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COMPARISON OF A.D.D. OF TRIANGULATIONS

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ABSTRACT

In this paper, making an appeal to computer we try to provide the measure of efficiency for triangulations through graphical represenation.

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1. Introduction. There are number of algorithms for comuting fixed points using triangulations and complementary pivoting techniques. Each algorithm requires a triangulation, a lebeling of the vertices of the trinagulation, and a particular starting point called as pivoting point.

The successful algorithms use one of the two methods to obtain better and better approximations of fixed points, without wasting previous approximations.

The first one is the restart method of Merril [4], independently developed by Kuhn and Mackinnon [3]. called "sandwich" method. If a fixed point of a mapping f on R^n is sought, one triangulates. $R^n \times [0,1]$ rather than R^n. All vertices lie in either $R^n \times \{0\}$ or $R^n \times \{1\}$. The latter are labelled according to the given mapping f, the former according to the constant map taking each x into $C \varepsilon R^n$. The point c can be so chosen as to be closed to a previous fixed point and provides a starting point for the algorithm.

The second method was introduced by Eaves [1] and extended by Eaves and Saigal [2]. Instead of triangulating S^n or R^n, one triangulates $S^n \times [1,\infty)$ or $R^n \times [1,\infty)$. Roughly, triangulations of finer and finer meshes are placed on the top of the another so that boundary simplices match. Then one generates an infinite path of simplices following fixed point of piecewise linear maps that approximates f better and better as the artificial coordinate tends to ∞. This process of deformation corresponds to the "homotopy" concept. Since we are not concerned with the maping *f,* we shall say such triangulations have continuous refinement of grid gize.

The efficiency of fixed point algorithm is very sensitive to the triangulation used. In this paper we are concerned with comparison of the theoretical measures of efficiency for different types of triangulations. The technique of the comparison is based on the concept of directional density of the triangulation in the direction d as defined by Todd [8]. The only other measures known are rather crude measure based on the number of simplices in the unit cube and the diameter, introduced by Saigal, Solow and Wolsey see [7].

In this paper, we have also tried to provide the measure of efficiency for such triangulations through graphical representations by using the computer.

2. Notations and Definitions. The following notations have been used in this paper :

R: Set of real numbers

Z: Set of all integers

E : Set of all even integers.

O: Set of all odd integers

N: Set of all positive integers $\{1,2,...\}$.

N^0: Set of all non-negative inegers $N \cup \{0\}$

R^n : n dimensional space, having co-ordinates indexed 1 through n.

R^{n+1}: $n+1$ dimensional space, with coordinates indexed 0 through n.

π: Group of permutation on $(1,2,...,n)$ and π_{n+1} group of permutation on $(0,1,2,...,n)$

u^i : i^{th} unit vector in R^n, iεN and $u = \sum_{i \in N} u^i$ is a vector of ones in R^{n+1}.

V^j: j^{th} unit vector in R^{n+1}, iεN_0 and $v = \sum_{j \in N} v^j$ is a vector of ones in $R^{n+\text{i}}$.

R_+^m : Non negative orthant of R^m i.e. $\{x \varepsilon R^m : x \geq 0\}$.

A: Set of $\{-1,+1\}$

B : Set of $\{0,1\}$.

Now we consider some standard definations and explanations which will be used in this paper.

2.1 Simplex. An (open) j-dimensional simplex $\sigma = \langle y^0, y^1, y^2, ..., y^j \rangle$ is the relative interior of the convex hull of $j+1$ affinely independent points $y^0, y^1, ..., y^j$ called its vertices. A simplex τ is a face of σ if its vertices are a subset of vertices of the σ. Again the closure of a (*j*–1)-dimensional face of the *j*-simplex σ is called a facet of σ. Two *j*-simplices are said to be adjacent if they share a common facet.

The diameter of σ is $\sup\{\|x-z\| : x, z\varepsilon\sigma\} = \max\{\|y^i - y^j\| : 0 \le i \le j\}$.

2.2. Triangulation. A triangulation G of an m-dimensional subset F of R^n is a collection of m-simplices that satisfies the following two conditions :

1. The simplices in G, together with all their faces form a partition of F; and
2. Each point of F has a neighbourhood meeting only a finite number of simplices.

We denote G^j the set of *j*-dimensional faces of simplices in G for $0 \le j \le m$. The members of G^0 is called the vertices of G-and the closures of members of G^{m-1} are called facets of G.

2.3. Pivot Rule. For a given simplex $\sigma\varepsilon G$ and a vertex y of σ, the rules for obtaining the simplex of G whose vertices include all vertices of σ except y are called the pivot rules of G.

2.4. Mesh. The mesh of a triangulation G is sup{diameter of $\sigma : \sigma\varepsilon G$ }. We shall use the euclidean norm throughout this paper.

2.5 Some Standard Regular Traingulations. We define some regular triangulations of R^n. We take the basic grid size of all triangulations to be unity, K,H and J_1 are triangulations of R^n. K (some times called K_1) is based on the standard subdivision of the cube. H is very closely related to K and J_1 is the "*Union Jack*" triangulation of R^n.

All these triangulations use integer vectors as vertices, let $K^0 = H^0 = J_1^0 = Z^n$. Also denote the set of central vertices of J_1 by $J_1^{0c} = \{ y\varepsilon J_1^0 : y_i$ is odd for each $i\varepsilon N$ }.

Now we give definitions of some standard trianglations given as below :

(1) *K triangulation* is the set of $\sigma = \langle y^0, \ldots, y^n \rangle$ where $y^0 = y; y^i = y^{i-1} + u^{\pi(i)},\ 1 \le i \le n$

(2) *H triangulation* is the set of $\sigma = \langle y^0, \ldots, y^n \rangle$ where $y^0 = y; y^i = y^{i-i} + q^{\pi(i)},'\ 1 \le i \le n$ where q^j is the *j*-th column of the $n \times n$ matrix

$$Q = \begin{bmatrix} -1 & & & 0 \\ +1 & \ddots & & \\ & \ddots & \ddots & \\ 0 & & +1 & -1 \end{bmatrix}$$

The simplex o is the set all $x \in R^n$ satisfying :

$$1 > -\sum_{1}^{\pi(1)}\left(x_{\pi(1)} - y_{\pi(1)}\right) > \ldots > -\sum_{1}^{\pi(n)}\left(x_{\pi(n)} - y_{\pi(n)}\right) > 0 .$$

(3) J_1 *trinagulation* is the set of $\sigma = \langle y^0, \ldots, y^n \rangle$ where $y^0 = y; y^i = y^{i-1} + s_{\pi(i)} u^{\pi(i)}$, $1 \le i \le n$ and y^0, j_1^{oc}, π is a permutation of N and $s \varepsilon R^n$ is a sign vector, where $s_i = \pm i$.

The simplex σ is the-set of all $s \varepsilon R^n$ satisfying :

$$1 > s_{\pi(1)}\left(x_{\pi(1)} - Y_{\pi(1)}\right) > \ldots > s_{\pi(n)}\left(x_{\pi(n)} - Y_{\pi(n)}\right) > 0 .$$

The triangulations by J_1 and by K of any unit cube with integral vertices are isomorphic upto orientation of the edges of the cube.

$K_2(m)$ and $J_2(m)$ are the triangulation of S^n having mesh ${}_\infty K_2(m) = m^{-1}$ and $mesh_\infty J_2(m) = 2m^{-1}$. Each $K_2(m)$ and $J_2(m)$ can be expressed as follows :

(4) $K_2(m) = \{\sigma : \sigma = k_2(y^0, \pi), \text{ if } \sigma \subseteq S^n\}$ is the collection of all such σ where $\sigma = \langle y^0, \ldots, y^n \rangle$ satisfying $y^i = y^{i-1} + m^{-1} q^{\pi(i)}$ for each $i \varepsilon N$ and where q^j is the jth column of the $(n+1) \times n$ matrix denoted by Q and expressed as

$$Q = \begin{bmatrix} -1 & & & 0 \\ +1 & \ddots & & \\ & \ddots & \ddots & \\ & & \ddots & -1 \\ 0 & & & +1 \end{bmatrix}$$

and its vertices are given by $K_2^0(m) = \{y \varepsilon S^n : my_i \varepsilon Z\}$ for each $i \varepsilon N_0$.

(5) $J_2(m) = \{\sigma : \sigma = j_2(y^0, \pi, s), \text{ if } \sigma \subseteq S^n\}$ is the collection of all such σ where $\sigma = \langle y^0, \ldots, y^n \rangle$ satisfying $y^i = y^{i-1} + m^{-1} s_{\pi(i)} q^{\pi(i)}$ for each $i \varepsilon N$ and $q^{\pi(i)}$ is as defined in (4) and its vertices are given by $J_2^0(m) = K_2^0(m)$ and centre vertices $J_2^{0c}(m) = \{y \varepsilon J_2^0(m) : my_i$ is even for $1 \le i < n$ and my_n is odd$\}$ for each $i \varepsilon N_0$

We now describe two triangulation with continuous refinement of grid size. The first J_3, based on J_1 was introduced by Todd [8], and K_3 is due to Eaves and

Saigal [2].

(6) K_3 triangulates $(0,1]\times R^n$. Let K_3^0 denotes set of vertices $\{y\varepsilon(0,1] \times R^n{:}y_0 = 2^{-k}$ for $0 \le k\varepsilon Z$ and $y_i / y_0 \varepsilon z$ for $i\varepsilon N\}$. A mapping $t : K_3^{0c} \to A^{n+1}$ is defined by $t_i(y)=0$ or if y_i/y_0 is odd and 1 otherwise. (Thus $t_0(y)$ always equals 0). Let π be a permutation of N_0 and K_3 be the set of $\sigma = k(y,\pi) = \langle y^{-1}, y^0, \ldots, y^n \rangle$ where

$$y^{-1} = y$$
$$y^i = y^{i-1} + y_0 v^{\pi(i)}, 0 \le i < j = \pi^{-1}(0)$$
$$y^j = y^{j-1} - y_0 \sum_{\ell=j}^{n} t_{\pi(i)}, v^{\pi(i)} + v^{0-} \sum_{k=j+1}^{n} v^{\pi(k)},$$
$$y^k = y^{k-1} + 2y_0 v^{\pi(k)} = j \le k \le n.$$

The simplex σ is the set of all $x \in R^{n+1}$ satisfying

$$y_0 \ge x_{\pi(0)} - y_{\pi(0)} + t_{\pi(0)}(x_0 - y_0) \ge \ldots \ge x_{\pi(\pi)} - Y_{\pi(\pi)} + t_{\pi(\pi)}(x_n - y_n) \ge y_0 - x_0.$$

(7) J_3 triangulates $(0,1]\times R^n$. Let J_3^0 denotes set of vertices $\{ y\varepsilon R^{n+1} : y_0 = 2^{-k}$ for $0 \le K\varepsilon Z$ and $y_i / y_0 \varepsilon z$ for $1 \le i \le n \}$.

Let $J_3^{0c} = \left\{ y\varepsilon J_3^0 : y_i / y_0 \varepsilon O \text{ for } 1 \le i \le n \right\}$ be the set of central vertices. A mapping $\omega : J_3^{0c} \to A^{n+1}$ is defined by $\omega_i(y) = -1$ or $+1$ according as y_i/y_0 is 1 or 3 mod 4. Let π is a permutation of N_0 and $s\varepsilon A^n$. Then J_3 is the set of simplex $\sigma = j_3(y,\pi,s) = \langle y^{-1}, y^0, \ldots, y^n \rangle$ where

$$y^{-1} = y,$$
$$y^i = y^{i-1} + y_0 s_{\pi(i)} v^{\pi(i)}, 0 \le i < j = \pi^{-1}(0),$$
$$y^j = y^{j-1} - y_0 \sum_{\ell=j}^{n} \omega_\ell(\pi) v^{\pi(\ell)},$$
$$y^k = y^{k-1} + 2y_0 \omega_\pi(k) v^{\pi(k)}, 1 \le k \le n.$$

The simplex σ is the set of all $x \in R^{n+1}$ satisfying :

$$y_0 \ge s_\pi(0)\left(x_\pi(1) - y_{\pi(1)}\right) \ge \ldots \ge s_{\pi(j-1)}\left(x_{\pi(j-1)} - y_{\pi(j-1)}\right) \ge x_0 - y_0$$
$$\ge \omega_{\pi(j+1)}\left(x_{\pi(j+1)} - y_{\pi(J+1)}\right) \ge \ldots \ge \omega_{\pi(n)}\left(x_{\pi(n)} - y_{\pi(n)}\right) \ge y_0 - x_0 .$$

(8) K_3' triangulates $(0,1]\times R^n$. The set of vertices of K_3', denoted by $K_3'^0$ is precisely J_3^0. We define $\mu : K_3'^0 \to B^{n+1}$ by

$\mu_i(y) = 1 - y_i / y_0 \bmod 2.$

Let n is a permutation of N_0 with $\pi(j) = 0$ and $\mu_\ell(y) = 0$ if $\pi^{-1}(\ell) > j$ then K_3' is the set of $\sigma = k(y,\pi) = \langle y^{-1}, y^0, ..., y^n \rangle$ where

$$y^{-1} = y$$
$$y^i = y^{i-1} + y_0 v^{\pi(i)}, 0 \le i < j$$
$$y^j = y^{j-1} - \sum_{i=0}^{j-1} \mu_c y_0 v^{\pi(i)} + y_0 v^0 - \sum_{k=j+1}^{n} y_0 v^{\pi(k)};$$
$$y^k = y^{k-1} + 2y_0 v^{\pi(k)} j < k \le n.$$

Let $z=x-y$, $\mu=\mu(y)$ and $w=z+z_0\mu$. Then simplex σ is the set of all $x \varepsilon R^{n+1}$ satisfying :

$$y_0 \ge w_{\pi(0)} \ge ... \ge w_{\pi(n)} \ge -w_0.$$

(9) Let A be a non-singular $n \times n$ matrix and G be any triangulation of R^n (here we consider $G=K$ or J_1). Let $AG= (A\sigma{:}\sigma\varepsilon G\}$ with $A\sigma=\{Ax{:}x\varepsilon\sigma\}$. Then *trinagulation* AK of R^n is the collection of simplices $\sigma=(y^0,...,y^n)$ such that all components of $A^{-1}y^0$ are integers and $y^i=y^{i-1}+a^{\pi(i)}$ where $a^{(j)}$ is the jth column of A. When A has 1's on the diagonal, -1's on the upper diagonal, and zeroes elsewhere, we have the H triangulation.

(10) J' is a triangulation of R^n defined as follows : The set of vertices of J' is the set of vector $v\varepsilon R^n$ with each component an integer, such that there is not precisely one even component or precisely one odd component. Each simplex $\sigma = j'(y,\pi,s)$ $= (y^0,...,y^n)$, where y is the vector each of the whose components is an even integer, $\pi = (\pi(1),...,\pi(n))$ is a permutation of $(1,2,...,n)$ and s is a sign vector (each $s_j = \pm 1$) with $s_{\pi(1)} = s_{\pi(n)} = 1$.

For convenience $s_{\pi(i)} e^{n(i)}$ is denoted by $\tilde{e}_i$, where $e^{n(i)}$ is the ith unit vector in R^n. For $n \ge 4$ the vertices of j' are defined by

$$y^0 = y$$
$$y^1 = y^0 + 2\tilde{e}^1,$$
$$y^2 = y^1 - \tilde{e}^1 + \tilde{e}^2,$$
$$y^j = y^{j-1} + \tilde{e}^i, 3 \le j \le n-2,$$
$$y^{n-1} = y^{n-2} + \tilde{e}^{n-1} - \tilde{e}^n,$$
$$y^n = y^{n-1} + 2\tilde{e}^n.$$

For $n=3$, each simplices of J' is of the form $\sigma = j'(y,\pi,s) = (y^0,y^1,y^2,y^3)$ where

$$y^0 = y,$$
$$y^1 = y^0 + 2e^{\pi(1)},$$
$$y^2 = y^1 - e^{\pi(i)} + s_{\pi(2)}e^{\pi(2)} + e^{\pi(3)},$$
$$y^3 = y^{2+}e^{\pi(3)}.$$

Now below we give three triangulations of $K._1$,$J.$ and $J._3$ of $R^n \times [0,1]$ given by Todd [8].

Let $R^n \times [0,1] = \{\bar{x} \varepsilon R^{n+1} : 0 \le \bar{x}^{n+1}\}$. Removal of the bar from a vector in $R^n \times [0,1]$ denotes its projection into R^n. Let $\bar{u}^1,...,\bar{u}^{n+1}$ be the unit vector in R^{n+1}; thus $u^1,...,u^n$ are the unit vectors in R^n.

(11) Suppose $\bar{y} \varepsilon Z^n \times \{0\}$ and π is a permutation of $\{1,2,..,n+1\}$. Then $k_1(\bar{y},\pi)$ denotes the (closed) simplex $\bar{y}^0,...,\bar{y}^{n+1}\}$, where

$$\bar{y}^0 = \bar{y},$$
$$\bar{y}^i = \bar{y}^{i-1} + \bar{u}^{\pi(i)}, 1 \le i \le n+1.$$

The triangulation K_1 is the set of all such $k_1(\bar{y},\pi)$'s.

(12) Suppose $\bar{y} \varepsilon Z^n \times \{1\}$ has all $\bar{y}^i$'s odd. Let π be a permutation of $\{1,2,...,n+1\}$ and let $\bar{s} \varepsilon R^n \times \{-1\}$ be a sign vector, each $\bar{s}_i$ is ± 1. Then $j_1(\bar{y},\pi,\bar{s})$ denotes the simplex $[\bar{y}^0,...,\bar{y}^{n+1}]$, where

$$\bar{y}^0 = \bar{y}$$
$$\bar{y}^i = \bar{y}^{i-1} + \bar{s}_{\pi(i)}\bar{u}^{\pi(i)}, 1 \le i \le n+1 \cdot$$

The triangulation J_1 is the set of all such $j_1(\bar{y},\pi,\bar{s})$'s.

(13) J_3 is the triangulation derived from J_3. Let $\bar{y} \varepsilon Z^n \times \{1\}$ has all $\bar{y}^i$'s odd. Let π be a permutation of $\{1,2,...,n+1\}$ with $\pi(j) = n+1$. Let $\bar{s} \varepsilon R^n \times \{-1\}$ be a sign vector such that, for $j \le k \le n+1, \bar{s}n(k)$ is -1 or +1 according as $\bar{y}^{\pi(k)}$ is 1 or 3 mod 4. Then $J_3(\bar{y},\pi,\bar{s})$ denotes the simplex $[\bar{y}^0,...,\bar{y}^{n+1}]$, where

$$\bar{y}^0 = \bar{y}$$
$$\bar{y}^i = \bar{y}^{i-1} + \bar{s}\pi(i)\bar{u}^{\pi(i)}, 1 \le i \le j.$$
$$\bar{y}^j = \bar{y}^{j-1} - \sum_{k=j+1}^{n+1} \bar{s}\pi(k)\bar{u}^{\pi(k)} + \bar{s}_{\pi+1}\bar{u}^{n+1},$$
$$\bar{y}^k = \bar{y}^{k-1} + 2\bar{s}\pi(k)\bar{u}^{\pi(k)}, j \le k \le n+1.$$

The triangulation $J_{\square 3}$ is the set of all such $J_3(\bar{y},\pi,\bar{s})$'s.

3. Measure of Efficiency for Triangulation. Todd [8] has indicated that the theoretical measure of the efficiency of different types of triangulations used to be compared by counting the number of simplices into which unit cube is divided. However, Todd [8] has also observed that many triangulations have different efficiency but yield the same number of simplices e.g. K, H or J_1 triangulations having $n!$ simplices and K'_3 or J_3 in $\{x \varepsilon R^{n+1} : 0 \le x \le 1, 1/2 \le x_0 \le 1\}$ having $(2^{n+1}-1)\ n!$ simplices.

Saigal Solow and Wolsey introduced the concept of diameter, for those triangulations that subdivided unit cube. This is the maximum over all pairs of facets of the triangulation lying in the facets of the cube as well as of the maximum number of simplices that form a path of simplices linking the two facets. They computed the diameter of K triangulation and an obvious extension to K_3 and found them compareble, Saigal [5] calculated the diameters of triangulations K and H and obtained results and suggesting that the number of iterations using K increases with n^2 while that using H increases with n^3. It is easy to see that the diameters of J_1 and K are equal.

The diameter of triangulation is a "worst best case" measure similar to the diameter of a polytope which is of interest in linear programming.

4. Main Result. First we shall give the concept of average directional density, then formula for obtaining different triangulations and numerical results for obtaining *a.d.d.* for different values of *n*. At last we shall give the computerised graph of average directional density of different triangulations.

4.1 Average Directional Density. Todd [8] proposed the concept of directional density as an alternative measure, for the comparison of efficiecy of the triangulations and is global in nature.

For $x, d \varepsilon R^n$ and $\lambda > 0$, let $[x, x+\lambda d]$ denote $\{x + \mu d : \mu \varepsilon [0,\lambda]\}$. Let G be a triangulation of R^n having mesh $\sqrt{n}$. Let $N(G,x,d,\lambda)$ denote the number of simplices of G intersecting $[x, x+\lambda d]$ divided by λ. Let $N(G,x,d,\lambda)$ denote the limit as $\rho \to \infty$, (if it exists) of the average of $N\{G,x,d,\lambda)$ for x uniformly distributed in $B(0,\rho) = \{x \varepsilon R^n : \|x\| < p\}$. Let $N(G,d)$ be the limit as $\lambda \to \infty$ (if it exists) of $N(G,d,\lambda)$. Finally, $N(G)$ is defined to the average of $N(G,d)$ for d uniformly distributed on $\partial B^n = \{d \varepsilon R^n : \|d\| = 1\}$.

We call $N(G,d)$ the directional density of G in direction d and $N(G\}$ the average directional density. It should be noted that $N(G,x,d,1)$ is the number of simplices met per unit step size. To eliminate the effect of starting point, we average over x lying in a large ball. To eliminate any effect on the ending point, we let $\lambda \to \infty$. Finally to get a measure independent of the direction d, we average over ∂B^n.

For any triangulation G of R^n, G^{n-2} is a countable collection of sets of dimension n–2. Thus for almost all $x, \{x + \lambda d : \lambda \varepsilon R\}$ meets no simplices of G^+ of dimension less than n–1. If x and $x+\lambda d$ lie in n-simplices of $G.[x,x+\lambda d]$ meet one more simplex of G than (n–1)-simplex of G. Thus the computation of N (G,x,λ,d) can be made by counting the number of points of $[x,x+d]$ lying in facet of G, for almost x.

From these observations the following results are obtained by Todd [8] and Vender Laan and Talman [19] :

(i) $N(k_1,d) = \sum_i |d_i| + \sum_{i<j} |d_i - d_j|$

(ii) $N(J_1,d) = \sum_i |d_i| + \sum_{i<j} \frac{1}{2}\left(|d_i + d_j| + |d_i - d_j|\right)$

(iii) $N(H_1,d) = \sum_{i \le j} \left|\sum_{i \le j} d_k\right|$

(iv) $N(J',d) = \sum_{i \le j} \frac{1}{2}\left(|d_i + d_j| + |d_i - d_j|\right)$

(v) Let $g_n = 2\Gamma(n/2)/(n-1)\sqrt{\pi}\Gamma((n-1)/2)$, then

(a) $N(K_1) = N(J_1) = \left(n + \sqrt{2}(n/2)g_n\right)$

(b) $N(H_1) = \sum_1 (n+1-i)\sqrt{(i)}g_n$

(c) $N(K\square) = 1/2\sqrt{(n+1)}\sum_i \sqrt{(n+1-i)}\sqrt{(i)}g_n$.

(d) $N(J') = (n/2)\sqrt{2}g_n$

(e) $N\left(\tilde{J}_1\right) = (1/4)\sqrt{(n+1)}\sum_k \sqrt{(n+1-k)}\sqrt{(k)} +$

$$(1/2)\sum_{1\le i\le j\le n}\left(3i+j+1-(i+j+1)\right)/(n+1)\Big)^{1/2}\Big]g_n$$

(f) $N(A*K) = \begin{cases} g_n\{n(n+1)/8\}^{1/2}\{n(n+2)\} & \text{if } n \text{ is even} \\ g_n\{n(n+1)/8\}^{1/2}(n+1) & \text{if } n \text{ is odd} \end{cases}$

Table 1 gives $N(G)/g_n$ for various values of a n and G equal to K, J_1, H□, H, $A*K$ and J'.

TABLE 1

The *a.d.d.* of the K or J_1, K□, H, $A*K$, J□$_1$ and J' triangulations for various values of n(mesh equal to $\sqrt{n}$)

TRIANGULATIONS *G*

n	K or J_1	K□	H	$A*K$	$\tilde{J}_1$	J'
1	1	0.7	1	1	0.7	0
2	3.41421	2.44949	3.4142	2.44949	2.698	1.4142
3	7.24264	5.4641	7.5604	4.89898	6.4587	4.2426
4	12.4853	9.94936	13.7047	7.74597	12.098	8.4852
5	19.1421	16.0797	22.089	11.619	20	14.1421
9	59.9117	60.0249	82.0105	33.541	77.6782	50.9116
15	163.492	197.748	269.433	87.6356	261.087	148.4924
20	288.701	392.453	534.189	151.987	524	268.7006
30	645.183	1044.16	1419.95	334.066	1406	615.1829
50	1782.41	3636.58	4942.12	910.357	4336	1732.412
100	7100.36	20108.5	27316.5	3588.52	27462	7000.357

4.2. Asymptotic Behaviour of A.D.D. of Different Triangulations. If G is any triangulation then Asymptotic behaviour of $N(G)/gn$ for $G = K, J_1, K$□, $\tilde{J}_1, H$ and AK are given as follows :

(i) For K or J_1 triangulation : $N(K)/g_n = \sim n^2/\sqrt{2} = N(J_1) = N(J')$,

(ii) For $K*$ triangulation : $N(K*)/g_n = \sim \pi n^{5/2}/16$,

(iii) For H triangulation : $N(H)/g_n = \sim 4n^{5/2}15$,

(iv) For $A*H$ triangulation : $N(A*K)/g_n = \sim n^2/\sqrt{8}$,

(v) For $J_1^{\bullet}$ triangulation : $N(J\bullet_1)/g_n = \sim 0\left(n^{5/2}\right)$.

In table 2 we compared asymptotic behaviour of different triangulations for different values of n.

TABLE 2

The asymptotic behaviour of *a.d.d.* of the K or J_1, $K\square$, H, A^*K and J' triangulations for various values of p (mesh equal to $\sqrt{n}$)

Tringulations *G*

n	K or J_1	$K\square$	H	A^*K	$\tilde{J}_1$
1	0.70711	0.19643	0.26667	0.35355	1
2	2.82843	1.11117	1.50849	1.41421	5.65685
3	6.36396	3.06202	4.15692	3.18198	15.5885
4	11.3137	6.28571	8.53333	5.65685	32
5	17.6777	10.9807	14.9071	8.83883	55.9017
9	57.2756	47.7321	64.8	28.6378	243
15	159.099	171.172	232.379	79.5495	871.421
20	282.843	351.382	477.028	141.421	1788.85
30	636.396	968.295	1314.53	318.198	4929.5
50	1767.77	3472.4	4714.05	883.883	17677.7
100	7071.07	19642.9	26666.7	3535.53	100000

6. Conclusions. In table 1 it has been observed that *acL.d* of H and $J\square_1$ trinagulations increases very fast as n increases and both remains nearly same for $n^{\wedge}30$. However, their values remain parallel for $30 \leq n \leq 50$. But values for $J\square_1$ become faster than H for $50 \leq n \leq 100$. For J' triangulation, in beginning its value increases very slow rate but as n increases its value pick up very fast and become very close to a.d.d. of K or J_1. Rate of increase of *a.d.d.* of A^*K triangulation is the slowest of all the triangulations. In the same way rate for $K\square$ is faster than K or J_1 but less than H and $J\square_1$.

From these observations one can conclude that those triangulations whose *a.d.d.* increases at faster rate are considered to be inferior than the other. On this basis H can be considered inferior among all the triangulations and A^*K is superior among all the triangulations.

Again if we take a graphical representation of tabulated values of a.d.d. for different triangulations, one can easily observed that for $n \leq 15$ there is not much deviation in the different graphs of *a.d.d.* for most of the triangulations. However, between $n=15$ and 20 the deviation between the value of H and K became faster, but that of H and $J\square$ remain the same, that is H and $J\square_1$ begins to deviate at the faster rate than $K\square$. Deviation of A^*K is not much from its previous values before $n=15$. In addition to this one can also easily observed that K (or J_1) and J' remain

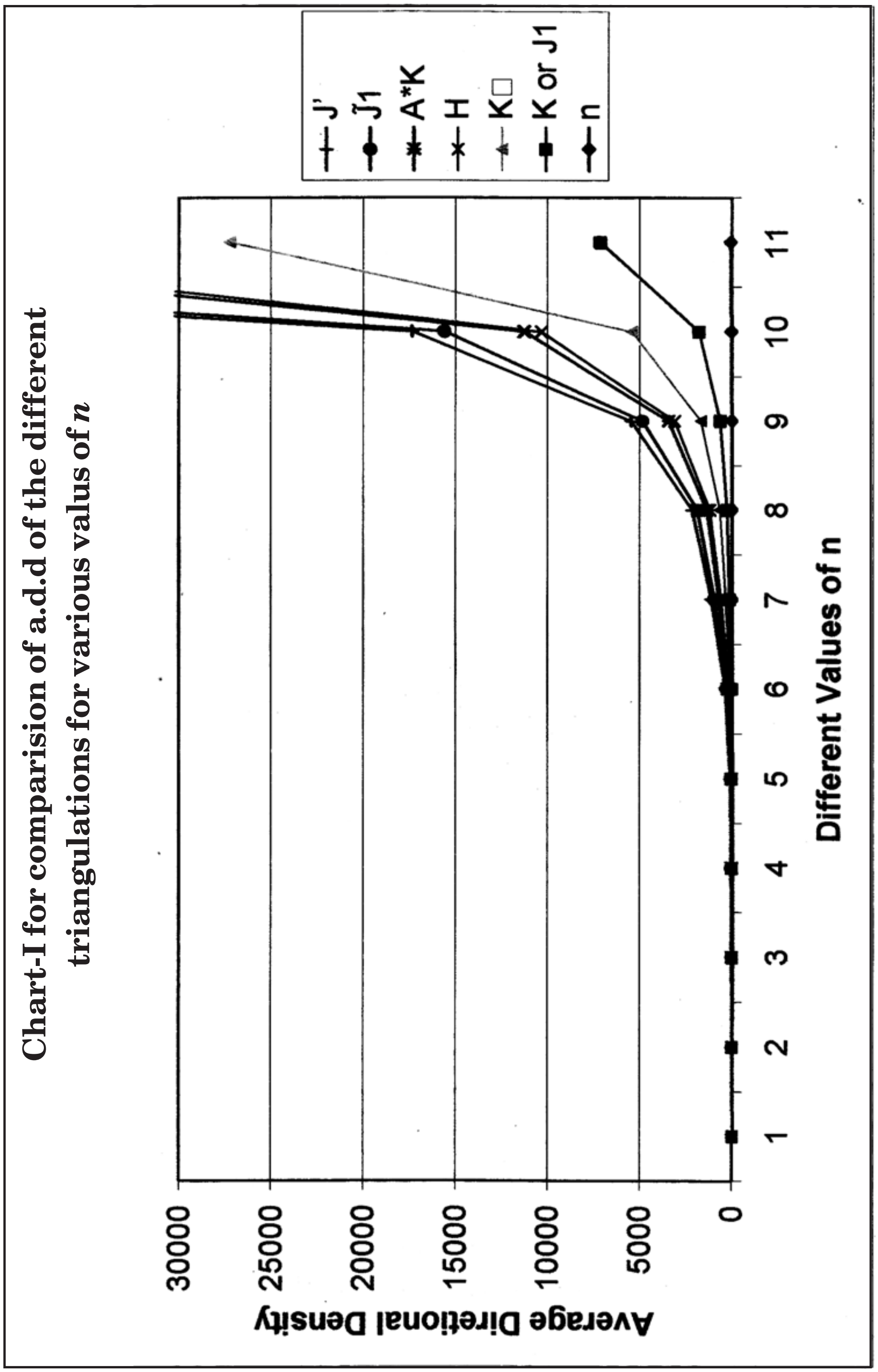
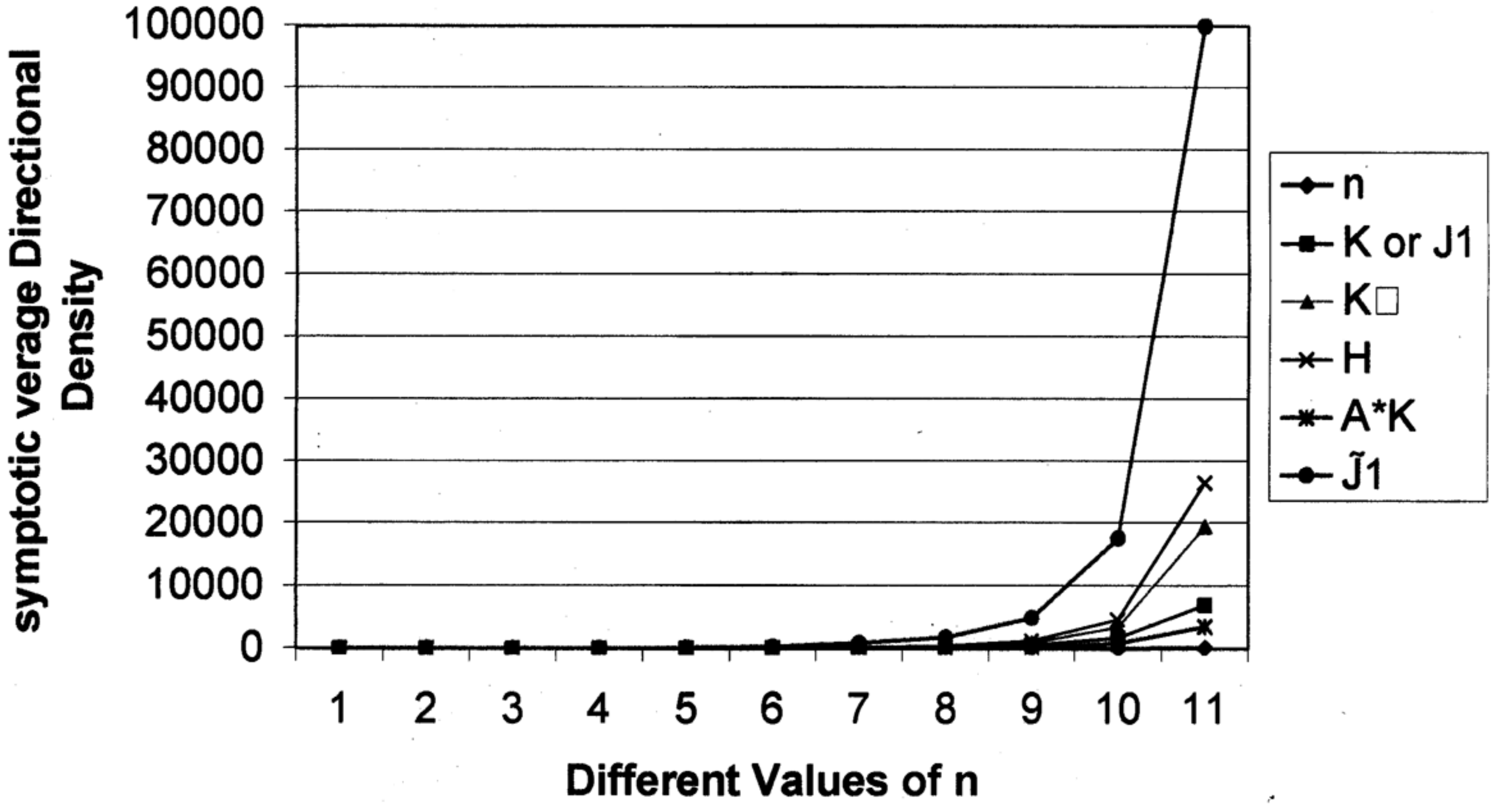


Chart-I for comparision of a.d.d of the different triangulations for various valus of *n*

Chart II-comparision of asymptotic a.d.d. of the different triangulations for various values of *n*



nearly same for values of $n \leq 20$ and their deviation is also less than H, $J\square_1$ and $K\square$, but greater than A^*K. However, from $n=30$ and onwards the values of *a.d.d.* for triangulations H and $J\square_1$ as well as $K\square$ steeply deviated from all other triangulations. Whereas the deviation between A^*K and that of K(or J_1) and J' is not much upto $n=30$ but deviation between them pick-up between 30 and 50 thereby the deviation of K (or J_1) and J' became steeper than A^*K. Though A^*K is superior of all the triangulations. But in practice K (for J_1) is found to be much convenient to determine the fixed point of given mapping.

Table-2 compares the asymptotic *a.d.d.* for different triangulations and chart-II gives their graphical representation of asymptotic *a.d.d.* based on tabulated value of table-2. Both tabulated values as well as graphical representation again highlight, A^*K as superior among all the triangulations.

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