**D_{RS}-TRIANGULATION BASED ON 2^{N}-RAY ALGORITHM**

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**ABSTRACT**

In order to improve the efficiency of the $2^N$-Ray algorithm; we propose a variant of the $D_{RS}$-Triangulation. A nice property of this triangulation is that it subdivides all the subsets, on which the $2^N$-Ray algorithm works, into simplices according to the $D_{RS}$-Triangulation. Numerical tests shows that $2^N$-Ray algorithm based on $D_{1/2}$-Triangulation is much more efficient.

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1. **Introduction.** Various triangulations have been studied by Dang [1], Dang and Talman [2], Scarf [4], Todd [6,7], Vander and Talman [8], Vertgeim [9]. The $2^n$-ray algorithm was proposed by Wright in [10] to compute solutions of nonlinear equations. The $2^n$-ray algorithm partitions $\mathbb{R}^n$ into $2^n$ cones which have the same vertex. Then a triangulation of $\mathbb{R}^n$ subdivides each cone into simplices. The $2^n$-ray algorithm starts at the vertex and leaves it along an edge of some cone. It follows a sequence of adjacent simplices with varying dimension. Under some mild conditions, the $2^n$-ray algorithm terminates at an $n$-dimensional simplex that yields an approximate solutions to the system of non-linear equations. Since these $2^n$ cones have $2^n$ edges, each of which is a ray, the $2^n$–ray algorithm has $2^n$ possible ray to leave the vertex.

Motivated by this work, we have tried to develop a new $D_{RS}$–triangulation. The $D_{RS}$-triangulation based on the $2^n$ ray algorithm is much more efficient than $2^n$ ray algorithm proposed by Wright [10] as well as $K_1$ triangulation introduced by Kuhn in [3] and $J_1$ triangulation given by Todd [5].

2. **Notations and Definitions.** The following notations have been used in this paper:

- $\mathbb{R}$: Set of real numbers,
- $\mathbb{Z}$: Set of all integers,
- $\mathbb{N}$: Set of positive integers (1,2,...,n),
- $\mathbb{N}^0$: Set of all non-negative integers $\mathbb{N} \cup \{0\}$,
$R^n$: $n$ dimensionally space, having co-ordinates indexed 1 through $n$,
$R^{n+1}$: $n+1$ dimensional space, with coordinates indexed 0 through $n$,
$\pi$: Group of permutation on $(1,2,...,n)$ and $\pi + 1$ group of permutation on $(0,1,2,...,n)$,
$u^i$: $i^{th}$ unit vector in $R^n$, $i \in N$ and $u = \sum_{i \in N} u^i$,
$R^n_+$: Non negative orthant of $R^n$ i.e. $\{x \in R^n; x \geq 0\}$.

Now we consider some standard definitions and explanations which will be used in this paper.

2.1 Standard Simplex. The standard $n$ dimensional closed simplex $S^n$ is the convex hull of $v^0, v^1, ..., v^n$ i.e. $S^n = \{x \in R^{n+1}; x^T x = 1\}$. $s^n_i$ denotes the face of $s^n$ opposite $v^i$ i.e. $S^n_i = \{x \in S^n; x_i = 0\}$ and boundary of $s^n$ is denoted by $\partial S^n = \cup_{i \in N} s^n_i$.

Again a $j$-dimensional simplex or $[j$-simplex] is the relative interior of the convex hull of $j+1$ affinely independent points $y^0, y^1, y^2, ..., y^j$, called its vertices. We write $\sigma = \{y^0, y^1, y^2, ..., y^j\}$. A simplex $\tau$ is a face of $\sigma$ if its vertices are a subset vertices of the $\sigma$. It is convenient to call the closure of a $(j-1)$ dimensional face of the $j$ simplex $\sigma$ as a facet of $\sigma$. Two $j$ simplices are said to be adjacent if they share a common facet.

2.2 Triangulation. A triangulation $G$ of $S^n$ is a collection of $n$ simplices and satisfies the following two conditions:
1. The simplices in $G$ together with all their faces form a partition of $S^n$ and
2. Each point of $S^n$ has a neighbourhood meeting only a finite number of simplices.

2(a) Pivot Rule. For a given simplices $G$ and a vertex $y$ of $\sigma$ the rules for obtaining the simplex of $G$ whose vertices include all vertices of $\sigma$ except $y$, are called the pivot rules of $G$.

2(b) Mesh. The mesh of a triangulation $G$ is $\sup_{\sigma \in G} diam \sigma$. We shall use the Euclidian norm though out this paper.

2.3 Definition. For each sign vector $s \in R^n$, let
$E(s) = \{x \in R^n; s_i x_i = \|x\| \text{ whenever } s_i \neq 0\} = \text{cone } \{t \in R^n; t \text{ is a sign vector, and } s_i \neq 0 \Rightarrow s_i = t_i\}$.

In case $s$ has $k$ non-zero components for $k>0$ than $E(s)$ is a polyhedral cone of dimension $n-k+1$. Also we have $E(0) = R^n$. Moreover, when $s \neq 0$ each $E(s) \cap B^n$. 
is a polyhedral of a cubical subdivisions of $B^n$ where $B^n$ donote the unit ball in $1\infty$ norm.

Wright [10] has also defined another subdivision of $R^n$ into closed convex cone as $n$-dimensional geometric form given as follows:

Let $C(s) = x \in R^n : \begin{cases} x_i = 0 & \text{if } s_i = 0 \\ s_i x_i \geq 0 & \text{if } s_i \neq 0 \end{cases}$

$= \text{cone } \{ s_i u_i : s_i \neq 0, \text{ for each sign vector } s \}$. If $s$ has $k$ non-zero components then $C(s)$ is an orthant of a $k$-dimensional coordinate subspace of $R^n$.

Wright [10] has proposed two type of $T$-triangulations of $R^n$ with the property that $E(t)$ is a subcomplex for every sign vector $t \in R^n$ for $t \neq 0$. The first triangulation is called a $K^1$ triangulation. This triangulation is obtained by taking the triangulation $K_1$ due to Kuhn [3] in the first orthant and the reflecting through coordinate hyperplane to triangulate the other orthant. A vector $v^1$ of a $n$-simplex $<v^1, v^{n+1}>$ of $K^1$ specified by choosing a sign vector $s$ with all the non-zero components, is a member of $C(s)$ when all its components are integrals and $\pi$ is a permutation of $\{1, 2, ..., n\}$, then $v^1 K^1$ is defined recursively as:

$V^{i+1} = V^i + S_{\pi(i)} U^{\pi(i)}$, for $i = 1, 2, ..., n$.

For $n = 2$ triangulation $K^1$ can be illustrated by following diagram.

The second triangulation $J_1$ is defined by Todd [5] and which can be illustrated for $n = 2$ by the following figure:

2.3.1. The $D_{RS}$-Triangulation.

Define $W^n = \{ x \in R^n : x_1 = \max x_i, i = 2, 3, ..., n \}$

taking a vector $Y = (y_1, y_2, ..., y_n)^T$, we have
\[ Y_i = \begin{cases} \left\lfloor x_i \right\rfloor & \text{if } x_i \text{ is even} \\ \left\lfloor x_i \right\rfloor + 1 & \text{otherwise}, \end{cases} \]

where \( \left\lfloor \alpha \right\rfloor \) is the greatest integer less than or equal to \( \alpha \). Let \( D \) be the set of all \( Y \in W^n \) where \( Y_i \) is defined above. If \( Y \in D \), we define
\[ I(y) = \{ i \in N; y_i = y_i \} \quad \text{and} \quad J(y) = \{ j \in N: y_j \geq y_j \}^T. \]

Let \( s = (s_1, s_2, \ldots, s_n)^T \) be a sign vector such that

1. For \( i \in N \), if \( y_i = 0 \) then \( s_i = 1 \), and if \( y_i \neq 0 \) then \( s_i = -1 \).

Let \( K(y, s) = \{ i \in I(y): s_i = 1 \} \).

Let \( \ell \) denote the number of elements in \( I(y) \) and \( h \) the number of elements in \( K(y, s) \), we take integer \( p \) such that

1. when \( h = 0 \), if \( \ell = n \) then \( p = 0 \) or 2,
2. when \( h > 0 \), if \( h = n \) then \( p = 0 \) and if \( h < n \) then \( 0 \leq p \leq n - 1 \).

Let \( \pi = \{ \pi(1), \pi(2), \ldots, \pi(n) \} \) be permutation of \( N \).

When \( h = 0 \), for \( j = 1, 2, \ldots, n \),
1. If \( j = 1 \), define
\[ g_i(j) = \begin{cases} -1 & \text{if } i \in I(y) \\ 0 & \text{otherwise} \end{cases} \quad \ldots (1) \]

for \( i = 1, 2, \ldots, n \).

2. If \( j \neq 1 \), we define
\[ g_i(j) = \begin{cases} S' & \text{if } i = j, \\ 0 & \text{otherwise} \end{cases} \quad \ldots (2) \]

for \( i = 1, 2, \ldots, n \).

When \( h > 0 \), for \( j = 1, 2, \ldots, n \),
1. If \( \pi(j) \in K(y, s) \), define
\[ g_i(\pi(j)) = \begin{cases} 1 & \text{if } i \in K(y, s) \text{ and } j \leq \pi^{-1}(i) \\ 0 & \text{otherwise} \end{cases} \quad \ldots (3) \]

for \( i = 1, 2, \ldots, n \),
2. If $\pi(j) \notin K(y,s)$, define

$$g_i(\pi(j)) = \begin{cases} s_n(j) & \text{if } i = \pi(j) \\ 0 & \text{otherwise} \end{cases}$$

...(4)

for $i=1,2,...,n$.

If $y$, $\pi$, $s$ and $p$ be as above, then vectors $y^0, y^1, ..., y^n$ are defined as follows:

for $p=0$, we have

$$y^0 = y$$

$$y^k = y + g(\pi(k)), k = 1,2,...,n$$

and for $p \geq 1$, we define

$$y^0 = y + s,$$

$$y^k = y^{k-1} - s_{s(k)}U^{(s(k))}, k = 1,2,...,p-1.$$  

$$y^k = y + g(\pi(k)), k = p,...,n.$$  

...(6)

The $y^0, y^1, ..., y^n$ vectors obtained from the above definition are affinely independent.

Thus their convex hull is a simplex. Let us denote this simplex by $D_{RS}(y,\pi,s,p)$ or $\langle y^0, y^1, ..., y^n \rangle$. Let $D_{RS}$ be the set of all such simplices. Then $D_{RS}$ is a triangulation of $W^n$. Note that simplices of the $D_{RS}$-triangulation can be represented in more than one way. Moreover, the triangulation of a whole cube in $W^n$ is the same as the $D_1$-triangulation.

To be more expatiate let us illustrate $D_{RS}$-triangulation of $W^n$ for $n=2$ and for $x \leq 4$. Obviously, we have that for $y_1 \leq 4$.

$$D = \left\{ (0,0,0)^T, (2,2,0)^T, (4,4,0)^T, (4,0,4)^T, (4,4,4)^T \right\}.$$  

...(7)

1. Let $y = (0,0,0)^T$. Then, $I(y) = \{i \in N : y_i = y_i \} = (1,2,3)$ and $i = 3$.

Then $s$ must be $(1,1,1)^T$. Thus

$$K(y,s) = \{i \in I(y) : s_i = 1 \} = (1,2,3)$$  

and $h = 3$. We have $p = 0$.

(a) Let $\pi = (2,3,1)$. Then $\pi^{-1} = (3,1,2)$ and by applying (3)

$$g(\pi(1)) = g(2) = (g_1(2), g_2(2), g_3(2))$$
\( (1,1,1)^T, \)
\[ g(\pi(2)) = g(3) = (1,0,1)^T, \]
\[ g(\pi(3)) = g(3) = (1,0,0)^T. \]

Therefore,
\[ y^0 = y = (0,0,0)^T, \]
\[ y^1 = y + g(\pi(1)) = (1,1,1)^T, \]
\[ y^2 = y + g(\pi(2)) = (1,0,1)^T, \]
\[ y^3 = y + g(\pi(3)) = (1,0,0)^T, \]
Let \( \sigma^1 = \{y^0, y^1, y^2, y^3\}. \)

(b) Let \( \pi = (3,2,1) \). Then \( \pi^{-1} = (3,2,1) \)
\[ g(\pi(1)) = g(3) = (1,1,1)^T, \]
\[ g(\pi(2)) = g(2) = (1,1,0)^T, \]
\[ g(\pi(3)) = g(1) = (1,0,0)^T, \]

Therefore,
\[ y^0 = y = (0,0,0)^T, \]
\[ y^1 = y + g(\pi(1)) = (1,1,1)^T, \]
\[ y^2 = y + g(\pi(2)) = (1,1,0)^T, \]
\[ y^3 = y + g(\pi(3)) = (1,0,0)^T, \]
Let \( \sigma^2 = \{y^0, y^1, y^2, y^3\}. \)

2. Let \( y = (2,2,0)^T \). Since \( I(y) = \{i \in \mathbb{N} : y_i = y_i\} = \{1,2\} \) and \( \ell = 2 \).

So \( s \) must be \((-1,-1,1)^T\). Thus \( k(y,s) = \{i \in I(y) ; s_i = 1\} = \phi \) and \( h = 0 \) while \( p \) can be any one of 0,1,2.

Now by applying (1) and (2), we have
\[ g(1) = (-1,-1,0) \]
\[ g(2) = (0,-1,0) \]
\[ g(3) = (0,0,1). \]
Now considering different values of $p$ and applying (6) and (7), we obtain the following simplices:

(a) For $p=0$. Let $\pi = (1,2,3)$. Therefore,
\[
y^0 = y = (2,2,0)^T,
\]
\[
y^1 = y + g(\pi(1)) = (1,1,0)^T,
\]
\[
y^2 = y + g(\pi(2)) = (2,1,0)^T,
\]
\[
y^3 = y + g(\pi(3)) = (2,2,1)^T. \text{ Let } \sigma^3 = \{y^0, y^1, y^2, y^3\}.\]

(b) For $p=1$. Let $\pi = (1,2,3)$. Therefore
\[
y^0 = y + s = (1,1,1)^T,
\]
\[
y^1 = y^0 - s_{\pi(1)}u^{(1)} = (2,1,1)^T,
\]
\[
y^2 = y + g(\pi(2)) = (2,1,0)^T,
\]
\[
y^3 = y + g(\pi(3)) = (2,2,1)^T. \text{ Let } \sigma^4 = \{y^0, y^1, y^2, y^3\}.\]

(c) For $p=2$. Let $\pi = (1,2,3)$. We have,
\[
y^0 = y + s = (1,1,1)^T,
\]
\[
y^1 = y^0 - s_{\pi(1)}u^{(1)} = (2,1,1)^T,
\]
\[
y^2 = y + g(\pi(2)) = (2,1,0)^T,
\]
\[
y^3 = y + g(\pi(3)) = (2,2,1)^T. \text{ Let } \sigma^5 = \{y^0, y^1, y^2, y^3\}.\]

3. Let $y = (4,4,0)^T$. Therefore, $I(y) = \{i\in\mathbb{N}; y_i = 1\} = \{1,2\}$ and $\ell = 2$.

We have that $s$ must be $(-1,-1,1)^T$. Thus $K(y,s) = \{i\in I(y); s_i = 1\} = \phi$ and $h = 0$.

We have that $p$ can be any one of 0,1,2. We also have
\[
g(1) = (-1,-1,0),
g(2) = (0,-1,0),
g(3) = (0,0,1).
\]

(a) For $p=0$, and $\pi = (1,2,3)$, we have
\[
y^0 = y = (4,4,0)^T,
\]
\[
y^1 = y + g(\pi(1)) = (3,3,0)^T,
\]
\[
y^2 = y + g(\pi(2)) = (4,3,0)^T,
\]
\[
y^3 = y + g(\pi(3)) = (4,4,1)^T. \text{ Let } \sigma^6 = \{y^0, y^1, y^2, y^3\}.\]
(b) For $p=1$, let $\pi = (1,2,3)$. Therefore
\[ y^0 = y + s = (3,3,1)^T, \]
\[ y^1 = y + g(\pi(1)) = (3,3,0)^T, \]
\[ y^2 = y + g(\pi(2)) = (4,3,0)^T, \]
\[ y^3 = y + g(\pi(3)) = (4,4,1)^T. \] Let $\sigma^7 = \langle y^0, y^1, y^2, y^3 \rangle$.

(c) For $p=2$ and $\pi = (1,2,3)$, we have
\[ y^0 = y + s = (3,3,1)^T, \]
\[ y^1 = y^0 - s_{\pi(1)} u_{\pi(1)} = (4,4,1)^T, \]
\[ y^2 = y + g(\pi(2)) = (4,3,0)^T, \]
\[ y^3 = y + g(\pi(3)) = (4,4,1)^T. \] Let $\sigma^8 = \langle y^0, y^1, y^2, y^3 \rangle$.

4. Let $y = (4,0,4)^T$. Since $I(\pi) = \{ i \in N : y_i = y_i \} = \{1,3\}$ and $\ell = 2$, so that $s$ must be $(-1,-1,1)^T$. Thus $K(y,s) = \{ i \in I(y) : s_i = 1 \} = \emptyset$ and $h=0$. we have that $p$ can be any one of $0,1,2$. We also have
\[ g(1) = (-1,0,-1), \]
\[ g(2) = (0,1,0), \]
\[ g(3) = (0,0,-1). \]

(a) For $p=0$, and $\pi = (1,2,3)$, we have
\[ y^0 = y = (4,0,4)^T, \]
\[ y^1 = y + g(\pi(1)) = (3,0,3)^T, \]
\[ y^2 = y + g(\pi(2)) = (4,1,4)^T, \]
\[ y^3 = y + g(\pi(3)) = (4,0,3)^T. \] Let $\sigma^9 = \langle y^0, y^1, y^2, y^3 \rangle$

(b) For $p=1$. Let $\pi = (1,2,3)$. Therefore
\[ y^0 = y + s = (3,1,3)^T, \]
\[ y^1 = y + g(\pi(1)) = (3,0,3)^T, \]
\[ y^2 = y + g(\pi(2)) = (4,1,4)^T, \]
\[ y^3 = y + g(\pi(3)) = (4,0,3)^T. \] Let $\sigma^{10} = \langle y^0, y^1, y^2, y^3 \rangle$

(c) For $p=2$ and $\pi = (1,2,3)$, we have
\[ y^0 = y + s = (3,1,3)^T, \]
\[ y^1 = y^0 - s_{\pi(1)}u^{\pi(1)} = (4,1,3)^T, \]
\[ y^2 = y + g(\pi(2)) = (4,1,4)^T, \]
\[ y^3 = y + g(\pi(3)) = (4,0,3)^T. \]

Let \( \sigma^{11} = \{y^0, y^1, y^2, y^3\} \)

5. Let \( y = (4,4,4)^T \). Since \( I(y) = \{i \in N; y_i = y \} = \{1,2,3\} \) and \( \ell = 3 \). We have that \( a \) must be \((-1,-1,-1)^T\). Thus \( K(y,s) = \phi \) and \( h = 0 \). we have that \( p \) can be any one of 0,1,2. We also have

\[ g(1) = (-1,-1,-1), \]
\[ g(2) = (0,-1,0), \]
\[ g(3) = (0,0,-1). \]

(a) For \( p=0 \). Let \( \pi = (1,2,3) \). Therefore

\[ y^0 = y = (4,4,4)^T, \]
\[ y^1 = y + g(\pi(1)) = (3,3,3)^T, \]
\[ y^2 = y + g(\pi(2)) = (4,3,4)^T, \]
\[ y^3 = y + g(\pi(3)) = (4,4,3)^T. \]

Let \( \sigma^{12} = \{y^0, y^1, y^2, y^3\} \).

(b) For \( p=1 \). Let \( \pi = (1,2,3) \). Therefore

\[ y^0 = y + s = (3,3,3)^T, \]
\[ y^1 = y + g(\pi(1)) = (3,3,3)^T, \]
\[ y^2 = y + g(\pi(2)) = (4,3,4)^T, \]
\[ y^3 = y + g(\pi(3)) = (4,4,3)^T. \]

Let \( \sigma^{13} = \{y^0, y^1, y^2, y^3\} \).

(c) For \( p=2 \). Let \( \pi = (1,2,3) \). We have,

\[ y^0 = y + s = (3,3,3)^T, \]
\[ y^1 = y^0 - s_{\pi(1)}u^{\pi(1)} = (4,3,3)^T, \]
\[ y^2 = y + g(\pi(2)) = (4,3,4)^T, \]
\[ y^3 = y + g(\pi(3)) = (4,4,3)^T. \]

Let \( \sigma^{14} = \{y^0, y^1, y^2, y^3\} \).

It can be seen that \( \{\sigma^i : i = 1,2,\ldots,14\} \) form a triangulation of \( W^3 \) for \( x \leq 4 \) as shown by the following figure:
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