{2}{g(z)}\int_0^z f(t)\,dt$$

to be in $*(\beta)$ if $f\in\ *(\alpha)$ and $g\in\ *(\gamma)$. They solved the converse problem also and determined sharp estimate for the function

$$f(z)=\frac{1}{2}\big(g(z)F(z)\big)'$$

to be in the class $*(\beta)$ where g and F are starlike functions of order γ and α respectively, with $\gamma+\alpha=1$. In 1979, Gupta [21] generalized the results of Al-Amiri[1] for the function f defined by

$$f(z)=\left\{(\gamma+\alpha)^{-1}z^{1-\gamma}\left(z^{\gamma}\left(F(z)\right)^{\alpha}\right)^{t}\right\}^{\frac{1}{\alpha}}\quad(\alpha,\gamma=1,2,\ldots). \tag{1.1}$$

In the same year, Yoshikawa and Yoshikai [52] extended the results of Ruscheweyh and Singh [40] on above functions and then applied it on Bazilevič functions. They obtained results of Bernardi [12] as a corollary. In 1983, Kumar [26] improved the result of Gupta [21] to the case when α is a positive real number and γ is a complex number such that $\gamma+\alpha\neq 0$.

In 1980, Bajpai and Dwivedii [5] extended Livingston [30] result for the function

$$F(z)=\frac{c+2}{z^{c+1}}\int_0^z t^{c-1}f(t)g(t)dt,\quad c\geq 0$$

and obtained sharp radius for $f\in\ *(\beta)$ when $F\in\ *(\beta)$ and $g\in\ (\alpha)$. In the same year, Karunakaran et al. [24] generalized the results of Bernardi [11,12] for functions f and F where

$$F(z)=\frac{c+1}{\left(g(z)\right)^{c}}\int_0^z t^{c-1}f(t)dt\quad c\geq 0$$

when $f\in\ *(\alpha)$, $F\in\ *(\beta)$ and $g\in\ *(\gamma)$. Gupta and Ahmed [20] worked on functions f and F connected by the relation

$$F(z)^{\alpha}=\frac{\alpha+c}{\left(g(z)\right)^{c}}\int_0^z\left(h(t)\right)^{c\text{-}1}\left(f(t)\right)^{\alpha}dt$$

where $\alpha,c\in\ $, $g\in\ *(\alpha)$ and $h\in\ *(\gamma)$. They obtained radius for the case $f\in\ *(\alpha)$ provided $F\in\ *(\beta)$ for $0\leq\alpha,\beta<1$.

During the same period, new classes were introduced and studied. Sakaguchi [41] in 1959 introduced the subclass $^{*}_{8}$ of starlike functions with respect to symmetric points consisting of $f \in$ satisfying

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)-f(-z)}\right) > 0 \text{ for } z \in \quad .$$

The class $^{*}_{8}$ contains the class of convex functions and is contained in the class of close-to-convex functions. In 1977, Das and Singh [15] introduced two subclasses of namely the class of convex functions and class $_{s}$ of close-to-convex functions with respect to symmetric points in . Thus a function f is in or $_{s}$ if

$$\operatorname{Re}\left(\frac{zf'(z)}{\left(f(z)-f(-z)\right)'}\right) > 0 \text{ or } \operatorname{Re}\left(\frac{zf'(z)}{g(z)-g(-z)}\right) > 0, \; z \in$$

for $g \in {}^{*}_{s}$. They studied behaviour of certain integral operators (analogous to Libera operator [28] on the member of $^{*}_{s}$, $_{s}$ and $_{s}$, thus giving closer view of these classes regarding their geometric aspects and other properties. In 1979, Thangmani [48] solved Livingston problem for functions F belonging to the class $^{*}_{s}$ (resp. $_{s}$ and $_{s}$) and obtained radius for f to be in respective classes and deduced some results of Livingston [30] as particular case. In 1980, Thangmani [49] obtained coefficient estimates, distortion theorems and radius of convexity for certain classes and deduced results of Sakaguchi [41] as particular cases.

In 1987, El-Ashwah and Thomas [4] introduced two subclasses $^{*}_{c}$ and $^{*}_{8c}$ of close-to-convex functions. The class $^{*}_{c}$ consists of functions f that are starlike with respect to conjugate points in satisfying

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)+\overline{f}(\overline{z})}\right) > 0, \quad z \in \quad .$$

In 1993, Kasi [25] found the sharp radius of the disk in which $f(z) = \frac{1}{2}\left(zF(z)\right)' \in S^{*}_{c}$ when $F \in {}^{*}_{c}$. In 1994, Shanmugam and Ravichandran [43] introduced the subclasses $_{c}$, $_{sc}$, $_{c}$ and $_{sc}$ of functions convex and close-to-convex with respect to conjugate and symmetric conjugate points. In the same year, Shanmugam and Ravichandran [42] also introduced the subclasses $^{*}_{c}(\alpha)$ (resp. $\in {}^{*}_{sc}(\alpha)$) and $_{c}(\alpha)$ (resp. $_{sc}(\alpha)$) of functions starlike and convex with respect to conjugate (resp. symmetric conjugate) points of order $\alpha(0 \le \alpha < 1)$ in . They also introduced the subclass $_c(\alpha,\beta)$ (resp. $_{sc}(\alpha,\beta)$ of of close-to-convex functions with respect to conjugate (resp. symmetric conjugate) points of order $\alpha(0 \le \alpha < 1)$ and type $\beta(0 \le \beta < 1)$. Thus, a function $f \in {}^{*}_{sc}(\alpha)$ if

$$\mathrm{Re}\frac{2zf'(z)}{f(z)+\bar{f}(\bar{z})} > \alpha \quad 0 \le \alpha < 1 \quad z \in \quad .$$

and ${}^{*}_{sc}(\alpha)$ provided

$$\mathrm{Re}\frac{2zf'(z)}{f(z)-\bar{f}(-\bar{z})} > \alpha \quad 0 \le \alpha < 1 \quad z \in$$

In [42], they have taken to be 2α. The classes $_c(\alpha)$, $_{sc}(\alpha)$, $_c(\alpha,\beta)$ and $_{sc}(\alpha,\beta)$ were also defined accordingly. They found a sharp radius r_0 such that if $F \in {}^{*}_{sc}(\alpha)$, $_c(\alpha)$ and $_c(\alpha,\beta)$ then

$$(1.2) \qquad f(z) = \frac{z^{1-m}\left[z^m\left(F(z)\right)\right]'}{m+1}$$

belongs to the class ${}^{*}_{c}(\alpha)$, $_c(\alpha)$ and $_c(\alpha,\beta)$ respetively in $|z| < r_0$. Radii for other classes were also obtained. The form (1.2) of *f* being a particular case of the function defined by Gupta [21] when $\alpha = 1$ and $\gamma = m$ in (1.1).

Let $S^{*}_{c,b}(\beta)(0 \le \beta < 1)$ be the class of functions $f \in {}^{*}_{c}(\beta)$ of the form

$$(1.3) \qquad f(z) = z + b(1-\beta)z^2 + a_3 z^3 + \dots \ .$$

It is no loss of generality to suppose $b \ge 0$. If this is not the case, then consider the function $w = e^{i\theta} f\left(e^{-i\theta} z\right)$ where $\theta = \arg b$. In this paper, a sharp radius $r_0 = r_0(\beta, m, b)$ has been obtained for the function *f* defined by (1.2) to be in the class ${}^{*}_{c,((m+2)b/(m+1))}$ provided $F \in {}^{*}_{c,b}(\beta)$. Also r_0 is obtained for the function *f* defined by (1.2) to be in the other similar classes.

2. Livingston Problem. In our first theorem, we investigate the Livingston problem for the class of starlike functions of order β with respect to conjugate points. In subsequent theorems, the problem is investigated for convex and close-to-convex of order β with respect to conjugate points. We make use of the following lemma in the proof of these results.

Lemma 2.1. [32] *Let* $b \ge 0$ *and* $0 \le \alpha < 1$. *If* $p(z) = 1 + b(1-\alpha)z + \sum_{n=2}^{\infty} b_n z^n$ *is analytic* *and* $\operatorname{Re} p(z) > \alpha$ *in* , *then*

$$\operatorname{Re} p(z) \ge \frac{1 + b\alpha|z| - (1-2\alpha)|z|^2}{1 + b|z| + |z|^2},$$

and

$$|p'(z)| \le \frac{\operatorname{Re}(p(z) - \alpha)}{1 - |z|^2}\left(\frac{b|z|^2 + 4|z| + b}{|z|^2 + b|z| + 1}\right)$$

for all $z \in$.

Theorem 2.2. *Suppose that* $0 \le \beta < 1, m \in \ := \{1,2,\ldots\}$ *and* $b \ge 0$. *If* $F \in {}^{*}_{c,b}(\beta)$ *then f defined by (1.2) is in* ${}^{*}_{c,\gamma}(\beta)$ *for* $|z| < r_0$ *where* $\gamma = ((m+2)/(m+1))b$ *and* r_0 *is the smallest positive root of the equation*

(2.1) $\quad r^4(1 - m - 2\beta) - br^3(m + \beta + 1) + r^2(2\beta - 6) + br(m + \beta - 1) + (m+1) = 0$

This result is sharp.

Proof. Define $P : \ \to \$ by

$$P(z) = \frac{2zF'(z)}{F(z) + \bar{F}(\bar{z})}.$$

Since $F \in {}^{*}_{c,b}(\beta)$, $\operatorname{Re} P(z) > \beta$ for all $z \in$. Using (1.2), we have

(2.2)
$$\frac{2zf'(z)}{f(z) + \bar{f}(\bar{z})} = \frac{\left(2z^{m+1}F'(z)\right)'}{\left(z^m\left(F(z) + \bar{F}(\bar{z})\right)\right)'} = \frac{\left(P(z)z^m\left(F(z) + \bar{F}(\bar{z})\right)\right)'}{\left(z^m\left(F(z) + \bar{F}(\bar{z})\right)\right)'}$$

$$= P(z) + P'(z)\frac{z^m K(z)}{\left(z^m K(z)\right)'} = P(z) + P'(z)\frac{z}{M(z)},$$

where

$$K(z) = \frac{F(z) + \bar{F}(\bar{z})}{2}$$

and

$$\frac{M(z)}{z} = \frac{\left(z^m K(z)\right)'}{z^m K(z)} = \frac{1}{z}\left(\frac{zK'(z)}{K(z)} + m\right),$$

so that

$$M(z) = \frac{zK'(z)}{K(z)} + m.$$

Now, since $F \in {}^{*}_{c,b}(\beta)$

$$\text{Re}\left(\frac{2zF'(z)}{F(z)+\bar{F}(\bar{z})}\right) > \beta. \tag{2.3}$$

In particular, replacing z by $\bar{z}$ in the above inequality, we have

$$\text{Re}\left(\frac{2\bar{z}F'(\bar{z})}{F(\bar{z})+\bar{F}(z)}\right) > \beta .$$

Using the fact that $\text{Re}\, z = \text{Re}\, \bar{z}$ it follows that

$$\text{Re}\left(\frac{2z\bar{F}'(\bar{z})}{F(z)+\bar{F}(\bar{z})}\right) > \beta . \tag{2.4}$$

On adding (2.3) and (2.4), we obtain

$$\text{Re}\left(\frac{2z\left(F'(z)+\bar{F}'(\bar{z})\right)}{F(z)+\bar{F}(\bar{z})}\right) > 2\beta$$

or equivalently

$$\text{Re}\left(\frac{zK'(z)}{K(z)}\right) > \beta.$$

Since $zK'(z)/K(z) = 1+b(1-\beta)z+\ldots$, Lemma 2.1 shows that

$$\text{Re}\frac{zK'(z)}{K(z)} \geq \frac{1+b\beta r-(1-2\beta)r^2}{1+br+r^2},$$

Hence

$$\begin{aligned}|M(z)| &\geq m+\text{Re}\left(\frac{zK'(z)}{K(z)}\right)\\ &\geq m+\frac{1+b\beta r-(1-2\beta)r^2}{1+br+r^2}\\ &= \frac{r^2(m+2\beta-1)+br(m+\beta)+(m+1)}{1+br+r^2}\end{aligned}$$

so that

$$\frac{|z|}{|M(z)|} \leq \frac{r\left(1+br+r^2\right)}{r^2(m+2\beta-1)+br(m+\beta)+(m+1)}. \tag{2.5}$$

To show $f \in {}^{*}_{c,\left(\frac{m+2}{m+1}\right)b}(\beta)$ in $|z| < r_0$ where r_0 is the smallest positive root of (2.1), we must show that

$$\operatorname{Re}\left(\frac{2zf'(z)}{f(z)+\bar{f}(\bar{z})}\right)>\beta \text{ in } |z|<r_0 .$$

In view of (2.2), it suffices to show that

$$\operatorname{Re}(P(z)-\beta)+\operatorname{Re}\left(P'(z)\frac{z}{M(z)}\right)>0 ,$$

or equivalently

$$\operatorname{Re}(P(z)-\beta)-|P'(z)|\frac{|z|}{|M(z)|}>0 . \tag{2.6}$$

Since $\operatorname{Re}P(z)>\beta$ and $P(z)=1+b(1-\beta)z+...,$ Lemma 2.1 gives

$$|P'(z)|\leq\frac{\operatorname{Re}(P(z)-\beta)}{(1-r^2)}\frac{(br^2+4r+b)}{(r^2+br+1)} .$$

Thus, inequality (2.6) is satisfied if

$$\operatorname{Re}(p(z)-\beta)\left(1-\frac{(br^2+4r+b)r}{(1-r^2)(r^2(m+2\beta-1)+br(m+\beta)+(m+1))}\right)>0 .$$

Using (2.5). Since $\operatorname{Re}(P(z))>\beta,$ this condition is equivalent to showing that

$$g(r)=r^4(1-m-2\beta)-br^3(m+\beta+1)+r^2(2\beta-6)+br(m+\beta-1)+(m+1)>0 .$$

If r_0 is the smallest positive root of $g(r)=0$ or (2.1), then $g(r)>0$ for $r<r_0$ and hence the result follows.

To see that the result is sharp for each m, consider the function

$$F(z)=\frac{z}{(1-bz+z^2)^{1-\beta}} .$$

It is easy to show that $F\in {}^{*}_{c,b}(\beta)$ and

$$f(z)=\frac{zF'(z)+mF(z)}{m+1}$$

$$=\frac{1}{m+1}\frac{z}{(z^2+bz+1)^{2-\beta}}\left[z^2(m+2\beta-1)-bz(m+\beta)+(m+1)\right]$$

so that $f'(z)=N(z)/(m+1)$ where

$$N(z)=(z^2-bz+1)^{\beta-3}\left[z^4(m+2\beta-1)(2\beta-1)\right.$$

$$-bz^3\left((3\beta-1)m+(4\beta^2-\beta-1)\right)+z^2\left(8\beta-6+b^2\beta(m+\beta)+2m\beta\right)$$

$$\left.+bz(1-m-(m+3)\beta)+(m+1)\right].$$

Consider

$$\frac{2zf'(z)}{f(z)+\bar{f}(\bar{z})}-\beta=\frac{2N(z)}{f(z)(m+1)}-\beta$$

$$=\frac{(1-\beta)\left[z^4(1-m-2\beta)+bz^3(m+\beta+1)+2z^2(\beta-3)-bz(m+\beta-1)+(m+1)\right]}{\left(1-bz+z^2\right)\left(z^2(m+2\beta-1)-bz(m+\beta)+(m+1)\right)}.$$

Therefor, we have

$$\frac{2zf'(z)}{f(z)+\bar{f}(\bar{z})}-\beta=0 \text{ for } z=-r_0.$$

Thus f is not in $^{*}_{c,\left(\frac{m+2}{m+1}\right)b}$ in any circle $|z|<r, r>r_0$, this completes the proof.

Remark 2.3. For b=2 and $\beta=2\alpha(0\le\alpha<1/2)$, (2.1) takes the form

$$\left((1-m-4\alpha)r^2+(4\alpha-4)r+(m+1)\right)(r+1)^2=0.$$

The second quadratic expression gives a negative root, thus, the required radius will be given by the smallest positive root of the first quadratic expression, which gives the result of [42, Theorem 2.2].

In [42], Shanmugam and Ravichandran also introduced the class $_c(\alpha)(0\le\alpha<1)$ related to the class $^*_c(\alpha)$ by the relation

$$f\in {}_c(\alpha) \text{ if and only if } zf'\in {}^*_c(\alpha).$$

Thus a function $f\in {}_c(\alpha)$ $(0\le\alpha<1)$ satisfies

$$\operatorname{Re}\left(\frac{2(zf'(z))}{f'(z)+\bar{f}'(\bar{z})}\right)>\alpha,\ Z\in$$

Let $_{c.b}(\beta)(0\le\beta<1)$ be the subclass of $_c(\beta)$ consisting of funcitons f of the form (1.3).

Theorem 2.4. *Suppose that* $0\le\beta<1$, $m\in$ *and* $b\ge0$. *If* $F\in {}_{c,b}(\beta)$ *then f defined by (1.2) is in* $_{c,\gamma}(\beta)$ *for* $|z|<r_0$ *where* $\gamma=((m+2)/(m+1))b$ *and* r_0 *is the smallest positive root of (2.1). The result is sharp for the function*

$$F(z)=\int_0^z\frac{1}{\left(1-bt+t^2\right)^{1-\beta}}dt.$$

Proof. The result follows from Theorem 2.2 and the fact that $f\in {}_{c,b}(\beta)$ in $|z|<r_0$ if and only if $zf'\in {}^*_{c,2b}(\beta)$ in $|z|<r_0$.

Recall that a function $f\in {}_c(\alpha,\beta), 0\le\alpha,\beta<1$ satisfy

$$\operatorname{Re}\left(\frac{2zf'(z)}{g(z)+\bar{g}(\bar{z})}\right)>\alpha,\ \ Z\in$$

for some $g\in {}^{*}_{c}(\beta)$. Let ${}_{c,b}(\alpha,\beta)(0\le\alpha,\beta<1)$ be the subclass of ${}_{c}(\alpha,\beta)$ consisting of functions *f* of the form (1.3).

Theorem 2.5. Let $F\in C_{c,b}(\alpha,\beta)$ with respect to $H\in {}^{*}_{C,b}(\beta)$ where $0\le\alpha,\beta<1$ and $b\ge 0$, then *f* defined by (1.2) is in ${}_{c,\gamma}(\alpha,\beta)$ with respect to $h\in {}^{*}_{c,\gamma}(\beta)$ for $|z|<r_0$ where $\gamma=((m+2)/(m+1))b$, r_0 is the smallest positive root of (2.1) and

(2.7) $$h(z)=\frac{z^{1-m}}{1+m}\left(z^{m}H(z)\right)'.$$

This result is sharp.

Proof. Considering the function $P: \ \to\ $ defined by

$$P(z)=\frac{2zF'(z)}{H(z)+\bar{H}(\bar{z})},$$

we note that $\operatorname{Re}P(z)>\alpha$ since $F\in {}_{c,b}(\alpha,\beta)$ and

$$\frac{2zf'(z)}{h(z)+\bar{h}(\bar{z})}=\frac{\left(2z^{m+1}F'(z)\right)'}{\left(z^{m}\left(H(z)+\bar{H}(\bar{z})\right)\right)'}=\frac{\left(P(z)z^{m}\left(H(z)+\bar{H}(\bar{z})\right)\right)'}{\left(z^{m}\left(H(z)+\bar{H}(\bar{z})\right)\right)'}$$

$$=P(z)+P'(z)\frac{z^{m}K(z)}{\left(z^{m}K(z)\right)'}=p(z)+P'(z)\frac{z}{M(z)},$$

where

$$K(z)=\frac{H(z)+\bar{H}(\bar{z})}{2}\ \text{ and }\ M(z)=\frac{zK'(z)}{K(z)}+m.$$

Since $H\in {}^{*}_{c,b}(\beta)$, the rest of the proof is similar to the proof of Theorem 2.2 and hence is omitted.

3. Livingston Problem for Other Classes. Let ${}^{*}_{s}(\alpha)(0\le\alpha<1)$ represents class of starlike functions with respect to symmetric points of order α, that is $f\in {}^{*}_{s}(\alpha)$ if

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)-f(-z)}\right)>\alpha,\quad z\in U.$$

For $0\le\alpha,\beta<1$ and $b\ge 0$, let ${}^{*}_{s,b}(\alpha)$, ${}_{s,b}(\alpha)$ and ${}_{s,b}(\alpha,\beta)$ be subclasses of ${}^{*}_{s}(\alpha)$,

$_s(\alpha)$ and $_s(\alpha,\beta)$ consisting of funcitons f of the form (1.3). For these classes, Livingston problem can be solved as seen by the following theorems :

Theorem 3.1. *Suppose that* $0 \le \beta < 1, m \in$ *and* $b \ge 0$. *If* $F \in {}^{*}_{s,b}(\beta)$ *then f defined by (1.2) is in* ${}^{*}_{s,\gamma}(\beta)$ *for* $|z| < r_0$ *where* $\gamma = ((m+2)/(m+1))b$ *and* r_0 *is the smallest positive root of (2.1). This result is sharp.*

Proof. *Taking the function*

$$P(z) = \frac{2zF(z)}{F(z) - F(-z)}, \qquad z \in$$

and proceeding in a similar way as in Theorem 2.2, we obtain the desired result.

Using the fact that $f \in {}_{s,b}(\beta)$ if and only if $zf' \in {}_{s,2b}(\beta)$ we deduce the following result:

Theorem 3.2. *Suppose that* $0 \le \beta < 1, m \in$ *and* $b \ge 0$. *If* $F \in {}_{s,b}(\beta)$ *then f defined by (1.2) is in* ${}_{s,\gamma}(\beta)$ *for* $|z| < r_0$ *where* $\gamma = ((m+2)/(m+1))b$ *and* r_0 *is the smallest positive root of* (2.1). *The result is sharp.*

Theorem 3.3. *Let* $F \in {}_{s,b}(\alpha,\beta)$ *with respect to* $H \in {}^{*}_{s,\gamma}(\beta)$ *where* $0 \le \alpha, \beta < 1$ *then f defined by* (1.2) *is in* ${}_{s,\gamma}(\alpha,\beta)$ *with respect to* $h \in {}^{*}_{s,\gamma}(\beta)$ *for* $|z| < r_0$, *where* $\gamma = ((m+2)/(m+1))b, r_0$ *is the smallest positive root of* (2.1) *and is given by* (2.7). *This result is sharp.*

Proof. Follows in a similar way as Theorem 2.5 by taking the function

$$P(z) = \frac{2zF'(z)}{H(z) - H(-z)}, \qquad z \in \ .$$

Finally, let ${}^{*}_{sc,b}(\alpha)$, ${}_{sc,b}$ and ${}_{sc,b}(\alpha,\beta)$ be subclasses of ${}^{*}_{sc}(\alpha)$, ${}_{sc}(\alpha)$ and ${}_{sc}(\alpha,\beta)$ consisting of functions f of the form (1.3) for $0 \le \alpha, \beta < 1$ and $b \ge 0$. The next three theorems discusses the Livingston problem for these classes.

Theorem 3.4. *Suppose that* $0 \le \beta < 1, m \in$ *and* $b \ge 0$. *If* $F \in {}^{*}_{sc,b}(\beta)$ *then f defined by (1.2) is in* ${}^{*}_{sc,\gamma}(\beta)$ *for* $|z| < r_0$ *where* $\gamma = ((m+2)/(m+1))b$ *and* r_0 *is the smallest positive root of* (2.1). *This result is sharp.*

Proof. The proof follows in a similar way as Theorem 2.2 by considering the function

$$P(z) = \frac{2zF'(z)}{F(z) - \overline{F}(-\overline{z})}, \quad z \in \ .$$

Applying Theorem 3.4 and using the fact that $f \in {}_{sc,b}(\beta)$ if and only if $zf' \in {}^{*}_{sc,2b}(\beta)$, we deduce the following result.

Theorem 3.5. *Suppose that* $0 \le \beta < 1, m \in$ *and* $b \ge 0$. *If* $F \in {}_{sc,b}(\beta)$, *then f defined by (1.2) is in* ${}_{sc,\gamma}(\beta)$ *for* $|z| < r_0$ *where* $\gamma = ((m+2)/(m+1))b$ *and* r_0 *is the smallest positive root of (2.1). This result is sharp.*

Similar result hold for class ${}_{sc,b}(\alpha,\beta)$ of close-to-convex funcitions with fixed second coefficient, as seen by the next theorem.

Theorem 3.6. *Let* $F \in {}_{sc,b}(\alpha,\beta)$ *with respect to* $H \in {}^{*}_{sc,b}(\beta)$ *where* $0 \le \alpha,\beta < 1$ *then f defined by (1.2) is in* ${}_{sc,b}(\alpha,\beta)$ *with respect to* $h \in {}^{*}_{sc,\gamma}(\beta)$ *for* $|z| < r_0$ *where* $\gamma = ((m+2)/(m+1))b, r_0$ *is the smallest positive root of (2.1) and h is given by (2.7). This result is sharp.*

Proof. It follows in a similar way as Theorem 2.5 by considering the function

$$P(z) = \frac{2zF'(z)}{H(z) + \bar{H}(-\bar{z})}, \qquad z \in$$

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FIXED POINT THEOREMS FOR R-WEAKLY COMMUTING MAPPING OF TYPE (*Ag*) MAPS IN FUZZY METRIC SPACE SATISFYING INTEGRAL TYPE INEQUALITY

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ABSTRACT

The aim of this paper is to prove fixed point theorems by using the property (S–B) and the notion of R-weak commuting mapping of type (Ag) in fuzzy metric space satisfying integral type inequality.

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Key Words and Phrases : Fixed point theorems, R-weakly, Commuting mappings of type Ag maps, Fuzzy metric space, Letesque-integral mapping.

1. Introduction. Pant [2] introduced the concept of R-weakly commuting maps in metric spaces. Later Pathak et al. [4] generalized this concept and gave the concept of R-weakly commuting mappings of type (A_g). Pant [3] defined the concept of R-weakly commuting maps of type (A_g) in the fuzzy metric space. In fact the main application of R-weakly commuting mappings of type (A_g) is in the study of common fixed points of noncompatible maps. It may be recalled that Khan [5] introduced the notion of compatibility and the study of common fixed points of compatible mappings emerged as an area of intense research activity during the last more than one decade. However, the study of non compatible mappings is also very interesting. The aim of the present paper is to obtain common fixed point theorems by using the property (S-B) and the notion of R-weakly commuting mapping of types (A_g) in fuzzy metric space satisfying integral type inequality.

2. Perliminaries.

Definition 1[2]. Two maps A and S are called R-weakly commuting at a point x if $d(ASx,SAx) \leq Rd(Ax,Sx)$ for some $R > 0$. A and S are called point wise R-weakly commuting on X if given x in X, there exists $R>0$ such that $d(ASx,SAx) \leq Rd(Ax,Sx)$.

Definition 2[8]. Two mappings A and S of a fuzzy metric space $(X,M,*)$ into itself are R-weakly commuting provided there exists some real number R such that $M(ASx,SAx,t) \geq M(Ax,Sx,t/R)$ for each $x \in X$ and $t>0$.

Definition 3[4]. Two self mappings A and S of a metric space (X,d) are called R-weakly commuting of type (A_g) if there exists some positie real number R such that $d(AAx,SAx) \le Rd(Ax,Sx)$ for all x in X.

Definition 4[3]. Two mappings A and S of a fuzzy metric space $(X,M,*)$ into itself are R-weakly commuting of type (A_g) provided there exists some real number R such that

$M(AAx,SAx,t) \ge M(Ax,Sx,t/R)$ for each $x \in X$ and $t>0$.

Defitions 5[7]. Two self mappings S and T of an fuzzy space $(X,M,*)$. We say that S and T satisfy the property $(S\text{-}B)$ if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z \text{ for some } z \in X.$$

Example 1[7]. Let $X = [0,+\infty)$. Consider $(X,M,*)$ be a fuzzy metric space, where

$M = \dfrac{t}{t+d(x,y)}$. Define $S,T : X \to X$ by $Tx = x/5$ and $Sx = 3x/5$, for all x in X.

Consider the sequence $\{x_n\} = \{1/n\}$. Clearly $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = 0$. Then S and T satisfy the property $(S\text{–}B)$.

3. Main Result.

Theorem 1. *Let $(X,M,*)$ be a fuzzy metric space and f and g be point wise R-weakly commuting self maps of type (A_g) of X satisfying the following conditions:*

(1) $f(X) \subset g(X)$

(2) *There exist a constant* $k \in (0,1)$ *such that*

$$\int_0^{M(fx,fy,kt)} \phi(t)dt \ge \int_0^{\min\{M(gx,gy,t),M(fx,gx,t),M(fy,gy,t),M(fx,gy,t)\}} \phi(t)dt,$$

where $\phi : R^+ \to R^+$ is Lebesgue-integrable mapping which is summable, non-negative and such that

$$\int_0^{\varepsilon} \phi(t)dt > 0 \text{ for each } \varepsilon > 0.$$

If f and g satisfy the property $(S\text{–}B)$ and the rane of either of $f(X)$ or $g(X)$ is a complete subspace of X, then f and g have a unique common fixed point.

Proof. Since f and g satisfy the property $(S\text{-}B)$, there exists a sequence $\{x_n\}$ in X such that (i)

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z \text{ for some } z \in X.$$

Since $z \in f(X)$ and $f(X) \subset g(X)$, there exists some point u in X such that $z=gu$ where $z = \lim_{n\to\infty} gx_n$. If $fu \neq gu$, the inequality

$$\int_0^{M(fx_n,fu,kt)} \phi(t)dt \geq \int_0^{\min\{M(gx_n,gu,t),M(fx_n,gx_n,t),M(fu,gu,t),M(fu,gx_n,t),M(fx_n,gu,t)\}} \phi(t)dt$$

Letting $n \to \infty$,

$$\int_0^{M(gu,fu,kt)} \phi(t)dt \geq \int_0^{\min\{M(gu,gu,t),M(gu,gu,t),M(fu,gu,t),M(fu,gu,t),M(gu,gu,t)\}} \phi(t)dt$$

$$\geq \int_0^{M(fu,gu,t)} \phi(t)dt.$$

Hence $fu=gu$. Since f and g are R-weakly commuting of type (A_g), there exists $R>0$ such that

$$M(ffu,gfu,t) \geq M(fu,gu,t/R) = 1.$$

That is $ffu = gfu$ and $ffu = fgu = gfu = ggu$. If $fu \neq ffu$, using (2) we have

$$\int_0^{M(fu,ffu,kt)} \phi(t)dt \geq \int_0^{\min\{M(gu,gfu,t),M(fu,gu,t),M(ffu,gfu,t),M(ffu,gfu,t),M(gu,ffu,t)\}} \phi(t)dt$$

$$\geq \int_0^{M(fu,ffu,t)} \phi(t)dt,$$

a contradiction. Hence $fu = ffu$ and $fu = ffu = fgu = gfu = ggu$.

Hence fu is a common fixed point of f and g. The case when $f(X)$ is a complete subspace of X is similar to the above case since $f(X) \subset g(X)$. Hence we have proved the theorem.

Example 1. Let $X = [2,20]$. Define $f,g : X \to X$ as

$f(x) = 2$ if $x = 2$ or >5, $\quad f(x) = 6$ if $2 < x \leq 5$.

$g(2) = 2$, $\quad g(x) = x + 4$ if $2 < x \leq 5$, $\quad g(x) = \dfrac{4x+10}{15}$ if $x>5$.

Also we define $M(fx,gy,t) = \dfrac{t}{t + d(fx,gy)}$ for every $x, y \in X$ and $t>0$.

Then f and g satisfy all the conditions of the above theorem and have a common fixed point at x=2. In this example $f(X)=\{2\}\cup\{6\}$ and $g(X)=[2,9]$.

It may be seen that $f(X)\subset g(X)$. It can be verified also that f and g are point wise R-weakly commuting maps of type (A_g) and satisfy the property (S-B).

Theorem 2. *Let (X,M,*) be a fuzzy metric space and f and g be non compatible point-wise R-weakly commuting self maps of type (A_g) of X satisfying the following conditions:*

(1) $f(X)\subset g(X)$,

(2) *There exist a constant* $k\in(0,1)$ *such that*

$$\int_0^{M(fx,fy,kt)}\phi(t)dt \geq \int_0^{\min\{M(gx,gy,t),M(fx,gx,t),M(fy,gy,t),M(fy,gy,t),M(fx,gy,t)\}}\phi(t)dt.$$

where $\phi: R^+ \to R^+$ *is Lebesque-integrable mapping which is summable, non-negative, and such that* $\int_0^{\varepsilon}\phi(t)dt>0$ *for each* $\varepsilon>0$.

If the range of f or g is a complete subspace of X, then f and g have a unique common fixed point and the fixed point is the point of discontinuity.

Proof. Since f and g are non-compatible maps, there exists a sequence $\{x_n\}$ in X such that (i) $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$ for some z in X. Either $\lim_{n\to\infty} M(fgx_n, gfx_n, t)\neq 1$ or the limit does not exist. Since $z\in f(x)$ and $f(X)\subset g(X)$, there exists some point u in X such that z=gu where $z=\lim_{n\to\infty} gx_n$ If $f(u)\neq g(u)$, the inequality

$$\int_0^{M(fx_n,fu,kt)}\phi(t)dt \geq \int_0^{\min.\{M(gx_n,gu,t),M(fx_n,gx_n,t),M(fu,gu,t),M(fu,gx_n,t),M(fx_n,gu,t)\}}\phi(t)dt.$$

Letting $n\to\infty$, we have $\int_0^{M(gu,fu,kt)}\phi(t)dt \geq \int_0^{M(gu,fu,t)}\phi(t)dt.$

Hence $fu=gu.$

Since f and g are R-weak commuting of type (A_g), there exists R>0 such that $M(ffu,gfu,t)\geq M(fu,gu,t/R)=1.$

That is, fu=gfu and ffu=fgu=gfu=ggu. If $fu\neq ffu$, using (2) we have

$$\int_0^{m(fu,ffu,kt)}\phi(t)\,dt \geq \int_0^{\min\{M(gu,gfu,t),M(fu,gu,t),M(ffu,gfu,t),M(ffu,gfu,t),M(gu,ffu,t)\}}\phi(t)dt$$

$$\geq \int_0^{M(fu,ffu,t)} \phi(t)\,dt,$$

a contradiction.

Hence $fu = ffu$ and $fu = ffu = fgu = gfu = ggu$.

Hence fu is a common fixed point of f and

The case when $f(X)$ is a complete subspace of X is similar to the above case since $f(X) \subset g(X)$.

Now we show that f and g are discontinuous at the common fixed point $z=fu=gu$. If possible, suppose f is continuous. Then considering the sequence $\{x_n\}$ of (i) we get $\lim_{n\to\infty} ffx_n = fz = z$. R-weak commutating of type (A_g) implies that,

$$M(ffx_n, gfx_n, t) \geq M(fx_n, gx_n, t/R) = 1,$$

which on letting $n \to \infty$ yields $\lim_{n\to\infty} gfx_n = fz = z$.

This, in turn, yields $\lim_{n\to\infty} M(fgx_n, gfx_n, t) = 1$. This contradicts the fact that $\lim_{n\to\infty} M(fgx_n, gfx_n, t)$ is either non zero or nonexistent for the sequence $\{x_n\}$ of (i)

Hence f is discontinuous at the fixed point.

Next, suppose that g is continuous, then for the sequence $\{x_n\}$ of (i),

We get $\lim_{n\to\infty} gfx_n = gz = z$ and $\lim_{n\to\infty} ggx_n = gz = z$.

In view of these limits, the inequality

$$\int_0^{M(fx_n, fgx_n, kt)} \phi(t)\,dt \geq \int_0^{\min(M(gx_n, ggx_n.t), M(fx_n, ggx_n.t), M(fgx_n, ggx_n.t), M(fgx_n gx_n.t), M(fx_n.ggx_n.t))} \phi(t)\,dt$$

yields a contradiction unless $\lim_{n\to\infty} fgx_n = fz = gz$ but $\lim_{n\to\infty} fgx_n = gz$ and $\lim_{n\to\infty} gfx_n = gz$ contradicts the fact that $\lim_{n\to\infty} M(fgx_n, gfx_n, t)$ is either nonzero or nonexistent. Thus both f and g are discontinuous at their common fixed point. Hence we have proved the theorem.

We now give an example to illustrate the above theorem.

Example 2. Let X=[2,20]. Define $f, g : X \to X$ *as*

$$f(x)\begin{pmatrix} 2, \text{ if } x = 2 \text{ or } > 5 \\ 6, \text{ if } 2 < x \leq 5 \end{pmatrix},$$

$$g(2)=2, g(x)=\begin{cases} 7, \text{ if } 2<x\leq 5 \\ \dfrac{4x+10}{15} \text{ if } x>5 \end{cases}.$$

Also we define

$$M(fx,gy,t)=\frac{t}{t+d(fx,gy)}, \text{ For every } x,y \in X \text{ and } t>0.$$

Then f and g satisfy all the conditions of the above theorem and have a common fixed point at x=2. in this example $f(x)=\{2\}\cup\{6\}$ and $g(x)=[2,6]\cup[7]$.

It may be seen that $f(X)\subset g(X)$. It can be verified also that f and g are point wise R-weakly commuting maps of type (A_g). To see that f and g are non compatiable, let us consider a sequence $\{x_n k=5+1/n : n>1\}$,

then $\lim_{n\to\infty} fx_n = 2, \lim_{n\to\infty} gx_n = 2, \lim_{n\to\infty} fgx_n = 6$ and $\lim_{n\to\infty} gfx_n = 2$.

Hence f and g are non compatible.

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ON SOME GENERALIZED FRACTIONAL INTEGRALS INVOLVING GENERALIZED SPECIAL FUNCTIONS OF SEVERAL VARIABLES

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ABSTRACT

In the present paper making an appeal to Saigo-Maeda opertors [21], we derive several left sided and right sided generalized fractional integrals involving multivariable *H*-function of Srivastava and Panda [14] to [16], generalized multiple hypergeometric function of Srivastava and Daoust ([9],[10]; also see Srivastava and Manocha [12,p.64 (18), (19),(20)]), generalized polynomials of Srivastava [8] and Srivastava-Garg [18], Srivastava-Singh [19] and generalized Fox-Wright function [2]. Some interesting special cases are also discussed to derive the results involving the product of many Srivastava polynomials [7], many Laguerre polynomials, many Hermite polynomials or many Jacobi polynomials, with multivariable *H*-function of Srivatava and Panda, generalized multiple hypergeometric function of Srivastava and Daoust or generalized Fox-Wright function.

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Key Words and Phrases : Fractional integrals, Saigo-Meda operators, Multivariable *H*-function of Srivastava and Panda, Multiple hypergeometric function of Srivastava and Daoust, Generalized Fox-Wright function.

1. Introduction. Recently Ram and Chandak [5] derived fractional derivative formulas for the product of multivariable polynomials due to Srivastava and Garg [18], the Fox-Wright generalized hypergeometric function ${}_p\Psi_q$ including its special cases introduced by Wright [24] by using generalized modified Saigo fractional derivative operator due to Samko [6] defined by

$$(1.1)\quad D_{0,x,m}^{\alpha,\beta,\eta} f(x) = \frac{d}{dz}\left(\frac{z^{-m(\beta-\eta)}}{\Gamma(1-\alpha)}\int_0^x \left(x^m - t^m\right)^{-\alpha}\right.$$

$$F\left[\beta-\alpha, 1-\eta; 1-\alpha; \left(1-t^m/x^m\right)\right],\ 0\le\alpha<1, \beta,\eta,x\in R, m\in N.$$

Actually,

(1.2) $D_{0,x,1}^{\alpha,\alpha,\eta} f(x) = D_x^{\alpha} f(x)$,

where D_x^{α} is familiar Riemann-Liouville fractional derivative operator defined by Miller and Ross [3].

For $0 \le \alpha < 1, m \in N, \beta, \eta, x \in R, k > \max[0, m(\beta - \eta)] - m$, Bhatt and Raina [1] established

(1.3) $D_{0,x,m}^{\alpha,\beta,\eta} x^k = \dfrac{\Gamma(1+k/m)\Gamma(1+\eta-\beta+k/m)}{\Gamma(1-\beta+k/m)\Gamma(1+\eta-\alpha+k/m)} x^{k-n,\beta}$.

Saigo [20] defined the operaors

(1.4) $\left(I_{0+}^{\alpha,\beta,\eta} f\right)x = \dfrac{x^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^x (n-t)^{\alpha-t} \, {}_2F_1\left(\alpha+\beta, -\eta; \alpha; (1-t/x)\right) f(t) dt \ \left(\mathrm{Re}(\alpha) > 0\right)$

$= \left(\dfrac{d}{dx}\right)^k \left(I_{0+}^{\alpha+k,\beta-k,\eta-k} f\right)x \quad \left(\mathrm{Re}(\alpha) \le 0; k = [\mathrm{Re}(-\alpha)] + 1\right)$,

(1.5) $I_{-}^{\alpha,\beta,\eta} f(x) = \dfrac{1}{\Gamma(\alpha)} \int_x^{\infty} (t-x)^{\alpha-1} t^{-\alpha-\beta} \, {}_2F_1(\alpha+\beta, -\eta; \alpha; 1-x/t) f(t) dt, \ \left(\mathrm{Re}(\alpha) > 0\right)$

$= \left(-\dfrac{d}{dt}\right)^k \left(I_{0+}^{\alpha+k,\beta-k,\eta} f\right)x, \quad \left(\mathrm{Re}(\alpha) \le 0; k = [\mathrm{Re}(-\alpha)] + 1\right)$.

Saigo [20] also introduced the operator:

(1.6) $\left(D_{0+}^{\alpha,\beta,\eta} f\right)x = \left(I_{0+}^{-\alpha,-\beta,\alpha+\eta} f\right)(x) = \left(\dfrac{d}{dx}\right)^k \left(I_{0+}^{-\alpha+k,-\beta-k,\alpha+\eta-k} f\right)(x)$

$\left(\mathrm{Re}(\alpha) > 0; k = [\mathrm{Re}(-\alpha)] + 1\right)$,

(1.7) $\left(D_{-}^{\alpha,\beta,\eta} f\right)x = \left(I_{-}^{-\alpha,-\beta,\alpha+\eta} f\right)(x) = \left(-\dfrac{d}{dx}\right)^k \left(I^{-\alpha+k,-\beta-k,\alpha+\eta} f\right)(x)$

$\left(\mathrm{Re}(\alpha) > 0; k = [\mathrm{Re}(-\alpha)] + 1\right)$.

Further Saigo-Maeda [21] introduced fractional calculus operators

(1.8) $\left(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} f\right)(x) = \dfrac{x^{-\alpha}}{\Gamma(\gamma)} \int_0^x (x-t)^{\gamma-1} t^{-\alpha'} F_3(\alpha, \alpha', \beta, \beta', \gamma; 1-t/x, 1-x/t)$

$f(t) d(t) \mathrm{Re}(\gamma) > 0$

$= \left(\dfrac{d}{dx}\right)^k \left(I_{0+}^{\alpha,\alpha',\beta+k,\beta',\gamma+k} f\right)(x) \quad \left(\mathrm{Re}(\gamma) \le 0; k = [-\mathrm{Re}(\gamma)] + 1\right)$, (1.9) $\left(I_{-}^{\alpha,\alpha',\beta,\beta',\gamma}f\right)(x)=\dfrac{x^{-\alpha'}}{\Gamma(\gamma)}\int_x^{\infty}(t-x)^{\gamma-1}t^{-\alpha}F_3(\alpha,\alpha',\beta,\beta',\gamma;1-x/t,1-t/x)$

$$f(t)dt \quad \mathrm{Re}(\gamma)>0$$

$$=\left(-\frac{d}{dx}\right)^k\left(I_{0+}^{\alpha,\alpha',\beta,\beta'+k,\gamma+k}f\right)(x) \quad \left(\mathrm{Re}(\gamma)\le 0;k=\left[-\mathrm{Re}(\gamma)\right]+1\right),$$

(1.10) $\left(D_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}f\right)(x)=\left(I_{0+}^{-\alpha',-\alpha,-\beta',-\beta,-\gamma}f\right)(x)$

$$=\left(\frac{d}{dx}\right)^k\left(I_{0+}^{-\alpha',-\alpha,-\beta'+k,-\beta,-\gamma+k}f\right)(x) \quad \left(\mathrm{Re}(\gamma)>0;k=\left[\mathrm{Re}(\gamma)\right]+1\right),$$

(1.11) $\left(D_{-\alpha}^{\alpha,\alpha',\beta,\beta',\gamma}f\right)(x)=\left(I_{-}^{-\alpha',-\alpha,-\beta',-\beta,-\gamma}f\right)(x)$

$$=\left(-\frac{d}{dx}\right)^k\left(I_{0+}^{-\alpha',-\alpha,-\beta',-\beta+k,-\gamma+k}f\right)(x) \quad \left(\mathrm{Re}(\gamma)>0;k=\left[\mathrm{Re}(\gamma)\right]+1\right).$$

In the present paper making an appeal to (1.8) and (1.9) we shall derive respectively left-sided and right-sided generalized fractional integrals involving the products of multivariable H-function of Srivastava-Panda ([14] to [16]), generalized multiple hypergeometric function of Srivastava and Daoust ([9],[10]), generalizd multivariable polynomials of Srivastava [8], Srivastava-Garg [18] and generalized Fox-Wright function [24]. Some interesting speical cases will also be discussed.

2. Left-Sided Generalized Fractional Integral Operator $I_0^{\alpha,\alpha',\beta,\beta',\gamma}$.

For $\mathrm{Re}(c)>0,\mathrm{Re}(s)>\max\left[\mathrm{Re}(-a'),\mathrm{Re}(-b'),\ \mathrm{Re}(a+b-c)\right]$, making an appeal to Prudnikovr et al. ([4], eqn. 8.4.51-2), we have

(2.1) $\int_0^{\infty}x^{s-1}(1-x)^{c-1}F_3(a,a',b,b';c;1-x,1-1/x)dx$

$$=\frac{\Gamma(c)\Gamma(s+a')\Gamma(s+b')\Gamma(s+c-a-b)}{\Gamma(s+a'+b')\Gamma(s+c-a)\Gamma(s+c-b)}.$$

Further making an appeal to (2.1), one can derive

(2.2) $\left(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}x^{\rho-1}\right)(x)=\dfrac{\Gamma(\rho)\Gamma(\rho+\gamma-\alpha-\alpha'-\beta)\Gamma(\rho+\beta'-\alpha')x^{\rho-\alpha-\alpha'+r-1}}{\Gamma(\rho+\gamma-\alpha-\alpha')\Gamma(\rho+\gamma-\alpha'-\beta)\Gamma(\rho+\beta')}$,

where $\mathrm{Re}(\gamma)>0,\mathrm{Re}(\rho)>\max\left[0,\mathrm{Re}(\alpha+\alpha'+\beta-\gamma),\mathrm{Re}(\alpha'-\beta')\right]$.

Now employing (2.2) we derive the various fractional integrals involving different generalized special functions :

3. Results Invoving Generalized Multiple Hypergeometric Function of Srivastava and Daoust .

(3.1) $\left(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1} S\left(z_1 t^{\sigma_1},\ldots,z_n t^{\sigma_n}\right)\right)(x) = x^{\rho-\alpha-\alpha'+\gamma-1}\overline{S}\left(z_1 x^{\sigma_1},\ldots,z_n x^{\sigma_n}\right),$

provided that $\mathrm{Re}(\gamma)>0, \mathrm{Re}(\rho)>\max\left[0,\mathrm{Re}(\alpha+\alpha'+\beta-\gamma),\mathrm{Re}(\alpha'-\beta')\right],$

$$1+\sum_{j=1}^{C}\psi_j^{(i)}+\sum_{j=1}^{D^{(i)}}\delta_j^{(i)}-\sum_{j=1}^{A}\theta_j^{(i)}-\sum_{j=1}^{B^{(i)}}\phi_j^{(i)}>0, i=1,\ldots,n.$$

$$S\left(x_1,\ldots,x_n\right)=S_{C:D';\ldots;D^{(n)}}^{A:B',\ldots,B^{(n)}}\left(\begin{matrix}\left[(a):\theta',\ldots,\theta^{(n)}\right]:\left[(b'):\phi'\right];\ldots;\left[\left(b^{(n)}\right):\phi^{(n)}\right];\\ \left[(c):\psi',\ldots,\psi^{(n)}\right]:\left[(d'):\delta'\right];\ldots;\left[\left(d^{(n)}\right):\delta^{(n)}\right];\end{matrix} x_1,\ldots,x_n\right)$$

is generalized multiple hypergeometric function of Srivatava and Daoust ([9], [10]) and

$$\overline{S}\left(x_1,\ldots,x_n\right)=S_{C+3:D';\ldots;D^{(n)}}^{A+3:B';\ldots;B^{(n)}}\left(\begin{matrix}\left[(a):\theta',\ldots,\theta^{(n)}\right],\left[\rho:\sigma_1,\ldots,\sigma_n\right],\\ \left[(c):\psi',\ldots,\psi^{(n)}\right],\left[\rho+\gamma-\alpha-\alpha':\sigma_1,\ldots,\sigma_n\right]\end{matrix}\right.$$

$$\left.\begin{matrix}\left[\rho+\gamma-\alpha-\alpha'-\beta:\sigma_1,\ldots,\sigma_n\right],\left[\rho+\beta'-\alpha':\sigma_1,\ldots,\sigma_n\right]: & \left[(b'):\phi'\right];\ldots;\left[\left(b^{(n)}\right):\phi^{(n)}\right];\\ \left[\rho+\gamma-\alpha'-\beta:\sigma_1,\ldots,\sigma_n\right],\left[\rho+\beta':\sigma_1,\ldots,\sigma_n\right]: & \left[(d'):\delta'\right];\ldots;\left[\left(d^{(n)}\right):\delta^{(n)}\right];\end{matrix} x_1,\ldots,x_n\right).$$

(3.2) $\left(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1} S\left(z_1 t^{\sigma_1},\ldots,z_n t^{\sigma_n}\right)\right) S_L^{h_1,\ldots,h_s}\left(y_1 t^{\lambda_1},\ldots,y_s t^{\lambda_s}\right)(x)$

$$=x^{\rho-\alpha-\alpha'+\gamma-1}\sum_{k_1,\ldots,k_s=0}^{h_1k_1+\ldots+h_sk_s\le L}(-L)_{h_1k_1+\ldots+h_sk_s} A\left(L;k_1,\ldots,k_s\right)\frac{y_1^{k_1}}{k_1!}\cdots\frac{y_s^{k_s}}{k_s!}x^{\lambda_1k_1+\ldots+\lambda_sk_s}\ \overline{\overline{S}}\left(z_1x^{\sigma_1},\ldots,z_nx^{\sigma_n}\right),$$

provided that $\mathrm{Re}(\gamma)>0, \mathrm{Re}(\rho)=\max\left(0,\mathrm{Re}\,\alpha+\alpha'+\beta-\gamma,\mathrm{Re}(\alpha'-\beta')\right),$

$$1+\sum_{j=1}^{C}\psi_j^{(i)}+\sum_{j=1}^{D^{(i)}}\delta_j^{(i)}-\sum_{j=1}^{A}\theta_j^{(i)}-\sum_{j=1}^{B^{(i)}}\phi_j^{(i)}>0, i=1,\ldots,n;$$

$$\overline{\overline{S}}\left(z_1,\ldots,z_n\right)=S_{C+3:D';\ldots;D^{(n)}}^{A+3:B';\ldots;B^{(n)}}\left(\begin{matrix}\left[(a):\theta',\ldots,\theta^{(n)}\right],\left[\rho+\sum_{i=1}^{s}\lambda_ik_i:\sigma_1,\ldots,\sigma_n\right],\\ \left[(c):\psi',\ldots,\psi^{(n)}\right],\left[\rho+\gamma-\alpha'-\beta+\sum_{i=1}^{s}\lambda_ik_i:\sigma_1,\ldots,\sigma_n\right],\end{matrix}\right.$$

$$\left[\rho+\gamma-\alpha-\alpha'-\beta+\sum_{i=1}^{s}\lambda_i k_i:\sigma_1,\ldots,\sigma_n\right],\left[\rho+\beta'-\alpha'+\sum_{i=1}^{s}\lambda_i k_i:\sigma_1,\ldots,\sigma_n\right]:$$

$$\left[\rho+\beta'+\sum_{i=1}^{s}\lambda_i k_i:\sigma_1,\ldots,\sigma_n\right],\left[\rho+\gamma-\alpha-\alpha'+\sum_{i=1}^{s}\lambda_i k_i:\sigma_1,\ldots,\sigma_n\right]:$$

$$\left.\begin{matrix}[(b'):\phi'];\ldots;\left[\left(b^{(n)}\right):\phi^{(n)}\right];\\ [(d'):\delta'];\ldots;\left[\left(d^{(n)}\right):\delta^{(n)}\right];\end{matrix} z_1,\ldots,z_n\right).$$

(3.3) $\left(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}t^{\rho-1}S\left(z_1t^{\sigma_1},\ldots,z_nt^{\sigma_n}\right)s_{n_1,\ldots,n_s}^{m_1,\ldots,m_s}\left(y_1t^{\lambda_1},\ldots,y_nt^{\lambda_n}\right)\right)(x)$

$$=x^{\rho-\alpha-\alpha'+\gamma-1}\sum_{k_1=0}^{[n_1/m_1]}\cdots\sum_{k_s=0}^{[n_s/m_s]}A\left(n_1,k_1;\ldots;n_s,k_s\right)\frac{(-n_1)_{m_1k_1}}{k_1!}\cdots\frac{(-n_s)_{m_sk_s}}{k_s!}$$

$$y_1^{k_1}\ldots y_s^{k_s}x^{\lambda_1k_1+\ldots+\lambda_sk_s}\bar{\bar{S}},$$

provided that $\mathrm{Re}(\gamma)>0,\mathrm{Re}(\rho)=\max\left[0,\mathrm{Re}(\alpha+\alpha'+\beta-\gamma),\mathrm{Re}(\alpha'-\alpha)\right]$

$$1+\sum_{j=1}^{C}\psi_j^{(i)}+\sum_{j=1}^{D^{(i)}}\delta_j^{(i)}-\sum_{j=1}^{A}\theta_j^{(i)}-\sum_{j=1}^{B^{(i)}}\phi_j^{(i)}>0, i=1,\ldots,n.$$

4. Special Cases of (3.3). For $A\left(n_1,k_1;\ldots;n_s,k_s\right)$

$= A\left(n_1,k_1\right)\ldots A\left(n_s,k_s\right),$

$S_{n_1,\ldots,n_s}^{m_1,\ldots,m_n}\left(x_1,\ldots,x_s\right)=S_{n_1}^{m_1}\left(x_1\right)\ldots S_{n_s}^{m_s}\left(x_s\right),$

therefore (3.3) gives

(4.1) $\left(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}t^{\rho-1}S\left(z_1t^{\sigma_1},\ldots,z_nt^{\sigma_n}\right)\prod_{i=1}^{s}S_{n_i}^{m_i}\left(y_it^{\lambda_i}\right)\right)(x)$

$$=x^{\rho-\alpha-\alpha'+\gamma-1}\sum_{k_1=0}^{[n_1/m_1]}\cdots\sum_{k_s=0}^{[n_s/m_s]}A_{n_1,k_1}\ldots A_{n_s,k_s}\left(-n_1\right)_{m_1k_1}\ldots\left(-n_s\right)_{m_sk_s}$$

$$\frac{y_1^{k_1}}{k_1!}\cdots\frac{\left(y_s\right)^{k_s}}{k_s!}x^{\lambda_1k_1+\ldots+\lambda_sk_s}\bar{\bar{S}}\left(z_1x^{\sigma_1},\ldots,z_nx^{\sigma_n}\right),$$

valid if all conditions of (3.3) are satisfied.

Case I Result Involving Hermite Polynomials. For each $m_i=2,A_{n_i,k_i}=(-1)k_i,$

we have $S_{n_i}^{2}\left(y_i\right)=y_i^{n_i/2}H_{n_i}\left(\frac{1}{2\sqrt{y_i}}\right), i=1,\ldots,s,$

therefore for Hermite polynomials [23, p. 106, Eqn. (5.5.4)] and [19,p.158], (4.1) reduces to

(4.2) $$I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1} S\left(z_1 t^{\sigma_1},\ldots,z_n t^{\sigma_n}\right) \prod_{i=1}^{s} \left(y_i t^{\lambda_i}\right)^{n_i/2} H_{n_i}\left(\frac{1}{2\sqrt{y_i t^{\lambda_i}}}\right)(x)$$

$$= x^{\rho-\alpha-\alpha'+\gamma-1} \sum_{k_1=0}^{[n_1/2]} \cdots \sum_{k_s=0}^{[n_s/2]} \prod_{i=1}^{s} (-1)^{k_i} \frac{(-n_i)_{2k_i} \left(y_i x^{\lambda_i}\right)^{k_i}}{k_i} \bar{\bar{S}}\left(z_1 x^{\sigma_1},\ldots,z_n x^{\sigma_n}\right)$$

valid if all conditions of (4.1) are satisfied

Case II Result Involving Laguerre Polynomials.

For each $m_i = 1, A_{n_i,k_i} = \frac{(1+\alpha_i)_{n_i}}{(1+\alpha_i)_{k_i}} \frac{1}{n_i!}$,

we have $s_{n_i}^{1}(y_i) \to L_{n_i}^{\alpha_i}(y_i), i = 1,\ldots,s.$

Therefore for Laguerre polynomials [23,p.100, eqn. (5.1.6)] and [19, p.158], (4.1) reduces to

(4.3) $$\left(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1} S\left(z_1 t^{\sigma_1},\ldots,z_n t^{\sigma_n}\right) \prod_{i=1}^{s} L_{n_i}^{(\alpha_i)}\left(y_i t^{\lambda_i}\right)\right)(x)$$

$$= x^{\rho-\alpha-\alpha'+\gamma-1} \prod_{i=1}^{s} \frac{(1+\alpha_i)_{n_i}}{n_i!} \sum_{k_1=0}^{n_1} \cdots \sum_{k_s=0}^{n_s} \prod_{i=1}^{s} \frac{(-n_i)_{k_i} \left(y_i x^{\lambda_i}\right)^{k_i}}{k_i!} \bar{\bar{S}}\left(z_1 x^{\sigma_1},\ldots,z_n x^{\sigma_n}\right)$$

valid if all conditions of (4.1) are satisfied.

Case III Results Involving Jacobi Polynomials. For each $m_i = 1$,

$$A_{n_i k_i} = \frac{(1+\alpha_i)_{n_i}}{n_i!} \frac{(1+\alpha_i+\beta_i+\eta_i)_{k_i}}{(1+\alpha_i)_{k_i}},$$

we have $S_{n_i}^{1}(y_i) \to P_{n_i}^{(\alpha_i,\beta_i)}(1-2y_i), i = 1,\ldots,s.$

Therefore, for Jacobi polynomials [23, p.68, eqn. (4.3.2)] and [19, p.159], (4.1) reduces to

(4.4) $$\left(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1} S\left(z_1 t^{\sigma_1},\ldots,z_n t^{\sigma_n}\right) \prod_{i=1}^{s} P_{n_i}^{(\alpha_i,\beta_i)}\left(-2y_i t^{\lambda_i}\right)\right)(x)$$

$$= x^{\rho-\alpha-\alpha'+\gamma-1} \prod_{i=1}^{s} \frac{(1+\alpha_i)_{n_i}}{n_i!} \sum_{k_1=0}^{n_1} \cdots \sum_{k_s=0}^{n_s} \prod_{i=1}^{s} \frac{(-n_i)_{k_i} (-1+\alpha_i+\beta_i+n_i)_{k_i}}{(1+\alpha_i)_{k_i}} \frac{\left(y_i x^{\lambda_i}\right)^{k_i}}{k_1!} \bar{\bar{S}}\left(z_1 x^{\sigma_1},\ldots,z_n x^{\sigma_n}\right)$$

where all conditions of (4.1) are satisfied.

5. Other Special Cases of (3.2) and (3.3). For $n=1$, $S^{A:B';...;B^{(n)}}_{C:D';...;D^{(n)}}$ reduses to generalized Fox-Wright function ${}_p\Psi_q$. Therefore (3.2) reduces to

(5.1) $\left(I^{\alpha,\alpha',\beta,\beta',\gamma}_{0+} t^{\rho-1} S^{h_1,...,h_s}_{L}\left(y_1 t^{\lambda_1},...,y_s t^{\lambda_s}\right) {}_p\Psi_q\left(zt^{\sigma}\right)\right)(x)$

$$= x^{\rho-\alpha-\alpha'+\gamma-1} \sum_{k_1,...,k_s=0}^{h_1k_1+...+h_sj_s\le L} (-L)_{h_1k_1+...+h_sk_s} A\left(L;k_1,...,k_s\right)\frac{y_1^{k_1}}{k_1!},...,\frac{y_s k_s}{k_s!} x^{\lambda_1 k_1+...+\lambda_s k_s} {}_{P+3}\overline{\Psi}_{q+3}\left(zx^{\sigma}\right)$$

where $a_i, b_j \in C; A_i > 0, B_j > 0, 1+\sum_{j=1}^{q} B_j - \sum_{i=1}^{\gamma} A_i \ge 0$, $A_i, B_j \in R\left(A_i, B_i \ne 0\right)$,

$i=1,...,p, j=1,...,q$; $\mathrm{Re}(\gamma)>0, \mathrm{Re}(\rho)=\min\left[0, \mathrm{Re}(\alpha+\alpha'+\beta-\gamma), \mathrm{Re}(\alpha'-\beta')\right]$ and

$$ {}_{p+3}\overline{\Psi}_{q+3}(z) = {}_{p+3}\Psi_{q+3}\left(\begin{matrix} (a_i, A_i)_{1,p}, \left(\rho+\sum_{i=1}^{s}\lambda_i k_i, \sigma\right), \\ (b_j, B_j)_{1,q}, \left(\rho+\gamma-\alpha'-\beta+\sum_{i=1}^{s}\lambda_i k_i, \sigma\right), \end{matrix}\right.$$

$$\left.\begin{matrix} \left(\rho+\gamma-\alpha-\alpha'-\beta+\sum_{i=1}^{s}\lambda_i k_i, \sigma\right), \left(\rho+\beta'-\alpha'+\sum_{i=1}^{s}\lambda_i k_i, \sigma\right) \\ \left(\rho+\beta'+\sum_{i=1}^{s}\lambda_i k_i, \sigma\right), \left(\rho+\gamma-\alpha-\alpha'+\sum_{i=1}^{s}\lambda_i k_i, \sigma\right) \end{matrix}\right| zx^{\sigma}\right),$$

while (3.3) reduces to

(5.2) $\left(I^{\alpha,\alpha',\beta,\beta',\gamma}_{0+} t^{\rho-1} S^{m_1,...,m_s}_{n_1,...,n_s}\left(y_1 t^{\lambda_1},...,y_s t^{\lambda_s}\right) {}_p\Psi_q\left(zt^{\sigma}\right)\right)(x)$

$$= x^{\rho-\alpha-\alpha'+\gamma} \sum_{k_1=0}^{[n_1/m_1]} \cdots \sum_{k_s=0}^{[n_s/m_s]} (-n_1)_{m_1k_1} \cdots (-n_s)_{m_sk_s} A\left(n_1,k_1;...;n_s,k_s\right)\frac{y_1^{k_1}}{k_1!}\cdots\frac{y_s^{k_s}}{k_s!} x^{\lambda_1 k_1+...+\lambda_s k_s}$$

$$ {}_{p+3}\overline{\Psi}_{q+3}\left(zx^{\sigma}\right),$$

where $a_i, b_j \in C, A_i > 0, B_j > 0, 1+\sum_{j=1}^{q} B_j - \sum_{i=1}^{p} A_i \ge 0, A_i, B_j \in R\left(A_i, B_j \ne 0\right), i=1,...,p;$

$j=1,...,q, \mathrm{Re}(\gamma)>0, \mathrm{Re}(\rho)=\min\left[0, \mathrm{Re}\,\alpha+\alpha'+\beta-\gamma, \mathrm{Re}(\alpha'-\beta')\right]$, m_i are positive integers, $A\left(n_1,k_1;...;n_s,k_s\right)$ are arbitrary constants real or complex.

For $A(n_1,k_1;...;n_s,k_s) = A_{n_1,k_1}...A_{n_s,k_s}$, we derive the following results involving the product of several generalized Srivastava polynomials $S_n^m(x)$ [30] of one variable and generalized Fox-Wright function [24]:

(5.3) $\left(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1} S_{n_1}^{m_1}\left(y_1 t^{\lambda_1}\right)...S_{n_s}^{m_s}\left(y_s t^{\lambda_s}\right) {}_p\Psi_q\left(zt^{\sigma}\right)\right)(x)$

$$= x^{\rho-\alpha-\alpha'+\gamma} \sum_{k_1=0}^{[n_1/m_1]} ... \sum_{k_s=0}^{[n_s/m_s]} (-n_1)_{m_1k_1} ...(-n_s)_{m_sk_s} A_{n_1,k_1},...,A_{n_s,k_s}$$

$$\frac{y_1^{k_1}}{k_1!}...\frac{y_s^{k_s}}{k_s!} x^{\lambda_1k_1+...+\lambda_sk_s} {}_{p+3}\overline{\Psi}_{q+3}\left(zx^{\sigma}\right).$$

6 Further Special Cases of (5.3).

Case I. Result involving Hermite polynomials. For each m_i=2, $A_{n_i,k_i} = (-1)^{k_i}$,

we have $S_{n_i}^2(y_i) = y_i^{n_i/2} H_{n_i}\left[1/\left(2\sqrt{y_i}\right)\right]$.

Therefore for Hermite polynomials [19, p.158; 23, p.106, eqn (5.5.4)], we derive from (5.3):

(6.1) $I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1} \prod_{i=1}^{s} \left(y_i t^{\lambda_i}\right)^{n_i/2} H_{n_i}\left(\frac{1}{2\sqrt{y_i t_i^{\lambda_i}}}\right) {}_p\Psi_q\left(zt^{\sigma}\right)(x)$

$$= x^{\rho-\alpha-\alpha'+\gamma-1} \sum_{k_1=0}^{[n_1/2]} ... \sum_{k_s=0}^{[n_s/2]} \prod_{i=1}^{s} (-n_i)_{2k_i} (-1)^{k_i} (y_i)^{k_i} x^{\lambda_i k_i} {}_{p+3}\overline{\Psi}_{q+3}\left(zx^{\sigma}\right).$$

Cases II. Result Involving Laguerre Polynomials. For each m_i=1,

$A_{n_i},k_i = \frac{(1+\alpha_i)_{n_i}}{(1+\lambda_i)_{k_i}} \frac{1}{n_i!}$, we have $S_{n_i}^1(y_i)$ and $L_{n_i}^{(\alpha_i)}(y_i), i = 1,...,s.$

Therefore (5.3) gives the result involving the product of many Laguerre polynomials [23, p.100, eqn. (5.1.6) and [19,p.158] and generalized Fox-Wright function

(6.2) $I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1} \prod_{i=1}^{s} L_{n_i}^{(\alpha_i)}\left(y_i t^{\lambda_i}\right) {}_p\Psi_q\left(zt^{\sigma}\right)(x)$

$$= x^{\rho-\alpha-\alpha'+\gamma-1} \prod_{i=1}^{s} \frac{(1+\alpha_i)}{n_i!}{}_{n_i} \sum_{k_1=0}^{n_1} ... \sum_{k_s=0}^{n_s} \prod_{i=1}^{s} \frac{(-n_i)_{k_i}}{(1+\lambda_i)_{k_i}} \frac{y_i^{k_i} x^{\lambda_i k_i}}{k_i!} {}_{p+3}\overline{\Psi}_{q+3}\left(zx^{\sigma}\right).$$

Case III. Result involving Jacobi polynomials. For each m_i=1,

$$A_{n_i,k_i} = \frac{(1+\alpha_i)_{n_i}}{n_i!}\frac{(1+\alpha_i+\beta_i+n_i)_{s_i}}{(1+\lambda_i)_{s_i}}, \text{ we have } S_{n_i}^1(y_i) \to P_{n_i}^{(\alpha_i,\beta_i)}(1-2y_i),\ i=1,\ldots,s,$$

Therefore for Jacobi polynomials [23, p.68, eqn. (4.3.2)] and [19,p.159], (7.5.3) reduces to

$$(6.3)\ I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1}\prod_{i=1}^{s} P_{n_i}^{(\alpha_i,\beta_i)}\left(1-2y_i t^{\lambda_i}\right){}_p\psi_q\left(zt^{\sigma}\right)(x)$$

$$= x^{\rho-\alpha-\alpha'+\gamma-1}\prod_{i=1}^{s}\frac{(1+\alpha_i)_{n_i}}{n_i!}\sum_{k_1=0}^{n_1}\cdots\sum_{k_s=0}^{n_s}\prod_{i=1}^{s}\frac{(-n_i)_{k_i}(1+\alpha_i+\beta_i+n_i)_{k_i}}{(1+\lambda_i)_{k_i}}\frac{y_i^{k_i}x^{\lambda_i k_i}}{k_i!}\ {}_{p+3}\overline{\psi}_{q+3}\left(zx^{\sigma}\right).$$

7. Results Involving Multivariable *H*-function of Srivastava and Panda. Making an appeal to (2.2), we derive

$$(7.1)\ \left(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1} H\left(z_1 t^{\sigma_1},\ldots,z_n t^{\sigma_n}\right)\right)(x) = x^{\rho-\alpha-\alpha'+\gamma-1}\overline{H}\left(zx^{\sigma_1},\ldots,z_n x^{\sigma_n}\right),$$

provided that $\mathrm{Re}(\gamma)>0, \mathrm{Re}(\rho)>\max\left[0,\mathrm{Re}(\alpha+\alpha'+\beta-\gamma),\mathrm{Re}(\alpha'-\beta')\right]\left[\arg z_i x^{\sigma_i}\right]<\frac{\pi}{2}\Delta_i$,

$$\Delta_i = -\sum_{j=\lambda+1}^{A}\theta_j^{(i)} + \sum_{j=1}^{\nu^{(i)}}\phi_j^{(i)} - \sum_{j=\nu^{(i)}+1}^{\beta^{(i)}}\phi_j^{(i)} - \sum_{j=1}^{C}\psi_j^{(i)} + \sum_{j=1}^{\mu^{(i)}}\delta_j^{(i)} - \sum_{j=\mu^{(i)}+1}^{D^{(i)}}\delta_j^{(i)} > 0,$$

$$\sum_{j=1}^{A}\theta_j^{(i)} - \sum_{j=1}^{C}\psi_j^{(i)} + \sum_{j=1}^{B^{(i)}}\phi_j^{(i)} - \sum_{j=1}^{D^{(i)}}\delta_j^{(i)} \le 0, i=1,\ldots,n;$$

$$H(z_1,\ldots,z_n) = H_{A,C:[B',D'];\ldots;\left[B^{(n)},D^{(n)}\right]}^{0,\lambda:(\mu',\nu');\ldots;\left(\mu^{(n)},\nu^{(n)}\right)}\left(\begin{matrix}\left[(a):\theta',\ldots,\theta^{(n)}\right]:\left[(b'):\phi'\right];\ldots;\left[\left(b^{(n)}\right):\phi^{(n)}\right];\\ \left[(c):\psi',\ldots,\psi^{(n)}\right]:\left[(d'):\delta'\right];\ldots;\left[\left(d^{(n)}\right):\delta^{(n)}\right];\end{matrix}\ z_1,\ldots,z_n\right)$$

is multivariable *H*-function of Srivastava and Panda ([14] to [16]),

$$\overline{H} = x^{\rho-\alpha-\alpha'+\gamma-1}H_{A+3,C+3:[B',D'];\ldots;\left[B^{(n)},D^{(n)}\right]}^{0,\lambda+3:(\mu',\nu');\ldots;\left(\mu^{(n)},\nu^{(n)}\right)}\left(\begin{matrix}\left[(a):\theta',\ldots,\theta^{(n)}\right],\left[1-\rho:\sigma_1,\ldots,\sigma_n\right],\\ \left[(c):\psi',\ldots,\psi^{(n)}\right],\left[1-\rho-\gamma+\alpha+\alpha':\sigma_1,\ldots,\alpha_n\right],\end{matrix}\right.$$

$$\begin{matrix}\left[1-\rho-\gamma+\alpha+\beta+\alpha':\sigma_1,\ldots,\sigma_n\right],\left[1-\rho-\beta'+\alpha':\sigma_1,\ldots,\sigma_n\right]:\\ \left[1-\rho-\gamma+\alpha'+\beta:\sigma_1,\ldots,\sigma_n\right],\left[1-\rho-\beta':\sigma_1,\ldots,\sigma_n\right]:\end{matrix}$$

$$\left.\begin{matrix}[(b'):\phi'];...;\left[\left(b^{(n)}\right):\phi^{(n)}\right]; \\ [(d'):\delta'];...;\left[\left(d^{(n)}\right):\delta^{(n)}\right];\end{matrix} z_1x^{\sigma_1},...,z_nx^{\sigma_n}\right).$$

(7.2) $\left(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1} S_L^{h_1,...,h_s}\left(y_1t^{\lambda_i},...,y_st^{\lambda_s}\right) H\left(z_1t^{\sigma_1},...,z_nt^{\sigma_n}\right)\right)(x)$

$$= x^{\rho-\alpha-\alpha'+\gamma-1} \sum_{k_1,...,k_s=0}^{h_1k_1+...+h_sk_s<L} (-L)_{h_1k_1+...+h_sk_s} A(L;k_1,...,k_s) \frac{y_1^{k_1}}{k_1!}...\frac{y_s^{k_s}}{k_s!} x^{\lambda_1k_1+...+\lambda_sk_s} \bar{\bar{H}}\left(zx^{\sigma_1},...,z_nx^{\sigma_n}\right),$$

provided that $\mathrm{Re}(\gamma)>0, \mathrm{Re}(\rho)=\min[0,\mathrm{Re}(\alpha+\alpha'+\beta-\gamma),\mathrm{Re}(\alpha'-\beta')]\ [\arg z_ix^{\sigma_i}]<\frac{\pi}{2}\Delta_i$

$$\Delta_i = -\sum_{j=\lambda+1}^{A}\theta_j^{(i)} + \sum_{j=1}^{\nu^{(i)}}\phi_j^{(i)} - \sum_{j=\nu^{(i)}+1}^{B^{(i)}}\phi_j^{(i)} - \sum_{j=1}^{C}\psi_j^{(i)} + \sum_{j=1}^{\mu^{(i)}}\delta_j^{(i)} - \sum_{j=\mu^{(i)}+1}^{D^{(i)}}\delta_j^{(i)} > 0,$$

$$\sum_{j=1}^{A}\theta_j^{(i)} - \sum_{j=1}^{C}\psi_j^{(i)} + \sum_{j=1}^{B^{(i)}}\phi_j^{(i)} - \sum_{j=1}^{D^{(i)}}\delta_j^{(i)} \le 0, i=1,...,n.$$

where

$$\bar{\bar{H}}\left(z_1x^{\sigma_1},...,z_nx^{\sigma_n}\right) = H^{0,\lambda+3\,:\,[\mu',\nu']\,;...;\,\left[\mu^{(n)},\nu^{(n)}\right]}_{A+3,C+3:[B',D']\,;...;\,\left[B^{(n)},D^{(n)}\right]}\left(\begin{matrix}\left[(a):\theta',...,\theta^{(n)}\right], \\ \left[(c):\psi',...,\psi^{(n)}\right],\end{matrix}\right.$$

$$\begin{matrix}\left[1-\rho-\sum_{j=1}^{s}\lambda_jk_j:\sigma_1,...,\sigma_n\right],\left[1-\rho-\gamma+\alpha+\beta+\alpha'-\sum_{j=1}^{s}\lambda_jk_j:\sigma_1,...,\sigma_n\right], \\ \left[1-\rho-\gamma+\alpha+\alpha'-\sum_{j=1}^{s}\lambda_jk_j:\sigma_1,...,\sigma_n\right],\left[1-\rho-\gamma+\alpha'+\beta-\sum_{j=1}^{s}\lambda_jk_j:\sigma_1,...,\sigma_n\right],\end{matrix}$$

$$\left.\begin{matrix}\left[1-\rho-\beta'+\alpha'-\sum_{j=1}^{s}\lambda_jk_j:\sigma_1,...,\sigma_n\right]:[(b'):\phi'];...;\left[\left(b^{(n)}\right):\left(\phi^{(n)}\right)\right]; \\ \left[1-\rho-\beta'-\sum_{j=1}^{s}\lambda_jk_j:\sigma_1,...,\sigma_n\right]:[(d'):\delta'];...;\left[\left(d^{(n)}\right):\left(\delta^{(n)}\right)\right];\end{matrix} z_1x_1^{\sigma_1},...,z_nx_n^{\sigma_n}\right).$$

For Srivastava polynomials, we can derive

(7.3) $\left(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1} H\left(z_1t^{\sigma_1},...,z_nt^{\sigma_n}\right) S_{n_1,...,n_s}^{m_1,...,m_s}\left(y_1t^{\lambda_1},...,y_st^{\lambda_s}\right)\right)(x)$

$$= x^{\rho-\alpha-\alpha'+\gamma-1}\sum_{k_1=0}^{[n_1/m_1]}\cdots\sum_{k_s=0}^{[n_s/m_s]} A(n_1,k_i;,\ldots,n_s,k_s)\frac{(-n_1)_{m_1k_1}}{k_1!}\cdots\frac{(-n_s)_{m_sk_s}}{k_s!}$$

$$\frac{y_1^{k_1}}{k_1!}\cdots\frac{y_s^{k_s}}{k_s!}x^{\lambda_1k_1+\ldots+\lambda_sk_s}\bar{\bar{H}}\left(zx^{\sigma_1},\ldots,z_nx^{\sigma_n}\right),$$

valid if $\operatorname{Re}(\gamma)>0,\operatorname{Re}(\rho)=\min\left[0,\operatorname{Re}(\alpha+\alpha'+\beta-\gamma),\operatorname{Re}(\alpha'-\beta')\right]\operatorname{Re}(\sigma_j)>0(j=1,\ldots,n)$,

$\operatorname{Re}(\lambda_i)>0,n_i\in N_0,m_i\in N,i=1,\ldots,s;N_0=N\cup\{0\}$. $\left|\arg\left(z_ix_i^{\sigma_i}\right)\right|<\Delta_i\pi/2$

$$\Delta_i=-\sum_{j=\lambda+1}^{A}\theta_j^{(i)}+\sum_{j=1}^{\nu^{(i)}}\phi_j^{(i)}-\sum_{j=\nu^{(i)}+1}^{\beta^{(i)}}\phi_j^{(i)}-\sum_{j=1}^{C}\psi_j^{(i)}+\sum_{j=1}^{\mu^{(i)}}\delta_j^{(i)}-\sum_{j=\mu^{(i)}+1}^{D^{(i)}}\delta_j^{(i)}>0,$$

$$\sum_{j=1}^{A}\theta_j^{(i)}-\sum_{j=1}^{C}\psi_j^{(i)}+\sum_{j=1}^{B^{(i)}}\phi_j^{(i)}-\sum_{j=1}^{D^{(i)}}\delta_j^{(i)}\le 0,i=1,\ldots,n.$$

8. Special Cases of (7.3).

(a) Result involving generalized Srivastava polynomials of one variable. For $A(n_1,k_1;\ldots;n_s,k_s)=A_{n_1,k_1}\ldots A_{n_s,k_s}$, we derive the result involving the products of generalized Srivastava polynomials of one variable [7]

$$(8.1)\quad\left(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}t^{\rho-1}H\left(z_1t^{\sigma_1},\ldots,z_nt^{\sigma_n}\right)\sum_{i=1}^{s}S_{n_i}^{m_i}\left(y_it^{\lambda_i}\right)\right)(x)$$

$$= x^{\rho-\alpha-\alpha'+\gamma-1}\sum_{k_1=0}^{[n_1/m_1]}\cdots\sum_{k_s=0}^{[n_s/m_s]} A_{n_1,k_1}\ldots A_{n_s,k_s}(-n_1)_{m_1k_1}\ldots(n_s)_{m_sk_s}x^{\lambda_1k_1+\ldots+\lambda_sk_s}$$

$$\frac{y_1^{k_1}}{k_1!}\cdots\frac{y_s^{k_s}}{k_s!}\bar{\bar{H}}\left(zx^{\sigma_1},\ldots,z_nx^{\sigma_n}\right),$$

(b) Result involving product of Hermite polynomials. For each m_i=2,

$$A_{n_i,k_i}=(-1)^{k_i};\ S_{n_i}^2(y_i)=y_i^{n_i/2}H_{n_i}\left(\frac{1}{2\sqrt{y_i}}\right),i=1,\ldots,s.$$

Therefore, we derive

$$(8.2)\quad\left(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}t^{\rho-1}H\left(z_1t^{\sigma_1},\ldots,z_nt^{\sigma_n}\right)\prod_{i=1}^{s}\left(y_it^{\lambda_i}\right)^{n_i/2}H_{n_i}\left(\frac{1}{2\sqrt{y_it^{\lambda_i}}}\right)\right)(x)$$

$$= x^{\rho-\alpha-\alpha'+\gamma-1} \sum_{k_1=0}^{[n_1/2]} \cdots \sum_{k_s=0}^{[n_s/2]} \prod_{i=1}^{s} (-1)^{k_i} \frac{(-n_i)_{2k_i}}{k_i!} \left(y_i x^{\lambda_i}\right)^{k_i} \bar{\bar{H}}\left(zx^{\sigma_1},\ldots,z_n x^{\sigma_n}\right).$$

(c) Result involving the product of Laguerre polynomials. For each $m_i=1$,

$A_{n_i,k_i} = \frac{(1+\alpha_i)_{n_i}}{(1+\alpha_i)_{k_i} n_i!}$, we have $s'_{n_i}(y_i) \to L_{n_i}^{(\alpha_i)}(y_i), i=1,\ldots,s.$

Therefore, we derive

$$(8.3) \left(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1} H\left(z_1 t^{\sigma_1},\ldots,z_n t^{\sigma_n}\right) \prod_{i=1}^{s} L_{n_i}^{(\alpha_i)}\left(y_i t^{\lambda_i}\right) \right)(x)$$

$$= x^{\rho-\alpha-\alpha'+\gamma-1} \prod_{i=1}^{s} \frac{(1+\alpha_i)_{n_i}}{n_i!} \sum_{k_1=0}^{n_1} \cdots \sum_{k_s=0}^{n_s} \prod_{i=1}^{s} (-n_i)_{k_i} y_i^{k_i} \frac{x^{\lambda_i k_i}}{k_i} \bar{\bar{H}}\left(z_1 x^{\sigma_1},\ldots,z_n x^{\sigma_n}\right).$$

(d) Result involving the product of Jacobi polynomials. For each $m_i = 1$,

$A_{n_i,k_i} = \frac{(1+\alpha_i)_{n_i}}{n_i!} \frac{(1+\alpha_i+\beta_i+n_i)_{k_i}}{(1+\alpha_i)_{k_i}}$, we have $S'_{n_i}(y_i) \to P_{n_i}^{(\alpha_i,\beta_i)}(1-2y_i), i=1,\ldots,s.$

Therefore, we derive

$$(8.4) \left(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1} H\left(z_1 t^{\sigma_1},\ldots,z_n t^{\sigma_n}\right) \prod_{i=1}^{s} P_{n_i}^{(\alpha_i,\beta_i)}\left(1-2y_i t^{\lambda_i}\right) \right)(x)$$

$$= x^{\rho-\alpha-\alpha'+\gamma-1} \prod_{i=1}^{s} \frac{(1+\alpha_i)_{n_i}}{n_i!} \sum_{k_1=0}^{n_1} \cdots \sum_{k_s=0}^{n_s} \prod_{i=1}^{s} (-n_i)_{k_i} \frac{(1+\alpha_i+\beta_i+\eta_i)_{k_i}}{(1+\alpha_i)_{k_i}}$$

$$\frac{\left(y_i x^{\lambda_i}\right)^{k_i}}{k_i!} \bar{\bar{H}}\left(z_1 x^{\sigma_1},\ldots,z_n x^{\sigma_n}\right).$$

9. Right-Sided Generalized Fractional Integral Operator $I_{-}^{\alpha,\alpha',\beta,\beta',\gamma}$.

Making an appeal to (2.1), we can derive

$$(9.1) \left(I_{-}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1}\right)(x) = \frac{\Gamma(1+\alpha+\alpha'-\gamma-\rho)\Gamma(1+\alpha+\beta'-\gamma-\rho)\Gamma(1-\beta-\rho)}{\Gamma(1-\rho)\Gamma(1+\alpha+\alpha'+\beta'-\gamma-\rho)\Gamma(1+\alpha-\beta-\rho)} x^{\rho-\alpha-\alpha'+\gamma-1}$$

provided that $\mathrm{Re}(\gamma) > 0, \mathrm{Re}(\rho) < 1 + \min\left[\mathrm{Re}(-\beta), \mathrm{Re}(\alpha+\alpha'-\gamma), \mathrm{Re}(\alpha+\beta'-\gamma)\right].$

(9.2) $\left(I_{-}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1} H\left(z_1 t^{\sigma_1},\ldots,z_n t^{\sigma_n}\right)\right)(x) = x^{\rho-\alpha-\alpha'+\gamma-1} H^{*}\left(z_1 x^{\sigma_1},\ldots,z_n x^{\sigma_n}\right)$,

where

$$H^{*}\left(z_1 x^{\sigma_1},\ldots,z_n x^{\sigma_n}\right)$$

$$= H^{0,\lambda+3:(\mu',\nu');\ldots;(\mu^{(n)},\nu^{(n)})}_{A+3;C+3:[B',D'];\ldots;[B^{(n)},D^{(n)}]}\left(\begin{matrix}\left[(a):\theta',\ldots,\theta^{(n)}\right],\left[1-\rho:\sigma_1,\ldots,\sigma_n\right],\\ \left[(c):\psi',\ldots,\psi^{(n)}\right],\left[1-\rho+\alpha+\alpha'-\gamma:\sigma_1,\ldots,\sigma_n\right],\end{matrix}\right.$$

$$\begin{matrix}\left[1-\rho+\alpha+\alpha'+\beta'-\gamma:\sigma_1,\ldots,\sigma_n\right],\left[1-\rho+\alpha-\beta:\sigma_1,\ldots,\sigma_n\right]:\\ \left[1-\rho+\alpha+\beta'-\gamma:\sigma_1,\ldots,\sigma_n\right],\left[1-\rho-\beta:\sigma_1,\ldots,\sigma_n\right]:\end{matrix}$$

$$\left.\begin{matrix}\left[(b'):\phi'\right];\ldots;\left[\left(b^{(n)}:\phi^{(n)}\right)\right];\\ \left[(d'):\delta'\right];\ldots;\left[\left(d^{(n)}:\delta^{(n)}\right)\right];\end{matrix} z_1 x^{\sigma_1},\ldots,z_n x^{\sigma_n}\right)$$

and $\mathrm{Re}(\gamma)>0,\mathrm{Re}(\rho)<1+\min\left[\mathrm{Re}(-\beta),\mathrm{Re}(\alpha+\alpha'-\gamma),\mathrm{Re}(\alpha+\beta'-\gamma)\right],\left|\arg\left(z_i x^{\sigma_i}\right)\right|<\frac{\pi}{2}\Delta_i$

$$\Delta_i = -\sum_{j=\lambda+1}^{A}\theta_j^{(i)} + \sum_{j=1}^{\nu^{(i)}}\phi_j^{(i)} - \sum_{j=\nu^{(i)}+1}^{B^{(i)}}\phi_j^{(i)} - \sum_{j=1}^{C}\psi_j^{(i)} + \sum_{j=1}^{\mu^{(i)}}\delta_j^{(i)} - \sum_{j=\mu^{(i)}+1}^{D^{(i)}}\delta_j^{(i)} > 0,$$

$$\prod_{j=1}^{A}\theta_j^{(i)} - \sum_{j=1}^{C}\psi_j^{(i)} + \sum_{j=1}^{B^{(i)}}\phi_j^{(i)} - \sum_{j=1}^{D^{(i)}}\delta_j^{(i)} \le 0, i=1,\ldots,n.$$

(9.3) $\left(I_{-}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1} H\left(z_1 t^{\sigma_1},\ldots,z_n t^{\sigma_n}\right) S_L^{h_1,\ldots,h_n}\left(y_1 t^{\lambda_1},\ldots,y_s t^{\lambda_s}\right)\right)(x)$

$$= x^{\rho-\alpha-\alpha'+\gamma-1}\sum_{k_1,\ldots,k_s=0}^{h_1k_1+\ldots+h_sk_s\le L}(-L)_{h_1k_1+\ldots+h_sk_s} A\left(L;k_1,\ldots,k_s\right)$$

$$\frac{y_1^{k_1}}{k_1!}\cdots\frac{y_s^{k_s}}{k_s!} x^{\lambda_1k_1+\ldots+\lambda_sk_s} H^{**}\left(z_1 x^{\sigma_1},\ldots,z_n x^{\sigma_n}\right),$$

where

$$H^{**}\left(z_1 x^{\sigma_1},\ldots,z_n x^{\sigma_n}\right) = H^{0,\lambda+3:(\mu',\nu');\ldots;(\mu^{(n)},\nu^{(n)})}_{A+3;C+3:[B',D'];\ldots;[B^{(n)},D^{(n)}]}\left(\begin{matrix}\left[(a):\theta',\ldots,\theta^{(n)}\right],\\ \left[(c):\psi',\ldots,\psi^{(n)}\right],\end{matrix}\right.$$

$$\left[1-\rho-\sum_{j=1}^{s}\lambda_j k_j:\sigma_1,\ldots,\sigma_n\right],\left[1-\rho+\alpha+\alpha'+\beta'-\gamma-\sum_{j=1}^{s}\lambda_j k_j:\sigma_1,\ldots,\sigma_n\right],$$

$$\left[1-\rho+\alpha+\alpha'-\gamma-\sum_{j=1}^{s}\lambda_j k_j:\sigma_1,\ldots,\sigma_n\right],\left[1-\rho+\alpha+\beta'-\gamma-\sum_{j=1}^{s}\lambda_j k_j:\sigma_1,\ldots,\sigma_n\right],$$

$$\left.\begin{matrix}\left[1-\rho+\alpha-\beta-\sum_{j=1}^{s}\lambda_j k_j:\sigma_1,\ldots,\sigma_n\right]:\left[(b'):\phi'\right];\ldots;\left[\left(b^{(n)}:\phi^{(n)}\right)\right]; \\ \left[1-\rho-\beta-\sum_{j=1}^{s}\lambda_j k_j:\sigma_1,\ldots,\sigma_n\right]:\left[(d'):\delta'\right];\ldots;\left[\left(d^{(n)}:\delta^{(n)}\right)\right];\end{matrix}\; z_1x^{\sigma_1},\ldots,z_nx^{\sigma_n}\right)$$

$$\mathrm{Re}(\gamma)>0,\mathrm{Re}(\rho)<1+\min\left[\mathrm{Re}(-\beta),\mathrm{Re}(\alpha+\alpha'-\gamma),\mathrm{Re}(\alpha+\beta'-\gamma)\right],\left|\arg\left(z_ix^{\sigma_i}\right)\right|<\pi\Delta_i/2$$

$$\Delta_i=-\sum_{j=\lambda+1}^{A}\delta_j^{(i)}+\sum_{j=1}^{\nu^{(i)}}\phi_j^{(i)}-\sum_{j=\nu^{(i)}+1}^{B^{(i)}}\phi_j^{(i)}-\sum_{j=1}^{C}\psi_j^{(i)}+\sum_{j=1}^{\mu^{(i)}}\delta_j^{(i)}-\sum_{j=\mu^{(i)}+1}^{D^{(i)}}\delta_j^{(i)}>0,$$

$$\sum_{j=1}^{A}\theta_j^{(i)}-\sum_{j=1}^{C}\psi_j^{(i)}+\sum_{j=1}^{\beta^{(i)}}\phi_j^{(i)}-\sum_{j=1}^{D^{(i)}}\delta_j^{(i)}\le 0, i=1,\ldots,n.$$

$\left(h_j\in N_j, j=1,\ldots,s\right), h_1,\ldots,h_s$ are arbitrary positive integers and the coefficients $A\left(L;k_1,\ldots,k_s\right)\left(L,k_j\in N_0; j=1,\ldots,s\right)$ are arbitrary constants real or complex.

For $n=1$, it reduces to Saxena et al. [22] for H-function of one variable.

(9.4) $\left(I_-^{\alpha,\alpha',\beta,\beta',\gamma}t^{\rho-1}H\left(z_1t^{\sigma_1},\ldots,z_nt^{\sigma_n}\right)S_{n_1,\ldots,n_s}^{m_1,\ldots,m_s}\left(y_1t^{\lambda_1},\ldots,y_st^{\lambda_s}\right)\right)(x)$

$$=x^{\rho-\alpha-\alpha'+\gamma-1}\sum_{k_1=0}^{[n_1/m_1]}\cdots\sum_{k_s=0}^{[n_s/m_s]}A\left(n_1,k_1;\ldots;n_s,k_s\right)\left(-n_1\right)_{m_1k_s}\ldots\left(-n_s\right)_{m_sk_s}$$

$$\frac{y_1^{k_1}}{k_1!}\cdots\frac{y_s^{k_s}}{k_s!}x^{\lambda_1k_1+\ldots+\lambda_sk_s}H^{**}\left(z_1x^{\sigma_1},\ldots,z_nx^{\sigma_n}\right),$$

where $S_{n_1,\ldots,n_s}^{m_1,\ldots,m_s}\left(x_1,\ldots,x_s\right)$ are multivariable generalized Srivastava polynomials [8]

For particular interest, if we choose $A\left(n_1,k_1;\ldots;n_s,k_s\right)=\prod_{i=1}^{s}A_{n_i,k_i}$ then (7.9.4) reduces to the following result involving the product of generlized Srivastava polynomials of one variable [7].

(9.5) $\left(I_-^{\alpha,\alpha',\beta,\beta',\gamma}t^{\rho-1}H\left(z_1t^{\sigma_1},\ldots,z_nt^{\sigma_n}\right)\prod_{i=1}^{s}S_{n_i}^{m_i}\left(y_it^{\lambda_i}\right)\right)(x)$

$$=x^{\rho-\alpha-\alpha'+\gamma-1}\sum_{k_1=0}^{[n_1/m_1]}\cdots\sum_{k_s=0}^{[n_s/m_s]}\prod_{i=1}^{s}\left(-n_i\right)_{m_ik_i}A_{n_i,k_i}\left(y_ix^{\lambda_i}\right)^{k_i}H^{**}\left(z_1x^{\sigma_1},\ldots,z_nx^{\sigma_n}\right).$$

10 Special Cases of (9.5)

(a) Result involving Hermite Polynomials. For $m_i=2, A_{n_i,k_i}=(-1)^{k_i}$, we have

$S^2_{m_i}(y_i) = y_i^{n_i/2} H_{n_i}\left(1/2\sqrt{y_i}\right), i = 1,\dots,s.$ Therefore, we derive

$$(10.1)\ \left(I_-^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1} H\left(z_1 t^{\sigma_1},\dots,z_n t^{\sigma_n}\right) \prod_{i=1}^{s} y_i^{n_i/2} t^{\lambda_i n_i/2} H_{n_i}\left(\frac{1}{2\sqrt{y_i t^{\lambda_i}}}\right)\right)(x)$$

$$= x^{\rho-\alpha-\alpha'+\gamma-1} \sum_{k_1=0}^{[n_1/2]} \cdots \sum_{k_s=0}^{[n_s/2]} \prod_{i=1}^{s} (-1)^{k_i} (-n_i)_{2k_i} \frac{\left(y_i x^{\lambda_i}\right)^{k_i}}{k_i!} H^{**}\left(z_1 x^{\sigma_1},\dots,z_n x^{\sigma_n}\right).$$

(b) Result involving Laguerre polynomials. For $m_i = 1$, $A_{n_i,k_i} = \frac{(1+\alpha_i)_{n_i}}{(1+\alpha_i)_{k_i}} \frac{1}{n_i!}$,

$S^1_{n_i}(y_i) \to L_{n_i}^{(\alpha_i)}(y_i), i = 1,\dots,s,$

we obtain

$$(10.2)\ \left(I_-^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1} H\left(z_1 t^{\sigma_1},\dots,z_n t^{\sigma_n}\right) \prod_{i=1}^{s} L_{n_i}^{(\alpha_i)}\left(y_i t^{\lambda_i}\right)\right)(x)$$

$$= x^{\rho-\alpha-\alpha'+\gamma-1} \prod_{i=1}^{s} \frac{(1+\alpha_i)}{n_i!} \sum_{k_1=0}^{n_1} \cdots \sum_{k_s=0}^{n_s} \prod_{i=1}^{s} \frac{(-n_i)_{k_i}}{(1+\alpha_i)_{k_i}} \frac{\left(y_i x^{\lambda_i}\right)^{k_i}}{k_i!} H^{**}\left(z_1 x^{\sigma_1},\dots,z_n x^{\sigma_n}\right).$$

(c) Result involving Jacobi Polynomials. For each $m_i = 1$,

$A_{n_i,k_i} = \frac{(1+\alpha_i)_{n_i}}{n_i!} \frac{(1+\alpha_i+\beta_i+n_i)_{s_i}}{(1+\alpha_i)_{s_i}}$, we have $S^1_{n_i}(y_i) \to P_{n_i}^{(\alpha_i,\beta_i)}(1-2y_i), i = 1,\dots,s.$

Therefore, we establish

$$(10.3)\ \left(I_-^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1} H\left(z_1 t^{\sigma_1},\dots,z_n t^{\sigma_n}\right) \prod_{i=1}^{s} P_{n_i}^{(\alpha_i,\beta_i)}\left(1-2y_i t^{\lambda_i}\right)\right)(x)$$

$$= x^{\rho-\alpha-\alpha'+\gamma-1} \prod_{i=1}^{s} \frac{(1+\alpha_i)_{n_i}}{n_i!} \sum_{k_1=0}^{n_i} \cdots \sum_{k_s=0}^{n_s} \prod_{i=1}^{s} \frac{(-n_i)_{k_i}}{(1+\alpha_i)_{k_i}} (1+\alpha_i+\beta_i+n_i)_{k_i} \frac{\left(y_i x^{\lambda_i}\right)^{k_i}}{k_i!} H^{**}\left(z_1 x^{\sigma_1},\dots,z_n x^{\sigma_n}\right).$$

11. Results Involving Multiple Hypergeometric Function of Srivastava and Daoust. We derive

$$(11.1)\ \left(I_-^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1} S\left(z_1 t^{\sigma_1},\dots,z_n t^{\sigma_n}\right)\right)(x) = x^{\rho-\alpha-\alpha'+\gamma-1} S^{*}\left(z_1 x^{\sigma_1},\dots,z_n x^{\sigma_n}\right),$$

where σ_i are real and negative, $\mathrm{Re}(\gamma)>0, \mathrm{Re}(\rho)<1+\min\left[\mathrm{Re}(\beta),\ \mathrm{Re}(\alpha+\alpha'-\gamma),\right.$

$\left.\mathrm{Re}(\alpha+\beta'-\gamma)\right],\ 1+\sum_{j=1}^{C}\psi_j^{(i)}+\sum_{j=1}^{D^{(i)}}\delta_j^{(i)}-\sum_{j=1}^{A}\theta_j^{(i)}-\sum_{j=1}^{B^{(i)}}\phi_j^{(i)}>0, i=1,..,n.$ and

$$S^*\left(z_1x^{\sigma_1},\ldots,z_nx^{\sigma_n}\right)=S^{A+3;B';\ldots;B^{(n)}}_{C+3:D';\ldots;D^{(n)}}\left(\begin{matrix}\left[(a):\theta',\ldots,\theta^{(n)}\right],\\ \left[(c):\psi',\ldots,\psi^{(n)}\right],\end{matrix}\right.$$

$$\begin{matrix}[1+\alpha+\alpha'-\gamma-\rho:-\sigma_1,\ldots,-\sigma_n],[1+\alpha+\beta'-\gamma-\rho:-\sigma_1,\ldots,-\sigma_n],\\ [1-\rho:-\sigma_1,\ldots,-\sigma_n],[1+\alpha+\alpha'+\beta'-\gamma-\rho:-\sigma_1,\ldots,-\sigma_n],\end{matrix}$$

$$\left.\begin{matrix}[1-\beta-\rho:-\sigma_1,\ldots,-\sigma_n]:\left[(b'):\phi'\right];\ldots;\left[\left(b^{(n)}:\phi^{(n)}\right)\right];\\ [1+\alpha-\beta-\rho:-\sigma_1,\ldots,-\sigma_n]:\left[(d'):\delta'\right];\ldots;\left[\left(d^{(n)}:\delta^{(n)}\right)\right];\end{matrix}\ z_1x^{\sigma_1},\ldots,z_nx^{\sigma_n}\right).$$

(11.2) $\left(I_-^{\alpha,\alpha',\beta,\beta',\gamma}t^{\rho-1}S\left(z_1t^{\sigma_1},\ldots,z_nt^{\sigma_n}\right)S_L^{h_1,\ldots,h_s}\left(y_1t^{\lambda_1},\ldots,y_st^{\lambda_s}\right)\right)(x)$

$$=x^{\rho-\alpha-\alpha'+\gamma-1}\sum_{k_1,\ldots,k_s=0}^{h_1k_1+\ldots+h_sk_s\le L}(-L)_{s_1k_1+\ldots+h_sk_s}A(L;k_1,\ldots,k_s)\prod_{i=1}^{s}\frac{\left(y_ix^{\lambda_i}\right)^{k_i}}{k_i!}S^{**}\left(z_1x^{\sigma_1},\ldots,z_nx^{\sigma_n}\right),$$

provided that $\mathrm{Re}(\gamma)>0, \mathrm{Re}(\rho)<1+\min\left[\mathrm{Re}(-\beta), \mathrm{Re}(\alpha+\alpha'-\gamma),\quad \mathrm{Re}(\alpha+\beta'-\gamma)\right],$

$1+\sum_{j=1}^{C}\psi_j^{(i)}+\sum_{j=1}^{D^{(i)}}\delta_j^{(i)}-\sum_{j=1}^{A}\theta_j^{(i)}-\sum_{j=1}^{B^{(i)}}\phi_j^{(i)}>0, i=1,\ldots,n.$

Similarly, we also derive

(11.3) $\left(I_-^{\alpha,\beta,\beta',\gamma}t^{\rho-1}S\left(z_1t^{\sigma_1},\ldots,z_nt^{\sigma_n}\right)S_{n_1,\ldots,n_s}^{m_1,\ldots,m_s}\left(y_1t^{\lambda_1},\ldots,y_nt^{\lambda_s}\right)\right)(x)$

$$=x^{\rho-\alpha-\alpha'+\gamma-1}\sum_{k_1=0}^{[n_1/m_1]}\cdots\sum_{k_s=0}^{[n_s/m_s]}(-n_1)_{m_1k_1}\ldots(-n_s)_{m_sk_s}A(n_1,k_1;\ldots;n_s,k_s)$$

$$\frac{y_1^{k_1}}{k_1!}\cdots\frac{y_s^{k_s}}{k_s!}S^{**}\left(z_1x^{\sigma_1},\ldots,z_nx^{\sigma_n}\right).$$

12. Special Cases of Generalized Multiple Hypergeometric Function of Srivastava and Daoust. For $n=1$, $S^{A:B',\ldots,B^{(n)}}_{C:D';\ldots;D^n}$ reduces to Fox-Wright generalized hypergeometric function ${}_p\Psi_q$ [24]. Therefore (11.2) reduces to

(12.1) $\left(I_{-}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1} {}_{p}\Psi_{q}\left(zt^{\sigma}\right) S_{L}^{h_1,\ldots,h_s}\left(y_s t^{\lambda_1},\ldots,y_s t^{\lambda_s}\right)\right)(x)$

$$= x^{\rho-\alpha-\alpha'+\gamma-1} \sum_{k_1,\ldots,k_s=0}^{h_1k_1+\ldots+h_sk_s\le L} (-L)_{h_1k_1+\ldots+h_sk_s} A(L:k_1,\ldots,k_s) \frac{y_1^{k_1}}{k_1!}\cdots\frac{y_s^{k_s}}{k_s!} x^{\lambda_1k_1+\ldots+\lambda_sk_s} {}_{p+3}\Psi_{q+3}^{*}\left(zx^{\sigma}\right)$$

where

$${}_{p+3}\Psi_{q+3}^{*}(z) = {}_{p+3}\Psi_{q+3}\left(\begin{matrix}(a_i,A_i)_{1,p},\left(1+\alpha+\alpha'-\gamma-\rho-\sum_{i=1}^{s}\lambda_ik_i,-\sigma\right),\\ (b_j,B_j)_{1,q},\left(1-\rho-\sum_{i=1}^{s}\lambda_ik_i,-\sigma\right),\end{matrix}\right.$$

$$\left.\begin{matrix}\left(1+\alpha+\beta'-\gamma-\rho-\sum_{i=1}^{s}\lambda_ik_i,-\sigma\right),\left(1-\beta-\rho-\sum_{i=1}^{s}\lambda_ik_i,-\sigma\right): \\ \left(1+\alpha+\alpha'+\beta'-\gamma-\rho-\sum_{i=1}^{s}\lambda_ik_i,-\sigma\right),\left(1-\beta-\rho-\sum_{i=1}^{s}\lambda_ik_i,-\sigma\right):\end{matrix}\middle| zx^{\sigma}\right)$$

and

$a_i,b_j \in C; A_i>0, B_j>0, 1+\sum_{j=1}^{q}B_j-\sum_{i=1}^{p}A_i \ge 0$, $A_i,B_j\in R\left(A_i,B_j\ne 0\right), i=1,\ldots,p;$

$j=1,..,q; \mathrm{Re}(\gamma)>0, \mathrm{Re}(\rho)<1+\min\left[\mathrm{Re}(-\beta),\mathrm{Re}(\alpha+\alpha'-\gamma),\ \mathrm{Re}(\alpha+\beta'-\gamma)\right]$, while σ is real greater than 0.

Also for multivariable generalized polynomials of Srivatava [8], we derive

(12.2) $\left(I_{-}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1} {}_{p}\Psi_{q}\left(zt^{\sigma}\right) S_{n_1,\ldots,n_s}^{m_1,\ldots,m_s}\left(y_1 t^{\lambda_1},\ldots,y_s t^{\lambda_s}\right)\right)(x)$

$$= x^{\rho-\alpha-\alpha'+\gamma-1} \sum_{k_1=0}^{[n_1/m_1]}\cdots\sum_{k_s=0}^{[n_s/m_s]} A(n_1,k_1;\ldots;n_s,k_s)(-n_1)_{m_1k_1}\cdots(-n_s)_{m_sk_s}$$

$$\frac{\left(y_1x^{\lambda_1}\right)^{k_1}}{k_1!}\cdots\frac{\left(y_sx^{\lambda_s}\right)^{k_s}}{k_s!} {}_{p+3}\Psi_{q+3}^{*}\left(zx^{\sigma}\right).$$

For $A(n_1,k_1;\ldots;n_s,k_s)=A_{n_1,k_1}\ldots A_{n_s,k_s}, S_{n_1,\ldots,n_s}^{m_1,\ldots,m_s}(x_1,\ldots,x_s)$ reduces to the product of generalized polynomials $S_n^m(x)$ of Srivastava [7]. Therefore, we derive

(12.3) $\left(I_{-}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1}\, {}_p\Psi_q\left(zt^{\sigma}\right)\prod_{i=1}^{s} S_{n_i}^{m_i}\left(y_i t^{\lambda_i}\right)\right)(x)$

$$= x^{\rho-\alpha-\alpha'+\gamma-1}\sum_{k_1=0}^{[n_1/m_1]}\cdots\sum_{k_s=0}^{[n_s/m_s]}\prod_{i=1}^{s} A_{n_i,k_i}\left(-n_i\right)_{m_1,k_i}\frac{\left(y_i x^{\lambda_i}\right)^{k_i}}{k_i\,!}\, {}_{p+3}\Psi^{*}_{q+3}\left(zx^{\sigma}\right).$$

13. Further Special Cases of (12.3).

(a) Results involving Hermite Polynomials. For $m_i = 2, A_{n_i,k_i} = (-1)^{k_i}$, we have

$$S_{n_i}^{2}\left(y_i\right) = y_i^{n_i/2} H_{n_i}\left(\frac{1}{2\sqrt{y_i}}\right), i = 1,\ldots,s.$$

Therefore, (7.12.3) reduces to

(13.1) $\left(I_{-}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1}\, {}_p\Psi_q\left(zt^{\sigma}\right)\prod_{i=1}^{s}\left(ty_i\right)^{n_i/2} H_{n_i}\left(\frac{1}{2\sqrt{y_i t^{\lambda_i}}}\right)\right)(x)$

$$= x^{\rho-\alpha-\alpha'+\gamma-1}\sum_{k_1=0}^{[n_1/2]}\cdots\sum_{k_s=0}^{[n_s/2]}\prod_{i=1}^{s}\frac{\left(-y_i x^{\lambda_i}\right)^{k_i}}{k_i\,!}\left(-n_i\right)_{2k_i}\, {}_{p+3}\Psi^{*}_{q+3}\left(zx^{\sigma}\right)$$

(b) Result involving Hermite polynomials. For $m_i = 1$, $A_{n_i,k_i} = \frac{\left(1+\alpha_i\right)_{n_i}}{\left(1+\alpha_i\right)_{k_i}}\frac{1}{n_i\,!}$,

$$S_{n_i}^{1}\left(y_i\right) \to L_{n_i}^{(\alpha_i)}\left(y_i\right), i = 1,\ldots,s.$$

Therefore, (7.12.3) gives

(13.2) $\left(I_{-}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1}\, {}_p\Psi_q\left(zt^{\sigma}\right)\prod_{i=1}^{s} L_{n_i}^{(\alpha_i)}\left(y_i t^{\lambda_i}\right)\right)(x)$

$$= x^{\rho-\alpha-\alpha'+\gamma-1}\prod_{i=1}^{s}\frac{\left(1+\alpha_i\right)_n}{n!}\sum_{k_1=0}^{n_1}\cdots\sum_{k_s=0}^{n_s}\prod_{i=1}^{s}\frac{\left(-n_i\right)_{k_i}}{k_i\,!}\, {}_{p+3}\Psi^{*}_{q+3}\left(zx^{\sigma}\right).$$

(c) Result involving Jacobi polynomials. For each $m_i = 1$, $A_{n_i,k_i} = \frac{\left(1+\alpha_i\right)_{n_i}}{n_i\,!}$

$\frac{(1+\alpha_i+\beta_i+n_i)_{s_i}}{(1+\alpha_i)_{s_i}}$, we have

$$S_{n_i}^{1}(y_i) \to P_{n_i}^{(\alpha_i,\beta_i)}(1-2y_i), i=1,\ldots,s.$$

Therefore (12.3) shows that

$$(13.3)\left(I_{-}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1}\, {}_p\Psi_q\left(zt^{\sigma}\right)\prod_{i=1}^{s} P_{n_i}^{(\alpha_i,\beta_i)}\left(1-2y_i t^{\lambda_i}\right)\right)(x)$$

$$= x^{\rho-\alpha-\alpha'+\gamma-1}\sum_{k_1=0}^{n_1}\cdots\sum_{k_s=0}^{n_s}\prod_{i=1}^{s}\frac{(1+\alpha_i)_{n_i}}{n_i}(-n_i)_{k_i}\frac{(1+\alpha_i+\beta_i+\eta_i)_{k_i}}{(1+\alpha_i)_{k_i}}$$

$$\frac{\left(y_i x^{\lambda_i}\right)^{k_i}}{k_i!}\, {}_{p+3}\Psi^{*}_{q+3}\left(zx^{\sigma}\right).$$

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NEW INTEGRAL OPERATOR ON MEROMORPHIC FUNCTIONS OF COMPLEX ORDER

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ABSTRACT

In this paper, we introduce new subclasses involving Hadamard product of meromorphic functions with complex order. We prove some properties of certain generalized integral operators for the aforementioned subclasses.

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1. Introduction. Let $\ \ = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk in the complex plane $\mathbb{C}$, $\ \ ^* = \ \ \setminus\{0\}$, the punctured open unit disk. Let Σ denote the class of meromorphic functions of the form

(1.1) $$f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n \quad (a_n \in \mathbb{C}),$$

which are analytic in $\ \ ^*$.

We denote by $\Sigma^*(\alpha)$ and $\Sigma_k(\alpha)$ the subclasses of the class Σ, which are defined for $0 \le \alpha < 1$ as follows:

$$\Sigma^*(\alpha) = \left\{ f : f \in \Sigma, -\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha \right\} (z \in \ \ ^*),$$

and

$$\Sigma_k(\alpha) = \left\{ f : f \in \Sigma, -\Re\left(\frac{zf''(z)}{f'(z)}\right) > \alpha \right\} \quad (z \in \ \ ^*),$$

Note that $\Sigma^*(\alpha)$ and $\Sigma_k(\alpha)$ are the well known subclasses of Σ consisting of meromorphic univalent functions which are respectively starlike and convex functions in $\quad *$ of order $\alpha(0 \le \alpha < 1)$.

For $f(z)$ given by (1.1) and $g(z)$ given by

$$g(z) = \frac{1}{z} + \sum_{n=0}^{\infty} b_n z^n \quad (b_n \in \mathbb{C}),$$

the Hadamard product (or convolution) of f and g is defined by

$$(f * g)(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n b_n z^n, (a_n, b_n \in \mathbb{C}, z \in \quad *).$$

Now, for f and g belong to Σ, we define the following new subclasses.

Definition 1.1 Let the functions $f, g \in \Sigma$ be analytic in $\quad *$. Then $f * g$ is in the class $\Sigma_k(\alpha, b)$ if it satisfies the inequality

$$\Re\left\{1 - \frac{1}{b}\left(\frac{z(f*g)'(z)}{(f*g)(z)} + 1\right)\right\} < \alpha, \tag{1.2}$$

where $b \in \mathbb{C} \setminus \{0\}$.

Definition 1.2 Let the functions $f, g \in \Sigma$ be analytic in $\quad *$. Then $f * g$ is in the class $\Sigma_k(\alpha, b)$ if it satisfies the inequality

$$\Re\left\{1 - \frac{1}{b}\left(\frac{z(f*g)''(z)}{(f*g)'(z)} + 2\right)\right\} < \alpha, \tag{1.3}$$

where $b \in \mathbb{C} \setminus \{0\}$. We note that $f, g \in \Sigma^*(\alpha, b)$ if and only if $-z(f*g)' \in \Sigma_k(\alpha, b)$.

Definition 1.3. Let the functions $f, g \in \Sigma$ be analytic in $\quad *$. Then $f * g$ is in the class $\Sigma \quad (\alpha, b)$ if it satisfies the inequality

$$\Re\left\{1 - \frac{1}{b}\left(\frac{z\left(z(f*g)''(z) + 3(f*g)'(z)\right)}{z(f*g)'(z) + 2(f*g)(z)} + 1\right)\right\} < \alpha, \tag{1.4}$$

where $b \in \mathbb{C} \setminus \{0\}$.

For $g(z)$ defined by

$$g(z)=\frac{z^2-z+1}{z(1-z)}=\frac{1}{z}+\sum_{n=0}^{\infty}z^n,$$

in Definitions 1.1, 1.2, and 1.3, we obtain classes of meromorphic functions of complex order introduced and studied by Mohammed and Darus [4]. An analogous to the integral operator defined by Frasin and Aouf [1] on the normalized analytic functions, we define the following integral operator on the space of meromorphic functions in the class Σ.

Definition 1.4 Given $f_i, g_i \in \Sigma, \gamma_i > 0$ for all $i=1,\dots,n, n\in\mathbb{N}$. We define the integral operator $_{\gamma_i}(f_1,\dots,f_n;g_1,\dots,g_n):\Sigma^n \to \Sigma$ by

(1.5) $$_{\gamma_i}(f_1,\dots,f_n;g_1,\dots,g_n)(z)=\frac{1}{z^2}\int_0^z \left(u(f_1*g_1)(u)\right)^{\gamma_1}\dots\left(u(f_n*g_n)(u)\right)^{\gamma_n}du$$

where $(f*g)(z)/z\neq 0,\ z\in {}^*$.

For the sake of simplicity, from now on we will write $_{\gamma_i}(z)$ instead of $_{\gamma_i}(f_1,\dots,f_n;g_1,\dots,g_n)(z)$.

Remark 1.1

(i) For $g_1=\dots=g_n=\frac{z^2-z+1}{z(1-z)}=\frac{1}{z}+\sum_{n=0}^{\infty}z^n$, we obtain the integral operator

$$_{\gamma_i}(z)=\frac{1}{z^2}\int_0^z\left(u(f_1(u))^{\gamma_1}\dots(uf_n(u))^{\gamma_n}\right)du.$$

introduced and studied by Mohammed and Darus [2].

(ii) For $g_1=\dots=g_n=-\frac{2z-1}{2(1-z)^2}=\frac{1}{z}-\sum_{n=0}^{\infty}nz^n$, we obtain the integral operator

$$_{\gamma_i}(z)=\frac{1}{z^2}\int_0^z\left(-u^2(f_1'(u))^{\gamma_1}\dots(-u^2f_n'(u))^{\gamma_n}\right)du,$$

also introduced and studied recently by Mohammed and Darus [3].

2 Main Results. In this section, we investigate some properties for the integral operator $_{\gamma_i}(z)$ defined by (1.5) of the subclasses given by 1.1, 1.2 and 1.3.

Theorem 2.1 Let $f_i*g_i\in\Sigma$. If $f_i*g_i\in\Sigma^*_{\alpha,b}$, then

$$_{\gamma_i}(z) \in \Sigma_F(\mu, b),$$

where $n \in \mathbb{N}, i \in \{1,...,n\}, b \in \mathbb{C} \setminus \{0\}$, and

$$\mu = 1 + \sum_{i=1}^{n} \gamma_i (\alpha_i - 1), \gamma_i > 0, 0 \le \alpha_i < 1.$$

Proof. A differentiation of $_{\gamma_i}(z)$ which is defined in (1.5), we obtain

(2.1) $$z^2 \, {}_{\gamma_i}'(z) + 2z \, {}_{\gamma_i}(z) = \left(z(f_1 * g_1)(z)\right)^{\gamma_1} ... \left(z(f_n * g_n)(z)\right)^{\gamma_n},$$

$$z^2 \, {}_{\gamma_i}''(z) + 4z \, {}_{\gamma_i}'(z) + 2 \, {}_{\gamma_i}(z)$$

$$= \sum_{i=1}^{n} \gamma_i \left(\frac{z(f * g)'_i(z) + (f_i * g_i)(z)}{z(f_i * g_i)(z)} \right) \left[\left(z(f_1 * g_1)(z)\right)^{\gamma_1} ... \left(z(f_n * g_n)(z)\right)^{\gamma_n} \right].$$

Then form (2.1), we obtain

(2.2) $$\frac{z^2 \, {}_{\gamma_i}''(z) + 4z \, {}_{\gamma_i}'(z) + 2 \, {}_{\gamma_i}(z)}{z^2 \, {}_{\gamma_i}'(z) + 2z \, {}_{\gamma_i}(z)} = \sum_{i=1}^{n} \gamma_i \left(\frac{(f_i * g_i)'(z)}{(f_i * g_i)(z)} + \frac{1}{z} \right).$$

By multiplying (2.2) with z we have

$$\frac{z^2 \, {}_{\gamma_i}''(z) + 4z \, {}_{\gamma_i}'(z) + 2 \, {}_{\gamma_i}(z)}{z^2 \, {}_{\gamma_i}'(z) + 2z \, {}_{\gamma_i}(z)} = \sum_{i=1}^{n} \gamma_i \left(\frac{z(f_i * g_i)'(z)}{(f_i * g_i)(z)} + 1 \right).$$

That is equivalent to

$$\frac{z\left(z \, {}_{\gamma_i}''(z) + 3 \, {}_{\gamma_i}'(z)\right)}{z \, {}_{\gamma_i}'(z) + 2 \, {}_{\gamma_i}(z)} + 1 = \sum_{i=1}^{n} \gamma_i \left(\frac{z(f_i * g_i)'(z)}{(f_i * g_i)(z)} + 1 \right).$$

So that, the above assertion can be written as

(2.3) $$1 - \frac{1}{b}\left(\frac{z\left(z \, {}_{\gamma_i}''(z) + 3 \, {}_{\gamma_i}'(z)\right)}{z \, {}_{\gamma_i}'(z) + 2 \, {}_{\gamma_i}(z)} + 1 \right) = \sum_{i=1}^{n} \gamma_i \left\{ 1 - \frac{1}{b}\left(\frac{z(f_i * g_i)'(z)}{(f_i * g_i)(z)} + 1 \right) \right\} + 1 - \sum_{i=1}^{n} \gamma_i .$$

Taking the real part of both terms of (2.3), we have

$$\Re\left\{1-\frac{1}{b}\left(\frac{z\left(z\ {}''_{\gamma_i}(z)+3\ {}'_{\gamma_i}(z)\right)}{z\ {}'_{\gamma_i}(z)+2\ {}_{\gamma_i}(z)}+1\right)\right\}=\sum_{i=1}^{n}\gamma_i\Re\left\{1-\frac{1}{b}\left(\frac{z(f_i*g_i)'(z)}{(f_i*g_i)(z)}+1\right)\right\}+1-\sum_{i=1}^{n}\gamma_i.$$

Since $f_i*g_i\in\Sigma^*(\alpha_i,b)$, we get

$$\Re\left\{1-\frac{1}{b}\left(\frac{z\left(z\ {}''_{\gamma_i}(z)+3\ {}'_{\gamma_i}(z)\right)}{z\ {}'_{\gamma_i}(z)+2\ {}_{\gamma_i}(z)}+1\right)\right\}<1+\sum_{i=1}^{n}\gamma_i\alpha_i-\sum_{i=1}^{n}\gamma_i,$$

this implies

$$\Re\left\{1-\frac{1}{b}\left(\frac{z\left(z\ {}''_{\gamma_i}(z)+3\ {}'_{\gamma_i}(z)\right)}{z\ {}'_{\gamma_i}(z)+2\ {}_{\gamma_i}(z)}+1\right)\right\}<1+\sum_{i=1}^{n}\gamma_i(\alpha_i-1).$$

Then

$${}_{\gamma_i}(z)\in\Sigma\ \ (\mu,b),\ \ \mu=1+\sum_{i=1}^{n}\gamma_i(\alpha_i-1).$$

This completes the proof.

Theorem 2.2 *Let* $f_i*g_i\in\Sigma$. *If* $f_i*g_i\in\Sigma_k(\alpha_i,b)$, *then*

$$(z)\in\Sigma\ \ (\mu,b),$$

where $n\in\mathbb{N},i\in\{1,...,n\},b\in\mathbb{C}\setminus\{0\}$, *and*

$$\mu=1+\sum_{i=1}^{n}\gamma_i(\alpha_i-1),\ \ \gamma_i>0,0\le\alpha_i<1.$$

Proof. A differentiation of ${}_{\gamma_i}(z)$, which is defined in (1.5), we obtain

(2.4) $$z^2\ {}'_{\gamma_i}(z)+2z\ {}_{\gamma_i}(z)=\left(-z^2(f_1*g_1)'(z)\right)^{\gamma_1}...\left(-z^2(f_n*g_n)'(z)\right)^{\gamma_n},$$

$$z^2\ {}''_{\gamma_i}(z)+4z\ {}'_{\gamma_i}(z)+2\ {}_{\gamma_i}(z)$$

$$=\sum_{i=1}^{n}\gamma_i\left(\frac{z^2(f_i*g_i)''(z)+2z(f_i*g_i)'(z)}{z(f_i*g_i)(z)}\right)\left[\left(-z^2(f_1*g_1)'(z)\right)^{\gamma_1}...\left(-z^2(f_n*g_n)'(z)\right)^{\gamma_n}\right].$$

Then from (2.4), we obtain

$$\frac{z^2 F_{\gamma_i}''(z)+4zF_{\gamma_i}'(z)+2F_{\gamma_i}(z)}{zF_{\gamma_i}'(z)+2F_{\gamma_i}(z)}=\sum_{i=1}^{n}\gamma_i\left(\frac{z(f_i*g_i)''(z)}{(f_i*g_i)'(z)}+2\right).$$

That is equivalent to

$$\frac{z\left(zF_{\gamma_i}''(z)+3F_{\gamma_i}'(z)\right)}{zF_{\gamma_i}'(z)+2F_{\gamma_i}(z)}+1=\sum_{i=1}^{n}\gamma_i\left(\frac{z(f_i*g_i)''(z)}{(f_i*g_i)'(z)}+2\right).$$

So that, the above assertion can be written as

$$(2.5)\qquad 1-\frac{1}{b}\left(\frac{z\left(zF_{\gamma_i}''(z)+3F_{\gamma_i}'(z)\right)}{2F_{\gamma_i}'(z)+2F_{\gamma_i}(z)}+1\right)=\sum_{i=1}^{n}\gamma_i\left\{1+\frac{1}{b}\left(\frac{z(f_i*g_i)''(z)}{(f_i*g_i)'(z)}+2\right)\right\}+1-\sum_{i=1}^{n}\gamma_i.$$

Taking the real part of both terms of (2.5), we have

$$\Re\left\{1-\frac{1}{b}\left(\frac{z\left(zF_{\gamma_i}''(z)+3F_{\gamma_i}'(z)\right)}{2F_{\gamma_i}'(z)+2F_{\gamma_i}(z)}+1\right)\right\}=\sum_{i=1}^{n}\gamma_i\Re\left\{1-\frac{1}{b}\left(\frac{z(f_i*g_i)''(z)}{(f_i*g_i)'(z)}+2\right)\right\}+1-\sum_{i=1}^{n}\gamma_i\,.$$

Since $f_i*g_i\in\Sigma_k(\alpha_i,b)$, we get

$$\Re\left\{1-\frac{1}{b}\left(\frac{z\left(zF_{\gamma_i}''(z)+3F_{\gamma_i}'(z)\right)}{zF_{\gamma_i}'(z)+2F_{\gamma_i}(z)}+1\right)\right\}<\sum_{i=1}^{n}\gamma_i\alpha_i+1-\sum_{i=1}^{n}\gamma_i,$$

this implies

$$\Re\left\{1-\frac{1}{b}\left(\frac{z\left(zF_{\gamma_i}''(z)+3F_{\gamma_i}'(z)\right)}{zF_{\gamma_i}'(z)+2F_{\gamma_i}(z)}+1\right)\right\}<1+\sum_{i=1}^{n}\gamma_i(\alpha_i-1).$$

Then

$$F_{\gamma_i}(z)\in\Sigma\ (\mu,b),\ \ \mu=1+\sum_{i=1}^{n}\gamma_i(\alpha_i-1).$$

This completes the proof.

Temark 2.1 The integral operator defined in (1.5) can be extended to p-valent type like the ones defined in [5] and [6].

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