LIE THEORY AND BASIC GAUSS POLYNOMIALS

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ABSTRACT

In the Present paper, an attempt has been made to bring basic hypergeometric functions within the purview of Lie theory by constructing a dynamical symmetry algebra of basic hypergeometric function \( _2\phi_1 \). Multiplier representation theory is then used to obtain generating function for basic analogue of Gauss polynomial. The results obtained in this paper are extensions of the results derived earlier by Miller [3] and Sarkar-Chatterjea [4].

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1. Introduction. The \( q \)-analogue of the Gauss functions or Heine's series [1] may be written as

\[
\phi_1(a,b;c;q;x) = \sum_{n=0}^{\infty} \left[ a; q, n \right] \left[ b; q, n \right] / \left[ c; q, n \right] [n; q]! \quad (c \neq 0, -1, -2, ...)
\]

where \( |q| < 1 \) and \( |x| < 1 \).

Here \( [a; q, n] \) and \( [n; q]! \) are respectively the basic Pochammer's symbol and basic factorial function defined as \( [a; q, n] = [a; q][a+1; q]...[a+n-1; q] \) and \( [n; q]! = [1; q][2; q]...[n; q] \).

The basic differential operator \( B_{q,x}^{-} \) is defined by [1] through the relation

\[
B_{q,x}^{-} \phi(x) = \frac{\phi(qx) - \phi(x)}{x(q-1)}.
\]  

2. The Dynamical Symmetry Algebra of \( _2\phi_1 \). The dynamical symmetry
algebra of the hypergeometric function has been defined by Miller [2]. We use the same technique to define the dynamical symmetry algebra of \( \phi_{21} \). Let
\[
\phi_{\alpha,\beta,\gamma,q} = \Gamma_q(\gamma - \alpha) \Gamma_q(\alpha) / \Gamma_q(\gamma) \phi_{\alpha,\beta,\gamma,q}[x; q; \gamma], \]
be the basis elements of a subspace of analytical functions of four variables \( x, s, u \) and \( t \), associated with Heine's basic hypergeometric function of Heine's series \( \phi_{21} \).

Introduction of variables \( s, u \) and \( t \) renders differential operators independent of parameters \( \alpha, \beta \) and \( \gamma \) and thus facilitates their repeated operation.

The dynamical symmetry algebra of \( \phi_{21} \) is a 15-dimensional complex Lie algebra isomorphic to \( sl(4) \), generated by twelve \( E^- \)-operators termed as raising or lowering operators in view of their effect of raising or lowering the corresponding suffix in \( \phi_{\alpha,\beta,\gamma,q} \). The \( E^- \)-operators are
\[
(i)\ E^-_{\alpha,q} = s^{-1} \left( x(1-x)B^\gamma_{q,x} + tB^\gamma_{q,t} - sB^\gamma_{q,s} - xuB^\gamma_{q,u} \right), \\
(ii)\ E^-_{\beta,\gamma,q} = u^{-1} t^{-1} \left( x(1-x)B^\gamma_{q,x} - xsB^\gamma_{q,s} + tB^\gamma_{q,t} - 1 \right). \]

The action of these operators on \( \phi_{\alpha,\beta,\gamma,q} \) is given by
\[
E^-_{\alpha,q} \phi_{\alpha,\beta,\gamma,q} = [\alpha - 1; q] \phi_{\alpha,\beta,\gamma,q}^{-1} \\
E^-_{\beta,\gamma,q} \phi_{\alpha,\beta,\gamma,q} = [\gamma - \alpha - 1; q] \phi_{\alpha,\beta,\gamma,q}^{-1} \]
for all \( \alpha, \beta, \gamma \).

The upper factor in each bracket is to be associated with plus sign and lower with minus sign. Twelve \( E^- \)-operators together with three maintenance operators \( J_a, J_b, J_c \) and Identity operator \( I \) form a basic for \( gl(4) \cong sl(4) \), where \( (I) \) is the 1-dimensional Lie algebra generated by 1.

Here
\[
J^\gamma_{a,q} = sB^\gamma_{q,x}, J^\gamma_{b,q} = uB^\gamma_{q,u}, J^\gamma_{c,q} = tB^\gamma_{q,t} \] and \( I^\gamma = 1 \)
with the results
\[
J^\gamma_{a,q} \phi_{\alpha,\beta,\gamma,q} = [\alpha; q] \phi_{\alpha,\beta,\gamma,q}, \\
J^\gamma_{b,q} \phi_{\alpha,\beta,\gamma,q} = [\beta; q] \phi_{\alpha,\beta,\gamma,q}, \\
J^\gamma_{c,q} \phi_{\alpha,\beta,\gamma,q} = [\gamma; q] \phi_{\alpha,\beta,\gamma,q},
\]
and
\[
I^\gamma \phi_{\alpha,\beta,\gamma,q} = \phi_{\alpha,\beta,\gamma,q}. \]

3. The Generating Functions for Basic Analogues of Gauss Polynomials. On comparing the results obtained by the action one parameter subgroup \( \left( exp_q aE^-_{\alpha,-\gamma,q} \right) \) generated by the operator \( E^-_{\alpha,q} \) defined in (2.2) on \( \phi_{\alpha,\beta,\gamma,q} \) defined in (2.1) and direct expansion, we get the identity...
\[
\left[ \frac{st}{(ax + st)} \right]^n \left[ \frac{a}{(st + 1)} \right]^{-1} \Phi_1[a, \beta; \gamma; q; x(a + st)/(ax + st)]
\]
\[
= \sum_{m=0}^{\infty} a^m [\gamma - m; q]_m / [m; q] ! \Phi_1[a q^{-m}, \beta; \gamma; q^{-m}; q; x] (st)^{-m} \quad \ldots \quad (3.1)
\]

Taking, \( \alpha \rightarrow 0, \beta \rightarrow \lambda + \mu + m - 1, \gamma \rightarrow q^{\lambda+m} \), \( st \rightarrow 1, a \rightarrow 1 \), we get

\[
(1 + x)^{1 - \lambda - \mu} [2; q]^{\lambda-1} = \sum_{m=0}^{\infty} [\gamma; q]_m / [m; q] ! \Phi_1[-m, \lambda + \mu + m - 1; \lambda; q; x] \quad \ldots \quad (3.2)
\]

By definition of basic Gauss polynomial [1]

\[
G_{m}^{\lambda, \mu} (q; x) = \Phi_1[-m, \lambda + \mu + m - 1; \lambda; q; x],
\]

where \( \lambda \neq 0, -1, -2, 3, \ldots \). \quad \ldots \quad (3.3)

Using (3.3) in (3.2), we get the generating function

\[
(1 + x)^{1 - \lambda - \mu} [2; q]^{\lambda-1} = \sum_{m=0}^{\infty} [\gamma; q]_m / [m; q] ! G_{m}^{\lambda, \mu} (q; x)
\]

for basic Gauss polynomials.

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**REFERENCES**


