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FLUCTUATING FLOW OF A VISCOELASTIC FLUID PAST AN INFINITE PLANE, POROUS WALL WITH CONSTANT SUCTION

by

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ABSTRACT

Fluctuating flow of a viscoelastic fluid, based on Rivlin-Ericksen constitutive equation, past an infinite plane, porous wall with constant suction is studied. The skin friction fluctuations have a phase-lag over the main stream velocity fluctuations for certain values of the viscoelastic parameter, while they have always the phase lead in the case of classical viscous fluid. The amplitude of the skin friction rises with the frequency. The unperturbed displacement thickness is found to differ in the viscoelastic case from that of the classical viscous case. Transient velocity profiles are drawn.

Introduction. Real Fluids (High polymer solutions, colloidal suspensions, liquid lubricants, pastes, paper pulp, condensed milk, and paint) possess both elastic and viscuous characteristics. Phenomena such as normal stress effects and variable viscosity are called non-Newtonian when they cannot be described by the classical relation (eq. (1) of Siddappa 1970). Lightbill (1954) investigated the effect of fluctuations of the external stream on the boundary layer of a two dimensional body in a viscous fluid. Stuart (1955) solved the Navier-stokes equations exactly for fluctuating flow of a viscous fluid past an infinite plane porous wall with constant suction. Since real fluids are viscoelastic, and since suction or injection can reduce drag by delaying separation and transitition from laminar to

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turbulent motions in the boundary layer, Stuart's work is here extended to the flow of viscoelastic fluids.

2. Formulation of the problem. Let x denote the distance along the two dimensional infinite, plane, porous wall, the equation to the wall being $y=0$, y the distance normal to it, u and v corresponding components of velocity, t the time, p the pressure, ρ the density, α the coefficient of kinematic viscosity, β the coefficient of kinematic viscoelasticity and γ the coefficient of kinematic crossviscosity. We consider an incompressible Rivlin-Ericksen fluid flow past an infinite, plane, porous wall.

For an incompressible Rivlin-Ericksen fluid flow past an infinite, plane, porous wall which is independent of x then the equation of motion and the continuity are

$$u_t+vu_y=-\frac{1}{\rho}p_x+\alpha u_{yy}+\beta(u_t+vu_y)_{yy} \tag{2.1}$$

$$v_t+vv_y=-\frac{1}{\rho}p_y+\alpha v_{yy}+2\beta[(v_t+vv_y)_{yy}+(u^2_y+v^2_y)_y]+\gamma(4v^2_y+u^2_y)_y \tag{2.2}$$

and

$$\frac{\partial v}{\partial y}=0. \tag{2.3}$$

The continuity equation (2.3) shows that v is a function of time only. In order to obtain a steady solution of the boundary-layer type it is known that v must be a negative, non-zero constant (v_w). In the unsteady case also we shall make this restriction (Stuart). Under this restriction, the equations of motion become

$$\frac{\partial u}{\partial t}+v_w\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\alpha\frac{\partial^2 u}{\partial y^2}+\beta\frac{\partial^2}{\partial y^2}\left(\frac{\partial u}{\partial t}+v_w\frac{\partial u}{\partial y}\right) \tag{2.4}$$

$$0=-\frac{1}{\rho}\frac{\partial p}{\partial y}+2(2\beta+\gamma)\left(\frac{\partial u}{\partial y}\right)\left(\frac{\partial^2 u}{\partial y^2}\right) \tag{2.5}$$

By integrating (2.5) we have

$$p(x, y, t)=\rho(2\beta+\gamma)\left(\frac{\partial u}{\partial y}\right)^2+g(x, t) \tag{2.6}$$

where $g(x, t)$ is a function added as a result of integration.

Further by considering (2.4) as $y\to\infty$, $u\to U(t)$ and the partial derivatives of u with respect to y tend to zero, we find that

$$\frac{du}{dt}=-\frac{1}{\rho}\frac{\partial p}{\partial x}=-\frac{1}{\rho}\frac{\partial g}{\partial x} \tag{2.7}$$

and so by integrating we have

$$g(x, t) = -\rho x \frac{dU}{dt} + h(t) \tag{2.8}$$

where $h(t)$ is at most a function of t.

Thus the pressure is given by

$$p(x, y, t) = \rho(2\beta+\gamma)\left(\frac{\partial u}{\partial y}\right)^2 - \rho x \frac{dU}{dt} + h(t) \tag{2.9}$$

It is clear from (2.9) that the pressure of the fluid depends on y in addition to x and t, unlike in the classical viscuous case. This is because of the viscoelastic nature of the fluid. The equation of our interest is

$$\frac{\partial u}{\partial t} + v_w \frac{\partial u}{\partial y} = \frac{dU}{dt} + \alpha \frac{\partial^2 u}{\partial y^2} + \beta \frac{\partial^2}{\partial y^2}\left(\frac{\partial u}{\partial t} + v_w \frac{\partial u}{\partial y}\right). \tag{2.10}$$

3. **Boundary Conditions.** We regard the viscoelastic flow as a limit of a flow through a pipe for which the boundary conditions are

(i) $$u(0, t) = 0$$

(ii) $$u(\infty, t) = U(t)$$

where $U(t)$ is the velocity at large distances from the wall,

(iii) $$\left(\frac{\partial u}{\partial y}\right)_{y=\infty} = 0.$$

The condition (iii) arises because of the symmetry about the axis of the pipe through the point at infinity.

4. **Solution of the problem.** We look for a periodic velocity of the form

$$\begin{aligned} U(t) &= U_0(1+\varepsilon e^{iwt}) \\ U(y, t) &= U_0[\varphi_0(y) + \varepsilon \varphi_1(y)\, e^{iwt}] \\ v &= v_w \end{aligned} \tag{4.1}$$

in which ε may be taken as positive by suitable choice of the origin of time. Substituting (4.1) in (2.10) and equating non-periodic and harmonic terms separately to zero, we have

$$v_w \frac{d\varphi_0}{dy} = \alpha \frac{d^2\varphi_0}{dy^2} + \beta v_w \frac{d^3\varphi_0}{dy^3} \tag{4.2}$$

and

$$iw\phi_1 + v_w \frac{d\varphi_1}{dy} = iw + \alpha \frac{d^2\varphi_1}{dy^2} + \beta iw \frac{d^2\varphi_1}{dy^2} + \beta v_w \frac{d^3\varphi_1}{dy^3}. \tag{4.3}$$

(4.1) gives the solution of the equation (2.10) whatever the magnitude of ε provided φ_0 and φ_1 satisfy (4.2) and (4.3). Thus the velocity field consists of a fluctuating component of flow parallel to the wall, with a constant component of velocity normal to the wall. The solution is physically real only if v_w is negative which corresponds to a suction through the surface.

With $\eta=\frac{|v_w|y}{\alpha^2}$ (4.2) and (4.3) become

$$k\frac{d^3\varphi_0}{d\eta^3}+\frac{d\varphi_0}{d\eta^2}+\frac{d\phi^0}{d\eta}=0 \tag{4.4}$$

$$k\frac{d^3\varphi_1}{d\eta^3}+(1-\bar{i}\lambda k)\frac{d^2\varphi_1}{d\eta^2}+\frac{d\varphi_1}{d\eta}-\bar{i}\lambda\varphi_1=-\bar{i}\lambda \tag{4.5}$$

where

$$\lambda=\frac{\omega\alpha}{|v_w|^2} \text{ and } k=-\frac{\beta|v_w|^2}{\alpha^2} \tag{4.6}$$

The boundary conditions are that

$$\varphi_0(0)=0,\ \varphi_0(\infty)=1,\ \frac{d\varphi_0(\infty)}{d\eta}=0 \tag{4.7}$$

and

$$\varphi_1(0)=0,\ \varphi_1(\infty)=1,\ \frac{d\varphi_1(\infty)}{d\eta}=0 \tag{4.8}$$

One solution of (4.4) subject to (4.7) is

$$\varphi_0(\eta)=1-exp\left(\frac{-1+\sqrt{1-4k}}{2k}\right)\eta,\ k\leqslant 0 \tag{4.9}$$

we note that when $k=0$ this becomes the asymptotic suction profile in the Newtonian fluid (Schlichting 1955)

$$\varphi_0(\eta)=1-e^{-\eta} \tag{4.10}$$

The unperturbed displacement thickness for $k<0$ is given by

$$\overset{*}{\delta}_0=\left(\frac{1+\sqrt{1-4k}}{2}\right)\frac{\alpha}{|v_w|} \tag{4.11}$$

$$\text{Therefore}\quad \lambda=\frac{4w\overset{*}{\delta}{}_0^2}{\alpha(1+\sqrt{1-4k})^2} \tag{4.12}$$

The other solution of (4.4) subject to (4.7) is

$$\varphi_0(\eta)=1-exp\left(\frac{-1-\sqrt{1-4k}}{2k}\right)\eta,\ 0<k\leqslant\tfrac{1}{4}. \tag{4.13}$$

The unperturbed displacement thickness is given by

$$\delta_0^{**}=\left(\frac{1-\sqrt{1-4k}}{2}\right)\frac{\alpha}{|v_w|},\ 0<k\leqslant\tfrac{1}{4}. \tag{4.14}$$

(4.9) is plotted against η for various values of k in Fig. 1 It is clear from the figure that the velocity profile for zero fluctuation is flattened and less steeper than the asymptotic suction profile (4.10). This is due to the viscoelastic nature of the fluid. It is also clear from (4.11) that the unperturbed displacement thickness increases from its classical case for $k\leqslant 0$. (4.13) is also plotted in fig. 2. This again illustrates the flattening effect. It is clear from (4.14) that the unperturbed displacement thickness decreases for $0<k\leqslant\frac{1}{4}$.

We see from the fig. 1 and fig. 2 that the velocity profiles under different conditions (ex $k=-50$, $k=+0{\cdot}07$) are almost identical. This is an interesting result.

The solution of (4.5) subject to (4.8) is

$$\varphi_1(\eta)=1-e^{-h\eta} \tag{4.15}$$

where $h=h_r+ih_i$ is given by the equation

$$kh^3-(1-i\lambda k)h^2+h+i\lambda=0. \tag{4.16}$$

We have solved this equation numerically. The tabulation of h for various values of k and λ is given in Table I.

The total velocity component parallel to the wall is given by

$$\frac{u}{U_0}=1-exp\left(\frac{-1+\sqrt{1-4k}}{2k}\right)\eta+\varepsilon(1-e^{-h\eta})\,e^{iwt} \tag{4.17}$$

The shear stress at the wall is given by

$$\frac{tw}{\frac{1}{2}\rho U_0|V_w|}=\frac{1-\sqrt{1-4k}}{2k}+\varepsilon\,|h|e^{i(wt+\theta)} \tag{4.18}$$

where

$$h=h_r+ih_i \tag{4.19}$$

and

$$\theta=\tan^{-1}\left(\frac{h_i}{h_r}\right) \tag{4.20}$$

Table I illustrates the trend of (h) and tan θ. We see from (4.18) and Table I that the skin friction has phase lead or phase lag according as θ is positive or negative over the velocity fluctuation at large distances from the plane. In the classical viscous case there is always a phase lead but no lag. The magnitude of the skin friction rises with λ because of the faction h. The equation (4.17) gives the transient velocity distribution, while an indication of its shape can be obtained from the skin friction formula.

Table I

k	λ	h_r	h_i	$\|h\|$	$\tan\theta$
0	0	1	0	1	0
0	·25	1·0493	·22754	1·07369	·21685
0	25	4·0533	3·5199	5·3670	·86791
−1	·25	1·4498	−·65779	1·592	−·45373
−1	25	1·9980	−50·029	50·069	−25·0398
−10	25	1·9999	158·12	158·13	79·604
−10	100	2·00004	632·462	632·465	316·225

provided $\varepsilon|h|\geqslant 1$, the shear stress i_w, is zero when

$$\cos(wt+\theta)=-\frac{(4k-1+\sqrt{1+4k})/2k}{\varepsilon|H|}$$

and is negative when

$$\cos(wt+\theta)<\frac{(4k-1+\sqrt{1-4k})/2k}{\varepsilon|h|}$$

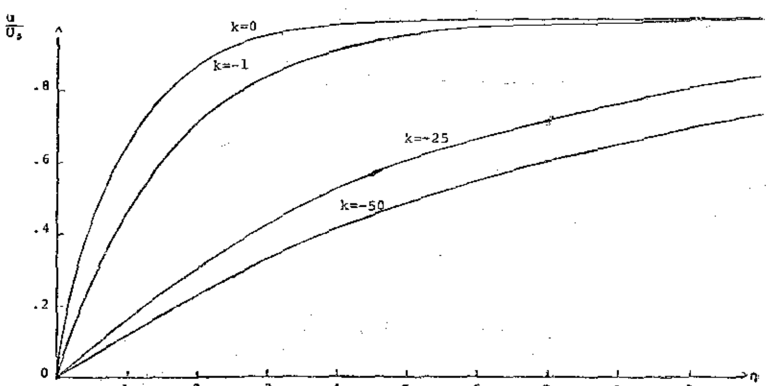

Fig. 4.1 Mean Velocity Profiles for $k \leq 0$

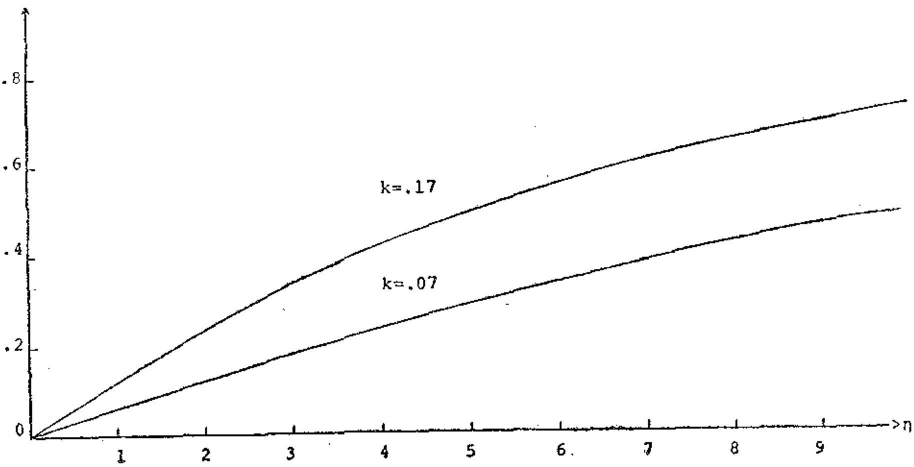

Fig. 2. Mean Velocity Profiles for $0<k\leqslant\frac{1}{4}$

The former case corresponds to a velocity profile with zero skin friction, while in the latter case the flow near the surface is in the negative x-direction. Since ε is not restricted in magnitude, the condition $\in|h|\geqslant 1$ can be obtained. Thus at certain times the flow near the surface is in the negative x-direction, while the flow in the main stream is always in the positive x-direction. Thus certain members of the class of transient velocity profiles are of a separation type (with zero skin friction) or separated type (with reverse flow near the surface) for some values of k as in the classical viscous fluid flow. The transient velocity profiles are illustrated in fig. 3 which illustrates also the flattening effect in the case of viscoelastic fluid.

For low frequencies (4.15) can be expanded in powers of λ to give

$$(4.21)\qquad \varphi_1=1-e^{-a\eta}-\frac{ia\eta\lambda e^{-a\eta}}{a-2}-\lambda^2\left\{\frac{(4-3a)\eta}{(a-2)^3}-\frac{\frac{1}{2}\,a^2\eta^2}{(a-2)^2}\right\}e^{-a\eta}+0(\lambda^3)$$

where $a=\dfrac{1-\sqrt{1-4k}}{2k}$ and $k\leqslant 0$

By putting $k=0$ i.e., $a=1$ for the classical viscous fluid we deduce

$$(4.22)\qquad \varphi_1=1-e^{-\eta}+i\eta\lambda e^{-\eta}+(\eta+\tfrac{1}{2}\eta^2)\,\lambda^2 e^{-\eta}+0(\lambda^3)$$

which agrees with Stuart's equation (3.16) of his paper.

The component of velocity parallel to the wall for low frequency is given by

$$(4.23)\qquad \frac{u}{U_0}=1-e^{-a\eta}+\varepsilon\left[1-e^{-a\eta}-\frac{ia\lambda\eta e^{-a\eta}}{a-2}\right.$$
$$\left.-\left\{\lambda^2\,\frac{(4-3a)}{(a-2)^3}\,\eta-\tfrac{1}{2}\,\frac{a^2\eta^2}{(a-2)^2}\right\}e^{-a\eta}+0(\lambda^3)\right]e^{i\omega t}$$

Thus the velocity fluctuation is, to first approximation in λ, the sum of the parts proportional to the instantaneous velocity and acceleration of the main stream.

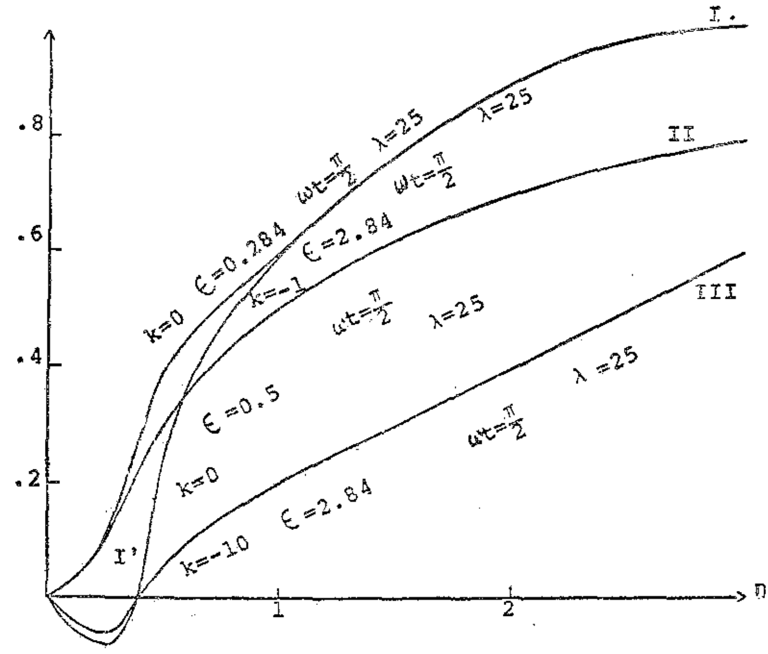

Fig. 3. Transient Velocity Profiles for k<0

I $k=0$	I′ $k=0$
II $k=-1$	III $k=-10$

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REFERENCES

[1] *Eringen, A.C.*, 1962, Nonlinear Theory of Continuous Media, McGraw-Hill, New York.

[2] *Goldstein, S.*, Modern Developments in Fluid Dynamics, Oxford, 1938.

[3] *Lighthill, M.J.*, Proc. Roy. Soc. **A 224** (1954).

[4] *Schlichting, H.*, Boundary Layer Theory, 2nd edition (translated by J. Kestin) New York, McGraw-Hill, 1960.

[5] *Siddappa, B. J. Miss.*, Acad. Sci. **15** (1970), 17-23.

[6] *Sidappa, B.* (1971), Ph.D. Thesis, University of Dublin, Ireland.

[7] *Stuart, J. T.*, Proc. Roy. Soc. **A 231** (1955), 116.